Landau damping in dynamical Lorentz gases

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Abstract

We analyse the Landau damping mechanism for variants of Vlasov equations, with a time dependent linear force term and a self-consistent potential that involves an additional memory effect. This question is directly motivated by a model describing the interaction of particles with their environment, through momentum and energy exchanges with a vibrating field. We establish the stability of homogeneous states. We bring out how the coupling influences the stability criterion, in comparison to the standard Vlasov case.

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Contents

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1 Introduction

In this work, we go back to the analysis of Landau damping mechanisms in kinetic equations. This effect has been brought out for the Vlasov equation of plasma physics in the pioneering work of L. Landau [23], and extended to gravitational models in astrophysics [25, 26], where it is thought to play a key role in the stability of galaxies. It can be interpreted as a stability statement about steady solutions, leading to a decay of the self-consistent force. A complete mathematical analysis of the Landau damping for non linear Vlasov equations has been performed in [27], and revisited later on in [6, 7] (see also [21]). Similar behaviors have been revealed for the 2D Euler system [5]. The phenomena are surprising since they describe damping mechanisms, counter-intuitive for *reversible* equations which apparently do not present any dissipative process.

The starting point of this contribution comes from an original model introduced by L. Bruneau and S. De Bièvre [8] describing the motion of a *single* classical particle interacting with its environment. The particle is described by its position $t \mapsto q(t) \in \mathbb{R}^d$, while the behavior of the environment is embodied into a scalar field $(t, x, z) \in (0, \infty)\times\mathbb{R}^d\times\mathbb{R}^n \mapsto \psi(t, x, z)$. The dynamic is modeled by the following set of differential equations

$$\begin{cases} \ddot{q}(t) = -\nabla V(q(t)) - \displaystyle\iint_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_1(q(t)-y)\ \sigma_2(z)\ \nabla_x\Psi(t,y,z)\,\mathrm{d}y\,\mathrm{d}z, \\ \partial^2_{tt}\Psi(t,x,z) - c^2\Delta_z\Psi(t,x,z) = -\sigma_2(z)\sigma_1(x-q(t)), \qquad x\in\mathbb{R}^d,\ z\in\mathbb{R}^n. \end{cases} \tag{1}$$

It corresponds to the intuition of a particle moving through an infinite set of n-dimensional elastic membranes, one for each position $x \in \mathbb{R}^d$. The physical properties of the membranes are characterized by the wave speed $c > 0$. The coupling between the particles and the environment is governed by two form functions σ_1, σ_2, which are both non negative, smooth and radially symmetric functions; they can be seen as determining the influence domain of the particle in each direction, the direction of particle's motion and the direction of wave propagation, respectively. It is therefore relevant to assume both form functions have a compact support. The particle exchanges its kinetic energy with the vibrations of the membranes. These mechanisms eventually act like a friction force since particle's energy is evacuated in the membranes, and, depending on the shape of the external potential $x \mapsto V(x)$, they determine the large time behavior of the particle. We refer the reader to [1, 11, 12, 13, 22, 29] for further studies of the system (1), that include numerical experiments and interpretation by means of random walks.

The system (1) can be generalized by considering a set of N particles going through the membranes. The mean field regime $N \to \infty$ leads to the following PDE system

$$\partial_t F + v \cdot \nabla_x F - \nabla_x (V + \Phi[\Psi]) \cdot \nabla_v F = 0, \qquad t \geq 0,\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d, \tag{2a}$$

$$(\partial_{tt}^2 \Psi - c^2 \Delta_z \Psi)(t, x, z) = -\sigma_2(z) \int_{\mathbb{R}^d} \sigma_1(x - y) \rho(t, y) \, \mathrm{d}y,\ t \geq 0,\ x \in \mathbb{R}^d,\ z \in \mathbb{R}^n, \tag{2b}$$

$$\rho(t, x) = \int_{\mathbb{R}^d} F(t, x, v) \, \mathrm{d}v, \tag{2c}$$

$$\Phi[\Psi](t, x) = \iint_{\mathbb{R}^d \times \mathbb{R}^n} \sigma_1(x - y) \sigma_2(z) \Psi(t, y, z) \, \mathrm{d}z \, \mathrm{d}y, \qquad t \geq 0,\ x \in \mathbb{R}^d, \tag{2d}$$

where now $(t, x, v) \mapsto F(t, x, v)$ is interpreted as the particles distribution function in phase space, $x \in \mathbb{R}^d$ being the position variable, and $v \in \mathbb{R}^d$ the velocity variable. The system (2a)–(2d) is completed by initial conditions

$$F\big|_{t=0} = F_0, \qquad (\Psi, \partial_t \Psi)\big|_{t=0} = (\Psi_0, \Psi_1). \tag{3}$$

We refer the reader to [17, 31] for the derivation of the N-particles system and the analysis of the mean field regime that leads to (2a)–(2d). The existence of solutions of (2a)–(2d) is investigated in [9]. Furthermore, asymptotic issues are also discussed that reveal an unexpected connection with the *gravitational* Vlasov-Poisson equation. This relation with another model of statistical physics can guide the intuition to analyze further mathematical properties of (2a)–(2d). In this spirit, the existence of equilibrium states and their stability is discussed in [2], adding in the kinetic model a dissipative effect with the Fokker–Planck operator, and in [10] where a variational approach is adopted for the collisionless model, following [19, 20, 34].

We wish to continue this analysis, adopting a different viewpoint. In [2, 10] the effect of a confining potential $x \mapsto V(x)$ is considered, which governs the shape of the equilibrium states. Here, we change the geometry of the problem, replacing the confining assumption on the external potential, by the assumption that particles' motion holds in the $d-$dimensional torus $\mathbb{T}^d$. In such a framework, like for the usual Vlasov-Poisson system, we can find space–homogeneous stationary solutions, and we wish to investigate their stability. This question is directly reminiscient to the well-known phenomena of damping brought out in plasma physics by L. Landau [23]: for the electrostatic Vlasov-Poisson system, it can be shown that the electric field of the linearized system decays exponentially fast. For gravitational interactions a similar discussion dates back to D. Lynden–Bell [25, 26]. In fact, Landau's analysis [23] was concerned with the linearized equation only. Of course the linearization procedure is questionable and the non linear dynamics might significantly depart form the linear behavior, as pointed out in [3]. A stunning analysis of the non linear problem in the analytic framework has been recently performed by C. Mouhot & C. Villani [27, 32]. A simplified analysis of the Landau damping has been proposed in [6]; we also refer the reader to [15] for results based on Sobolev regularity (with a definition of the force which involves only a finite number of Fourier modes, though). The Landau damping around homogeneous solutions has also been investigated in the whole space $\mathbb{R}^d$ [7], thus dealing with a set of particles having an infinite mass. See also [21] for an alternative approach that uses integration along phase-space characteristics. We wish to address these issues for the system (2a)–(2d), still when $V = 0$. The analysis of the non-linear equations is quite involved; it requires a complex functional framework and fine estimates in order to control the non linear effects, the so–called "plasma echoes", that can break the damping mechanisms observed on the linearized model. By the way, it has been recently shown that insufficient regularity of the perturbation can annihilate the damping mechanisms, and the proof (which, though, is very specific to the coupling with the Poisson equation; it is not clear that the argument applies for more regular convolution kernels) precisely uses the role of the plasma echoes against damping [4]. Nevertheless it turns out that identifying stability conditions for the linearized problem plays a central role in the analysis of the non linear stability, see [27, Condition **(L)**]. Beyond their interest for the specific model (2a)–(2d) of particles interacting with their environment, the results we are going to discuss can be thought of with some generality. Indeed, as we shall detail below, the equation for the particle distribution function can be recast as follows

$$\partial_t F + v \cdot \nabla_x F - \nabla_x \Phi_I \cdot \nabla_v F - \nabla_x \Phi_S \cdot \nabla_v F = 0,$$

where the potential splits into two parts, that both induce new issues compared to the case of the "standard" Vlasov system (hereafter simply refered to as the "Vlasov equation"):

- $\Phi_I(t, x)$ does not depend on F: this is a *linear* contribution in the equation. The damping then relies on suitable time-decay properties, here related to the dispersion properties of the free wave equation.
- the self-consistent potential $\Phi_S(t, x)$ is defined by a convolution with respect to space, combined with a half-convolution with respect to time

$$\Phi_S(t, x) = -\int_0^t \int \Sigma(x - y) p_c(t - s) \rho(s, y) \, \mathrm{d}y \, \mathrm{d}s.$$

> Then the Landau damping relies on properties of the kernel Σ, which is quite similar to the analysis of the Vlasov case, but also on decay properties of the kernel p_c.

The discussion is organized as follows. We start by checking that we can find homogeneous solutions in Section 2. We also introduce different, but complementary, ways to think of the equations and we make a series of comments explaining how the problem differs from the usual Vlasov system. We complete this preliminary section by paying a specific attention to the properties of the kernel p_c, depending on the dimension n, which play a crucial role in the analysis. In Section 3, which is the heart of this work, we turn to the linearized problem. The analysis of the linearized equation reduces to study a certain integral equation, satisfied by the Fourier coefficients of the macroscopic density. That the damping occurs relies on a stability criterion on the kernel of this Volterra equation, which, at least, can be verified when c, the speed of wave propagation, is large enough. Next, we briefly explain the method for proving the non linear Landau damping for the free space problem, for which the functional framework is less intricate, in Section 4.1. We present how the main arguments should be adapted for the torus in Section 4.2. We further discuss the stability criterion in Section 5, in the spirit of the Penrose criterion. Quite surprisingly, we are led to an intricate expression, much more complicated than for the Vlasov model. Nevertheless, these expressions allows us to establish some conclusions close to what is known on the gravitational Vlasov case. We also propose several interpretations of criteria that lead to (un)stable solutions. The interested reader will find fully detailed arguments in [33], and numerical illustrations in [18].

2 Preliminaries

In what follows, $\mathbb{X}^d$ stands indifferently for $\mathbb{T}^d$ or $\mathbb{R}^d$, and for given functions $\phi : x \in \mathbb{X}^d \mapsto \phi(x)$ and $g : v \in \mathbb{R}^d \mapsto g(v)$, we denote

$$\langle \varphi \rangle_{\mathbb{X}^d} = \int_{\mathbb{X}^d} \varphi(x)\,\mathrm{d}x, \qquad \langle g \rangle_{\mathbb{R}^d} = \int_{\mathbb{R}^d} g(v)\,\mathrm{d}v,$$

where $\mathrm{d}x$ is either the usual Lebesgue measure on $\mathbb{X}^d = \mathbb{R}^d$ or the normalized Lebesgue measure on $\mathbb{X}^d = \mathbb{T}^d$. We shall also use indifferently the notation $\widehat{\cdot}$ for the Fourier coefficients of a $\mathbb{T}^d-$periodic function

$$\varphi : \mathbb{T}^d \to \mathbb{R}, \qquad \widehat{\varphi}(k) = \int_{\mathbb{T}^d} e^{-ik\cdot x}\varphi(x)\,\mathrm{d}x \text{ for } k \in \mathbb{Z}^d,$$

or the Fourier transform over $\mathbb{R}^m$ (with $m = d$ or $m = n$)

$$\varphi : \mathbb{R}^m \to \mathbb{R}, \qquad \widehat{\varphi}(\xi) = \int_{\mathbb{R}^m} e^{-ix\cdot\xi}\varphi(x)\,\mathrm{d}x \text{ for } \xi \in \mathbb{R}^m.$$

We equally use the same notation for a function ϕ depending on $x \in \mathbb{X}^d$ and $v \in \mathbb{R}^d$

$$\widehat{\varphi}(k,\xi) = \iint_{\mathbb{X}^d\times\mathbb{R}^m} e^{-ik\cdot x}e^{-i\xi\cdot v}\varphi(x,v)\,\mathrm{d}v\ \ \mathrm{d}x,$$

for $\xi \in \mathbb{R}^m$ and either $k \in \mathbb{Z}^d$ (case $\mathbb{X}^d = \mathbb{T}^d$) or $k \in \mathbb{R}^d$ (case $\mathbb{X}^d = \mathbb{R}^d$). In the sequel, we shall use the shorthand notation $k \in \mathbb{X}^{\star d}$ to encompass these two situations. Throughout the paper, we shall use the notations

$$\langle x \rangle = \sqrt{1+x^2}$$

and, given a real number s, s^+ means $s+\epsilon$ for $\epsilon > 0$ arbitrarily small. We write $A \lesssim B$ when we can find a constant $C > 0$ such that $A \leq CB$. Here, A, B are in general functions of time, space, velocity, or their associated Fourier variables; it is thus understood that C is uniform over these variables. In certain circumstances, we write $A \lesssim_r B$ to emphasize the fact that the constant C depends on the parameter r.

2.1 Rewriting the equations

Due to the linearity of the wave equation, the solution of (2b) can be split into a contribution that depends only on the initial condition (Ψ_0, Ψ_1) and a contribution that depends only on ρ, see [9, Eq. (6)–(8)]. Accordingly, we split the potential into

$$\Phi = \Phi_I + \Phi_S,$$

where Φ_I depends only on (Ψ_0, Ψ_1) as follows

$$\Phi_I(t,x) = \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^n \times \mathbb{X}^d} \sigma_1(x-y) \left(\widehat{\Psi_0}(y,\zeta) \cos(c|\zeta|t) + \widehat{\Psi_1}(y,\zeta) \frac{\sin(c|\zeta|t)}{c|\zeta|} \right) \widehat{\sigma_2}(\zeta) \, \mathrm{d}y \, \mathrm{d}\zeta \tag{4}$$

and the coupling term reads

$$\begin{aligned} \Phi_S(t,x) &= -\int_0^t p_c(t-s) \Sigma \star \rho(s,x) \, \mathrm{d}s, \\ \Sigma &= \sigma_1 \star \sigma_1, \\ p_c(t) &= \int_{\mathbb{R}^n} \frac{\sin(c|\zeta|t)}{c|\zeta|} \; |\widehat{\sigma}_2(\zeta)|^2 \frac{\mathrm{d}\zeta}{(2\pi)^n}. \end{aligned} \tag{5}$$

The properties of the function $t \mapsto p_c(t)$, collected in Lemma 2.3 below, play a crucial role in the asymptotic analysis of (2a)–(2d).

2.2 Homogeneous solutions

Let $\rho_0 > 0$ and let $v \mapsto M(v)$ be a given function such that $\int_{\mathbb{R}^d} M(v) \, \mathrm{d}v = 1$. We claim that

$$\mathscr{M} : (x,v) \in \mathbb{X}^d \times \mathbb{R}^d \longmapsto \mathscr{M}(x,v) = \rho_0 M(v)$$

is a stationary solution of (2a)–(2d), associated to a spatially homogeneous potential Φ, when starting from spatially homogeneous data for the wave equation. On the torus, since M and $\mathrm{d}x$ are normalized, ρ_0 is the mass of the solution $\mathscr{M}$. With $F = \mathscr{M}$, the right hand side of the wave equation (2b) becomes

$$-\sigma_2(z) \iint_{\mathbb{X}^d \times \mathbb{R}^d} \sigma_1(x-y) \mathscr{M}(y,v) \, \mathrm{d}v \; \mathrm{d}y = -\sigma_2(z) \; \langle \sigma_1 \rangle_{\mathbb{X}^d} \langle \mathscr{M} \rangle_{\mathbb{R}^d},$$

which depends only on the variable $z \in \mathbb{R}^n$. Therefore, considering space-homogeneous initial data $(x, z) \mapsto (\Psi_0^H(z), \Psi_1^H(z))$, the solution of the wave equation

$$\partial_{tt}^2 \Psi^H - c^2 \Delta_z \Psi^H = -\sigma_2(z) \langle \sigma_1 \rangle_{\mathbb{X}^d} \langle \mathscr{M} \rangle_{\mathbb{R}^d}$$

is given by the inverse Fourier transform of

$$\widehat{\Psi}^H(t, \xi) = \widehat{\Psi_0^H}(\xi) \cos(c|\xi|t) + \widehat{\Psi_1^H}(\xi) \frac{\sin(c|\xi|t)}{c|\xi|} - \frac{1 - \cos(c|\xi|t)}{c^2|\xi|^2}\, \widehat{\sigma}_2(\xi) \langle \sigma_1 \rangle_{\mathbb{X}^d} \langle \mathscr{M} \rangle_{\mathbb{R}^d},$$

and it does not depend on the space variable x. Accordingly, the associated potential

$$\Phi[\Psi^H](t, x) = \langle \sigma_1 \rangle_{\mathbb{X}^d} \iint_{\mathbb{R}^n} \sigma_2(z) \Psi^H(t, z) \,\mathrm{d}z$$

does not depend on x. We obtain

$$(\partial_t + v \cdot \nabla_x) \mathscr{M} = 0 = \nabla_x \Phi[\Psi^H] \cdot \nabla_v \mathscr{M},$$

and finally $(\mathscr{M}, \Psi^H)$ is a homogeneous solution of (2a)–(2d). We bring the attention of the reader to the fact that, in the case $\mathbb{X}^d = \mathbb{R}^d$, the homogeneous solutions have infinite mass and infinite energy.

Remark 2.1 (Stationary solutions) *A specific case of interest corresponds to* stationary *solutions. Let us associate to $\mathscr{M}$, the function*

$$\Psi_{\mathrm{eq}}(z) = \frac{1}{c^2} \Gamma(z) \langle \sigma_1 \rangle_{\mathbb{X}^d} \langle \mathscr{M} \rangle_{\mathbb{R}^d},$$

where Γ is the solution of $\Delta_z \Gamma(z) = \sigma_2(z)$. It defines a stationary solution Ψ_{eq} for the wave equation (2c) *(with initial data $\Psi_0^H = \Psi_{\mathrm{eq}}$ and $\Psi_1^H = 0$). The associated potential thus reads*

$$\iint_{\mathbb{X}^d \times \mathbb{R}^n} \sigma_1(x - y) \sigma_2(z) \Psi_{\mathrm{eq}}(z) \ \mathrm{d}x \,\mathrm{d}z = \langle \sigma_1 \rangle_{\mathbb{X}^d} \int_{\mathbb{R}^n} \sigma_2(z) \Psi_{\mathrm{eq}}(z) \,\mathrm{d}z,$$

which does not depend on the space variable $x \in \mathbb{X}^d$, nor on the time variable t.

2.3 Equations for the fluctuations

Given a space-homogeneous solution $(\mathscr{M}, \Psi^H)$, we expand the solution as

$$F(t, x, v) = \mathscr{M}(v) + f(t, x, v), \qquad \Psi(t, x, z) = \Psi^H(t, z) + \psi(t, x, z). \tag{6}$$

The fluctuations (f, ψ) satisfy

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \Phi[\psi] \cdot \nabla_v (\mathscr{M} + f) = 0, \tag{7a}$$

$$\Phi[\psi](t, x) = \iint_{X^d \times \mathbb{R}^n} \sigma_1(x - y) \sigma_2(z) \psi(t, y, z) \,\mathrm{d}y \,\mathrm{d}z, \tag{7b}$$

$$\partial_{tt}^2 \psi - c^2 \Delta_z \psi = -\sigma_2(z) \int_{\mathbb{R}^d} \sigma_1(x - y) \varrho(t, y) \,\mathrm{d}y, \tag{7c}$$

$$\varrho(t, x) = \int_{\mathbb{R}^d} f(t, x, v) \,\mathrm{d}v, \tag{7d}$$

completed by the initial conditions

$$f(0,x,v) = f_0(x,v), \qquad (\psi(0,x,z), \partial_t\psi(0,x,z)) = (\psi_0(x,z), \psi_1(x,z)). \tag{8}$$

As said above, it can be convenient to set $\psi(t,x,z) = \psi_I(t,x,z) + \psi_S(t,x,z)$, with the contribution from the initial data

$$\widehat{\psi}_I(t,x,\xi) = \widehat{\psi_0}(x,\xi)\cos(c|\xi|t) + \widehat{\psi_1}(x,\xi)\frac{\sin(c|\xi|t)}{c|\xi|}$$

and the self-consistent contribution

$$\widehat{\psi}_S(t,x,\xi) = -\int_0^t \frac{\sin(c|\xi|[t-\tau])}{c|\xi|}\widehat{\sigma}_2(\xi)\sigma_1 \star \varrho(\tau,x)\,\mathrm{d}\tau.$$

Plugging this into the expression of the potential, we get

$$\Phi[\psi](t,x) = \sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))\,(x),$$

where we have set

$$\mathscr{F}_I(t,x) = \int_{\mathbb{R}^n} \sigma_2(z)\psi_I(t,x,z)\,\mathrm{d}z$$

and

$$\mathscr{G}_\varrho(t,x) = \int_0^t p_c(t-\tau)\varrho(\tau,x)\,\mathrm{d}\tau.$$

Hence, the evolution equation for the fluctuation f can be recast as

$$\partial_t f + v\cdot\nabla_x f - \nabla\sigma_1 \star (\mathscr{F}_I - \sigma_1 \star \mathscr{G}_\varrho)\cdot\nabla_v(\mathscr{M} + f) = 0. \tag{9}$$

Finally, let us introduce

$$g(t,x,v) = f(t,x+tv,v),$$

which allows us to get rid of the advection operator. We remark that

$$\partial_t g(t,x,v) = (\partial_t + v\cdot\nabla_x)f(t,x+tv,v)$$

and

$$(\nabla_v f)(t,x+tv,v) = \nabla_v\Big[f(t,x+tv,v)\Big] - t\nabla_x f(t,x+tv,v) = (\nabla_v - t\nabla_x)g(t,x,v).$$

Thus, (9) becomes

$$\partial_t g(t,x,v) = \nabla\sigma_1 \star (\mathscr{F}_I - \sigma_1 \star \mathscr{G}_\varrho)\,(t,x+tv)\cdot(\nabla_v - t\nabla_x)(\mathscr{M} + g)(t,x,v), \tag{10a}$$

$$g(0,x,v) = f_0(x,v). \tag{10b}$$

The following rough statement gives the flavor of the result we wish to justify.

Theorem *We assume that the data $\sigma_1, \sigma_2, \psi_0, \psi_1, f_0$ are smooth enough. We assume, furthermore, that the analog of the* **(L)***-condition for the Vlasov-Wave equation holds. If, initially, the fluctuation is small enough, then, we can find an asymptotic profile g^∞ so that $g(t) - g^\infty$ and the applied force $\nabla\sigma_1 \star (\mathscr{F}_I - \sigma_1 \star \mathscr{G}_\varrho)$ tend to 0 as $t \to \infty$.*

The precise statements are given in Theorem 4.4 (case $\mathbb{X}^d = \mathbb{R}^d$) and Theorem 4.16 (case $\mathbb{X}^d = \mathbb{T}^d$) Let us make a few comments to announce the forthcoming analysis.

- The stability condition (**L**) (see Section 5), like for the usual Vlasov equation, imposes that a certain symbol cannot reach the value 1. In particular, the stability condition holds provided the wave speed c is large enough, see Proposition 3.10.
- The functional framework is a bit intricate. Roughly speaking, we distinguish two types of results, depending whether we work with analytic functions and regularity measured by means of Gevrey spaces (for the torus, the result applies only in this framework), or with functions having enough Sobolev regularity (the result on $\mathbb{R}^d$ applies in this context, and we can also establish the damping for the *linearized* problems in both cases $\mathbb{X}^d = \mathbb{R}^d$ and $\mathbb{X}^d = \mathbb{T}^d$).
- Typically the smallness assumption is imposed on a certain space X (of Gevrey or Sobolev type), but the damping holds in slightly "less regular" spaces Y, with $X \subset Y$. The rate of convergence depends on the functional framework (Gevrey vs. Sobolev) and how far Y is from X.
- For the problem on $\mathbb{R}^d$, we shall need to assume $d \geq 3$; the method breaks down in smaller dimensions, for reasons that already appeared for the Vlasov-Poisson system [7].

For the usual Vlasov equation, the main ingredients to justify the Landau damping can be recapped as follows:

- the transport operator induces a phase mixing phenomena, which is a source of decay for the macroscopic density ϱ;
- when linearizing the system around the homogeneous solution, the Fourier modes of ϱ decouple, leading to a Volterra equation for the Fourier transform of the density. It permits to identify a stability criterion, that depends on the homogeneous solution and on the potential so that the linear dynamics induced by the force term does not annihilate the effects of the phase mixing;
- it remains to control the non linear effects, with the plasma echoes that tend to contribute against the phase mixing.

Technically, in order to address this program, one assumes the smallness of the data and justifies uniform boundedness with respect to time, and, eventually, the Landau damping. In particular, the echoes should be controlled by means of the underlying norms. Rewriting the potential with (4)–(5), we realize that the system (2a)–(2d) substantially differs from the usual Vlasov system dealt with in [27] and [6, 7] in the following aspects:

- there is an additional term $\nabla_x \Phi_I \cdot \nabla_v F$, with a force *independent* on the particles density. This linear perturbation could drive the solution far from the homogeneous state $\mathscr{M}$;
- the self-consistent potential Φ_S involves a half-convolution with respect to the time variable, inducing a sort of memory effect. In particular, the function p_c dramatically influences the expression of the stability criterion.

As we shall see, the analysis of the linearized problem, and the stability criterion, sensibly differ from the Vlasov case. Nevertheless, this linearized analysis remains at the heart of the proof of the Landau damping: once the Landau damping established for the linearized equation, the arguments of [27] and [6, 7] can be adapted to handle the nonlinear problem. Furthermore, we will also bring out the analogies with the gravitational Vlasov-Poisson problem, in terms of conditions of the equilibrium profile. We address both the confined case $\mathbb{X}^d = \mathbb{T}^d$ and the free space problem $\mathbb{X}^d = \mathbb{R}^d$, underlying the differences needed depending on the technical framework.

2.4 The kernel p_c

As said above, the decay properties of the kernel p_c, consequences of the dispersion properties of the wave equations, are crucial for the analysis. When $n \geq 3$, p_c is integrable and satisfies

$$\int_0^\infty p_c(t)\,\mathrm{d}t = \frac{\kappa}{c^2}, \qquad \text{with} \qquad \kappa = \int_{\mathbb{R}^n} \frac{|\widehat{\sigma}_2(\zeta)|^2}{|\zeta|^2}\,\mathrm{d}\zeta < \infty,$$

see [9, Lemma 4.4]. The following statement strengthens this result, depending on the dimension $n \geq 2$ and the assumptions on the form function σ_2. Roughly speaking, we distinguish the case of odd dimensions $n \geq 3$ where the necessary estimates are consequences of the Huygens' principle, and even dimensions where the dispersion effects are weaker. Similar considerations apply when dealing with the term $\mathscr{F}_I$.

Lemma 2.3 *Let $n \geq 2$ and let σ_2 belong to the Besov space $B_1^{n-1,1}$.*

(i) There exists a constant $C(\sigma_2) > 0$ such that

$$|p_c(t)| \leq \frac{C(\sigma_2)}{c\langle ct\rangle^{\frac{n-1}{2}}}.$$

(ii) Moreover, if $|\sigma_2(z)| \lesssim \langle z\rangle^{-m_2}$ with $m_2 > n+(n-1)/2$, then there exists a constant $C(\sigma_2) > 0$ such that

$$|p_c(t)| \leq \frac{C(\sigma_2)}{c\langle ct\rangle^{n-1}}.$$

Let $n \geq 3$ be an odd *integer.*

(iii) Suppose that $|\sigma_2(z)| \lesssim \langle z\rangle^{-m_2}$ for some $m_2 > n+\alpha$, with $\alpha > 0$. Then there exists a constant $C(\sigma_2) > 0$ such that

$$|p_c(t)| \leq \frac{C(\sigma_2)}{c\langle ct\rangle^{\alpha}}.$$

(iv) Let $\lambda > 0$. If $|\sigma_2(z)| \lesssim \exp(-\lambda_2|z|)$ for some $\lambda_2 > \lambda$, then there exists a constant $C(\sigma_2) > 0$ such that

$$|p_c(t)| \leq \frac{C(\sigma_2)\,e^{-\lambda|ct|}}{c}.$$

(v) If $\sigma_2 \in C_c^0(\mathbb{R}^n)$ with $\operatorname{supp}(\sigma_2) \subset B(0, R_2)$, then p_c has a compact support included in $[0, \frac{2R_2}{c}]$ and it satisfies

$$|p_c(t)| \leq C \frac{\|\sigma_2\|_{L^{2n/(n+2)}} \|\sigma_2\|_{L^2}}{c},$$

for a certain constant $C > 0$.

The decay of p_c is intimately connected to the energy dissipation mechanisms through the vibration of the medium, which are at the heart of the qualitative properties of the model introduced in [8]. In dimension $n = 1$, a direct computation by means of D'Alembert formula shows that

$$p_c(t) = \frac{1}{2c} \int_{-\infty}^{+\infty} \sigma_2(z) \left(\int_{z-ct}^{z+ct} \sigma_2(s) \, \mathrm{d}s \right) \mathrm{d}z \xrightarrow[t \to \infty]{} \frac{1}{2c} \|\sigma_2\|_{L^1_z}^2 > 0.$$

Hence, in this case $p_c \notin L^1(0, \infty)$, there is no loss of memory at all; numerical simulations indeed confirm that there is no damping phenomena [18]. Similarly, working in the torus $\mathbb{T}^n$ for the wave equation leads to

$$p_c(t) = \sum_{\ell \neq 0} \frac{|\widehat{\sigma}_2(\ell)|^2}{c\ell} \sin(c\ell t) + |\widehat{\sigma}_2(0)|^2 t.$$

It shows that there is no possible energy dispersion mechanism in this geometry.

As we shall see later on the rate of the Landau damping is directly related to the decay rate of p_c. If even dimensions n are considered the best decay rate provided by Lemma 2.3 leads to $|p_c(t)| \lesssim \langle t \rangle^{-(n-1)}$. However, the Landau damping also requires some regularity on the Cauchy data for the Vlasov equation. For instance, the analysis of the non linear Landau damping in $\mathbb{R}^d$, inspired from [7], leads to suppose that the data lies in the Sobolev space H^{36} (which might be sub-optimal, see [7, Remark 1]). This imposes a constraint on the decay of p_c, which amount to a condition on the dimension n for the wave equation (like $n-1 \geq 36$, see **(H1)** and **(A1)**–**(A2)**). Then, one may wonder to identify minimal regularity assumptions to obtain the Landau damping. The alternative proof of [21], which is less demanding in terms of regularity, could be adapted in order to extend the result in this direction. It is easier to discuss the linearized problem, for which we obtain $n \geq 6$ (see Remark 3.5). We point out that when n is odd the only condition is $n \geq 3$, for both the linear and the non linear cases.

Proof. The proof relies on dispersion estimates for the wave equation, that we shall use in several places. Let us denote $(\dot{W}, W)$ the group of the wave equation (with propagation speed $c = 1$): we write the solution of the Cauchy problem

$$\begin{cases} (\partial_{tt}^2 - c^2 \Delta_z) \Upsilon(t, z) = 0, \\ (\Upsilon, \partial_t \Upsilon)\big|_{t=0} = (\Upsilon_0, \Upsilon_1). \end{cases} \tag{11}$$

as $\Upsilon(t, \cdot) = \dot{W}(ct)\Upsilon_0 + \frac{1}{c} W(ct)\Upsilon_1$. In terms of Fourier variable, $\dot{W}(t)$ corresponds to the multiplication by $\cos(|\zeta|t)$ and $W(t)$ to the multiplication by $\sin(|\zeta|t)/|\zeta|$:

$$\widehat{\dot{W}(ct)\Upsilon_0}(\zeta) = \cos(c|\zeta|t)\widehat{\Upsilon}(\zeta) \qquad \text{and} \qquad \frac{1}{c}\widehat{W(ct)\Upsilon_1}(\zeta) = \frac{\sin(c|\zeta|t)}{c|\zeta|} \widehat{\Upsilon}(\zeta).$$

Therefore, p_c can be cast as

$$p_c(t) = \frac{1}{c}\int_{\mathbb{R}^n} \sigma_2\, W(ct)\sigma_2\,\mathrm{d}z.$$

The dispersion estimates rely on the operators $U^{\pm}(t)$ defined by

$$\widehat{U^{\pm}\Upsilon}(\zeta) = e^{\pm i|\zeta|t}\,\widehat{\Upsilon}(\zeta).$$

Indeed, since $\dot{W}(t) = (U^+ + U^-)/2$ and $W(t) = (U^+ - U^-)/(2i\sqrt{-\Delta_z})$ an estimate with $U^{\pm}(t)$ can be translated into an estimate for $\dot{W}(t)$ and $W(t)$. The basic estimate states as follows (see e. g. [16, Proof of Proposition 3.1] and the references therein): if Υ has its Fourier transform supported in $\{\zeta \in \mathbb{R}^n \mid 2^{j-1} \le |\zeta| \le 2^{j+1}\}$, then

$$\|U^{\pm}(t)\Upsilon\|_{L^\infty_z} \le C\,\min\left(2^{nj}, 2^{\frac{n+1}{2}j}|t|^{-\frac{n-1}{2}}\right)\|\Upsilon\|_{L^1_z}. \tag{12}$$

Estimate (12) can be refined as follows, see [30, Proof Of Lemma 3.2],

$$\begin{aligned} &|U^{\pm}(t)\Upsilon(z)| \\ &\quad\le C_N\,\min\left(2^{nj}, 2^{\frac{n+1}{2}j}|t|^{-\frac{n-1}{2}}, 2^{(\frac{n+1}{2}-N)j}|t|^{-\frac{n-1}{2}}\big|\,|t|-|z|\,\big|^{-N}\right)\|\Upsilon\|_{L^1_z}, \end{aligned} \tag{13}$$

where N can be any integer. Such an estimate can be seen as a generalization of Huygens' principle which holds only in odd dimensions: it tells us that $U^{\pm}(t)\Upsilon$ reaches its maximum next to the cone $t = |z|$. In order to use these estimates, we introduce a sequence $\varphi_j \in \mathcal{S}(\mathbb{R}^n)$ such that $\sum_j \hat{\varphi}_j(\zeta) = 1$ and for any $j \in \mathbb{Z}$, $\mathrm{supp}(\widehat{\varphi}_j) \subset \{\zeta \mid 2^{j-1} \le |\zeta| \le 2^{j+1}\}$. We set $\Upsilon_j = \varphi_j \star \Upsilon$ so that $\Upsilon = \sum_j \Upsilon_j$ and thanks to (12) we get

$$\|U^{\pm}(t)\Upsilon\|_{L^\infty_z} \le C\,\min\left(\sum_{j\in\mathbb{Z}} 2^{nj}\|\Upsilon_j\|_{L^1_z}, |t|^{-\frac{n-1}{2}}\sum 2^{\frac{n+1}{2}j}\|\Upsilon_j\|_{L^1_z}\right), \tag{14}$$

where $\sum_j 2^{sj}\|\Upsilon_j\|_{L^1_z}$ is nothing but the $\dot{B}^{s,1}_1$-norm of Υ. We refer the reader to [16] for a thorough introduction to Besov spaces: the homogeneous Besov spaces $\dot{B}^{s,1}_1$ satisfy a scale invariance property but there is no obvious embedding relations between $\dot{B}^{s,1}_1$ and $\dot{B}^{s',1}_1$ for $s \ge s'$ (if $s' \ge 0$, $2^{sj} \ge 2^{s'j}$ for $j \ge 0$ but $2^{sj} < 2^{s'j}$ for $j < 0$). In order to make use of a single functional space, we prefer to work with the non homogeneous Besov spaces $B^{s,1}_1$: we have $B^{s,1}_1 \subset \dot{B}^{s,1}_1$ for $s \ge 0$ and $B^{s,1}_1$ embeds into $B^{s',1}_1$ for $s \ge s'$. Therefore, we get

$$\|U^{\pm}(t)\Upsilon\|_{L^\infty_z} \le C\,\min\left(1, |t|^{-\frac{n-1}{2}}\right)\|\Upsilon\|_{B^{n,1}_1} \lesssim \langle t\rangle^{-\frac{n-1}{2}}\|\Upsilon\|_{B^{n,1}_1}. \tag{15}$$

Similarly, from (13) we get

$$\begin{aligned} &|U^{\pm}(t)\Upsilon(z)| \\ &\quad\le C_N\,\min\left(\|\Upsilon\|_{\dot{B}^{n,1}_1}, |t|^{-\frac{n-1}{2}}\|\Upsilon\|_{\dot{B}^{\frac{n+1}{2},1}_1}, |t|^{-\frac{n-1}{2}}\big|\,|t|-|z|\,\big|^{-N}\|\Upsilon\|_{\dot{B}^{\frac{n+1}{2}-N,1}_1}\right). \end{aligned} \tag{16}$$

Note that we do not work with Besov space with negative regularity index s (which would imply irrelevant conditions on $\xi = 0$). Assuming $N \le (n+1)/2$, we are led to

$$|U^{\pm}(t)\Upsilon(z)| \le C_N\,\min\left(1, |t|^{-\frac{n-1}{2}}, |t|^{-\frac{n-1}{2}}\big|\,|t|-|z|\,\big|^{-N}\right)\|\Upsilon\|_{B^{n,1}_1}. \tag{17}$$

We can now finish the proof of Lemma 2.3. Since $p_c(t) = \frac{1}{c}(\int \sigma_2 W(ct)\sigma_2\,\mathrm{d}z)$, we have $|p_c(t)| \leq \frac{1}{c}\|\sigma_2\|_{L^1_z}\|W(ct)\sigma_2\|_{L^\infty_z}$. By applying (a variant with an extra factor $1/2^{j-1}$ of) (12), we obtain

$$\|W(ct)\varphi_j \star \sigma_2\|_{L^\infty_z} \leq \frac{C}{2^{j-1}} \min\left(2^{nj}, 2^{\frac{n+1}{2}j}|ct|^{-\frac{n-1}{2}}\right) \|\varphi_j \star \sigma_2\|_{L^1_z}.$$

Summing over $j \in \mathbb{Z}$ yields

$$|p_c(t)| \leq \frac{K}{c\langle ct\rangle^{\frac{n-1}{2}}}\|\sigma_2\|_{L^1_z}\|\sigma_2\|_{B^{n-1,1}_1},$$

which proves (i). Estimate (ii) uses the refined estimate (13) which gives, for any $N \in \mathbb{N}$,

$$\begin{aligned}&|W(ct)\varphi_j \star \sigma_2(z)| \\ &\quad \leq \frac{C_N}{2^{j-1}} \min\left(2^{nj}, 2^{\frac{n+1}{2}j}|ct|^{-\frac{n-1}{2}}, 2^{(\frac{n+1}{2}-N)j}|ct|^{-\frac{n-1}{2}}\big|\,|ct|-|z|\,\big|^{-N}\right) \|\varphi_j \star \sigma_2\|_{L^1_z}.\end{aligned}$$

With $N = (n-1)/2$ and summing over $j \in \mathbb{Z}$, we get

$$|p_c(t)| \leq \frac{2C_N}{c}\left(\int_{\mathbb{R}^n} |\sigma_2(z)| \min\left(1, |ct|^{-\frac{n-1}{2}}, |ct|^{-\frac{n-1}{2}}\big|\,|ct|-|z|\,\big|^{-\frac{n-1}{2}}\right) \mathrm{d}z\right) \|\sigma_2\|_{B^{n-1,1}_1}.$$

We have

$$\begin{aligned}&\int_{\mathbb{R}^n} |\sigma_2(z)| \min\left(1, |ct|^{-\frac{n-1}{2}}, |ct|^{-\frac{n-1}{2}}\big|\,|ct|-|z|\,\big|^{-\frac{n-1}{2}}\right) \mathrm{d}z \\ &\qquad\qquad \lesssim \int_{\mathbb{R}^n} |\sigma_2(z)| \min\left(\langle ct\rangle^{-\frac{n-1}{2}}, \langle |ct|\,\big|\,|ct|-|z|\,\big|\rangle^{-\frac{n-1}{2}}\right) \mathrm{d}z.\end{aligned}$$

We split the integration domain into the ball $B(0,|ct|/2)$ and its complementary and we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n} |\sigma_2(z)| \min\left(\langle ct\rangle^{-\frac{n-1}{2}}, \langle |ct|\,\big|\,|ct|-|z|\,\big|\rangle^{-\frac{n-1}{2}}\right) \mathrm{d}z \\ &\quad = \int_{B(0,\frac{|ct|}{2})} |\sigma_2(z)|\,\langle |ct|\,\big|\,|ct|-|z|\,\big|\rangle^{-\frac{n-1}{2}}\,\mathrm{d}z + \int_{\complement B(0,\frac{|ct|}{2})} |\sigma_2(z)|\,\langle ct\rangle^{-\frac{n-1}{2}}\,\mathrm{d}z \\ &\quad \leq \int_{B(0,\frac{|ct|}{2})} |\sigma_2| \left\langle \frac{|ct|^2}{2}\right\rangle^{-\frac{n-1}{2}} \mathrm{d}z + \langle ct\rangle^{-\frac{n-1}{2}}\left(\int_{\complement B(0,\frac{|ct|}{2})} |\sigma_2(z)|\,\mathrm{d}z\right) \\ &\quad \lesssim \left\langle \frac{|ct|}{2}\right\rangle^{-(n-1)} \|\sigma_2\|_{L^1_z} + \langle ct\rangle^{-\frac{n-1}{2}}\left\langle \frac{|ct|}{2}\right\rangle^{-\frac{n-1}{2}}\left(\int_{\complement B(0,\frac{|ct|}{2})} |\sigma_2(z)|\langle z\rangle^{\frac{n-1}{2}}\,\mathrm{d}z\right).\end{aligned}$$

The assumption on σ_2 ensures that the last integral is finite

We turn to the specific case of odd dimensions. The role of the Huygens principle appears clearly with the estimate (v). Indeed the support assumption on σ_2 implies, when n is odd, that

$$\text{if } ct \geq R_2 + |z| \text{ then } W(t)\sigma_2(z) = 0.$$

Therefore, when $t \geq \frac{2R_2}{c}$, the product $\sigma_2(z)\,W(ct)\sigma_2(z)$ vanishes (see Fig. 1) and $p_c(t) = 0$. Bearing in mind that $n \geq 3$, Hölder inequality yields

$$|p_c(t)| \leq \frac{1}{c}\|\sigma_2\|_{L^{2n/(n+2)}}\|W(ct)\sigma_2\|_{L^{2n/(n-2)}}.$$

We conclude by combining the Sobolev embedding inequality, see e. g. [24, Lemma 8.3],

$\|W(ct)\sigma_2\|_{L^{2n/(n-2)}} \leq C_S\|\nabla_z W(ct)\sigma_2\|_{L^2}$, and the energy conservation for the wave equation which implies

$$\|\nabla_z W(ct)\sigma_2\|_{L^2}^2 \leq \|\partial_s(W(s)\sigma_2)\big|_{s=ct}\|_{L^2}^2 + \|\nabla_z W(ct)\sigma_2\|_{L^2}^2 \leq \|\sigma_2\|_{L^2}^2.$$

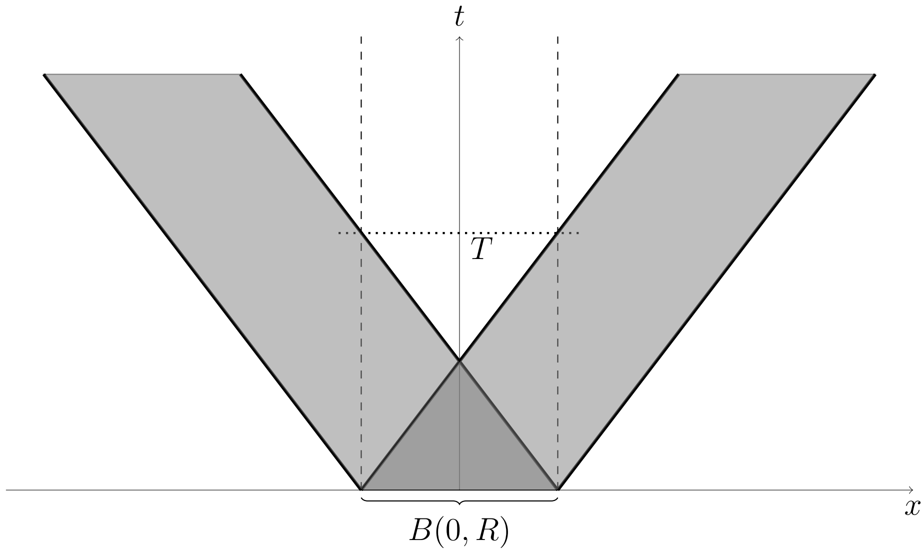

Figure 1: Propagation cone: the signal emanating from the ball $B(0,R)$ cannot be felt in this ball after time T

We turn to the proof of (iii). Consider $t > 0$ and $0 < R < ct$. We split as follows

$$\sigma_2 = \sigma_2 \mathbf{1}_{|z|\leq R} + \sigma_2 \mathbf{1}_{|z|>R} := u_1 + u_2.$$

By linearity of the wave equation, we can write

$$p_c(t) = \frac{1}{c}\int_{\mathbb{R}^n} \sigma_2\, W(ct)u_1\,\mathrm{d}z + \frac{1}{c}\int_{\mathbb{R}^n} \sigma_2\, W(ct)u_2\,\mathrm{d}z.$$

Since u_1 is supported in $B(0,R)$, the support of $W(ct)u_1$ lies in $\{z \,|\, ct - R \leq |z| \leq ct + R\}$. Since $ct - R > 0$, the first integral is dominated as follows (we already know from the proof of (i) that $\|W(ct)u_1\|_{L^\infty_z} \lesssim \|\sigma_2\|_{B_1^{n-1,1}}$)

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \sigma_2\, W(ct)u_1\,\mathrm{d}z\right| &= \langle ct - R\rangle^{-\alpha}\left|\int_{\complement B(0,ct-R)} \langle ct-R\rangle^{\alpha}\sigma_2(z)\, W(t)u_1(z)\,\mathrm{d}z\right| \\ &\lesssim \langle ct-R\rangle^{-\alpha}\left(\int_{\complement B(0,ct-R)} \langle z\rangle^{\alpha}|\sigma_2(z)|\,\mathrm{d}z\right)\|\sigma_2\|_{B_1^{n-1,1}}.\end{aligned}$$

By virtue of the assumptions on σ_2, the right hand side is finite. The integral with u_2 can be estimated by using Plancherel's formula, which yields

$$\int_{\mathbb{R}^n} \sigma_2\, W(ct)u_2\,\mathrm{d}z = \int_{\mathbb{R}^n} \widehat{\sigma}_2(\zeta)\frac{\sin(c|\zeta|t)}{|\zeta|}\widehat{u}_2(\zeta)\,\mathrm{d}\zeta = \int_{\mathbb{R}^n} u_2\, W(ct)\sigma_2\,\mathrm{d}z.$$

It leads to

$$\left|\int_{\mathbb{R}^n} \sigma_2\, W(ct) u_2 \,\mathrm{d}z\right| = \left|\int_{\mathbb{R}^n} u_2\, W(ct)\sigma_2 \,\mathrm{d}z\right| \lesssim \left(\int_{\mathbb{R}^n} |\sigma_2(z)|\mathbf{1}_{|z|>R}\,\mathrm{d}z\right) \|\sigma_2\|_{B_1^{n-1,1}}$$
$$= \langle R\rangle^{-\alpha}\left(\int_{\mathbb{R}^n} \langle R\rangle^{\alpha}|\sigma_2(z)|\mathbf{1}_{|z|>R}\,\mathrm{d}z\right)\|\sigma_2\|_{B_1^{n-1,1}}$$
$$\leq \langle R\rangle^{-\alpha}\left(\int_{\mathbb{R}^n} \langle z\rangle^{\alpha}|\sigma_2(z)|\,\mathrm{d}z\right)\|\sigma_2\|_{B_1^{n-1,1}},$$

which is finite too. We have proved that

$$|p_c(t)| \lesssim \frac{1}{c}\left(\langle ct-R\rangle^{-\alpha} + \langle R\rangle^{-\alpha}\right)$$

and we conclude by setting $R = ct/2$. Item (iv) is justified similarly, just replacing the polynomial weights by exponential weights. ■

Analogous conclusions apply to $\mathscr{F}_I$ which can be cast as

$$\mathscr{F}_I(t,x) = \int_{\mathbb{R}^n} \sigma_2(z)\Big(\dot{W}(ct)\Psi_0(x,z) + \frac{1}{c}W(ct)\Psi_1(x,z)\Big)\,\mathrm{d}z.$$

3 Linearized Landau Damping

3.1 The linearized system

In the expansion (6), let us assume that the fluctuations f and ψ remain small, so that we neglect the quadratic term (with respect to the perturbations) $\nabla_x\Phi[\psi]\cdot\nabla_v f$ in the evolution equations (note in particular that this assumes the smallness of the initial fluctuations (ψ_0,ψ_1)). We are thus led to the following linearized system

$$\partial_t f + v\cdot\nabla_x f - \nabla_x\phi\cdot\nabla_v\mathscr{M} = 0, \qquad t\geq 0,\ x\in\mathbb{X}^d,\ v\in\mathbb{R}^d, \tag{18a}$$
$$\phi(t,x) = \iint_{\mathbb{X}^d\times\mathbb{R}^n} \sigma_1(x-y)\psi(t,y,z)\sigma_2(z)\,\mathrm{d}z\ \ \mathrm{d}y, \qquad t\geq 0,\ x\in\mathbb{X}^d \tag{18b}$$
$$\partial_{tt}^2\psi - c^2\Delta_z\psi = -\sigma_2(z)\int_{\mathbb{X}^d}\sigma_1(x-y)\varrho(t,y)\ \ \mathrm{d}y, \qquad t\geq 0,\ x\in\mathbb{X}^d,\ z\in\mathbb{R}^n, \tag{18c}$$
$$\varrho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \qquad t\geq 0,\ x\in\mathbb{X}^d. \tag{18d}$$

The system is completed by initial conditions

$$f\big|_{t=0} = f_0, \qquad (\psi,\partial_t\psi)\big|_{t=0} = (\psi_0,\psi_1). \tag{19}$$

The expected result can be explained as follows: let us assume that the fluctuation does not provide additional mass: $\iint f(0,x,v)\,\mathrm{d}v\,\mathrm{d}x = 0$, and, to fix ideas, $\psi_0 = 0$ and $\psi_1 = 0$. In such a case, linearized Landau damping asserts that ϱ converges strongly to 0, while f converges weakly to 0, as $t\to\infty$. Moreover, the potential ϕ also vanishes for large times. We are going to establish that such a behavior holds for the system (18)–(19).

We start by applying the Fourier transform, with respect to x and v to (18a). It yields

$$(\partial_t - k\cdot\nabla_\xi)\widehat{f}(t,k,\xi) = -k\cdot\xi\ \widehat{\phi}(t,k)\ \widehat{\mathscr{M}}(\xi).$$

The equation can be integrated along characteristics, which leads to the following Duhamel formula

$$\widehat{f}(t,k,\xi) = \widehat{f}_0(k,\xi+tk) - \int_0^t (\xi+(t-\tau)k)\cdot k\ \widehat{\phi}(\tau,k)\ \widehat{\mathscr{M}}(\xi+(t-\tau)k)\,\mathrm{d}\tau. \tag{20}$$

We turn to the expression of the Fourier coefficients of the potential. We remind the reader that we can split the potential into

$$\phi = \phi_I + \phi_S,$$

where ϕ_I depends only on (ψ_0,ψ_1) as follows

$$\phi_I(t,x) = \iint_{\mathbb{X}^d\times\mathbb{R}^n} \sigma_1(x-y)\sigma_2(z)\underbrace{\left(\dot{W}(ct)\psi_0(y,z) + \frac{1}{c}W(ct)\psi_1(y,z)\right)}_{=\psi_I(t,y,z)}\,\mathrm{d}y\,\mathrm{d}z \tag{21}$$

and the coupling term reads

$$\phi_S(t,x) = -\int_0^t p_c(t-\tau)\Sigma\star\varrho(\tau,x)\,\mathrm{d}\tau.$$

Plugging the expression of $\phi=\phi_I+\phi_S$ into (20), we obtain

$$\begin{aligned}
\widehat{f}(t,k,\xi) &= \widehat{f}_0(k,\xi+tk) - \int_0^t (\xi+(t-\tau)k)\cdot k\ \widehat{\phi}_I(\tau,k)\ \widehat{\mathscr{M}}(\xi+(t-\tau)k)\,\mathrm{d}\tau\\
&\quad +|\widehat{\sigma_1}(k)|^2\int_0^t (\xi+(t-\tau)k)\cdot k\ \left(\int_0^\tau p_c(\tau-s)\widehat{\varrho}(s,k)\,\mathrm{d}s\right)\ \widehat{\mathscr{M}}(\xi+(t-\tau)k)\,\mathrm{d}\tau\\
&= \widehat{f}_0(k,\xi+tk) - \int_0^t (\xi+(t-\tau)k)\cdot k\ \widehat{\phi}_I(\tau,k)\ \widehat{\mathscr{M}}(\xi+(t-\tau)k)\,\mathrm{d}\tau\\
&\quad +|\widehat{\sigma_1}(k)|^2\int_0^t \left(\int_s^t p_c(\tau-s)(\xi+k(t-\tau))\cdot k\ \widehat{\mathscr{M}}(\xi+(t-\tau)k)\,\mathrm{d}\tau\right)\ \widehat{\varrho}(s,k)\,\mathrm{d}s\\
&= \widehat{f}_0(k,\xi+tk) - \int_0^t (\xi+(t-\tau)k)\cdot k\ \widehat{\phi}_I(\tau,k)\ \widehat{\mathscr{M}}(\xi+(t-\tau)k)\,\mathrm{d}\tau\\
&+|\widehat{\sigma_1}(k)|^2\int_0^t \left(\int_0^{t-s} p_c(\tau)(\xi+(t-[\tau+s])k)\cdot k\ \widehat{\mathscr{M}}(\xi+(t-[\tau+s])k)\,\mathrm{d}\tau\right)\ \widehat{\varrho}(s,k)\,\mathrm{d}\varsigma.
\end{aligned}$$

We are led to an integral equation for the (Fourier coefficients of) the macroscopic density by considering this relation for $\xi=0$. Let us set

$$a(t,k) = \widehat{f}_0(k,tk) - |k|^2\int_0^t \widehat{\phi}_I(\tau,k)\ (t-\tau)\widehat{\mathscr{M}}((t-\tau)k)\,\mathrm{d}\tau \tag{22}$$

and

$$\mathscr{K}(t,k) = |k|^2\ |\widehat{\sigma_1}(k)|^2\int_0^t p_c(\tau)\,(t-\tau)\widehat{\mathscr{M}}((t-\tau)k)\,\mathrm{d}\tau. \tag{23}$$

Then, we obtain an integral equation for the fluctuation of the macroscopic density

$$\widehat{\varrho}(t,k) = a(t,k) + \int_0^t \mathscr{K}(t-s,k)\widehat{\varrho}(s,k)\,\mathrm{d}s. \tag{24}$$

The analysis of this relation makes use of the Laplace transform

$$\varphi : (0,\infty) \to \mathbb{C}, \qquad \mathscr{L}\varphi(\omega) = \int_0^{+\infty} e^{-\omega t}\varphi(t)\,\mathrm{d}t \text{ for } \omega \in \mathbb{C},$$

which is well defined for $\mathrm{Re}(\omega)$ large enough.

3.2 Linearized Landau damping in finite regularity

The linearized Landau damping holds with an algebraic rate provided the solution ϱ of (24) satisfies

$$|\widehat{\varrho}(t,k)| \leq C\langle tk\rangle^{-m} \tag{25}$$

(see for instance [27, section 3]) for a certain $m > 0$. For Volterra equations like (24) we can establish (see [6, Lemma 4.1], [7, Proposition 2.2]) mode-by-mode estimates in L^2_t norm: for any k

$$\int_0^{+\infty} \langle tk\rangle^{2m} |\widehat{\varrho}(t,k)|^2 \,\mathrm{d}t \leq C_{LD}^2 \int_0^{+\infty} \langle tk\rangle^{2m} |a(t,k)|^2 \,\mathrm{d}t, \tag{26}$$

where $C_{LD} > 0$ does not depend on k. From such an L^2_t estimate, we get an L^∞_t estimate as follows

$$\begin{aligned}
\langle tk\rangle^m |\widehat{\varrho}(t,k)| &\leq \langle tk\rangle^m |a(t,k)| + \left| \int_0^t \langle (t-\tau)k + \tau k\rangle^m \mathscr{K}_k(t-\tau,k)\widehat{\varrho}(\tau,k)\,\mathrm{d}\tau \right| \\
&\leq \langle tk\rangle^m |a(t,k)| + \left(\int_0^t \langle \tau k\rangle^{2m} |\mathscr{K}(\tau,k)|^2 \,\mathrm{d}\tau \right)^{1/2} \left(\int_0^t \langle \tau k\rangle^{2m} |\widehat{\varrho}(\tau,k)|^2 \,\mathrm{d}\tau \right)^{1/2} \\
&\leq \langle tk\rangle^m |a(t,k)| + C_{LD} \left(\int_0^t \langle \tau k\rangle^{2m} |\mathscr{K}(\tau,k)|^2 \,\mathrm{d}\tau \right)^{1/2} \left(\int_0^t \langle \tau k\rangle^{2m} |a(\tau,k)|^2 \,\mathrm{d}\tau \right)^{1/2},
\end{aligned}$$

where we are left with the task of verifying that

$$\begin{cases}
\sup\limits_{\substack{t\geq 0\\ k\in\mathbb{X}^{\star d}\setminus\{0\}}} \langle tk\rangle^m |a(t,k)| < +\infty, \\
\sup\limits_{k\in\mathbb{X}^{\star d}\setminus\{0\}} \left(\int_0^{+\infty} \langle \tau k\rangle^{2m} |\mathscr{K}(\tau,k)|^2 \,\mathrm{d}\tau \right) \left(\int_0^{+\infty} \langle \tau k\rangle^{2m} |a(\tau,k)|^2 \,\mathrm{d}\tau \right) < +\infty
\end{cases} \tag{27}$$

hold. We are going to identify conditions on $a(t,k)$ and $\mathscr{K}(t,k)$ such that (26) applies and to justify that (27) is satisfied. We refer the reader to [7, Proof of Proposition 2.2] for a proof of the following claim.

Lemma 3.1 *Let $\mathscr{K}$ satisfy*

$$\inf_{k\in\mathbb{X}^{\star d}\setminus\{0\}} \left|1-\mathscr{L}\mathscr{K}(\omega,k)\right| \geq \kappa > 0 \quad \textit{for } \mathrm{Re}(\omega)\geq 0, \tag{$\mathbf{L}$}$$

and for any $0 \leq j \leq m$:

$$\sup_{\substack{k\in\mathbb{X}^{\star d}\setminus\{0\}\\ \mathrm{Re}(\omega)\geq 0}} \left(|k|^j \left|\partial_\omega^j \mathscr{L}\mathscr{K}(\omega,k)\right|\right) < +\infty.$$

Then there exists a constant $C_{LD}>0$, which does not depend on k, such that the solutions of (24) *satisfy* (26).

Estimate (26) makes sense when $t\mapsto \langle tk\rangle^m a(t,k)$ is square integrable, a property that needs to be carefully checked in the current framework.
Condition ($\mathbf{L}$) gives rise to a stability criterion on the stationary profile $\mathscr{M}$. Since the operator $\mathscr{K}$ involves the kernel p_c the detailed condition substantially differs from the usual Vlasov case. That this statement applies for our purpose relies on the following assumptions:

(H1) $n > m + \frac{5}{2}$,

(H2) $\sigma_2 \in B_1^{n-1,1}$ and $|\sigma_2(z)| \leq C_2\langle z\rangle^{-m_2}$ with $m_2 > \frac{3n-1}{2}$,

(H3) $\sup_{k\in\mathbb{X}^{\star d}} \left(\left\|\widehat{\psi}_0(k)\right\|_{B^{n,1}_{1,(z)}} + \left\|\widehat{\psi}_1(k)\right\|_{B^{n-1,1}_{1,(z)}}\right) < +\infty,$

(H4) $|\widehat{\sigma}_1(k)| \leq C_1\langle k\rangle^{-m_1}$ with $m_1 > m+1$,

(H5) $\left|\widehat{\mathscr{M}}(\xi)\right| \leq C\langle\xi\rangle^{-\bar{m}}$ with $\bar{m} > m+2$ and $\left|\widehat{f}_0(k,\xi)\right| \leq C_0\langle\xi\rangle^{-m_0}$ with $m_0 > m+\frac{1}{2}$.

Proposition 3.2 *Assume **(H1)**–**(H5)**.*
(i) *There exists a constant $A>0$ such that for any $0\leq j\leq m$, $k\in\mathbb{X}^{\star d}\setminus\{0\}$ and $\omega\in\mathbb{C}$ with $\mathrm{Re}(\omega)\geq 0$, we have*

$$|k|^j \left|\partial_\omega^j \mathscr{L}\mathscr{K}(\omega,k)\right| \leq A.$$

(ii) *For any $k\in\mathbb{X}^{\star d}\setminus\{0\}$,*

$$\int_0^{+\infty} |k|\langle tk\rangle^{2m} \left|a(t,k)\right|^2 \,\mathrm{d}t < +\infty.$$

(iii) (27) *holds.*

The regularity of the data σ_1, $\mathscr{M}$ and f_0 is controlled by assumptions **(H4)**–**(H5)**: the higher the algebraic decay rate m requested on the Fourier modes of ϱ, see (25), the higher the regularity on the data. Assumption **(H1)** tunes the dimension n for the wave equation: the decay of the Fourier modes of ϱ is limited by the dispersion of the wave equation, which is stronger as n increases.
However, as indicated in Lemma 2.3, for odd n the Huygens principle and the decay of σ_2 imply strengthened decay properties on p_c. Accordingly, Proposition 3.2 applies replacing **(H1)**–**(H3)** by

(H1') $n \geq 3$ is odd,

(H2') $\sigma_2 \in B_1^{n-1,1}$ and $|\sigma_2(z)| \leq C_2 \langle z \rangle^{-m_2}$ with $m_2 > n + m + \frac{3}{2}$

(H3')
- $\sup_{k \in \mathbb{X}^{\star d}} \left(\left\| \widehat{\psi}_0(k) \right\|_{B^{n,1}_{1,(z)}} + \left\| \widehat{\psi}_1(k) \right\|_{B^{n-1,1}_{1,(z)}} \right) < +\infty,$
- there exists a constant $C > 0$ such that

$$\sup_{k \in \mathbb{X}^{\star d}} \left(\left| \widehat{\psi}_0(k, z) \right| + \left| \widehat{\psi}_1(k, z) \right| \right) \leq C \langle z \rangle^{-m_2}.$$

Hypothesis **(H2)** or **(H2')** can be relaxed. Indeed, the decay imposed in **(H2)**, **(H2')** on σ_2 allows us to apply the refined dispersion estimates described in the proof of Lemma 2.3. Nevertheless, we can simply use the standard estimates as in Lemma 2.3-i). Then, the decay of p_c is slower and, as a counterpart, the dimension n in **(H1)** is more constrained. Proposition 3.2 applies replacing **(H1)**–**(H2)** by

(H1") $n > 2m + 4$,

(H2") $\sigma_2 \in B_1^{n-1,1}$.

Before proving Proposition 3.2 let us detail a useful statement.

Lemma 3.3 *Let $\alpha > 1$ and $\beta \geq 0$. For any $\gamma \geq 0$ such that $\gamma \leq \beta$ et $\gamma < \alpha - 1$, we have*

$$\int_0^t \langle t - \tau \rangle^{-\alpha} \langle \tau k \rangle^{-\beta} \, \mathrm{d}\tau \lesssim \langle k \rangle^{\gamma} \langle tk \rangle^{-\gamma}. \tag{28}$$

Proof. We split the integral

$$\int_0^t \langle t - \tau \rangle^{-\alpha} \langle \tau k \rangle^{-\beta} \, \mathrm{d}\tau = \int_0^{t/2} + \int_{t/2}^t \langle t - \tau \rangle^{-\alpha} \langle \tau k \rangle^{-\beta} \, \mathrm{d}\tau$$
$$\leq \int_0^{t/2} \langle t - \tau \rangle^{-\alpha} \, \mathrm{d}\tau + \int_{t/2}^t \langle t - \tau \rangle^{-\alpha} \left\langle \frac{tk}{2} \right\rangle^{-\beta} \, \mathrm{d}\tau.$$

The second integral is dominated by

$$\int_{t/2}^t \langle t - \tau \rangle^{-\alpha} \left\langle \frac{tk}{2} \right\rangle^{-\beta} \, \mathrm{d}\tau \lesssim \langle tk \rangle^{-\beta} \int_0^{+\infty} \langle u \rangle^{-\alpha} \, \mathrm{d}u$$

which is finite provided $\alpha > 1$. For the first integral we observe that, for any $0 \leq \tau \leq t/2$,

$$\langle tk \rangle = \left\langle \frac{t}{k} 2k \right\rangle \leq \left\langle \frac{t}{2} \right\rangle \langle 2k \rangle \leq \langle t - \tau \rangle \langle 2k \rangle,$$

holds, and we infer that

$$\int_0^{t/2} \langle t - \tau \rangle^{-\alpha} \, \mathrm{d}\tau \leq \frac{\langle 2k \rangle^{\gamma}}{\langle tk \rangle^{\gamma}} \int_0^{+\infty} \langle u \rangle^{\gamma - \alpha} \, \mathrm{d}u.$$

The right hand side is finite when $\gamma < \alpha - 1$, which finishes the proof. ∎

Proof of Proposition 3.2. (i) We start from

$$\partial_\omega^j \mathscr{L}\mathscr{K}(\omega,k) = |k|\,|\widehat{\sigma}_1(k)|^2 \int_0^{+\infty} (-t)^j e^{-\omega t} \left(\int_0^t p_c(\tau)|k|(t-\tau)\widehat{\mathscr{M}}([t-\tau]k)\,\mathrm{d}\tau \right)\,\mathrm{d}t.$$

Permuting integrals and with the change of variables $u = t-\tau$, we get

$$\begin{aligned}
|k|^j &\left|\partial_\omega^j \mathscr{L}\mathscr{K}(\omega,k)\right| \\
&\leq |k|\,|\widehat{\sigma}_1(k)|^2 \int_0^{+\infty} \left(\int_0^{+\infty} |(u+\tau)k|^j\,|p_c(\tau)|\,|uk| \left|\widehat{\mathscr{M}}(uk)\right|\,\mathrm{d}u \right)\mathrm{d}\tau \\
&\lesssim |\widehat{\sigma}_1(k)|^2 \left(\int_0^{+\infty} |\tau k|^j\,|p_c(\tau)|\,\mathrm{d}\tau \right)\left(\int_0^{+\infty} |uk|^{j+1}\left|\widehat{\mathscr{M}}(uk)\right|\,\mathrm{d}u|k| \right) \\
&= |k|^j\,|\widehat{\sigma}_1(k)|^2 \left(\int_0^{+\infty} |\tau|^j\,|p_c(\tau)|\,\mathrm{d}\tau \right)\left(\int_0^{+\infty} |s|^{j+1}\left|\widehat{\mathscr{M}}\left(\frac{k}{|k|}s\right)\right|\,\mathrm{d}s \right).
\end{aligned}$$

By **(H4)**, $|k|^j|\widehat{\sigma}_1(k)|^2$ is bounded. Then **(H2)** allows us to apply Lemma 2.3 and we deduce that $|p_c(t)| \lesssim \langle t\rangle^{-(n-1)}$. Owing to **(H1)** the second factor is finite. Finally, **(H5)** implies that the last factor is finite too and remains uniformly bounded with respect to k. We point out that the mechanisms of this estimate differs substantially from the standard Vlasov case, where the decay rate improves with the mode. Here p_c does not not carry any frequency k, but the power of $|k|$ are controlled by the decay assumptions on $\widehat{\sigma}_1$.

(ii) The term to be estimated can be cast as (we use $\langle tk\rangle \lesssim \langle \tau k\rangle\langle (t-\tau)k\rangle$) :

$$\begin{aligned}
\int_0^{+\infty} &\langle tk\rangle^2\,|a(t,k)|^2\,\mathrm{d}t \\
&\lesssim \int_0^{+\infty} \langle tk\rangle^{2m}\left|\widehat{f}_0(k,tk)\right|^2\mathrm{d}t + \int_0^{+\infty} \langle tk\rangle^{-(1^+)}\Big| \int_0^t \langle \tau k\rangle^{m+\frac{1}{2}^+}|k|\widehat{\phi}_I(\tau,k) \\
&\qquad\qquad \times \langle (t-\tau)k\rangle^{m+\frac{1}{2}^+}(t-\tau)|k|\widehat{\mathscr{M}}([t-\tau]k)\,\mathrm{d}\tau\Big|^2\,\mathrm{d}t \\
&\lesssim \frac{1}{|k|}\int_0^{+\infty} \langle u\rangle^{2m}\left|\widehat{f}_0(k,\frac{k}{|k|}u)\right|^2\mathrm{d}u + \frac{1}{|k|}\left(\int_0^{+\infty} \langle \tau k\rangle^{2m+1^+}|k|\left|\widehat{\phi}_I(\tau,k)\right|^2\mathrm{d}\tau \right) \\
&\qquad \times \left(\int_0^{+\infty} \langle sk\rangle^{2m+3^+}\left|\widehat{\mathscr{M}}(sk)\right|^2|k|\,\mathrm{d}s \right)\left(\int_0^{+\infty} \langle u\rangle^{-(1^+)}\,\mathrm{d}u \right).
\end{aligned}$$

Using **(H5)** we infer

$$\frac{1}{|k|}\int_0^{+\infty} \langle u\rangle^{2m}\left|\widehat{f}_0(k,\frac{k}{|k|}u)\right|^2\mathrm{d}u \lesssim \frac{1}{|k|}\int_0^{+\infty}\langle u\rangle^{-1^+}\,\mathrm{d}t \lesssim \frac{1}{|k|},$$

and

$$\int_0^{+\infty} \langle sk\rangle^{2m+3^+}\left|\widehat{\mathscr{M}}(sk)\right|^2|k|\,\mathrm{d}s \lesssim \int_0^{+\infty}\langle u\rangle^{-(1^+)}\,\mathrm{d}t \lesssim 1,$$

It remains to justify that

$$\int_0^{+\infty} \langle \tau k\rangle^{2m+1^+}|k|\left|\widehat{\phi}_I(\tau,k)\right|^2\mathrm{d}\tau$$

is finite for any $k \in \mathbb{X}^{\star d}\setminus\{0\}$. To this end we observe that the dispersion induced by the wave equation ensures

$$\left|\widehat{\phi}_I(\tau,k)\right| \lesssim |\widehat{\sigma}_1(k)| \left(\|\sigma_2\|_{L^1_z} + C_2\right) \left(\|\widehat{\psi}_0(k)\|_{B^{n,1}_{1,(z)}} + \frac{1}{c}\|\widehat{\psi}_1(k)\|_{B^{n-1,1}_{1,(z)}}\right) \frac{1}{\langle c\tau\rangle^{n-1}}. \quad (29)$$

This follows from

$$\widehat{\phi}_I(\tau,k) = \widehat{\sigma}_1(k) \int_{\mathbb{R}^n} \sigma_2(z) \left(\dot{W}(c\tau)(\widehat{\psi}_0(k)) + \frac{1}{c} W(c\tau)(\widehat{\psi}_1(k))\right)(z)\,\mathrm{d}z$$

and reasoning as in the proof of Lemma 2.3-(ii). We conclude that

$$\begin{aligned}
&\int_0^{+\infty} \langle \tau k\rangle^{2m+1^+} |k| \left|\widehat{\phi}_I(\tau,k)\right|^2 \mathrm{d}\tau \\
&\quad\lesssim |k|\,|\widehat{\sigma}_1(k)|^2 \left(\|\widehat{\psi}_0(k)\|_{B^{n,1}_{1,(z)}} + \frac{1}{c}\|\widehat{\psi}_1(k)\|_{B^{n-1,1}_{1,(z)}}\right) \int_0^{+\infty} \frac{\langle\tau\rangle^{2m+1^+}\langle k\rangle^{2m+1^+}}{\langle c\tau\rangle^{2(n-1)}}\,\mathrm{d}\tau \\
&\qquad\lesssim \langle k\rangle^{2m+2^+} |\widehat{\sigma}_1(k)|^2 \left(\|\widehat{\psi}_0(k)\|_{B^{n,1}_{1,(z)}} + \frac{1}{c}\|\widehat{\psi}_1(k)\|_{B^{n-1,1}_{1,(z)}}\right) \int_0^{+\infty} \frac{\langle\tau\rangle^{2m+1^+}}{\langle c\tau\rangle^{2(n-1)}}\,\mathrm{d}\tau.
\end{aligned}$$

That this quantity is bounded uniformly with respect to k is a consequence of **(H1)**, **(H3)** and **(H4)**.

(iii) We have obtained

$$\int_0^{+\infty} \langle tk\rangle^{2m} |a(t,k)|^2\,\mathrm{d}t \lesssim \frac{1}{|k|},$$

where the factor $1/|k|$ comes from a change of variables. We justify similarly that $\sup_{t,k} \langle tk\rangle^m |a(t,k)| < \infty$. (There is no factor $1/|k|$ is this estimate.) It remains to study

$$\sup_k \left(\int_0^{+\infty} \langle tk\rangle^{2m} |\mathscr{K}(t,k)|^2\,\mathrm{d}t\right)\left(\int_0^{+\infty} \langle tk\rangle^{2m} |a(t,k)|^2\,\mathrm{d}t\right)$$

and to show that

$$\int_0^{+\infty} \langle tk\rangle^{2m} |\mathscr{K}(t,k)|^2\,\mathrm{d}t \lesssim |k|.$$

Observe that

$$\mathscr{K}(t,k) = |k||\widehat{\sigma}_1(k)|^2 \int_0^t p_c(t-\tau)\,\tau|k|\widehat{\mathscr{M}}(\tau k)\,\mathrm{d}\tau.$$

Based on **(H2)**, **(H5)** and Lemma 2.3, we write

$$\left|\int_0^t p_c(t-\tau)\,\tau|k|\widehat{\mathscr{M}}(\tau k)\,\mathrm{d}\tau\right| \lesssim \int_0^t \langle t-\tau\rangle^{-(n-1)}\langle \tau k\rangle^{-(\bar{m}-1)}\,\mathrm{d}\tau.$$

Lemma 3.3 allows us to dominate this quantity by $\langle k\rangle^{\gamma}\langle tk\rangle^{-\gamma}$ for any $\gamma \geq 0$ such that $\gamma \leq \bar{m}-1$ and $\gamma < n-2$. In particular, with **(H1)** and **(H5)** it applies with $\gamma = m + 1^+/2$. We conclude that

$$\int_0^{+\infty} \langle tk\rangle^{2m} |\mathscr{K}(t,k)|^2\,\mathrm{d}t \lesssim |k| \left(\sup_k \langle k\rangle^{2m+1^+} |\widehat{\sigma}_1(k)|^4\right) \int_0^{+\infty} \langle tk\rangle^{-(1^+)} |k|\,\mathrm{d}t \lesssim |k|$$

which ends the proof. ∎

We can now state the results for linearized Landau damping in finite regularity on the torus or the whole space. For the sake of conciseness we only mention the case of $\mathbb{R}^d$ (see [33] for further results).

Proposition 3.4 (Linearized Landau damping on $\mathbb{R}^d$ with finite regularity) *Let $\mathbb{X}^d = \mathbb{R}^d$ and $m > 0$. Let us assume **(H1)**–**(H5)** and* (**L**). *There exists a constant $C > 0$ such that for every $k \in \mathbb{R}^d \setminus \{0\}$ and for every $t \geq 0$,*

$$|\widehat{\varrho}(t,k)| \leq C\langle tk\rangle^{-m}.$$

Moreover, if m is large enought, then, as $t \to +\infty$, the fluctuation of spatial density $\varrho(t)$, the force terme $\nabla_x\phi$ and the fluctuation of media $\psi(t)$ converge strongly to 0. To be more specific:

- *If $m > d/2$, then for every $r \in [0, m - \frac{d}{2})$ there exists a constant $C_r > 0$ such that*

$$\|\varrho(t)\|_{H^r_x} \leq C_r \langle t\rangle^{-\frac{d}{2}}.$$

- *If $m > (d+2)/2$, then for every $r \in [0, m_1 - \frac{d+2}{2})$ there exists a constant $\bar{C}_r > 0$ such that*

$$\|\nabla_x\phi_I(t)\|_{H^r_x} \leq \bar{C}_r \langle t\rangle^{-(n-1)}$$

  and for every $r \in [0, 2m_1 - \frac{d+2}{2})$ there exists a constant $\bar{C}'_r$ such that

$$\|\nabla_x\phi_S(t)\|_{H^r_x} \leq \bar{C}'_r \langle t\rangle^{-\frac{d+2}{2}}.$$

- *If $m > d/2$ and $n > d+3$, then for every $r \in [0, m_1 - \frac{d}{2})$ there exists a constant $\widetilde{C}_r > 0$ such that*

$$\left\|\psi(t) - \dot{W}(ct)\psi_0 - \frac{1}{c}\,W(ct)\psi_1\right\|_{L^\infty_z H^r_x} \leq \widetilde{C}_r \langle t\rangle^{-\frac{d}{2}}.$$

Remark 3.5 *Let us detail a few examples:*

(i) For the density, with $d = 3$, $n \geq 5$, $m = 2$, $m_0 = 3$, $m_1 = 4$, $m_2 > (3n-1)/2$ and $\bar{m} = 5$, we get

$$\|\varrho(t)\|_{L^2_x} \lesssim \langle t\rangle^{-\frac{3}{2}}.$$

Moreover, with $d = 3$, $n \geq 8$, $m = 5$, $m_0 = 6$, $m_1 = 7$, $m_2 > (3n-1)/2$ and $\bar{m} = 8$, we obtain

$$\|\varrho(t)\|_{L^\infty_x} \lesssim \|\varrho(t)\|_{H^3_x} \lesssim \langle t\rangle^{-\frac{3}{2}}.$$

(ii) For the force, with $d = 3$, $n \geq 6$, $m = 3$, $m_0 = 4$, $m_1 = 5$, $m_2 > (3n-1)/2$ and $\bar{m} = 6$, we get

$$\|\nabla_x\phi(t)\|_{L^2_x} \lesssim \langle t\rangle^{-\frac{5}{2}}.$$

Moreover, with $d = 3$, $n \geq 6$, $m = 3$, $m_0 = 4$, $m_1 = 6$, $m_2 > (3n-1)/2$ and $\bar{m} = 6$, we obtain

$$\|\nabla_x\phi(t)\|_{L^\infty_x} \lesssim \|\nabla_x\phi(t)\|_{H^3_x} \lesssim \langle t\rangle^{-\frac{5}{2}}.$$

(iii) For the vibration field, with $d = 3$, $n \geq 7$, $m = 2$, $m_0 = 3$, $m_1 = 4$, $m_2 > (3n-1)/2$ and $\bar{m} = 5$, we get

$$\left\| \psi(t) - \dot{W}(ct)\psi_0 - \frac{1}{c}\, W(ct)\psi_1 \right\|_{L^\infty_z L^2_x} \lesssim \langle t \rangle^{-\frac{3}{2}}.$$

Moreover, with $d = 3$, $n \geq 7$, $m = 2$, $m_0 = 3$, $m_1 = 5$, $m_2 > (3n-1)/2$ and $\bar{m} = 5$, we have

$$\begin{aligned}\left\| \psi(t) - \dot{W}(ct)\psi_0 - \tfrac{1}{c}\, W(ct)\psi_1 \right\|_{L^\infty_z L^\infty_x} \\ \lesssim \left\| \psi(t) - \dot{W}(ct)\psi_0 - \tfrac{1}{c}\, W(ct)\psi_1 \right\|_{L^\infty_z H^3_x} \lesssim \langle t \rangle^{-\frac{3}{2}}.\end{aligned}$$

Remark 3.6 *As explained in Proposition 3.2, the decay of $\widehat{\varrho}(t,k)$ is directly related to the dispersion of the wave equation, and thus on n. This explains the constraints on the dimension n. Nevertheless, when $n \geq 3$ is odd, we can obtain the time decay of $\widehat{\varrho}(t,k)$ without further restrictions on n. Accordingly, with **(H1')–(H3')** et **(H4)–(H5)** the convergence to 0 of the density fluctuation ϱ and the force $\nabla_x \phi$ can be established. However, constraints appear when considering the fluctuation of the medium ψ: with the norms we are using, we need $n > d+3$. In dimension $d = 3$, this excludes $n = 3$ and $n = 5$. This restriction can be relaxed by considering instead the supremum over a ball $B(0,R)$ of finite radius. For instance, in dimension $d = 3$ with $n = 3$, assuming **(H1')–(H3')** and **(H4)–(H5)**, we can show that, for any $0 < R < \infty$*

$$\sup_{z \in B(0,R)} \left\| \psi(t,z) - \dot{W}(ct)\psi_0(z) - \frac{1}{c}\, W(ct)\psi_1(z) \right\|_{H^r_x} \leq C_R \langle t \rangle^{-1}.$$

where $C_R > 0$ blows up as $R \to +\infty$. Further details on this issue can be found in the proof of Proposition 3.4.

Proof of Proposition 3.4. Owing to **(H1)–(H5)** we can apply Proposition 3.2 and Lemma 3.1. Proposition 3.2 ensures that (27) holds and from this, we can exhibit $C > 0$, independent of k, such that for any $k \in \mathbb{R}^d \setminus \{0\}$,

$$\langle tk \rangle^m \, |\widehat{\varrho}(t,k)| \leq C.$$

That $\varrho(t)$ converges to 0 is a consequence of

$$\begin{aligned}\|\varrho(t)\|^2_{H^r_x} \simeq \|\varrho(t)\|^2_{L^2_x} + \|\varrho(t)\|^2_{\dot{H}^r_x} &= \int_{\mathbb{R}^d} |\widehat{\varrho}(t,k)|^2 \, \mathrm{d}k + \int_{\mathbb{R}^d} |k|^{2r} \, |\widehat{\varrho}(t,k)|^2 \, \mathrm{d}k \\ &\lesssim \frac{1}{t^d} \int_{\mathbb{R}^d} \langle tk \rangle^{-2m} \, t^d \, \mathrm{d}k + \frac{1}{t^{d+2r}} \int_{\mathbb{R}^d} |tk|^{2r} \langle tk \rangle^{-2m} \, t^d \, \mathrm{d}k \\ &= \frac{1}{t^d} \int_{\mathbb{R}^d} \langle x \rangle^{-2m} \, \mathrm{d}x + \frac{1}{t^{d+2r}} \int_{\mathbb{R}^d} |x|^{2r} \langle x \rangle^{-2m} \, \mathrm{d}x,\end{aligned}$$

where all integrals are finite provided $2r - 2m < -d$, that is $r < m - d/2$.

Next, we estimate both terms of $\nabla_x \phi = \nabla_x \phi_I + \nabla_x \phi_S$. We have

$$\|\nabla_x \phi_I(t)\|^2_{H^r_x} \simeq \int_{\mathbb{R}^d} |k|^2 \left| \widehat{\phi}_I(t,k) \right|^2 \, \mathrm{d}k + \int_{\mathbb{R}^d} |k|^{2r+2} \left| \widehat{\phi}_I(t,k) \right|^2 \, \mathrm{d}k,$$

and, as noticed when proving Proposition 3.2, $\widehat{\phi}_I(t,k)$ satisfies (29). It follows that

$$\|\nabla_x \phi_I(t)\|^2_{H^r_x} \lesssim_c \left(\int_{\mathbb{R}^d} |k|^2 |\widehat{\sigma}_1(k)|^2 \,\mathrm{d}k + \int_{\mathbb{R}^d} |k|^{2r+2} |\widehat{\sigma}_1(k)|^2 \,\mathrm{d}k \right) \langle t\rangle^{-2(n-1)},$$

where the two integrals are finite, due to **(H4)**, when $r < m_1 - 1 - d/2$. Next, we apply Lemma 2.3-(ii):

$$\begin{aligned}
\|\nabla_x \phi_S(t)\|^2_{H^r_x} &\simeq \int_{\mathbb{R}^d} |k|^2 \left|\widehat{\phi}_S(t,k)\right|^2 \,\mathrm{d}k + \int_{\mathbb{R}^d} |k|^{2r+2} \left|\widehat{\phi}_S(t,k)\right|^2 \,\mathrm{d}k \\
&= \int_{\mathbb{R}^d} (|k|^2 + |k|^{2r+2}) |\widehat{\sigma}_1(k)|^4 \left| \int_0^t p_c(t-\tau) \widehat{\varrho}(\tau,k) \,\mathrm{d}\tau \right|^2 \,\mathrm{d}k \\
&\lesssim_c \int_{\mathbb{R}^d} (|k|^2 + |k|^{2r+2}) |\widehat{\sigma}_1(k)|^4 \left| \int_0^t \langle t-\tau\rangle^{-(n-1)} \langle \tau k\rangle^{-m} \,\mathrm{d}\tau \right|^2 \,\mathrm{d}k.
\end{aligned}$$

By Lemma 3.3, for any $\gamma \geq 0$ such that $\gamma \leq m$ and $\gamma < n-2$, we get

$$\int_0^t \langle t-\tau\rangle^{-(n-1)} \langle \tau k\rangle^{-m} \,\mathrm{d}\tau \lesssim \langle k\rangle^{\gamma} \langle tk\rangle^{-\gamma},$$

and we conclude with

$$\begin{aligned}
\|\nabla_x \phi_S(t)\|^2_{H^r_x} &\lesssim_c \int_{\mathbb{R}^d} (|k|^2 + |k|^{2r+2}) |\widehat{\sigma}_1(k)|^4 \langle k\rangle^{2\gamma} \langle tk\rangle^{-2\gamma} \,\mathrm{d}k \\
&\lesssim \left(\sup_k \langle k\rangle^{2r+2\gamma} |\widehat{\sigma}_1(k)|^4 \right) t^{-(d+2)} \int_{\mathbb{R}^d} |tk|^2 \langle tk\rangle^{-2\gamma} t^d \,\mathrm{d}k \\
&= \left(\sup_k \langle k\rangle^{2r+2\gamma} |\widehat{\sigma}_1(k)|^4 \right) t^{-(d+2)} \int_{\mathbb{R}^d} |x|^2 \langle x\rangle^{-2\gamma} \,\mathrm{d}x.
\end{aligned}$$

The last integral is finite when $2 - 2\gamma < -d$, that is $\gamma > (d+2)/2$ and the supremum over k is finite too provided $2r + 2\gamma \leq 4m_1$, that is $r \leq 2m_1 - \gamma$.

We turn to ψ. We have

$$\psi(t) - \dot{W}(ct)\psi_0 - \frac{1}{c} W(ct)\psi_1 = -\frac{1}{c} \int_0^t W(c[t-\tau]) \sigma_2 \, \sigma_1 \star \varrho(\tau) \,\mathrm{d}\tau.$$

Hence, for any $z \in \mathbb{R}^n$, we obtain

$$\begin{aligned}
&\|\psi(t,z) - \dot{W}(ct)\psi_0(z) - \frac{1}{c} W(ct)\psi_1(z)\|^2_{H^r_x} \\
&\qquad \simeq \int_{\mathbb{R}^d} \left(1 + |k|^{2r}\right) |\widehat{\sigma}_1(k)|^2 \left| \frac{1}{c} \int_0^t W(c[t-\tau]) \sigma_2(z) \, \widehat{\varrho}(\tau,k) \,\mathrm{d}\tau \right|^2 \,\mathrm{d}k.
\end{aligned}$$

We combine the dispersion estimate (15) to (25) and we arrive at

$$\left| \frac{1}{c} \int_0^t W(c[t-\tau]) \sigma_2(z) \, \widehat{\varrho}(\tau,k) \,\mathrm{d}\tau \right| \lesssim_c \int_0^t \langle t-\tau\rangle^{-\frac{n-1}{2}} \langle \tau k\rangle^{-m} \,\mathrm{d}\tau.$$

Lemma 3.3 allows us to obtain, for any $\gamma \geq 0$ such that $\gamma \leq m$ and $\gamma < (n-1)/2 - 1$,

$$\int_0^t \langle t-\tau\rangle^{-\frac{n-1}{2}} \langle \tau k\rangle^{-m} \,\mathrm{d}\tau \lesssim \langle k\rangle^{\gamma} \langle tk\rangle^{-\gamma}.$$

We deduce that

$$
\begin{aligned}
&\|\psi(t,z) - \dot{W}(ct)\psi_0(z) - \frac{1}{c}\, W(ct)\psi_1(z)\|^2_{H^r_x} \\
&\qquad\qquad\lesssim_c \int_{\mathbb{R}^d} \left(1 + |k|^{2r}\right) |\widehat{\sigma}_1(k)|^2 \langle k\rangle^{2\gamma} \langle tk\rangle^{-2\gamma}\,\mathrm{d}k \\
&\qquad\qquad\lesssim \left(\sup_k \langle k\rangle^{2r+2\gamma} |\widehat{\sigma}_1(k)|^2\right) t^{-d} \int_{\mathbb{R}^d} \langle tk\rangle^{-2\gamma}\, t^d\,\mathrm{d}k \\
&\qquad\qquad\qquad = \left(\sup_k \langle k\rangle^{2r+2\gamma} |\widehat{\sigma}_1(k)|^2\right) t^{-d} \int_{\mathbb{R}^d} \langle x\rangle^{-2\gamma}\,\mathrm{d}x.
\end{aligned}
$$

The last integral is finite when $\gamma > d/2$ (this imposes $m > d/2$ and $n > d+3$). The supremum over k is finite provided $2r + 2\gamma \le 2m_1$, that is $r \le m_1 - \gamma$.

The estimate in Remark 3.6 is obtained by restricting to the z's in the ball $B(0, |ct|/4)$. We apply the refined estimate (17), gathered to (25). We get

$$
\begin{aligned}
&\left| \frac{1}{c} \int_0^t W\big(c[t-\tau]\big)\sigma_2(z)\, \widehat{\varrho}(\tau,k)\,\mathrm{d}\tau \right| \\
&\qquad\qquad \lesssim \frac{1}{c}\, |k|^{-\frac{1}{2}} \int_0^t \big\langle c|t-\tau| \cdot \big|c|t-\tau| - |z|\,\big|\big\rangle^{-\frac{n-1}{2}} \langle \tau k\rangle^{-m}\,\mathrm{d}\tau.
\end{aligned}
$$

We proceed as for proving Lemma 3.3: for any $\gamma \ge 0$ we obtain

$$
\begin{aligned}
&\int_0^t \big\langle c|t-\tau| \cdot \big|c|t-\tau| - |z|\,\big|\big\rangle^{-\frac{n-1}{2}} \langle \tau k\rangle^{-m}\,\mathrm{d}\tau \\
&\quad \lesssim \frac{\langle 2k\rangle^{\gamma}}{\langle tk\rangle^{\gamma}} \int_0^{t/2} \big\langle c|t-\tau| \cdot \big|c|t-\tau| - |z|\,\big|\big\rangle^{-\frac{n-1}{2}} \langle t/2\rangle^{\gamma}\,\mathrm{d}\tau \\
&\qquad\qquad + \int_{t/2}^t \big\langle c|t-\tau| \cdot \big|c|t-\tau| - |z|\,\big|\big\rangle^{-\frac{n-1}{2}} \langle tk/2\rangle^{-m}\,\mathrm{d}\tau \\
&\quad \le \frac{\langle 2k\rangle^{\gamma}}{\langle tk\rangle^{\gamma}} \int_0^{t/2} \big\langle c|t-\tau| \cdot \big|c|t-\tau| - |z|\,\big|\big\rangle^{-\frac{n-1}{2}} \langle t-\tau\rangle^{\gamma}\,\mathrm{d}\tau \\
&\qquad\qquad + \langle tk/2\rangle^{-m} \int_{t/2}^t \big\langle c|t-\tau| \cdot \big|c|t-\tau| - |z|\,\big|\big\rangle^{-\frac{n-1}{2}}\,\mathrm{d}\tau \\
&\quad = \frac{\langle 2k\rangle^{\gamma}}{\langle tk\rangle^{\gamma}} \int_{t/2}^t \big\langle cu \cdot \big|cu - |z|\,\big|\big\rangle^{-\frac{n-1}{2}} \langle u\rangle^{\gamma}\,\mathrm{d}u \\
&\qquad\qquad + \langle tk/2\rangle^{-m} \int_0^{t/2} \big\langle cu \cdot \big|cu - |z|\,\big|\big\rangle^{-\frac{n-1}{2}}\,\mathrm{d}u.
\end{aligned}
$$

First, $ct/2 \le cu \le ct$ and $0 \le |z| \le ct/4$ imply $|cu - |z|\,| \ge ct/4 \ge cu/2$ so that

$$
\big\langle cu \cdot \big|cu - |z|\,\big|\big\rangle^{-1} \le \left\langle \frac{c^2u^2}{2} \right\rangle^{-1} \lesssim_c \langle u\rangle^{-2}.
$$

We thus deduce that

$$
\int_{t/2}^t \big\langle cu \cdot \big|cu - |z|\,\big|\big\rangle^{-\frac{n-1}{2}} \langle u\rangle^{\gamma}\,\mathrm{d}u \lesssim \int_0^{+\infty} \langle u\rangle^{-(n-1)} \langle u\rangle^{\gamma}\,\mathrm{d}u
$$

which is finite when $\gamma < n-2$. Second, we have

$$\int_0^{t/2} \langle cu \cdot |cu - |z|\,|\rangle^{-\frac{n-1}{2}}\,\mathrm{d}u \lesssim_c \int_{\mathbb{R}} \langle u \cdot |u - |z|\,|\rangle^{-\frac{n-1}{2}}\,\mathrm{d}u.$$

As $|u| \to +\infty$, we have

$$\langle u \cdot |u - |z|\,|\rangle^{-\frac{n-1}{2}} \simeq_{|z|} \langle u\rangle^{-(n-1)}$$

which is finite provided $n \geq 3$. However, we should make precise how it depends on $|z|$. To this end, we write

$$\begin{aligned}\int_{\mathbb{R}} \langle u \cdot |u - |z|\,|\rangle^{-\frac{n-1}{2}}\,\mathrm{d}u &= \int_{\mathbb{R}} \langle (u+|z|/2)\cdot(u-|z|/2)\rangle^{-\frac{n-1}{2}}\,\mathrm{d}u \\ &= \int_{\mathbb{R}} \left\langle u^2 - |z|^2/4\right\rangle^{-\frac{n-1}{2}}\,\mathrm{d}u = \int_{\mathbb{R}} \left(\frac{\langle u^2\rangle}{\langle u^2 - |z|^2/4\rangle}\right)^{\frac{n-1}{2}} \left\langle u^2\right\rangle^{-\frac{n-1}{2}}\,\mathrm{d}u.\end{aligned}$$

A mere function analysis shows that, for any $a \geq 0$.

$$x \mapsto \frac{\langle x\rangle^2}{\langle x-a\rangle^2}$$

reaches its maximum over $[0,+\infty)$ for $x = (a+\sqrt{a^2+4})/2$, which leads to

$$\left(\frac{\langle u^2\rangle}{\langle u^2 - |z|^2/4\rangle}\right)^{\frac{n-1}{2}} \lesssim |z|^{n-1}.$$

It follows that

$$\int_{\mathbb{R}} \langle u \cdot |u - |z|\,|\rangle^{-\frac{n-1}{2}}\,\mathrm{d}u \lesssim |z|^{n-1} \int_{\mathbb{R}} \langle u\rangle^{-(n-1)}\,\mathrm{d}u \lesssim |z|^{n-1}.$$

Therefore, when $n \geq 3$, for any $\gamma \in [0, n-2)$ and $z \in B(0, ct/4)$, we have

$$\left|\frac{1}{c}\int_0^t W(c[t-\tau])\sigma_2(z)\,\widehat{\varrho}(\tau,k)\,\mathrm{d}\tau\right| \lesssim_c \langle k\rangle^{\gamma}\langle tk\rangle^{-\gamma} + |z|^{n-1}\langle tk\rangle^{-m}.$$

We infer that

$$\begin{aligned}&\|\psi(t,z) - \dot{W}(ct)\psi_0(z) - \frac{1}{c}W(ct)\psi_1(z)\|^2_{H^r_x} \\ &\quad\lesssim_c \int_{\mathbb{R}^d} \left(1+|k|^{2r}\right)|\widehat{\sigma}_1(k)|^2 \left(\langle k\rangle^{2\gamma}\langle tk\rangle^{-2\gamma} + |z|^{2(n-1)}\langle tk\rangle^{-2m}\right)\mathrm{d}k \\ &\quad\lesssim \frac{\langle z\rangle^{2(n-1)}}{t^d}\left(\sup_k \langle k\rangle^{2r+2\gamma}|\widehat{\sigma}_1(k)|^2\right)\int_{\mathbb{R}^d}\left(\langle tk\rangle^{-2\gamma} + \langle tk\rangle^{-2m}\right) t^d\,\mathrm{d}k \\ &\quad= \frac{\langle z\rangle^{2(n-1)}}{t^d}\left(\sup_k \langle k\rangle^{2r+2\gamma}|\widehat{\sigma}_1(k)|^2\right)\int_{\mathbb{R}^d}\left(\langle x\rangle^{-2\gamma} + \langle x\rangle^{-2m}\right)\mathrm{d}x\end{aligned}$$

where the last integral is finite when $\gamma, m > d/2$. When n is even, we can use **(H1')**–**(H3')** instead: the condition on m imposes regularity on the data but no further restriction on n. Such restriction arise from the condition on γ: we already have $\gamma \in [0, n-2)$. To be more specific, we have $n > (d+4)/2$. For $d = 1$ this holds for any $n \geq 3$; but, for for $d = 2$ or for the most relevant case $d = 3$, we should assume $n \geq 4$ and $n \geq 5$, respectively. Nonetheless, it is equally possible to make use of the decay of

$\widehat{\sigma}_1$ in order to obtain a singularity which remains integrable at 0 and and gives more integrability at $+\infty$. The price to be paid is the strengthening of the regularity of σ_1 and, more importantly, a reduced convergence rate for large times. To be specific, we get

$$\begin{aligned}
&\|\psi(t,z) - \dot{W}(ct)\psi_0(z) - \frac{1}{c}\,W(ct)\psi_1(z)\|^2_{H^r_x} \\
&\quad\lesssim_c \int_{\mathbb{R}^d} \left(1+|k|^{2r}\right) |\widehat{\sigma}_1(k)|^2 \left(\langle k\rangle^{2\gamma}\langle tk\rangle^{-2\gamma} + |z|^{2(n-1)}\langle tk\rangle^{-2m}\right) \mathrm{d}k \\
&\quad= \int_{\mathbb{R}^d} \left(|k|^{d-1} + |k|^{2r+d-1}\right) |k|^{-(d-1)}|\widehat{\sigma}_1(k)|^2 \left(\langle k\rangle^{2\gamma}\langle tk\rangle^{-2\gamma} + |z|^{2(n-1)}\langle tk\rangle^{-2m}\right) \mathrm{d}k \\
&\quad\lesssim \frac{\langle z\rangle^{2(n-1)}}{t} \left(\sup_k \langle k\rangle^{2r+2\gamma+d-1}|\widehat{\sigma}_1(k)|^2\right) \int_{\mathbb{R}^d} |tk|^{-(d-1)} \left(\langle tk\rangle^{-2\gamma} + \langle tk\rangle^{-2m}\right) t^d\,\mathrm{d}k \\
&\quad= \frac{\langle z\rangle^{2(n-1)}}{t} \left(\sup_k \langle k\rangle^{2r+2\gamma+d-1}|\widehat{\sigma}_1(k)|^2\right) \int_{\mathbb{R}^d} |x|^{-(d-1)} \left(\langle x\rangle^{-2\gamma} + \langle x\rangle^{-2m}\right) \mathrm{d}x.
\end{aligned}$$

The last integral is finite when $\gamma > 1/2$. This is compatible with the condition $\gamma < n-2$ provided $n \geq 3$. It is possible to optimize this approach in order to find a sharp decay rate. ■

3.3 Linearized Landau damping in analytic regularity

That the linearized Landau damping holds with an exponential rate relies, from (24), on an estimate on ϱ like

$$|\widehat{\varrho}(t,k)| \leq C\, e^{-\lambda|tk|} \tag{30}$$

(see [27, section 3]) for some $\lambda > 0$. To this end we shall use the analog in analytic regularity of Lemma 3.1.

Lemma 3.7 *Suppose that $\mathscr{L}\mathscr{K}(\omega|k|,k)$is well-defined on $k \in \mathbb{X}^{\star d}\setminus\{0\}$ et $\omega \in \{z \in \mathbb{C} \mid \mathrm{Re}(z) > -\Lambda\}$ for a certain $\Lambda > 0$. We also suppose that*

$$\inf_{k\in\mathbb{X}^{\star d}\setminus\{0\}} \left|1 - \mathscr{L}\mathscr{K}(\omega|k|,k)\right| \geq \kappa > 0 \quad \textit{for } \mathrm{Re}(\omega) > -\Lambda, \tag{$\mathbf{L}'$}$$

is fulfilled. Then, for any $0 < \lambda < \Lambda$ we can find $C_{LD} > 0$, which does not depend on k, such that any solution of (24) *satisfies, for any $k \in \mathbb{X}^{\star d}\setminus\{0\}$,*

$$\int_0^{+\infty} e^{2\lambda|tk|}\,|\widehat{\varrho}(t,k)|^2\,\mathrm{d}t \leq C_{LD}^2 \int_0^{+\infty} e^{2\lambda|tk|}\,|a(t,k)|^2\,\mathrm{d}t. \tag{31}$$

We refer the reader to [32, Proof of Lemma 3.5] or [6, Section 4] for details on this statement. It allows us to derive the following estimate in L^∞_t norm

$$\begin{aligned}
e^{\lambda|tk|}\,|\widehat{\varrho}(t,k)| &\leq e^{\lambda|tk|}\,|a(t,k)| \\
&\qquad + C_{LD}\left(\int_0^{+\infty} e^{2\lambda|\tau k|}\,|\mathscr{K}(\tau,k)|^2\,\mathrm{d}\tau\right)^{1/2} \left(\int_0^{+\infty} e^{2\lambda|\tau k|}\,|a(\tau,k)|^2\,\mathrm{d}\tau\right)^{1/2}.
\end{aligned}$$

It remains to check that the data satisfy

$$\begin{cases} \sup\limits_{\substack{t\geq 0\\ k\in\mathbb{X}^{\star d}\setminus\{0\}}} e^{\lambda|tk|}\,|a(t,k)| < +\infty, \\ \sup\limits_{k\in\mathbb{X}^{\star d}\setminus\{0\}} \left(\int_0^{+\infty} e^{2\lambda|\tau k|}\,|\mathscr{K}(\tau,k)|^2\,\mathrm{d}\tau\right)\left(\int_0^{+\infty} |k|e^{2\lambda|\tau k|}\,|a(\tau,k)|^2\,\mathrm{d}\tau\right) < +\infty. \end{cases} \tag{32}$$

In order to apply Lemma 3.7 and to check that (32) holds, we assume

(K1) $n \geq 3$ is odd,

(K2) $\sigma_2 \in C^0_c(\mathbb{R}^n)$ with $\mathrm{supp}(\sigma_2) \subset B(0,R_2)$,

(K3) we have $\mathrm{supp}(\psi_0,\psi_1) \subset \mathbb{X}^d \times B(0,R_I)$, for some $0 < R_I < \infty$, and

$$\sup_{k\in\mathbb{X}^{\star d}} \left\{\int_{\mathbb{R}^n} \left(|\widehat{\psi}_1(k,z)|^2 + c^2|\nabla_z\widehat{\psi}_0(k,z)|^2\right)\mathrm{d}z\right\} = \mathscr{E}_I < \infty,$$

(K4) the function $\sigma_1 : \mathbb{X}^d \to (0,\infty)$ is radially symmetric and real analytic, and in particular (see [32, Proposition 3.16]) there exists $C_1, \lambda_1 > 0$ such that, for any $k \in \mathbb{X}^{\star d}$, $|\widehat{\sigma}_1(k)| \leq C_1\, e^{-\lambda_1|k|}$.

(K5) there exists $C_0, \lambda_0 > 0$ such that for any $\xi \in \mathbb{R}^d$, $k \in \mathbb{X}^{\star d}$ we have

$$|\widehat{\mathscr{M}}(\xi)| \leq C\, e^{-\bar{\lambda}|\xi|}, \qquad |\widehat{f}_0(k,\xi)| \leq C_0\, e^{-\lambda_0|\xi|}.$$

Namely, we assume analytic regularity on the data with **(K4)** and **(K5)**. Note that **(K4)** is not a strong restriction in the present context, contrarily to what it could be for the Vlasov case, since for this model σ_1 is naturally smooth. Moreover, physically the form function σ_1 would naturally be compactly supported (the support being interpreted as the "domain of influence" of the particle), which does not make sense in the analytic framework. Thus, we should here think σ_1 as a peaked bump function. We also bear in mind the fact that σ_1 is radially symmetric: its Fourier coefficients are real and we have $\widehat{\sigma_1 \star \sigma_1}(k) = |\widehat{\sigma_1}(k)|^2 \geq 0$. These assumptions, together with the finite speed of propagation for the wave equation, allow us to control the "initial data" contribution in (22) and the kernel (23). Let us explain the role of **(K3)** for the associated contribution to (21) in (22). In (21), ψ_I is the solution of the wave equation on $\mathbb{R}^n$, starting form initial data (ψ_0,ψ_1). The space variable $x \in \mathbb{X}^d$ appears only as a parameter in this equation. Assumption **(K3)** means that the Fourier transform (with respect to the parameter) of the initial data has finite and uniformly bounded energy. When $\mathbb{X}^d = \mathbb{T}^d$, **(K3)** holds under the condition

$$\iint_{\mathbb{X}^d\times\mathbb{R}^n} \left(|\psi_1(x,z)|^2 + c^2|\nabla_z\psi_0(x,z)|^2\right)\mathrm{d}z\ \mathrm{d}x = \mathscr{E}_I < \infty,$$

which implies that the Fourier coefficients of the energy lies in $\ell^2(\mathbb{Z}^d)$, and thus in $\ell^\infty(\mathbb{Z}^d)$. This assumption is quite natural since this quantity is involved in the global energy balance for (2a)–(2d), see [9, 10, 31]. Working in $\mathbb{R}^d$, this has to be replaced by condition **(K3)**.

A naive intuition would relate the damping rate to the decay rate of p_c. In finite regularity, we indeed obtained a polynomial damping rate assuming the polynomial decay of p_c. The analytic framework is more demanding and it is not enough to assume the exponential decay of p_c. The proof of Lemma 3.9 below will make the role of the stronger assumptions **(K1)-(K2)** clear.

Proposition 3.8 *Suppose **(K1)–(K5)**. The quantity $\mathscr{L}\mathscr{K}(\omega|k|, k)$ is well-defined for any $\omega \in \mathbb{C}$ such that* $\mathrm{Re}(\omega) > -\bar{\lambda}$ *and* (32) *holds for any $\lambda > 0$ such that*

$$\lambda < \min\Big(\lambda_0, \bar{\lambda}, \frac{c\lambda_1}{R_2}, \frac{c\lambda_1}{R_I + R_2}\Big).$$

The statement follows from a direct application of the following claim, and reproducing the computations of the proof of Proposition 3.2.

Lemma 3.9 *Suppose **(K1)–(K5)**.*

(i) Let $a(t,k)$ be defined by (22). *Then, there exists $\alpha > 0$ such that for every $0 < \lambda < \min(\lambda_0, \bar{\lambda}, c\lambda_1/(R_I + R_2))$, $|a(t,k)| \le \alpha e^{-\lambda|k|t}$ holds for any $t \ge 0$, $k \in \mathbb{X}^{\star d}$.*

(ii) Let $\mathscr{K}(t,k)$ be defined by (23). *Then, there exists $C > 0$ such that for every $0 < \lambda < \min(\bar{\lambda}, c\lambda_1/R_2)$, $|\mathscr{K}(t,k)| \le C e^{-\lambda|k|t}$ holds for any $t \ge 0$, $k \in \mathbb{X}^{\star d}$.*

Proof. We start with the proof of (i). First of all, assumption **(K5)** tells us that

$$|\widehat{f}_0(k, tk)| \le C_0\, e^{-\lambda_0 t|k|}$$

and since

$$|a(t,k)| \lesssim |\widehat{f}_0(k, tk)| + |k|^2 \int_0^t \Big|\widehat{\phi}_I(\tau, k)\Big|\ (t-\tau)\ \Big|\widehat{\mathscr{M}}((t-\tau)k)\Big|\ \mathrm{d}\tau,$$

we only have to deal with second term. Then, relation (21) can be recast as

$$\phi_I(t,x) = \int_{\mathbb{X}^d} \sigma_1(x-y)\left(\int_{\mathbb{R}^n} \sigma_2(z)\psi_I(t,x,z)\,\mathrm{d}z\right)\mathrm{d}y$$

with ψ_I the solution of the free wave equation

$$\begin{aligned} &(\partial^2_{tt} - c^2\Delta_z)\psi_I = 0,\\ &(\psi_I, \partial_t\psi_I)\big|_{t=0} = (\psi_0, \psi_1). \end{aligned}$$

Assumptions **(K1)** and **(K3)** allow us to make use of Huygens' principle which tells us that

$$\mathrm{supp}(\psi_I(t,x,\cdot)) \subset \big\{z \in \mathbb{R}^n,\ ct - R_I \le |z| \le ct + R_I\big\}.$$

Therefore, by virtue of **(K2)**, the product $\sigma_2(z)\psi_I(t,x,z)$ vanishes when $t \ge \frac{R_I + R_2}{c} = S_0$ for any $x \in \mathbb{X}^d$, $z \in \mathbb{R}^n$ (see Fig. 1). Hence, ϕ_I is supported in $[0, S_0] \times \mathbb{X}^d$ and we can write

$$\widehat{\phi}_I(\tau, k) = \widehat{\sigma}_1(k)\left(\int_{\mathbb{R}^n} \sigma_2\, \widehat{\psi}_I(\tau, k)\,\mathrm{d}z\right)\mathbf{1}_{t \le S_0}.$$

Moreover, thanks to Sobolev's embedding, energy conservation for the wave equation and assumption **(K3)**, we have

$$\left|\int_{\mathbb{R}^n} \sigma_2\, \widehat{\psi}_I(\tau,k)\,\mathrm{d}z\right| \le \|\sigma_2\|_{L_z^{\frac{2n}{n+2}}} \|\widehat{\psi}_I(\tau,k)\|_{L_z^{\frac{2n}{n-2}}}$$
$$\lesssim \|\sigma_2\|_{L_z^{\frac{2n}{n+2}}} \|\nabla_z \widehat{\psi}_I(\tau,k)\|_{L^2_z} \le \frac{1}{c}\|\sigma_2\|_{L_z^{\frac{2n}{n+2}}} \left(\|\partial_t \widehat{\psi}_I(\tau,k)\|^2_{L^2_z} + c^2 \|\nabla_z \widehat{\psi}_I(\tau,k)\|^2_{L^2_z}\right)^{\frac12}$$
$$= \frac{1}{c}\|\sigma_2\|_{L_z^{\frac{2n}{n+2}}} \left(\|\widehat{\psi}_1(k)\|^2_{L^2_z} + c^2 \|\nabla_z \widehat{\psi}_0(k)\|^2_{L^2_z}\right)^{\frac12} \le \frac{1}{c}\|\sigma_2\|_{L_z^{\frac{2n}{n+2}}} \sqrt{\mathscr{E}_I}.$$

From these two facts, and thanks to **(K4)**–**(K5)**, we can eventually conclude as follows: for every $0<\lambda<\min(\bar\lambda, \lambda_1/S_0)$,

$$|k|^2 \int_0^t \left|\widehat{\Phi}_I(\tau,k)\right| (t-\tau) \left|\widehat{\mathscr{M}}((t-\tau)k)\right| \mathrm{d}\tau \lesssim |k|^2 e^{-\lambda_1|k|} \int_0^{S_0} |t-\tau| e^{-\bar\lambda(t-\tau)|k|}\,\mathrm{d}\tau$$
$$= |k|^2 e^{-\lambda_1|k|} \int_0^{S_0} |t-\tau| e^{-\lambda(t-\tau)|k|} e^{-(\bar\lambda-\lambda)(t-\tau)|k|}\,\mathrm{d}\tau \le S_0^2 \left(\sup_k |k|^2 e^{-(\lambda_1-\lambda S_0)|k|}\right) e^{-\lambda|tk|}.$$

Accordingly, $a(t,k)$ is dominated by $\mathscr{O}(e^{-\lambda|k|t})$, uniformly with respect to k, for $0<\lambda<\min(\lambda_0, \bar\lambda, \lambda_1/S_0)$. (Note that S_0 behaves like $1/c$; as c becomes large, only λ_0 and $\bar\lambda$ are relevant in this condition.)

We turn now on the estimate on $\mathscr{K}$. With **(K4)**, **(K5)** and Lemma 2.3 (we use **(K1)** and **(K2)** in order to apply this lemma), we can estimate $\mathscr{K}$ as follows: for every $0<\lambda<\min(\bar\lambda, c\lambda_1/R_2)$,

$$|\mathscr{K}(t,k)| \le |k|^2 |\widehat{\sigma_1}(k)|^2 \int_0^{\frac{2R_2}{c}} |p_c(\tau)|\,(t-\tau) \left|\widehat{\mathscr{M}}((t-\tau)k)\right| \mathrm{d}\tau$$
$$\lesssim |k|^2 e^{-2\lambda_1|k|} \int_0^{\frac{2R_2}{c}} (t-\tau) e^{-\lambda(t-\tau)|k|} e^{-(\bar\lambda-\lambda)|k|}\,\mathrm{d}\tau \lesssim \left(\sup_k |k|^2 e^{-2(\lambda_1-\frac{R_2}{c}\lambda)|k|}\right) e^{-\lambda|tk|}$$

which tells us that $\mathscr{K}(t,k)$ is dominated by $\mathscr{O}(e^{-\lambda|k|t})$, uniformly with respect to k, provided $0<\lambda<\min\left(\bar\lambda, \frac{c\lambda_1}{R_2}\right)$. ∎

Hence, assuming **(K1)**–**(K5)** and (**L**′), the solution of (18)–(19) satisfies (30). We deduce the convergence of the fluctuation of density $\varrho(t)$, force $\nabla_x\phi(t)$, and medium $\psi(t)$ (with exponential rate on the torus and polynomial rate for the free space problem), like in Proposition 3.4 and [27, Theorem 3.1].

3.4 Stability criterion for large wave speeds

We turn to investigate the "**(L)**-condition" made on the Laplace transform of $\mathscr{K}$ (see (**L**) and (**L**′)), where

$$\mathscr{L}\mathscr{K}(\omega,k) = |\widehat{\sigma_1}(k)|^2\ \mathscr{L}p_c(\omega)\mathscr{L}(|k|^2 t \widehat{\mathscr{M}}(kt))(\omega).$$

In fact, for the Vlasov equation, such a property holds under a smallness assumption, see [27, Condition (a) in Proposition 2.1]. Here, this condition can be rephrased by means of a condition on the wave speed $c \gg 1$. The latter confirms the intuition that the damping is related to the ability to evacuate the particles energy through the membranes, see [8]. (It also raises the issue to determine whether or not there exist stable equilibrium for $c \ll 1$.) A similar smallness condition on $1/c$ appears in the asymptotic statements for a single particle [8, Theorem 2, 3 & 4], for the analysis of the relaxation to equilibrium for the Vlasov-Wave-Fokker-Planck model [2, Theorem 2.3], and the stability analysis in [10]. Moreover, as mentioned in the Introduction, up to a suitable c-dependent rescaling of the coupling, the regime $c \to \infty$ leads to the usual Vlasov system [8], and it can be checked that the stability criterion for large c's is consistent to the condition exhibited for the Vlasov equation. The role of the wave speed c on the damping phenomena is investigated on numerical grounds in [18].

Proposition 3.10 (Stability criterion for large c's) *(i) Assume **(H1)–(H2)** and **(H4)–(H5)**. There exists $c_0 > 0$ such that if $c > c_0$ then condition* $(\mathbf{L})$ *is fulfilled.*
*(ii) Assume **(K1)–(K2)** and **(K4)–(K5)**. There exists $c_0 > 0$ such that if $c > c_0$ then condition* $(\mathbf{L}')$ *is fulfilled.*

Proof. We only detail the proof of (ii), the former item being justified by a similar approach. Let $0 < \Lambda < \min(\bar{\lambda}, c\lambda_1/R_2)$ and let ω be a complex number such that $\mathrm{Re}(\omega) > -\Lambda$. On the one hand, we have, for any $k \neq 0$,

$$\Big|\mathscr{L}\big(|k|^2 t\widehat{\mathscr{M}}(tk)\big)(\omega|k|)\Big| = \Big|\int_0^\infty s\widehat{\mathscr{M}}\Big(\frac{k}{|k|}s\Big)e^{-\omega s}\,\mathrm{d}s\Big| \lesssim \int_0^\infty se^{-\bar{\lambda}s}e^{\Lambda s}\,\mathrm{d}s \lesssim 1.$$

On the other hand, Lemma 2.3 allows us to estimate the Laplace transform of the kernel p_c as follows

$$|\mathscr{L}p_c(\omega|k|)| \le \|p_c\|_{L^\infty}\int_0^{2R_2/c} e^{\Lambda|k|s}\,\mathrm{d}s \lesssim \frac{1}{c}\frac{e^{\frac{2R_2}{c}\Lambda|k|}}{c}.$$

Owing to **(K4)**, we obtain

$$|\widehat{\sigma_1}(k)|^2|\mathscr{L}p_c(\omega|k|)| \lesssim \frac{1}{c^2}e^{-2(\lambda_1-\frac{R_2}{c}\Lambda)|k|}.$$

We observe that the right hand side tends to 0 as $c \to \infty$. Therefore, for any $\kappa \in (0,1)$, provided c is large enough, we have

$$\sup_{k\neq 0}|\mathscr{L}\mathscr{K}(\omega|k|,k)| \le 1-\kappa$$

for any $\omega \in \mathbb{C}$ with $\mathrm{Re}(\omega) > -\Lambda$, which implies $\inf_{k\neq 0}|\mathscr{L}\mathscr{K}(\omega|k|,k) - 1| \ge \kappa > 0$. ■

Section 5 provides a thorough discussion of the stability criterion, beyond the mere assumption of large wave speeds c.

4 Nonlinear Landau Damping

In this Section, we briefly explain how the non linear Landau damping can be justified. We consider two distincts geometrical and functional frameworks: the free space problem can be handled by working with Sobolev spaces [7], while on the torus the dispersion effects of the transport operator do not operate and we work with analytic regularity [6, 27]. We point out the new difficulties compared to the Vlasov case and explain how the arguments can be adapted for our purposes. Fully detailed proofs and further comments can be found in [33].

4.1 The free space problem

We shall see that the damping in $\mathbb{R}^d$ occurs with a restriction on the space dimension: we should assume $d \geq 3$. As in [7], the analysis in the whole space relies on dispersive phenomena attached to the free transport operator; these effects are indeed strong enough to dominate the plasma echoes when $d \geq 2$, and a further technical restriction arises in the bootstrap argument, that leads to impose $d \geq 3$.

4.1.1 Functional framework

We shall make use of Sobolev-type spaces. For $s \in \mathbb{R}$, $m \in \mathbb{N} \setminus \{0\}$, we denote

$$H^s(\mathbb{R}^m) = \Big\{ u : \mathbb{R}^m \to \mathbb{R},\ \int_{\mathbb{R}^m} \langle x \rangle^{2s} |\widehat{u}(x)|^2 \,\mathrm{d}x \Big\}.$$

Given x and y in $\mathbb{R}^d$, x, y stands for the vector in $\mathbb{R}^{2d}$ that results from the concatenation of x and y. Consequently, we can set $\langle x, y\rangle = (1+|x|^2+|y|^2)^{1/2}$. With $\alpha = (\alpha_1, \dots \alpha_d) \in \mathbb{N}^d$, we introduce the differential operator

$$\mathrm{D}_\xi^\alpha = (-i\partial_{\xi_1}^{\alpha_1}) \cdots (-i\partial_{\xi_d}^{\alpha_d}).$$

For $s \geq 0$, H^s stands for the standard Sobolev space. We shall make use of the norms introduced in [7]. We deal with functions $f : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, and for $P \in \mathbb{N}$, $s \geq 0$, we denote

$$\|f(t)\|_{H_P^s}^2 = \sum_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|\leq P}} \|(x, v) \mapsto v^\alpha f(t, x, v)\|_{H^s}^2 = \sum_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|\leq P}} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \langle k, \xi\rangle^{2s} \left|\mathrm{D}_\xi^\alpha \widehat{f}(t, k, \xi)\right|^2 \,\mathrm{d}k \,\mathrm{d}\xi. \tag{33}$$

It is also convenient to consider

$$\begin{aligned}
\|\langle t\nabla_x, \nabla_v\rangle f(t)\|_{H_P^s}^2 &= \sum_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|\leq P}} \|(x, v) \mapsto \langle t\nabla_x, \nabla_v\rangle v^\alpha f(t, x, v)\|_{H^s}^2 \\
&= \sum_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|\leq P}} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \langle tk, \xi\rangle^2 \langle k, \xi\rangle^{2s} \left|\mathrm{D}_\xi^\alpha \widehat{f}(t, k, \xi)\right|^2 \,\mathrm{d}k \,\mathrm{d}\xi
\end{aligned}$$

(there is a slight abuse of notation here since the right hand side is actually *equivalent* to the definition of $\|\langle t\nabla_x, \nabla_v\rangle f(t)\|_{H_P^s}^2$ based on (33)) and

$$\begin{aligned}
\Big\| |\nabla_x|^\delta f(t)\Big\|_{H_P^s}^2 &= \sum_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|\leq P}} \|(x, v) \mapsto |\nabla_x|^\delta v^\alpha f(t, x, v)\|_{H^s}^2 \\
&= \sum_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|\leq P}} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |k|^{2\delta} \langle k, \xi\rangle^{2s} \left|D_\xi^\alpha \widehat{f}(t, k, \xi)\right|^2 \,\mathrm{d}k \,\mathrm{d}\xi.
\end{aligned}$$

We shall also use L^∞-type estimate on Fourier transforms; we set

$$\left\|\widehat{\langle\nabla_{x,v}\rangle^s f}\right\|_{L^\infty_{(t)}L^\infty_{(k,\xi)}} = \sup_{t\in[0,T]}\left(\sup_{k,\xi\in\mathbb{R}^d}\left\{\langle k,\xi\rangle^s\left|\widehat{f}(t,k,\xi)\right|\right\}\right).$$

For a function $(t,x)\in(0,\infty)\times\mathbb{R}^d\mapsto\varrho(t,x)\in\mathbb{R}$ we introduce the modified Sobolev norm

$$\int_{\mathbb{R}^d}|k|\langle k,tk\rangle^{2s}|\widehat{\varrho}(t,k)|^2\,\mathrm{d}k = \|A_s(t)\widehat{\varrho}(t)\|_{L^2_{(k)}},$$

where we have set

$$A_s(t,k) = |k|^{1/2}\langle k,tk\rangle^s,$$

and we shall also use

$$\|A_s\widehat{\varrho}\|_{L^2_{(k,t)}} = \int_0^T\int_{\mathbb{R}^d}|k|\langle k,tk\rangle^{2s}|\widehat{\varrho}(t,k)|^2\,\mathrm{d}k\,\mathrm{d}t,$$

and

$$\|A_s\widehat{\varrho}\|_{L^\infty_{(k)}L^2_{(t)}} = \sup_{k\in\mathbb{R}^d}\left(\int_0^T|k|\langle k,tk\rangle^{2s}|\widehat{\varrho}(t,k)|^2\right)^{1/2}.$$

The norms defined on the macroscopic density ϱ equally apply to the kinetic quantity g, replacing $\widehat{\varrho}(t,k)$ by $\widehat{g}(t,k,tk)$.

We go back to the formulation (9). Compared to the usual Vlasov equation, the expression of the potential $\Phi[\psi]$ now involves the contribution of the initial data $\mathscr{F}_I$, and the self-consistent part $\mathscr{G}_\varrho$ presents a memory effect, through the kernel p_c. It is convenient to think of the problem with some generality on these quantities. Thus, let us collect the hypothesis on the data of the problem: $\mathscr{F}_I$, p_c and σ_1. We refer the reader to the previous section in order to translate these assumption on the original data σ_2, ψ_0 and ψ_1.

(A1) There exists an exponent $\alpha_I>0$ sufficiently large such that

$$\sup_{k\in\mathbb{R}^d}\left|\widehat{\mathscr{F}_I}(t,k)\right|\lesssim\langle t\rangle^{-\alpha_I},$$

(A2) There exists an exponent $\alpha_c>0$ sufficiently large such that

$$|p_c(t)|\lesssim\langle t\rangle^{-\alpha_c},$$

(A3) $\sigma_1\in\mathscr{S}(\mathbb{R}^d)$: for any $\alpha\geq 0$ we have

$$\lim_{|k|\to+\infty}\langle k\rangle^\alpha|\widehat{\sigma}_1(k)|=0.$$

This formulation of the hypothesis has the advantage of pushing the generality of the result, both on the "linear" perturbation due to the data through $\mathscr{F}_I$ and on the memory effects in the self-consistent potential through p_c. The following claims are crucial for our purposes: roughly speaking, they explain why the situation is not very different from the Vlasov case, once the role of $\mathscr{F}_I(t)$ and p_c well understood, and it justifies that the approach of [7] is robust enough to be adapted. Note that **(A1)** is the assumption that makes the constants $C_1(\mathscr{F}_I)$ and $C_2(\mathscr{F}_I)$ below meaningful.

Proposition 4.1 *Let **(A1)–(A3)** be fulfilled. Then for any $0 < T < \infty$ and any $s \geq 0$ such that $s < \alpha_I - 1/2$ and $s < (\alpha_c - 1)/2$, the following three estimates hold*

$$\left\| A_s \widehat{\sigma}_1 \left(\widehat{\mathscr{F}_I} - \widehat{\sigma}_1 \widehat{\mathscr{G}_\varrho} \right) \right\|^2_{L^2_{(t)} L^2_{(k)}} \lesssim C_1(\mathscr{F}_I) + \|A_s \widehat{\varrho}\|^2_{L^2_{(t)} L^2_{(k)}}, \tag{34a}$$

$$\left\| A_s \widehat{\sigma}_1 \left(\widehat{\mathscr{F}_I} - \widehat{\sigma}_1 \widehat{\mathscr{G}_\varrho} \right) \right\|^2_{L^\infty_{(k)} L^2_{(t)}} \lesssim C_1(\mathscr{F}_I) + \|A_s \widehat{\varrho}\|^2_{L^\infty_{(k)} L^2_{(t)}}, \tag{34b}$$

$$\begin{aligned} &\sup_{t\in[0,T]} \sup_{k\in\mathbb{R}^d} \langle k, tk\rangle^s |\widehat{\sigma}_1(k)| \left| \widehat{\mathscr{F}_I}(t,k) - \widehat{\sigma}_1(k) \widehat{\mathscr{G}_\varrho}(t,k) \right| \\ &\qquad\qquad \lesssim C_2(\mathscr{F}_I) + \sup_{t\in[0,T]} \sup_{k\in\mathbb{R}^d} \langle k, tk\rangle^s \left| \widehat{\varrho}(t,k) \right|, \end{aligned} \tag{34c}$$

with

$$C_1(\mathscr{F}_I) = \int_0^{+\infty} \langle t\rangle^{2s} \sup_k \left| \widehat{\mathscr{F}_I}(t,k) \right|^2 \, \mathrm{d}t \ \ \textit{and} \ \ C_2(\mathscr{F}_I) = \sup_{t,k} \langle t\rangle^s \left| \widehat{\mathscr{F}_I}(t,k) \right|.$$

Remark 4.2 *We shall use the following variant of the statement : for any polynomial $k \mapsto P(k)$, we have*

$$\left\| P A_s \widehat{\sigma}_1 \left(\widehat{\mathscr{F}_I} - \widehat{\sigma}_1 \widehat{\mathscr{G}_\varrho} \right) \right\|^2_{L^2_{(t)} L^2_{(k)}} \lesssim C_1(\mathscr{F}_I) + \|A_s \widehat{\varrho}\|^2_{L^2_{(t)} L^2_{(k)}}, \tag{35a}$$

$$\left\| P A_s \widehat{\sigma}_1 \left(\widehat{\mathscr{F}_I} - \widehat{\sigma}_1 \widehat{\mathscr{G}_\varrho} \right) \right\|^2_{L^\infty_{(k)} L^2_{(t)}} \lesssim C_1(\mathscr{F}_I) + \|A_s \widehat{\varrho}\|^2_{L^\infty_{(k)} L^2_{(t)}}, \tag{35b}$$

$$\begin{aligned} &\sup_{t\in[0,T]} \sup_{k\in\mathbb{R}^d} \langle k, tk\rangle^s P(k) |\widehat{\sigma}_1(k)| \left| \widehat{\mathscr{F}_I}(t,k) - \widehat{\sigma}_1(k) \widehat{\mathscr{G}_\varrho}(t,k) \right| \\ &\qquad\qquad \lesssim C_2(\mathscr{F}_I) + \sup_{t\in[0,T]} \sup_{k\in\mathbb{R}^d} \langle k, tk\rangle^s \left| \widehat{\varrho}(t,k) \right|, \end{aligned} \tag{35c}$$

These estimates can be justified since σ_1 lies in the Schwartz class and thus $P(k)\widehat{\sigma}_1(k)$ remains a function with fast decay.

Proof. In order to prove (34a), we analyse separately the contribution from $\widehat{\mathscr{F}_I}$ and $\widehat{\mathscr{G}_\varrho}$ as follows

$$\begin{aligned} \left\| A_s \widehat{\sigma}_1 \left(\widehat{\mathscr{F}_I} - \widehat{\sigma}_1 \widehat{\mathscr{G}_\varrho} \right) \right\|^2_{L^2_{(t)} L^2_k} &\lesssim \underbrace{\int_0^T \int_{\mathbb{R}^d_k} |k| \langle k, tk\rangle^{2s} |\widehat{\sigma}_1(k)|^2 |\widehat{\mathscr{F}_I}(t,k)|^2 \, \mathrm{d}k \, \mathrm{d}t}_{=\mathrm{I}} \\ &\quad + \underbrace{\int_0^T \int_{\mathbb{R}^d_k} |k| \langle k, tk\rangle^{2s} |\widehat{\sigma}_1(k)|^4 |\widehat{\mathscr{G}_\varrho}(t,k)|^2 \, \mathrm{d}k \, \mathrm{d}t}_{=\mathrm{II}}. \end{aligned}$$

For I, by using $\langle k, tk\rangle^2 \leq \langle k\rangle^2 \langle t\rangle^2$, we readily obtain

$$\mathrm{I} \leq \left(\int_{\mathbb{R}^d_k} |k| \langle k\rangle^{2s} |\widehat{\sigma}_1(k)|^2 \, \mathrm{d}k \right) \left(\int_0^{+\infty} \langle t\rangle^{2s} \sup_k \left| \widehat{\mathscr{F}_I}(t,k) \right|^2 \, \mathrm{d}t \right).$$

For II we start by applying Cauchy-Schwarz' inequality

$$
\begin{aligned}
|\widehat{\mathscr{G}}_\varrho(t,k)|^2 &= \left|\int_0^t p_c(t-\tau)\varrho(\tau,k)\,\mathrm{d}\tau\right|^2 \\
&\leq \left(\int_0^t |p_c(t-\tau)|\,\mathrm{d}\tau\right)\left(\int_0^t |p_c(t-\tau)||\widehat{\varrho}(\tau,k)|^2\,\mathrm{d}\tau\right).
\end{aligned}
$$

Going back to II, we are led to

$$
\mathrm{II} \leq \|p_c\|_{L^1}\int_0^T\int_0^t |p_c(t-\tau)|\left(\int_{\mathbb{R}^d_k} |k|\langle k,\tau k\rangle^{2s}\frac{\langle k,tk\rangle^{2s}}{\langle k,\tau k\rangle^{2s}}|\widehat{\sigma}_1(k)|^4|\widehat{\varrho}(t,k)|^2\,\mathrm{d}k\right)\mathrm{d}\tau\ \mathrm{d}t.
$$

A simple study of function shows that (for $t\geq\tau$)

$$
\sup_{k\in\mathbb{R}^d}\frac{\langle k,tk\rangle^{2s}}{\langle k,\tau k\rangle^{2s}} \leq \frac{\langle t\rangle^{2s}}{\langle \tau\rangle^{2s}}.
$$

Since $|\widehat{\sigma}_1(k)| \leq \|\sigma_1\|_{L^1}\lesssim 1$, and using Fubini's theorem, we obtain

$$
\begin{aligned}
\mathrm{II} &\lesssim \|p_c\|_{L^1}\int_0^T\left(\int_\tau^T |p_c(t-\tau)|\frac{\langle t\rangle^{2s}}{\langle \tau\rangle^{2s}}\|A_s\widehat{\varrho}(\tau)\|^2_{L^2_{(k)}}\,\mathrm{d}t\right)\mathrm{d}\tau \\
&\lesssim \|p_c\|_{L^1}\int_0^T \|A_s\widehat{\varrho}(\tau)\|^2_{L^2_{(k)}}\left(\int_0^{T-\tau}|p_c(u)|\frac{\langle u+\tau\rangle^{2s}}{\langle\tau\rangle^{2s}}\,\mathrm{d}u\right)\mathrm{d}\tau.
\end{aligned}
$$

Since $\langle u+\tau\rangle^{2s}\lesssim\langle u\rangle^{2s}\langle\tau\rangle^{2s}$, we arrive at

$$
\mathrm{II}\lesssim \|p_c\|_{L^1}\left(\int_0^{+\infty}\langle u\rangle^{2s}|p_c(u)|\,\mathrm{d}u\right)\|A_s\widehat{\varrho}\|^2_{L^2_{(t)}L^2_{(k)}}.
$$

It ends the proof of (34a).

Estimate (34b) follows the same strategy: for $k\in\mathbb{R}^d$, we split as follows

$$
\begin{aligned}
&\int_0^T |k|\langle k,tk\rangle^{2s}|\widehat{\sigma}_1(k)|^2\left|\widehat{\mathscr{F}}_I(t,k)-\widehat{\sigma}_1(k)\widehat{\mathscr{G}}_\varrho(t,k)\right|^2\,\mathrm{d}t \\
&\qquad\leq \underbrace{\int_0^T |k|\langle k,tk\rangle^{2s}|\widehat{\sigma}_1(k)|^2|\widehat{\mathscr{F}}_I(t,k)|^2\,\mathrm{d}t}_{=\mathrm{J}}+\underbrace{\int_0^T |k|\langle k,tk\rangle^{2s}|\widehat{\sigma}_1(k)|^4|\widehat{\mathscr{G}}_\varrho(t,k)|^2\,\mathrm{d}t}_{=\mathrm{JJ}}.
\end{aligned}
$$

Proceeding as above, we obtain

$$
\mathrm{J}\leq\left(\sup_{k\in\mathbb{R}^d}|k|\langle k\rangle^{2s}|\widehat{\sigma}_1(k)|^2\right)\left(\int_0^{+\infty}\langle t\rangle^{2s}\sup_k\left|\widehat{\mathscr{F}}_I(t,k)\right|^2\,\mathrm{d}t\right)
$$

and

$$
\begin{aligned}
\mathrm{JJ} &\lesssim \|p_c\|_{L^1}\int_0^T\left(\int_\tau^T |p_c(t-\tau)|\frac{\langle t\rangle^{2s}}{\langle\tau\rangle^{2s}}|k|\langle k,\tau k\rangle^{2s}|\widehat{\varrho}(\tau,k)|^2\,\mathrm{d}t\right)\mathrm{d}\tau \\
&\lesssim \|p_c\|_{L^1}\left(\int_0^{+\infty}\langle u\rangle^{2s}|p_c(u)|\,\mathrm{d}u\right)\left(\int_0^T |k|\langle k,\tau k\rangle^{2s}|\widehat{\varrho}(\tau,k)|^2\,\mathrm{d}\tau\right).
\end{aligned}
$$

We proceed with a slightly different approach for (34c) when dealing with the contribution involving $\widehat{\mathscr{G}_\varrho}$. For any $t \in [0,T]$ and $k \in \mathbb{R}^d$, we write

$$\begin{aligned}\langle k, tk\rangle^s |\widehat{\sigma}_1(k)| \left|\widehat{\mathscr{F}_I}(t,k) - \widehat{\sigma}_1(k)\widehat{\mathscr{G}_\varrho}(t,k)\right| \\ \lesssim \left(\sup_{k\in\mathbb{R}^d} \langle k\rangle^s |\widehat{\sigma}_1(k)|\right)\left(\sup_{t\in[0,T]} \langle t\rangle^s \sup_k \left|\widehat{\mathscr{F}_I}(t,k)\right|\right) + \langle k, tk\rangle^s |\widehat{\mathscr{G}_\varrho}(t,k)|.\end{aligned}$$

Since

$$\begin{aligned}\langle k, tk\rangle^s |\widehat{\mathscr{G}_\varrho}(t,k)| &\leq \int_0^t |p_c(t-\tau)| \frac{\langle k, tk\rangle^s}{\langle k, \tau k\rangle^s} \langle k, \tau k\rangle^s |\widehat{\varrho}(\tau,k)| \,\mathrm{d}\tau \\ &\lesssim \left(\int_0^t |p_c(t-\tau)| \frac{\langle t\rangle^s}{\langle \tau\rangle^s} \,\mathrm{d}\tau\right)\left(\sup_{\tau\in[0,T]} \sup_{k\in\mathbb{R}^d} \langle k, \tau k\rangle^s |\widehat{\varrho}(\tau,k)|\right),\end{aligned}$$

it suffices to observe that

$$\int_0^t |p_c(t-\tau)| \frac{\langle t\rangle^s}{\langle \tau\rangle^s} \,\mathrm{d}\tau < \infty$$

by virtue of **(A2)**. ■

Proposition 4.3 *Let **(A1)**–**(A3)** be fullfiled. Assume that $\mathscr{M} \in H^{\tilde{s}}_P$ with $P > d/2$ and $\tilde{s} \geq 0$. Then for any $s \geq 0$ such that $s < \tilde{s} - 2d$ and $s < \alpha_I - 1$, we have*

$$\left\|(t,k) \mapsto A_s(t,k) \int_0^t \widehat{\nabla_x \sigma_1}(k)\widehat{\mathscr{F}_I}(\tau,k)\widehat{\nabla_v \mathscr{M}}\big((t-\tau)k\big) \,\mathrm{d}\tau\right\|_{L^2_{(t)} L^2_{(k)}} \lesssim \int_0^{+\infty} \langle t\rangle^{2s+1^+} \sup_k \left|\widehat{\mathscr{F}_I}(t,k)\right|^2 \mathrm{d}t \tag{36a}$$

$$\left\|(t,k) \mapsto A_s(t,k) \int_0^t \widehat{\nabla_x \sigma_1}(k)\widehat{\mathscr{F}_I}(\tau,k)\widehat{\nabla_v \mathscr{M}}\big((t-\tau)k\big) \,\mathrm{d}\tau\right\|_{L^\infty_{(k)} L^2_{(t)}} \lesssim \int_0^{+\infty} \langle t\rangle^{2s+1^+} \sup_k \left|\widehat{\mathscr{F}_I}(t,k)\right|^2 \mathrm{d}t \tag{36b}$$

Proof. First, let us introduce the following notation

$$I(t,k) = A_s(t,k) \int_0^t \widehat{\nabla_x \sigma_1}(k)\widehat{\mathscr{F}_I}(\tau,k)\widehat{\nabla_v \mathscr{M}}\big((t-\tau)k\big) \,\mathrm{d}\tau$$

and estimate for every $k \in \mathbb{R}^d$ the $L^2_{(t)}$ norm of $t \mapsto I(t,k)$. By using the relations

$\langle k, tk\rangle \lesssim \langle k, \tau k\rangle\langle [t-\tau]k\rangle$ and $\langle k, \tau k\rangle \leq \langle k\rangle\langle \tau\rangle$, we obtain

$$\int_0^T |I(t,k)|^2 \,\mathrm{d}t \lesssim |k|^3 |\widehat{\sigma}_1(k)|^2 \int_0^T \langle tk\rangle^{-(1^+)} \times \left(\int_0^t \langle \tau k\rangle^{\frac{1}{2}^+} \langle k, \tau k\rangle^s \left|\widehat{\mathscr{F}}_I(\tau,k)\right| \langle (t-\tau)k\rangle^{s+\frac{1}{2}^+} \left|\widehat{\nabla_v \mathscr{M}}((t-\tau)k)\right|\right)^2 \mathrm{d}t$$

$$\lesssim |k||\widehat{\sigma}_1(k)|^2 \int_0^T \langle tk\rangle^{-(1^+)} \left(\int_0^{+\infty} \langle \tau k\rangle^{1^+} \langle k, \tau k\rangle^{2s} \left|\widehat{\mathscr{F}}_I(\tau,k)\right|^2 \mathrm{d}\tau\right) \times \left(\int_0^{+\infty} \langle (t-\tau)k\rangle^{2s+1^+} \left|\widehat{\nabla_v \mathscr{M}}((t-\tau)k)\right|^2 |k| \,\mathrm{d}\tau\right) |k| \,\mathrm{d}t$$

$$\lesssim |k|\langle k\rangle^{2s+1^+} |\widehat{\sigma}_1(k)|^2 \left(\int_0^{+\infty} \langle \tau\rangle^{2s+1^+} \sup_k |\widehat{\mathscr{F}}_I(\tau,k)|^2 \,\mathrm{d}\tau\right) \times \left(\int_0^{+\infty} \langle u\rangle^{2s+1^+} \left|\widehat{\nabla_v \mathscr{M}}(u\frac{k}{|k|})\right|^2 \mathrm{d}u\right) \int_0^T \langle u\rangle^{-(1^+)} \,\mathrm{d}u.$$

Since $\mathscr{M} \in H^{\tilde{s}}_P$, we have $\xi \mapsto \langle \xi\rangle^{\tilde{s}} \widehat{\mathscr{M}}(\xi) \in H^P$, where $P > d/2$, and Sobolev's embedding yields $|\widehat{\mathscr{M}}(\xi)| \lesssim \|\widehat{\mathscr{M}}\|_{H^P} \langle \xi\rangle^{-\tilde{s}}$. Then, as soon as $s < \tilde{s} - (1^+)$, this ensures that the integral involving $\mathscr{M}$ is uniformly bounded with respect to k. Eventually **(A3)** ensures that both $L^2_{(k)} L^2_{(t)}$ and $L^\infty_{(k)} L^2_{(t)}$-norm of $I(t,k)$ are dominated as asserted. ■

The analysis of the Landau Damping, as it is already clear for the linearized problem, relies heavily on the formulation of the problem by means of the Fourier variables. Let us collect the useful formula from which the reasoning starts. Integrating (10a)–(10b) over $[0,t]$, we get

$$g(t,x,v) = f_0(x,v) + \int_0^t \nabla_x \sigma_1 \star (\mathscr{F}_I - \sigma_1 \star \mathscr{G}_\varrho)(\tau, x+\tau v) \cdot (\nabla_v - \tau \nabla_x)(\mathscr{M}(v) + g(\tau,x,v)) \,\mathrm{d}\tau.$$

We check that

$$\int_{\mathbb{R}^{2d}} u(x+\tau v, v) e^{-ik\cdot x} e^{-i\xi\cdot v} \,\mathrm{d}v \,\mathrm{d}x = \int_{\mathbb{R}^{2d}} u(y,v) e^{-ik\cdot y} e^{-i(\xi - \tau k)\cdot v} \,\mathrm{d}v \,\mathrm{d}x = \widehat{u}(k, \xi - \tau k).$$

We also bear in mind that $\widehat{\mathbf{1}(v)}(\xi) = \delta(\xi = 0)$ and $\widehat{\mathbf{1}(x)}(k) = \delta(k = 0)$. We thus obtain

$$\begin{aligned}
\widehat{g}(t,k,\xi) &= \widehat{f}_0(k,\xi) \\
&\quad - \int_0^t \int_{\mathbb{R}^{2d}} n\widehat{\sigma_1}(n)(\widehat{\mathscr{F}_I} - \widehat{\sigma_1}\widehat{\mathscr{G}_\varrho})(\tau,n)\delta(\zeta = \tau n) \cdot (\xi - \zeta)\widehat{\mathscr{M}}(\xi - \zeta)\delta(n = k) \,\mathrm{d}n \,\mathrm{d}\zeta \,\mathrm{d}\tau \\
&\quad - \int_0^t \int_{\mathbb{R}^{2d}} n\widehat{\sigma_1}(n)(\widehat{\mathscr{F}_I} - \widehat{\sigma_1}\widehat{\mathscr{G}_\varrho})(\tau,n)\delta(\zeta = \tau n) \\
&\qquad\qquad \cdot(\xi - \zeta - \tau(k-n))\widehat{g}(\tau, k-n, \xi - \zeta) \,\mathrm{d}n \,\mathrm{d}\zeta \,\mathrm{d}\tau \\
&= \widehat{f}_0(k,\xi) - \int_0^t k\widehat{\sigma_1}(k)(\widehat{\mathscr{F}_I} - \widehat{\sigma_1}\widehat{\mathscr{G}_\varrho})(\tau,\tau k) \cdot (\xi - \tau k)\widehat{\mathscr{M}}(\xi - \tau k) \,\mathrm{d}\tau \\
&\quad - \int_0^t \int_{\mathbb{R}^d} n\widehat{\sigma_1}(n)(\widehat{\mathscr{F}_I} - \widehat{\sigma_1}\widehat{\mathscr{G}_\varrho})(\tau,n) \cdot (\xi - \tau k)\widehat{g}(\tau, k-n, \xi - \tau n) \,\mathrm{d}n \,\mathrm{d}\tau.
\end{aligned} \tag{37}$$

Eventually, the macroscopic density is evaluated by

$$\begin{aligned}\widehat{\varrho}(t,k) &= \int_{\mathbb{R}^{2d}} f(t,x,v)e^{-ik\cdot x}\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{R}^{2d}} g(t,x-tv,v)e^{-ik\cdot x}\,\mathrm{d}v\,\mathrm{d}x\\ &= \int_{\mathbb{R}^{2d}} g(t,y,v)e^{-ik\cdot y}e^{-itk\cdot v}\,\mathrm{d}v\,\mathrm{d}y = \widehat{g}(t,k,tk).\end{aligned}$$

Going back to (37) with $\xi = tk$, we arrive at

$$\begin{aligned}\widehat{\varrho}(t,k) = \widehat{f}_0(k,tk) - \int_0^t k\widehat{\sigma_1}(k)(\widehat{\mathscr{F}_I} - \hat{\sigma}_1\widehat{\mathscr{G}_\varrho})(\tau,\tau k)\cdot(t-\tau)k\widehat{\mathscr{M}}((t-\tau)k)\,\mathrm{d}\tau\\ - \int_0^t\int_{\mathbb{R}^d} n\widehat{\sigma_1}(n)(\widehat{\mathscr{F}_I} - \widehat{\sigma_1}\widehat{\mathscr{G}_\varrho})(\tau,n)\cdot((t-\tau)k)\widehat{g}(\tau,k-n,tk-\tau n)\,\mathrm{d}n\,\mathrm{d}\tau \quad (38)\end{aligned}$$

4.1.2 Main result

We are ready now to state the main result about the non linear Landau damping. As said above, the proof makes the constraint $d \geq 3$ on the space dimension appear.

Theorem 4.4 (Landau damping in $\mathbb{R}^d$) *Let $d \geq 3$. Suppose **(A1)–(A3)**. There exists universal constants ε_0, $R_0 > 0$ and $r \in (0, R_0)$ such that if $s > R_0$,*

$$\sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\leq P}} \|x^\alpha f_0\|^2_{H^s_P} \leq \varepsilon_0^2 \qquad \int_0^{+\infty}\langle t\rangle^{2s}\sup_k \left|\widehat{\mathscr{F}_I}(t,k)\right|^2\,\mathrm{d}t \leq \varepsilon_0^2, \quad \sup_{t,k}\langle t\rangle^s\left|\widehat{\mathscr{F}_I}(t,k)\right| \leq \varepsilon_0,$$

and $\mathscr{M} \in H^{\tilde{s}}_P(\mathbb{R}^d_v)$ with $P > d/2$ and $\tilde{s} \geq s + 2d$ satisfies (**L**)*, then, the unique solution g of* (10a)–(10b) *is globally defined. Moreover, there exists $g^\infty \in H^r_P$ such that*

$$\begin{aligned}\|g(t) - g^\infty\|_{H^\sigma_P} &\lesssim \varepsilon_0\langle t\rangle^{-\frac{d}{2}} \quad \text{for } 0\leq\sigma\leq r, &(39a)\\ |\widehat{g}(t,k,tk)| &\lesssim \varepsilon_0\langle k,tk\rangle^{-(r+d+2)} &(39b)\\ \|\langle\nabla_x\rangle^\sigma\nabla\sigma_1\star(\mathscr{F}_I(t) - \sigma_1\star\mathscr{G}_g(t))\|_{L^\infty(\mathrm{d}x)} &\lesssim \varepsilon_0\langle t\rangle^{-d-1} \quad \text{for } \sigma\geq 0 &(39c)\end{aligned}$$

holds.

Remark 4.5 *Estimate* (39c) *holds because σ_1 is assumed to be in the Schwartz class; this assumption can be relaxed at the price of introducing constraints on the regularity exponent σ.*

Estimate (39b) *provides a decay of $\widehat{\varrho}(t,k)$ with rate $\langle k,tk\rangle^{-(r+d+2)}$; the statement can be completed by the convergence to 0 of the fluctuations ψ of the medium state, see Proposition 3.4.*

The proof of the Landau Damping in fact relies on a bootstrap estimate, see [7, Proposition 2.5], which states as follows.

Proposition 4.6 (Bootstrap) *Let the hypothesis of Theorem 4.4 be fulfilled and let $0 < \delta < 1/2$. There exists real numbers $2(d+1)+1 < s_1 < s_2 < s_3 < s_4 < s$ and*

$K_1, ..., K_5 \geq 1$ *such that, for any* $g \in C^0([0,T], H^s_P)$ *solution of* (10a)–(10b) *on the time interval* $[0,T]$ *verifying*

$$\begin{aligned}
\|\langle t\nabla_x, \nabla_v\rangle g(t)\|^2_{H^{s_4}_P} &\leq 4K_1\varepsilon^2\langle t\rangle^5, && (40a)\\
\|A_{s_4}\widehat{\varrho}\|^2_{L^2_{(t)}L^2_{(k)}} &\leq 4K_2\varepsilon^2, && (40b)\\
\||\nabla_x|^\delta g(t)\|^2_{H^{s_3}_P} &\leq 4K_3\varepsilon^2, && (40c)\\
\|A_{s_2}\widehat{\varrho}\|^2_{L^\infty_{(k)}L^2_{(t)}} &\leq 4K_4\varepsilon^2, && (40d)\\
\|\widehat{\langle\nabla_{x,v}\rangle^{s_1} g}(t)\|_{L^\infty_{(k,\xi)}} &\leq 4K_5\varepsilon, && (40e)
\end{aligned}$$

for $0 < \varepsilon \leq \varepsilon_0$ *small enough, the following estimates hold on* $[0,T]$

$$\begin{aligned}
\|\langle t\nabla_x, \nabla_v\rangle g(t)\|^2_{H^{s_4}_P} &\leq 2K_1\varepsilon^2\langle t\rangle^5, && (41a)\\
\|A_{s_4}\widehat{\varrho}\|^2_{L^2_{(t)}L^2_{(k)}} &\leq 2K_2\varepsilon^2, && (41b)\\
\||\nabla_x|^\delta g(t)\|^2_{H^{s_3}_P} &\leq 2K_3\varepsilon^2, && (41c)\\
\|A_{s_2}\widehat{\varrho}\|^2_{L^\infty_{(k)}L^2_{(t)}} &\leq 2K_4\varepsilon^2, && (41d)\\
\|\widehat{\langle\nabla_{x,v}\rangle^{s_1} g}(t)\|_{L^\infty_{(k,\xi)}} &\leq 2K_5\varepsilon. && (41e)
\end{aligned}$$

Remark 4.7 *We shall see within the proof how the* s_i*'s are chosen, according to some compatibility conditions. This choice determines the possible value for* R_0 *that arises in Theorem 4.4 as a threshold for the Sobolev regularity in which the damping is evaluated. To be specific, Proposition 4.6 holds for* $s > s_4 + 2d$ *and* $s_i > s_{i-1} + 2d$ *and in Theorem 4.4, we can set*

$$R_0 = s_4 + 2d, \qquad r = s_1 - d - 2.$$

The condition on ε_0 *imposes a smallness constraint on the initial perturbation.*

Remark 4.8 *It might be surprising that the half-convolution with respect to time plays a relatively weak role in this statement, compared to the Vlasov case. At first sight, we would suspect that the memory effect changes a lot the control of the force terms, or that it imposes further restrictions. In fact, the heart of the proof relies on the estimates in Proposition 4.1, and the main impact of the memory term is rather on the stability condition, where it completely modifies, in a quite intricate way, the expression of the symbol* $\mathscr{L}\mathscr{K}$. *This can be seen as a confirmation of the robustness of the approach designed in [27, 6, 7].*

The proof of the Landau damping from the bootstrap follows closely [7]; full details can be found in [33]. The bootstrap argument in itself is adapted from [7] by taking advantage of the analogies with the Vlasov equation. There are two main differences that require some care: the additional term $\mathscr{F}_I(t)$ should be controlled with the bootstrap norms and all quantities where $\|\varrho(t)\|$ arises in [7] should be controlled here by $\|\mathscr{G}_\varrho\|$. Both $\|\mathscr{F}_I(t)\|$ and the estimates of $\|\mathscr{G}_\varrho\|$ by $\|\varrho(t)\|$ should be evaluated by using the norms involved in Proposition 4.6. These issues are the motivation for Proposition 4.1 and Proposition 4.3. For instance, let us detail this strategy for the estimate of $A_{s_4}\widehat{\varrho}$ in the $L^2_{(k)}L^2_{(t)}$ norm. The other estimates proceed similarly, by combining the arguments of [7] to Propositions 4.1 and 4.3, see [33].

4.1.3 Estimate of the $L^2_{(k)}L^2_{(t)}$ norm of $A_{s_4}\widehat{\varrho}$.

The estimate of $A_{s_4}\widehat{\varrho}$ is a consequence of the following two claims, for which we refer the reader to [7, Section 2.3 and 3]. The former is a version of Lemma 3.1 adapted to the norms of the bootstrap.

Proposition 4.9 (Linearized damping on $\mathbb{R}^d$) *Let the assumptions of Theorem 4.4 be fulfilled. We consider a family of functions $\{t \in [0,T] \mapsto a(t,k),\ k \in \mathbb{R}^d\}$. We suppose that, for any $k \in \mathbb{R}^d$,*

$$\int_0^T |k|\langle k, tk\rangle^{2s}|a(t,k)|^2 \,\mathrm{d}t < +\infty,$$

holds. Then, we can find a constant C_{LD} (which does not depend on k and T) such that any solution $(t,k) \mapsto \phi(t,k)$ of the system

$$\begin{aligned}\phi(t,k) &= a(t,k) + \int_0^t \mathscr{K}(t-\tau,k)\phi(\tau,k)\,\mathrm{d}\tau \\ &= a(t,k) + \int_0^t |\widehat{\sigma}_1(k)|^2|k|^2(t-\tau)\widehat{\mathscr{M}}([t-\tau]k)\left(\int_0^\tau p_c(\tau-\sigma)\phi(\sigma,k)\,\mathrm{d}\sigma\right)\mathrm{d}\tau,\end{aligned}$$

on $[0,T]$ satisfies the following estimate: for any $k \in \mathbb{R}^d$

$$\int_0^T |k|\langle k, tk\rangle^{2s}|\phi(t,k)|^2\,\mathrm{d}t \le C_{LD}\int_0^T |k|\langle k, tk\rangle^{2s}|a(t,k)|^2\,\mathrm{d}t.$$

The second estimate is concerned with the time-response kernel

$$\bar{K}(t,\tau,k,n) = \frac{|k|^{1/2}|n|^{1/2}|k(t-\tau)|}{\langle n\rangle^2}\left|\widehat{g}(t,k-n,tk-\tau n)\right|.$$

which is a crucial quantity for the analysis of the echo phenomena. It leads to the constraint on s_1 involved in Proposition 4.6. Technically, this statement is substantially different when $\mathbb{X}^d = \mathbb{T}^d$ or when $\mathbb{X}^d = \mathbb{R}^d$. In the torus, the proof needs analytic regularity but is free of constraint on the space dimension d (see [6, Section 6]). For the free space problem, the argument relies on dispersion mechanisms of the transport operator which are strong enough only when $d \ge 2$; in this situation it is thus possible to work in finite regularity.

Proposition 4.10 *Let $0 < T < \infty$. Let $s_1 > 2(d+1)+1$. The following two estimates hold*

$$\sup_{t\in[0,T]}\sup_{k\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}\bar{K}(t,\tau,k,n)\,\mathrm{d}n\,\mathrm{d}\tau \lesssim \sup_{\tau\in[0,T]}\sup_{k,\xi\in\mathbb{R}^d}\langle k,\xi\rangle^{s_1}\left|\widehat{g}(\tau,k,\xi)\right|$$

and

$$\sup_{\tau\in[0,T]}\sup_{n\in\mathbb{R}^d}\int_\tau^T\int_{\mathbb{R}^d}\bar{K}(t,\tau,k,n)\,\mathrm{d}k\,\mathrm{d}t\lesssim\sup_{\tau\in[0,T]}\sup_{k,\xi\in\mathbb{R}^d}\langle k,\xi\rangle^{s_1}\left|\widehat{g}(\tau,k,\xi)\right|.$$

Remark 4.11 *The factor $1/\langle n\rangle^2$ in the kernel $\bar{K}$ comes from the convolution kernel used in [7]. Here, since σ_1 is Schwartz class, this factor can be replaced by $1/\langle n\rangle^m$ with $m\in\mathbb{N}$ as large as we wish.*

We follow closely the arguments of [7], up to the perturbation due to $\mathscr{F}_I$ and $\mathscr{G}_g$; as pointed out above, these perturbations do not substantially modify the analysis, owing to Proposition 4.1 and Proposition 4.3.

We start from the expression of $\widehat{\varrho}(t,k)$ in (38) and we apply Proposition 4.9 in order to estimate the $L^2_{(t)}$ norm of $A_{s_i}\widehat{\varrho}$ (with $i\in\{2,4\}$). We get

$$\begin{aligned}\|A_{s_i}\widehat{\varrho}(\cdot,k)\|^2_{L^2_{(t)}}\lesssim&\int_0^T|k|\langle k,tk\rangle^{2s_i}|\widehat{f}_0(k,tk)|^2\,\mathrm{d}t\\&+\int_0^T\left|\int_0^t|k|^{1/2}\langle k,tk\rangle^{s_4}k\,\widehat{\sigma}_1(k)\widehat{\mathscr{F}_I}(\tau,k)\cdot[t-\tau]k\widehat{\mathscr{M}}([t-\tau]k)\,\mathrm{d}\tau\right|^2\mathrm{d}t\\&+\int_0^T\Big|\int_0^t\int_{\mathbb{R}^d_n}|k|^{1/2}\langle k,tk\rangle^{s_4}n\,\widehat{\sigma}_1(n)\left(\widehat{\mathscr{F}_I}(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(\tau,n)\right)\\&\qquad\cdot[t-\tau]k\,\widehat{g}(\tau,k-n,tk-\tau n)\,\mathrm{d}\tau\,\mathrm{d}n\Big|^2\mathrm{d}t.\end{aligned}\tag{42}$$

Integrating (42) with respect to k yields

$$\begin{aligned}\|A_{s_4}\widehat{\varrho}\|^2_{L^2_{(k)}L^2_{(t)}}\lesssim&\int_{\mathbb{R}^d}\int_0^T|k|\langle k,tk\rangle^{2s_4}\left|\widehat{f_0}(k,tk)\right|^2\mathrm{d}k\,\mathrm{d}t\\&+\int_{\mathbb{R}^d}\int_0^t\left|\int_0^t|k|^{1/2}\langle k,tk\rangle^{s_4}k\widehat{\sigma}_1(k)\widehat{\mathscr{F}_I}(\tau,k)\cdot(t-\tau)k\widehat{f}^0([t-\tau]k)\,\mathrm{d}\tau\right|^2\mathrm{d}k\,\mathrm{d}t\\&+\int_{\mathbb{R}^d}\int_0^T\Big|\int_0^t\int_{\mathbb{R}^d}|k|^{1/2}\langle k,tk\rangle^{s_4}n\widehat{\sigma}_1(n)\left(\widehat{\mathscr{F}_I}(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(\tau,n)\right)\\&\qquad\cdot(t-\tau)k\widehat{g}(\tau,k-n,tk-\tau n)\,\mathrm{d}\tau\,\mathrm{d}n\Big|^2\mathrm{d}k\,\mathrm{d}t.\end{aligned}$$

We denote the three terms in the right hand side as CT1, CT2 and NLT, respectively (for "constant term 1 and 2, non linear term"). In what follows, we are going to split the discussion according to the estimate NLT $\lesssim$ NLTT + NLTR, where NLTT (for transport) and NLTR (for reaction) stand for the contributions that arise from the following decomposition

$$\langle k,tk\rangle^{s_4}\lesssim\langle k-n,tk-\tau n\rangle^{s_4}+\langle n,\tau n\rangle^{s_4}.$$

Estimate on CT1 **and** CT2. Thanks to [7, Lemma 2.6] we have

$$\mathrm{CT1}\lesssim\sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|\le P}}\|(x,v)\mapsto x^\alpha f_0(x,v)\|^2_{H^s_P}\le\varepsilon^2.$$

In Proposition 4.3, we already obtained CT2 $\lesssim\varepsilon^2$.

Estimate on NLTT. As said above, having Proposition 4.1 at hand permits us to readily adapt the arguments of [7]. The Cauchy-Schwarz inequality yields

$$\begin{aligned}\mathrm{NLTT} \le \int_{\mathbb{R}^d}\int_0^T \Big(\int_0^t \int_{\mathbb{R}^d} \langle\tau\rangle^{5/2}|n||\widehat{\sigma}_1(n)||\widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)|\,\mathrm{d}\tau\,\mathrm{d}n\Big)\\ \times\Big(\int_0^t\int_{\mathbb{R}^d} \langle\tau\rangle^{-5/2}|n||\widehat{\sigma}_1(n)|\,\Big|\widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\Big|\,k|\langle k-n, tk-\tau n\rangle^{2s_4}\\ \times |(t-\tau)k|^2|\widehat{g}(\tau,k-n,tk-\tau n)|^2\,\mathrm{d}\tau\,\mathrm{d}n\Big)\,\mathrm{d}k\,\mathrm{d}t.\end{aligned}$$

Now, (34c) and (40e) ensure that

$$\langle n,\tau n\rangle^{s_1}|\widehat{\sigma}_1(n)||\widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)| \lesssim (1+K_5)\varepsilon.$$

Since $|n|\langle\tau\rangle \le \langle n,\tau n\rangle$, we get

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^d} \langle\tau\rangle^{5/2}|n||\widehat{\sigma}_1(n)||\widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)|\,\mathrm{d}\tau\,\mathrm{d}n\\ \lesssim \Big(\int_0^t \langle\tau\rangle^{5/2}\int_{\mathbb{R}^d_n} |n|\langle n,\tau n\rangle^{-s_1}\,\mathrm{d}n\ \mathrm{d}\tau\Big)(1+K_5)\varepsilon\\ \lesssim \Big(\int_0^{+\infty}\langle\tau\rangle^{5/2-d-1}\,\mathrm{d}\tau\Big)(1+K_5)\varepsilon \lesssim (1+K_5)\varepsilon\end{aligned}$$

where the last estimate assumes the condition $5/2-d-1<-1$, that is $d>5/2$. This is one of the constraints on the space dimension d which imply that the analysis applies only when $d\ge 3$.

Going back to NLTT we are led to (by using $(|t-\tau)k| \le \langle \tau(k-n), tk-\tau n\rangle)$

$$\begin{aligned}
\mathrm{NLTT} &\lesssim (1+K_5)\varepsilon \int_{\mathbb{R}^d}\int_0^T \Big(\int_0^t \int_{\mathbb{R}^d} \langle\tau\rangle^{+5/2}|n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right| \\
&\quad \times \langle\tau\rangle^{-5}|k|\langle k-n, tk-\tau n\rangle^{2s_4}\langle \tau(k-n), tk-\tau n\rangle^2 |\widehat{g}(\tau, k-n, tk-\tau n)|^2 \,\mathrm{d}\tau\,\mathrm{d}n\Big)\,\mathrm{d}k\,\mathrm{d}t \\
&\lesssim (1+K_5)\varepsilon \int_{\mathbb{R}^d}\int_0^T \langle\tau\rangle^{+5/2}|n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right| \\
&\quad \times \Big(\int_\tau^T \int_{\mathbb{R}^d} \langle\tau\rangle^{-5}|k|\langle k-n, tk-\tau n\rangle^{2s_4}\langle \tau(k-n), tk-\tau n\rangle^2 \\
&\quad \times |\widehat{g}(\tau, k-n, tk-\tau n)|^2 \,\mathrm{d}t\,\mathrm{d}k\Big)\,\mathrm{d}n\,\mathrm{d}\tau \\
&\lesssim (1+K_5)\varepsilon \left(\int_{\mathbb{R}^d}\int_0^T \langle\tau\rangle^{+5/2}|n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right| \,\mathrm{d}n\,\mathrm{d}\tau\right) \\
&\quad \times \Big(\sup_{0\le\tau\le T}\sup_{n\in\mathbb{R}^d} \langle\tau\rangle^{-5}\int_{\mathbb{R}^d}\int_{-\infty}^{+\infty} \langle k-n, tk-\tau n\rangle^{2s_4}\langle \tau(k-n), tk-\tau n\rangle^2 \\
&\quad \times |\widehat{g}(\tau, k-n, tk-\tau n)|^2 |k| \,\mathrm{d}t\ \mathrm{d}k\Big) \\
&\lesssim (1+K_5)^2\varepsilon^2 \Big(\sup_{0\le\tau\le T}\sup_{n\in\mathbb{R}^d} \langle\tau\rangle^{-5}\int_{\mathbb{R}^d}|k|\int_{-\infty}^{+\infty} \big|\langle \tau(k-n), tk-\tau n\rangle \langle k-n, tk-\tau n\rangle^{s_4} \\
&\quad \times \widehat{g}(\tau, k-n, tk-\tau n)\big|^2 \,\mathrm{d}t\,\mathrm{d}k\Big).
\end{aligned}$$

With two changes of variables and by applying [7, Lemma 2.8], we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d}|k|\int_{-\infty}^{+\infty} |\langle \tau(k-n), tk-\tau n\rangle \langle k-n, tk-\tau n\rangle^{s_4}\widehat{g}(\tau, k-n, tk-\tau n)|^2 \,\mathrm{d}t\,\mathrm{d}k \\
&= \int_{\mathbb{R}^d}\int_{-\infty}^{+\infty} \left|\langle \tau(k-n), t\frac{k}{|k|}-\tau n\rangle \langle k-n, t\frac{k}{|k|}-\tau n\rangle^{s_4}\widehat{g}(\tau, k-n, tk-\tau n)\right|^2 \,\mathrm{d}t\,\mathrm{d}k \\
&\le \sup_{\omega\in\mathbb{S}^{d-1}}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{-\infty}^{+\infty} |\langle \tau(k-n), t\omega+x\rangle \langle k-n, t\omega+x\rangle^{s_4}\widehat{g}(\tau, k-n, t\omega+x)|^2 \,\mathrm{d}t\,\mathrm{d}k \\
&\le \sup_{\omega\in\mathbb{S}^{d-1}}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{-\infty}^{+\infty} |\langle \tau k, t\omega+x\rangle \langle k, t\omega+x\rangle^{s_4}\widehat{g}(\tau, k-n, t\omega+x)|^2 \,\mathrm{d}t\,\mathrm{d}k \\
&\lesssim \|\langle \tau\nabla_x, \nabla_v\rangle g(\tau)\|^2_{H^{s_4}_P}\,.
\end{aligned}$$

Finally, combining this with (40a) we obtain

$$\mathrm{NLTT} \lesssim (1+K_5)^2 K_1\varepsilon^4.$$

Estimate on NLTR. We make the time-response kernel $\bar{K}$ appear:

$$\mathrm{NLTR} = \int_{\mathbb{R}^d}\int_0^T \Big(\int_0^t \int_{\mathbb{R}^d} \bar{K}(t,\tau,k,n)\langle n,\tau n\rangle^{s_4}|n|^{1/2}\langle n\rangle^2|\widehat{\sigma}_1(n)| \times \left|\widehat{\mathscr{F}_I}(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(\tau,n)\right| \mathrm{d}\tau\,\mathrm{d}n\Big)^2 \mathrm{d}k\,\mathrm{d}t.$$

Then, Cauchy-Schwarz' inequality and Fubini's theorem allow us to obtain

$$\begin{aligned}
\mathrm{NLTR} &\lesssim \int_{\mathbb{R}^d}\int_0^T \left(\int_0^t\int_{\mathbb{R}^d}\bar{K}(t,\tau,k,n)\,\mathrm{d}\tau\,\mathrm{d}n\right)\Big(\int_0^t\int_{\mathbb{R}^d}\bar{K}(t,\tau,k,n) \\
&\qquad \times\langle n,\tau n\rangle^{2s_4}|n|\langle n\rangle^4|\widehat{\sigma}_1(n)|^2\left|\widehat{\mathscr{F}_I}(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(\tau,n)\right|^2\mathrm{d}\tau\,\mathrm{d}n\Big)\,\mathrm{d}k\,\mathrm{d}t \\
&\lesssim \left(\sup_{t\in[0,T]}\sup_{k\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}\bar{K}(t,\tau,k,n)\,\mathrm{d}\tau\,\mathrm{d}n\right)\int_0^T\int_{\mathbb{R}^d}\left(\int_\tau^T\int_{\mathbb{R}^d}\bar{K}(t,\tau,k,n)\,\mathrm{d}t\,\mathrm{d}k\right) \\
&\qquad \times\langle n,\tau n\rangle^{2s_4}|n|\langle n\rangle^4|\widehat{\sigma}_1(n)|^2\left|\widehat{\mathscr{F}_I}(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(\tau,n)\right|^2\mathrm{d}\tau\,\mathrm{d}n \\
&\lesssim \left(\sup_{t\in[0,T]}\sup_{k\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}\bar{K}(t,\tau,k,n)\,\mathrm{d}\tau\,\mathrm{d}n\right)\left(\sup_{\tau\in[0,T]}\sup_{n\in\mathbb{R}^d}\int_\tau^T\int_{\mathbb{R}^d}\bar{K}(t,\tau,k,n)\,\mathrm{d}t\,\mathrm{d}k\right) \\
&\qquad \times\int_0^T\int_{\mathbb{R}^d}\langle n,\tau n\rangle^{2s_4}|n|\langle n\rangle^4|\widehat{\sigma}_1(n)|^2\left|\widehat{\mathscr{F}_I}(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(\tau,n)\right|^2\mathrm{d}\tau\,\mathrm{d}n.
\end{aligned}$$

By using (34a) and (40b), we obtain

$$\int_0^T\int_{\mathbb{R}^d}\langle n,\tau n\rangle^{2s_4}|n|\langle n\rangle^4|\widehat{\sigma}_1(n)|^2\left|\widehat{\mathscr{F}_I}(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(\tau,n)\right|^2\mathrm{d}\tau\,\mathrm{d}n \lesssim (1+K_2)\varepsilon^2.$$

Gathering this with Lemma 4.10 and (40e), we are led to

$$\mathrm{NLTR} \lesssim (1+K_2)K_5^2\varepsilon^4.$$

Recap. We have shown that, if g is a solution of (10a)–(10b) satisfying (40a)–(40e) on $[0,T]$, then

$$\|A_{s_4}\widehat{\varrho}\|^2_{L^2_{(k)}L^2_{(t)}} \lesssim \left(1+(1+K_5)^2K_1\varepsilon^2+(1+K_2)K_5^2\varepsilon^2\right)\varepsilon^2.$$

Let us denote C_1 the constant hidden in the symbol $\lesssim$ of this estimate. Choosing $K_2 \geq C_1$ and $\varepsilon \ll 1$ so that

$$(1+K_5)^2K_1\varepsilon^2+(1+K_2)K_5^2\varepsilon^2 \leq 1$$

allows us to conclude that (41b) holds.

4.2 Periodic framework

The dispersive effect which has been used for proving the Landau damping on $\mathbb{R}^d$ does not exist on the torus. For this reason, in order to control the echoes, we shall work in the analytic framework, following [6]. For the Vlasov-Poisson problem, the analysis of [4] is a hint that this regularity could be necessary. As a counterpart of this regularity, there is no restriction on the space dimension d.

The proof still relies on a bootstrap argument, see [6]. There are two main arguments, like on $\mathbb{R}^d$: firstly, the force term $\nabla\sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))$ can be controlled, in suitable norms, by the macroscopic density $\varrho(t)$, and, secondly, the contribution associated to the initial data $\int_0^t \nabla\sigma_1 \star \mathscr{F}_I(\tau, x+\tau v)\cdot\nabla_v \mathscr{M}(v)\,\mathrm{d}\tau$ does not perturb too much the bootstrap property (here, we refer the reader to the remarks made when analyzing the whole space problem).

4.2.1 Functional framework

We start by introducing several Gevrey norms. Let $g : (0,\infty)_t \times \mathbb{T}^d_x \times \mathbb{R}^d_v \to \mathbb{R}$. The Gevrey norm $\|\cdot\|_{\mathcal{G}^{\lambda,\sigma;s}}$ is defined by

$$\|g(t)\|^2_{\mathcal{G}^{\lambda,\sigma;s}} = \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^{2\sigma} e^{2\lambda\langle k,\xi\rangle^s} \left|\widehat{g}(t,k,\xi)\right|^2 \mathrm{d}\xi$$

and we also need the Gevrey norm $\|\cdot\|_{\mathcal{F}^{\lambda,\sigma;s}}$ given by

$$\|g(t)\|^2_{\mathcal{F}^{\lambda,\sigma;s}} = \sum_{k\in\mathbb{Z}^d} \langle k,tk\rangle^{2\sigma} e^{2\lambda\langle k,tk\rangle^s} \left|\widehat{g}(t,k,tk)\right|^2.$$

Let $\varrho : \mathbb{R}_t \times \mathbb{T}^d_x \to \mathbb{R}$. The Gevrey norm $\|\cdot\|_{\mathcal{F}^{\lambda,\sigma;s}}$ reads

$$\|\varrho(t)\|^2_{\mathcal{F}^{\lambda,\sigma;s}} = \sum_{k\in\mathbb{Z}^d} \langle k,tk\rangle^{2\sigma} e^{2\lambda\langle k,tk\rangle^s} \left|\widehat{\varrho}(t,k)\right|^2.$$

In what follows, we always assume $\lambda,\sigma \geq 0$ and $0 < s \leq 1$.

As a warm-up, we observe that, with $g(t,x,v) = f(t,x+tv,v)$ and $\varrho(t,x) = \int f(t,x,v)\,\mathrm{d}v$, we have

$$\|\varrho(t)\|_{\mathcal{F}^{\lambda,\sigma;s}} = \|g(t)\|_{\mathcal{F}^{\lambda,\sigma;s}}.$$

Moreover, assuming $\sigma > d/2$ we have a $\sigma-$ring property: with $h(t,x,v) = \varrho(t,x+tv)g(t,x,v)$, we have

$$\|h(t)\|_{\mathcal{G}^{\lambda,\sigma;s}} \lesssim \|\varrho(t)\|_{\mathcal{F}^{\lambda,\sigma;s}} \|g(t)\|_{\mathcal{G}^{\lambda,\sigma;s}}.$$

Finally, we shall also need the following Gevrey norm: for $P \in \mathbb{N}$, we define the norm $\|\cdot\|_{\mathcal{G}^{\lambda,\sigma;s}_P}$ of a function $(t,x,v) \mapsto g(t,x,v)$ by

$$\begin{aligned}\|g(t)\|^2_{\mathcal{G}^{\lambda,\sigma;s}_P} &= \sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|\leq P}} \|(x,v)\mapsto v^\alpha g(t,x,v)\|^2_{\mathcal{G}^{\lambda,\sigma;s}} \\ &= \sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|\leq P}} \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^{2\sigma} e^{2\lambda\langle k,\xi\rangle^s} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right|^2 \mathrm{d}\xi.\end{aligned}$$

The $\sigma-$ring estimate equally applies to this norm.

From now on, we assume that

$$\sigma > d/2, \qquad P > d/2, \qquad 0 < s \leq 1.$$

We shall consider the parameter λ as a function of the time variable $\lambda : t \mapsto \lambda(t) \in (0, \infty)$, continuous and decreasing.

In contrast to what we did for the problem on $\mathbb{R}^d$, we do not express general conditions on $\mathscr{F}_I$ and p_c. Instead, we shall use the same assumptions as in the case of the linearized Landau damping. For the sake of convenience, let us recall them here.

(K1) $n \geq 3$ is odd,

(K2) $\sigma_2 \in C_c^0(\mathbb{R}^n)$ with $\mathrm{supp}(\sigma_2) \subset B(0, R_2)$.

(K3) $\mathrm{supp}(\psi_i) \subset \mathbb{T}^d \times B(0, R_I)$, $i = 1, 2$ and

$$\mathscr{E}_I = \iint_{\mathbb{T}^d \times \mathbb{R}^n} \Big(|\psi_1(x,z)|^2 + c^2 |\nabla_z \psi_0(x,z)| \Big) \,\mathrm{d}x \,\mathrm{d}z < +\infty.$$

(K4) $\sigma_1 : \mathbb{T}^d \to \mathbb{R}_+$ is radially symmetry and analytic; in particular there exist $C_1, \lambda_1 > 0$ such that $|\widehat{\sigma}_1(k)| \leq C_1 \exp(-\lambda_1 |k|)$ holds for any $k \in \mathbb{Z}^d$.

Note theat assumption **(K5)** on $\mathscr{M}$ and f_0 will be replaced by $\mathscr{M}, f_0 \in \mathcal{G}_P^{\widetilde{\lambda_0},0;s}$.

As a consequence of **(K1)** and **(K2)** the kernel p_c has a compact support: $\mathrm{supp}(p_c) \subset [0, 2R_2/c]$, see Lemma 2.3. By virtue of **(K2)** and **(K3)**, $\mathscr{F}_I$ is compactly supported too: $\mathrm{supp}(\mathscr{F}_I) \subset [0, (R_I + R_2)/c]$, as pointed out in the proof of Lemme 3.9. In what follows, the following parameters will play an important role

$$2R_2/c, \qquad S_0 = (R_I + R_2)/c.$$

The following statement, analog for the torus of Proposition 4.1, is a crucial ingredient to justify the boostrap property.

Proposition 4.12 *Let **(K1)–(K4)** be fulfilled. Let $t \mapsto \lambda(t) > 0$ be a continuous and decreasing function. For any $\sigma \geq 0$ and $0 < s \leq 1$, we get*

$$\|\nabla\sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \lesssim \mathscr{E}_I \mathbf{1}_{0 \leq t \leq S_0} + \int_0^t |p_c(t-\tau)| \, \|\varrho(\tau)\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}} \,\mathrm{d}\tau, \quad (43)$$

Consequently, the following estimates hold

$$\|\nabla\sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \lesssim \mathscr{E}_I + \int_0^t \|\varrho(\tau)\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}} \,\mathrm{d}\tau, \quad (44a)$$

$$\sup_{\tau \in [0,t]} \|\nabla\sigma_1 \star (\mathscr{F}_I(\tau) - \sigma_1 \star \mathscr{G}_\varrho(\tau))\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}} \lesssim \mathscr{E}_I + \sup_{\tau \in [0,t]} \|\varrho(\tau)\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}}, \quad (44b)$$

$$\int_0^t \|\nabla\sigma_1 \star (\mathscr{F}_I(\tau) - \sigma_1 \star \mathscr{G}_\varrho(\tau))\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}} \,\mathrm{d}\tau \lesssim \mathscr{E}_I + \int_0^t \|\varrho(\tau)\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}} \,\mathrm{d}\tau. \quad (44c)$$

Remark 4.13 *The following observations will be useful:*

i) In the specific case $s = 1$ we shall need a further assumption on $\lambda(0)$: for this situation, we assume $\lambda(0) < C(\lambda_1, 2R_2/c, S_0) = \min(\lambda_1/\langle S_0\rangle, 2\lambda_1/\langle 2R_2/c\rangle)$.

ii) In contrast to the analysis of the Vlasov-Poisson problem, a control of $\int \|\varrho\| \,\mathrm{d}\tau$ ensures a pointwise control of the force term. This fact, which can be seen as a kind of regularizing effect of the half-time-convolution, simplifies the proof of the bootstrap property.

iii) Like for the whole space problem, the exponential decay of $\widehat{\sigma}_1(k)$ can be used to absorb any polynomial with respect to k that arises in the estimates, see Remark 4.2.

Proof. We estimate separately the contributions from $\mathscr{F}_I$ and $\mathscr{G}_\varrho$:

$$\|\nabla\sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \lesssim \|\nabla\sigma_1 \star \mathscr{F}_I(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} + \|\nabla\Sigma \star \mathscr{G}_\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}.$$

For the former, we use $\mathrm{supp}(\mathscr{F}_I) \subset [0, S_0] \times \mathbb{T}^d$ and the estimate (see the proof of Lemma 3.9)

$$|k|\,|\widehat{\sigma}_1(k)|\,|\widehat{\mathscr{F}}_I(t,k)| \leq C_1|k|e^{-\lambda_1|k|}\|\sigma_2\|_{L^{2n/(n+2)}}\sqrt{\mathscr{E}_I}\mathbf{1}_{0\leq t\leq S_0}. \tag{45}$$

We obtain

$$\begin{aligned}\|\nabla\sigma_1 \star \mathscr{F}_I(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} &\lesssim \left(\sum_{k\in\mathbb{Z}^d}\langle k, tk\rangle^{2\sigma}e^{2\lambda(t)\langle k,tk\rangle^s}|k|^2e^{-2\lambda_1|k|^2}\right)\mathscr{E}_I\mathbf{1}_{0\leq t\leq S_0}\\ &\lesssim \left(\sum_{k\in\mathbb{Z}^d}\langle k\rangle^{2\sigma}\langle S_0\rangle^{2\sigma}e^{2\lambda(0)\langle k\rangle^s\langle S_0\rangle^s}|k|^2e^{-2\lambda_1|k|^2}\right)\mathscr{E}_I\mathbf{1}_{0\leq t\leq S_0}.\end{aligned}$$

When $0 < s < 1$ the sum is finite; when $s = 1$ we should impose the additional condition $\lambda_1 > \lambda(0)\langle S_0\rangle$.

For the latter, we apply the Cauchy-Schwarz inequality, so that

$$\begin{aligned}\|\,\nabla\Sigma \star \mathscr{G}_\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} &= \sum_{k\in\mathbb{Z}^d}\langle k, tk\rangle^{2\sigma}e^{2\lambda(t)\langle k,tk\rangle^s}|k|^2|\widehat{\sigma}_1(k)|^4\left|\int_0^t p_c(t-\tau)\widehat{\varrho}(\tau,k)\,\mathrm{d}\tau\right|^2\\ &\leq \|p_c\|_{L^1}\int_0^t |p_c(t-\tau)|\left(\sum_{k\in\mathbb{Z}^d}\langle k, tk\rangle^{2\sigma}e^{2\lambda(t)\langle k,tk\rangle^s}|k|^2|\widehat{\sigma}_1(k)|^4|\widehat{\varrho}(\tau,k)|^2\right)\mathrm{d}\tau\\ &= \|p_c\|_{L^1}\int_0^t |p_c(t-\tau)|\left(\sum_{k\in\mathbb{Z}^d}I_k(t,\tau)\langle k, \tau k\rangle^{2\sigma}e^{2\lambda(t)\langle k,\tau k\rangle^s}|\widehat{\varrho}(\tau,k)|^2\right)\mathrm{d}\tau.\end{aligned}$$

It follows that

$$I_k(t,\tau) = |k|^2|\widehat{\sigma}_1(k)|^4\frac{\langle k, tk\rangle^{2\sigma}}{\langle k, \tau k\rangle^{2\sigma}}e^{2(\lambda(t)-\lambda(\tau)\langle k,tk\rangle^s}e^{\lambda(\tau)(\langle k,tk\rangle^s-\langle k,\tau k\rangle^s)}.$$

Therefore if $I_k(t,\tau)$ is bounded uniformly with respect to k, t and τ, then we get

$$\|\,\nabla\Sigma \star \mathscr{G}_\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \lesssim \int_0^t |p_c(t-\tau)|\,\|\varrho(\tau)\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}}\,\mathrm{d}\tau.$$

We are left with the task of justify a uniform bound on $I_k(t,\tau)$. To this end, we remember that p_c has a compact support: we can restrict the time integration to

$0 \leq t-\tau \leq 2R_2/c$. For $t \geq \tau$, a simple analysis of function shows that

$$\sup_{k\in\mathbb{Z}^d} \frac{\langle k, tk\rangle^{2\sigma}}{\langle k, \tau k\rangle^{2\sigma}} \leq \frac{\langle t\rangle^{2\sigma}}{\langle \tau\rangle^{2\sigma}} \leq \langle t-\tau\rangle^{2\sigma} \leq \langle 2R_2/c\rangle^{2\sigma}.$$

Since $t \mapsto \lambda(t)$ is decreasing, we have $\exp(2(\lambda(t)-\lambda(\tau))\langle k, tk\rangle^s) \leq 1$. Finally, with $0 < s \leq 1$, we have (see [6, Lemma 3.2])

$$|\langle x\rangle^s - \langle y\rangle^s| \leq \langle x-y\rangle^s,$$

so that $\langle k, tk\rangle^s - \langle k, \tau k\rangle^s \leq \langle (t-\tau)k\rangle^s \leq \langle \frac{2R_2}{c} k\rangle^s$ and $\exp(2\lambda(\tau)\left(\langle k, tk\rangle^s - \langle k, \tau k\rangle^s\right)) \leq \exp(2\lambda(0)\langle \frac{2R_2}{c}\rangle^s \langle k\rangle^s)$. We conclude with

$$I_k(t,\tau) \leq C_1^4 |k|^2 e^{-4\lambda_1|k|} \langle 2R_2/c\rangle^{2\sigma} e^{2\lambda(0)\langle \frac{2R_2}{c}\rangle^s \langle k\rangle^s},$$

when $0 < s < 1$, while for $s = 1$ we further assume $4\lambda_1 > 2\lambda(0)\langle 2R_2/c\rangle$. ∎

We turn to the estimate of the force term $\int_0^t \nabla\sigma_1 \star \mathscr{F}_I(\tau, x+\tau v)\cdot \nabla_v \mathscr{M}(v)\,\mathrm{d}\tau$ by means of the norms involved in the bootstrap.

Proposition 4.14 *Let **(K1)**–**(K4)**. Assume that $\mathscr{M} \in \mathcal{G}_P^{\widetilde{\lambda_0},0;s}$ for some integer $P > d/2$. Let $t \mapsto \lambda(t) > 0$ be continuous, decreasing, and such that $\lambda(0) < \widetilde{\lambda_0}$. Then for any $\sigma \geq 0$ and $0 < s \leq 1$, we have*

$$\int_0^T \left\| \int_0^t \nabla\sigma_1 \star \mathscr{F}_I(\tau, x+\tau\, v)\cdot \nabla_v \mathscr{M}(v)\,\mathrm{d}\tau \right\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \mathrm{d}t \lesssim \mathscr{E}_I. \tag{46}$$

Remark 4.15 *Again, when $s = 1$ a constraint on $\lambda(0)$ like $\lambda(0) < C'(\lambda_1, S_0) = \lambda_1/\langle S_0\rangle$ should be imposed.*

Proof. We start with

$$\begin{aligned}
&\int_0^T \left\| \int_0^t \nabla\sigma_1 \star \mathscr{F}_I(\tau, x+\tau\, v)\cdot \nabla_v \mathscr{M}(v)\,\mathrm{d}\tau \right\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \mathrm{d}t \\
&\quad\leq \int_0^T \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \Big(\int_0^t \langle k, tk\rangle^\sigma e^{\lambda(t)\langle k,tk\rangle^s} |k|\, |\widehat{\sigma}_1(k)| \left|\widehat{\mathscr{F}_I}(\tau,k)\right| \\
&\qquad\qquad \times \left|(t-\tau)k\right| \left|\widehat{\mathscr{M}}((t-\tau])k)\right| \mathrm{d}\tau\Big)^2 \mathrm{d}t,
\end{aligned}$$

and we define $I(t,k)$ as follow

$$I(t,k) = \int_0^t \langle k, tk\rangle^\sigma e^{\lambda(t)\langle k,tk\rangle^s} |k|\, |\widehat{\sigma}_1(k)| \left|\widehat{\mathscr{F}_I}(\tau,k)\right| \left|(t-\tau)k\right| \left|\widehat{\mathscr{M}}((t-\tau])k)\right| \mathrm{d}\tau.$$

For any $k \neq 0$, we have $\langle t\rangle \leq \langle k, tk\rangle$, and since λ is decreasing, we obtain

$$\begin{aligned}
I(t,k) \leq \langle t\rangle^{-1} &\int_0^t \langle k, \tau k\rangle^{\sigma+1} e^{\lambda(\tau)\langle k,\tau k\rangle^s} |k|\, |\widehat{\sigma}_1(k)| \left|\widehat{\mathscr{F}_I}(\tau,k)\right| \\
&\times \langle [t-\tau]k\rangle^{\sigma+1} e^{\lambda(\tau)\langle [t-\tau]k\rangle^s} |t-\tau|\, |k| \left|\widehat{\mathscr{M}}([t-\tau]k)\right| \mathrm{d}\tau.
\end{aligned}$$

Since $\|\xi \mapsto \exp(\widetilde{\lambda_0}\langle\xi\rangle^s)\widehat{\mathscr{M}}(\xi)\|_{H^P} \lesssim \|\mathscr{M}\|_{\mathcal{G}_P^{\widetilde{\lambda_0},0;s}}$ and $P > d/2$, the Sobolev embedding $H^P \hookrightarrow C^0$ ensures that

$$|\widehat{\mathscr{M}}(\xi)| \lesssim e^{-\widetilde{\lambda_0}\langle\xi\rangle^s}.$$

Then, by using (45), we arrive at

$$\begin{aligned}I(t,k) &\lesssim \langle t\rangle^{-1}\langle k\rangle^{\sigma+1}\langle S_0\rangle^{\sigma+1}e^{\lambda(0)\langle k\rangle^s\langle S_0\rangle^s}|k|e^{-\lambda_1|k|}\\ &\qquad\times\left(\int_0^t \langle[t-\tau]k\rangle^{\sigma+1}e^{\lambda(0)\langle[t-\tau]k\rangle^s}|t-\tau|\,|k|e^{-\widetilde{\lambda_0}\langle[t-\tau]k\rangle^s}\,\mathrm{d}\tau\right)\sqrt{\mathscr{E}_I}.\end{aligned}$$

Since $\lambda(0) < \widetilde{\lambda_0}$ we have

$$\int_0^t \langle[t-\tau]k\rangle^{\sigma+1}e^{\lambda(0)\langle[t-\tau]k\rangle^s}|t-\tau|\,|k|e^{-\widetilde{\lambda_0}\langle[t-\tau]k\rangle^s}\,\mathrm{d}\tau \le \int_{\mathbb{R}}\langle u\rangle^{\sigma+2}e^{-(\widetilde{\lambda_0}-\lambda(0))\langle u\rangle^s}\,\mathrm{d}u \lesssim 1.$$

Therefore, when $0 < s < 1$ we obtain $\int_0^T\sum_k I(t,k)^2\,\mathrm{d}t \lesssim \mathscr{E}_I$ and for $s = 1$ we conclude similarly at the price of a constraint like $\lambda_1 > \lambda(0)\langle S_0\rangle$. ∎

4.2.2 Main result

That the Landau damping holds on the torus can be formulated as follows.

Theorem 4.16 (Landau damping in $\mathbb{T}^d$) *Let **(K1)**–**(K4)** be fullfield. Let $P > d/2$ be an integer, $0 < s \le 1$ be a real number and $\mathscr{M}, f_0 \in \mathcal{G}_P^{\widetilde{\lambda_0},0;s}$ with $\widetilde{\lambda_0} > 0$. We also assume (without any loss of generality) that the space average of $\int f_0\,\mathrm{d}v$ is equal to 0. There exists a universal constant ε_0, such that if*

$$\|f_0\|_{\mathcal{G}_P^{\widetilde{\lambda_0},\sigma;s}} \le \varepsilon_0 \ \ ; \ \ \mathscr{E}_I \le \varepsilon_0^2$$

and $\mathscr{M}$ satisfies $(\mathbf{L})$, then, the unique solution g of (10a)–(10b) is globally defined. To be more specific, for any $0 < \lambda' < \widetilde{\lambda_0}$, we have $g \in C^0(\mathbb{R}_+;\mathcal{G}^{\lambda',0;s})$ and there exists an asymptotic density $g^\infty \in \mathcal{G}^{\lambda',0;s}$, the space average of which vanishes, such that

$$\|g(t) - g^\infty\|_{\mathcal{G}^{\lambda',0;s}} \lesssim \varepsilon_0 e^{-\frac{1}{2}(\widetilde{\lambda_0}-\lambda')\langle t\rangle^s}, \tag{47a}$$

$$\|\varrho(t)\|_{\mathcal{F}^{\lambda',0;s}} \lesssim \varepsilon_0 e^{-\frac{1}{2}(\widetilde{\lambda_0}-\lambda')\langle t\rangle^s}, \tag{47b}$$

$$\|\nabla\sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))\|_{\mathcal{F}^{\lambda',0;s}} \lesssim \varepsilon_0 e^{-\frac{1}{2}(\widetilde{\lambda_0}-\lambda')\langle t\rangle^s}. \tag{47c}$$

Remark 4.17 *When $s = 1$ the constraint on λ' becomes*

$$\lambda' < \min\Big(\widetilde{\lambda_0}, \frac{\lambda_1}{\langle S_0\rangle}, \frac{2\lambda_1}{\langle 2R_2/c\rangle}\Big).$$

Remark 4.18 *Estimate (47b) can be rephrased as a decay of $\widehat{\varrho}(t,k)$ like $\exp(-\lambda'\langle tk\rangle^s)$. This can be used to establish also that fluctuation of the medium ψ tends to 0, see Proposition 3.4).*

Like for the problem set on $\mathbb{R}^d$, the proof relies on a bootstrap argument, which, in this context, states as follows.

Proposition 4.19 (Bootstrap) *Let the assumptions of Theorem 4.16 be fulfilled. Let $\alpha_0 = (\widetilde{\lambda_0} + \lambda')/2$ and $\sigma > d/2 + 6$. There exists a function $\lambda : \mathbb{R}_+ \to (\alpha_0, \widetilde{\lambda_0})$, continuous and decreasing, a real $\beta > 2$ and constants $K_1, K_2, K_3, K_4 > 0$ such that if g is a solution of* (10a)–(10b) *on the time interval* $[0, T]$ *verifying*

$$\|g(t)\|^2_{\mathcal{G}_P^{\lambda(t),\sigma+1;s}} \leq 4K_1\langle t\rangle^7\varepsilon^2 \tag{48a}$$

$$\|g(t)\|^2_{\mathcal{G}_P^{\lambda(t),\sigma-\beta;s}} \leq 4K_2\varepsilon^2 \tag{48b}$$

$$\int_0^T \|\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t \leq 4K_3\varepsilon^2 \tag{48c}$$

for $0 < \varepsilon \leq \varepsilon_0$ small enough, then g also satisfies, on $[0, T]$, the estimates

$$\|g(t)\|^2_{\mathcal{G}_P^{\lambda(t),\sigma+1;s}} \leq 2K_1\langle t\rangle^7\varepsilon^2 \tag{49a}$$

$$\|g(t)\|^2_{\mathcal{G}_P^{\lambda(t),\sigma-\beta;s}} \leq 2K_2\varepsilon^2 \tag{49b}$$

$$\int_0^T \|\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t \leq 2K_3\varepsilon^2 \tag{49c}$$

$$\|\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \leq 2K_4\langle t\rangle\varepsilon^2 \tag{49d}$$

Remark 4.20 *The role of (49d) is a bit different from its analog for the Vlasov-Poisson problem. Indeed, the interest of this estimate is to provide a pointwise control on the force term. However, here, as said above, such a control can be obtained by estimating $\int \|\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t$. Consequently (49c) is enough to finish the proof, without using (49d) and the proof slightly simplifies. Nevertheless, we keep (49d) in the statement since it is useful to justify (47b).*

The justification of the bootstrap follows the same approach than for the problem on $\mathbb{R}^d$. Since the structure of the Vlasov-Wave equation is close to the structure of the Vlasov-Poisson equation, we can perform the same estimates than in [6]. The price to be paid is to replace terms of the form $\|\varrho(t)\|_{\mathcal{F}}$ by

$$\|\nabla\sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))\|_{\mathcal{F}}\,. \tag{50}$$

Then all the difficulty consists in controlling (50) by means of $\|\varrho(t)\|_{\mathcal{F}}$. Since Proposition 4.12 allows us to perform this kind of estimate, we have a complete proof of the Proposition 4.19 by applying this strategy. Details can be found in [33].

5 Discussion of the stability criterion

In this section we come back to the stability criteria (**L**) and (**L′**) which are absolutely crucial for justifying the Landau damping. We already know that a large wave speed guarantees the damping, see Proposition 3.10. Nevertheless, we may also wonder, for a given wave speed c, whether or not an equilibrium $\mathscr{M}$ is stable or unstable.

5.1 Towards a Landau-Penrose criterion

For the usual Vlasov equation, a "practical" condition on the equilibrium $\mathscr{M}$ — the Penrose criterion, see [27, Condition (c) in Proposition 2.1] — can be exhibited to ensure the linearized stability. By following a similar approach we expect to find a criterion with the same flavor for the Vlasov-Wave problem. However we shall see that the half-convolution with respect to time that defines p_c makes the criterion much more intricate.

Throughout this section we assume that

$$\sigma_1 \text{ and } \sigma_2 \text{ are radially symmetric,}$$

which makes the computation more explicit. With a slight abuse, we shall use the same notation for radially symmetric functions and their radial representation. As a warm-up, let us briefly recall why it suffices to check that $\omega \in i\mathbb{R} \mapsto \mathscr{L}(\omega|k|, k) \in \mathbb{C}$ never crosses the real-axis beyond 1, see details in [32, Section 3.4] for the Vlasov-Poisson equation and [33] for the Vlasov-wave model.

The first step of the reasoning consists in showing that it is sufficient to check that $\mathscr{L}\mathscr{K}(\omega|k|, k) \neq 1$ for every k and $\omega \in \mathbb{C}$ with $\mathrm{Re}(\omega) \geq 0$. Let us distinguish four different cases, depending if $\mathbb{X}^d = \mathbb{T}^d$ or $\mathbb{R}^d$ and depending if we are considering $(\mathbf{L})$ or $(\mathbf{L}')$.

First case: $\mathbb{X}^d = \mathbb{T}^d$ **and** $(\mathbf{L})$**.** In this case we check that $\mathscr{L}\mathscr{K}((\alpha + i\beta)|k|, k)$ converges to 0 when $|k| \to +\infty$, uniformly with respect to $\alpha + i\beta$ and it converges to 0 when $\alpha \to +\infty$, uniformly with respect to k and β. Moreover, thanks to the Riemann-Lebesgue Lemma, we can also prove that $\mathscr{L}\mathscr{K}((\alpha+i\beta)|k|, k)$ converges to 0 when $|\beta| \to +\infty$. There is *a priori* no reason for the latter convergence to be uniform with respect to k and α. However, since we consider an infimum over all $k \in \mathbb{Z}^d \setminus \{0\}$, the first convergence ensures us that we can restrict to a finite number of modes k and the convergence when $|\beta| \to +\infty$ is indeed uniform with respect to k. We can also justify that this convergence is uniform with respect to α. To this end, we show that $\alpha \mapsto \mathscr{L}\mathscr{K}(\alpha + i\beta)|k|, k)$ is uniformly continuous with respect to k and β. Since the convergence of $\mathscr{L}\mathscr{K}$ to 0 when $\alpha \to +\infty$ is uniform with respect to β, we can consider α in a compact subset of $(0, \infty)$ and then (by uniform continuity) only a finite number of α's. Now, we know that outside of a compact of $\{\omega \in \mathbb{C}\,,\, \mathrm{Re}(\omega) \geq 0\} \times \mathbb{Z}^d \setminus \{0\}$ the application $(\omega, k) \mapsto \mathscr{L}\mathscr{K}(\omega|k|, k)$ is far from 1. Since in a compact of this set there is a finite number of modes k and since the application $\omega \mapsto \mathscr{L}\mathscr{K}(\omega|k|, k)$ is continuous, condition $(\mathbf{L})$ is satisfied if and only if $\mathscr{L}\mathscr{K}(\omega|k|, k) \neq 1$ for every $k \in \mathbb{Z}^d \setminus \{0\}$ and every $\omega \in \mathbb{C}$ such that $\mathrm{Re}(\omega) \geq 0$.

Second case: $\mathbb{X}^d = \mathbb{R}^d$ **and** $(\mathbf{L})$**.** This case is not far from the previous one, we only have to understand what happens when k *lives in a continuum space* like $\mathbb{R}^d \setminus \{0\}$. If we fix some $\delta > 0$ arbitrarily small and if we only consider the infimum over $\{|k| \geq \delta\}$, then we can follow the same strategy, up to the fact that we have now to justify the uniform continuity of $k \mapsto \mathscr{L}\mathscr{K}((\alpha + i\beta)|k|, k)$ with respect to β.

Next, we study what happens when k goes to 0 (this point is irrelevant for the usual Vlasov case: since the potential is singular at 0 the symbol $\mathscr{L}\mathscr{K}$ can not reach 1 when $k \to 0$). It is not possible to extend $k \mapsto \mathscr{L}\mathscr{K}(\omega|k|, k)$ by continuity at 0, but for every sequence $(k_n)_{n\in\mathbb{N}}$ such that $k_n \to 0$, up to a sub-sequence, we can assume that $(k_n/|k_n|)_{n\in\mathbb{N}}$ converges to a certain σ^∞. Then we are led to

$$\lim_{n\to+\infty} \mathscr{L}\mathscr{K}(\omega|k_n|, k_n) = |\widehat{\sigma}_1(0)|^2 \left(\int_0^{+\infty} p_c(t)\,\mathrm{d}t\right)\left(\int_0^{+\infty} e^{-\omega u}\, u\, \widehat{\mathscr{M}}(u\sigma^\infty)\,\mathrm{d}u\right).$$

Since $\int_0^\infty p_c\,\mathrm{d}t = \kappa/c^2$, we conclude that (**L**) is satisfied if and only if for every $k \in \mathbb{R}^d \setminus \{0\}$, $\sigma \in \mathbb{S}^{d-1}$, $\omega \in \mathbb{C}$ with $\mathrm{Re}(\omega) \geq 0$,

$$\mathscr{L}\mathscr{K}(\omega|k|, k) \neq 1 \qquad \textbf{and} \qquad \mathcal{L}(\omega,\sigma) = \frac{\kappa}{c^2}|\widehat{\sigma}_1(0)|^2 \left(\int_0^{+\infty} e^{-\omega u}\, u\, \widehat{\mathscr{M}}(u\sigma)\,\mathrm{d}u\right) \neq 1.$$

Third case: $\mathbb{X}^d = \mathbb{T}^d$ **and** $(\mathbf{L}')$**.** In this case, we show the uniform continuity with respect to k and β of $\alpha \mapsto \mathscr{L}\mathscr{K}\big((\alpha+i\beta)|k|, k\big)$ when α lies in an interval of the form $(-\lambda, +\infty)$ with $\lambda > 0$. Then, if the criterion $(\mathbf{L}')$ is satisfied for a certain $\kappa > 0$ for all $\omega = \alpha + i\beta$ with $\alpha \geq 0$, we can find $0 < \Lambda < \lambda$ such that (possibly replacing κ by $\kappa/2$) criterion $(\mathbf{L}')$ is satisfied for all $\omega = \alpha + i\beta$ with $\alpha > -\Lambda$.
From that point we can apply the arguments of the first case in order to conclude that $(\mathbf{L}')$ is satisfied if and only if $\mathscr{L}\mathscr{K}(\omega|k|, k) \neq 1$ for every $k \in \mathbb{Z}^d \setminus \{0\}$ and $\omega \in \mathbb{C}$ with $\mathrm{Re}(\omega) \geq 0$.
Fourth case: $\mathbb{X}^d = \mathbb{R}^d$ **and** $(\mathbf{L}')$**.** By combining the arguments of the third and second cases we obtain that $(\mathbf{L}')$ is satisfied if and only if for every $k \in \mathbb{R}^d \setminus \{0\}$, $\sigma \in \mathbb{S}^{d-1}$, $\omega \in \mathbb{C}$ with $\mathrm{Re}(\omega) \geq 0$,

$$\mathscr{L}\mathscr{K}(\omega|k|, k) \neq 1 \qquad \textbf{and} \qquad \mathcal{L}(\omega,\sigma) \neq 1.$$

The second step of the argument consists in applying Rouché's theorem in order to compute the number of zeros of $\omega \mapsto \mathscr{L}\mathscr{K}(\omega|k|, k) - 1$ in a certain compact of $\{\omega \in \mathbb{C}\,,\, \mathrm{Re}(\omega) \geq 0\}$ (note that is possible to justify that $\omega \mapsto \mathscr{L}\mathscr{K}(\omega|k|, k)$ is holomorphic). To be more specific, the previous step allows us to find a radius $\Omega > 0$ such that $\mathscr{L}\mathscr{K}$ is far from 1 for every k and $\omega \in \mathbb{C}$ with $\mathrm{Re}(\omega) \geq 0$ and $|\omega| \geq \Omega$. If we assume, for every k, that $\omega \mapsto \mathscr{L}\mathscr{K}(\omega|k|, k)$ never achieves the value 1 on the imaginary axis, then Rouché's theorem tells us that the number of zeros of $\omega \mapsto \mathscr{L}\mathscr{K}(\omega|k|, k) - 1$ is equal to

$$N = \frac{1}{2i\pi}\int_{\Gamma_\Omega} \frac{\partial_\omega \mathscr{L}\mathscr{K}(\omega|k|, k)}{\mathscr{L}\mathscr{K}(\omega|k|, k) - 1}\,\mathrm{d}\omega$$

where $\Gamma_\Omega = C_\Omega \cup [-i\Omega, i\Omega]$ with $C_\Omega = \{\Omega e^{i\theta}\,,\, \theta \in [\pi/2, 3\pi/2]\}$. We split the integral over the path Γ_Ω into a contribution over C_Ω and an other contribution over $[-i\Omega, i\Omega]$ and we let Ω go to $+\infty$: we can justify that the integral over C_Ω goes to 0 and we eventually obtain

$$N = \frac{1}{2i\pi}\int_{\mathscr{L}\mathscr{K}(i|k|\mathbb{R}, k)} \frac{1}{z-1}\,\mathrm{d}z.$$

Since $\mathscr{L}\mathscr{K}(i\beta|k|, k) \to 0$ when $\beta \to \pm\infty$, $\mathscr{L}\mathscr{K}(i|k|\mathbb{R}, k) \cup \{0\}$ is a closed path in $\mathbb{C}$ (which does not cross 1) and we deduce that $\mathscr{L}\mathscr{K}(i\omega|k|, k) \neq 1$ for every k and $\omega \in \mathbb{C}$ with $\mathrm{Re}(\omega) \geq 0$ if and only if $\mathscr{L}\mathscr{K}(i\beta|k|, k) \neq 1$ for every k and $\beta \in \mathbb{R}$ **and** the winding number of the path $\mathscr{L}\mathscr{K}(i|k|\mathbb{R}, k) \cup \{0\}$ around 1 is equal to 0. This formulation eventually allows us to obtain the announced sufficient (but not necessary) criterion: if for every k and $\beta \in \mathbb{R}$

$$\mathrm{Im}\left(\mathscr{L}\mathscr{K}(i\beta|k|,k)\right) = 0 \qquad \Longrightarrow \qquad \mathrm{Re}\left(\mathscr{L}\mathscr{K}(i\beta|k|,k)\right) < 1,$$

then the linear stability criterion is satisfied.

Remark 5.1 *For $\mathbb{X}^d = \mathbb{R}^d$ the second step has to be performed also on the symbol $\mathcal{L}$. Then the complete sufficient condition is: if for every $k \in \mathbb{R}^d \setminus \{0\}$ and $\sigma \in \mathbb{S}^{d-1}$, $\beta \in \mathbb{R} \mapsto \mathscr{L}\mathscr{K}(i\beta|k|,k)$* ***and*** *$\beta \in \mathbb{R} \mapsto \mathcal{L}(i\beta,\sigma)$ never crosses the real-axis beyond 1, then the linear stability criterion is satisfied.*

5.2 Computations of Laplace transforms for the Penrose criterion

In order to find an expression for the stability criterion, we compute $\mathscr{L}\mathscr{K}(\omega|k|,k)$ on the imaginary axis: namely, with $\beta \in \mathbb{R}$, we consider

$$\begin{aligned}\mathscr{L}\mathscr{K}(i\beta|k|,k) &= \lim_{\substack{\alpha\to 0\\ \alpha>0}} \mathscr{L}\mathscr{K}\left((\alpha+i\beta)|k|,k\right)\\ &= \rho_0|\widehat{\sigma_1}(k)|^2 \Big\{\lim_{\substack{\alpha\to 0\\ \alpha>0}} \mathscr{L}p_c\big((\alpha+i\beta)|k|\big)\Big\}\Big\{\lim_{\substack{\alpha\to 0\\ \alpha>0}} \mathscr{L}\big(t|k|^2\widehat{M}(tk)\big)\big((\alpha+i\beta)|k|\big)\Big\}.\end{aligned}$$

where

$$v \mapsto \mathscr{M}(v) = \rho_0 M(v), \qquad \rho_0 > 0, \qquad \int M(v)\,\mathrm{d}v = 1.$$

The computation of the Laplace transform of $t \mapsto t|k|^2\,\widehat{M}(tk)$ is based on the Plemelj formula; see [14, Example 5.2], which leads to (see [27, Proposition 2.1])

$$\lim_{\substack{\alpha\to 0\\ \alpha>0}} \mathscr{L}\big(t|k|^2\widehat{M}(kt)\big)\big((\alpha+i\beta)|k|,k\big) \;=\; -\mathrm{P.V.}\int_{\mathbb{R}} \frac{\mu'_{k/|k|}(r)}{r+\beta}\,\mathrm{d}r - i\pi\mu'_{k/|k|}(-\beta),$$

where P.V. denotes the usual *principal value operator* and where $\mu_{k/|k|}$ is the one-dimensional marginal of M defined by

$$\mu_{k/|k|}(r) = \int_{v_\perp\cdot k=0} M\Big(r\frac{k}{|k|} + v_\perp\Big)\,\mathrm{d}v_\perp.$$

Next, the Laplace transform of p_c can be determined by using the classical result [28, Formula (VI,2;13)]

$$\mathscr{L}\big(\mathbf{1}_{t\geq 0}\sin(\theta t)\big)(\omega) = \frac{\theta}{\omega^2+\theta^2}, \qquad \text{for } \mathrm{Re}(\omega) > 0.$$

For $\alpha > 0$, $\beta \in \mathbb{R}$, we thus get (we recall that p_c is defined by (5))

$$\mathscr{L}p_c\big((\alpha+i\beta)|k|\big) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{|\widehat{\sigma}_2(\zeta)|^2}{(\alpha+i\beta)^2|k|^2 + c^2|\zeta|^2}\,\mathrm{d}\zeta.$$

Since σ_2 is radially symmetric, its Fourier transform is radially symmetric too and we can write

$$\mathscr{L}p_c\big((\alpha+i\beta)|k|\big) = \frac{|\mathbb{S}^{n-1}|}{(2\pi)^n}\int_0^{+\infty} \frac{r^{n-1}|\widehat{\sigma}_2(r)|^2}{(\alpha^2-\beta^2)|k|^2+c^2r^2+2i\alpha\beta|k|^2}\,\mathrm{d}r.$$

In order to compute this integral we will apply the following Plemelj-like formula.

Lemma 5.2 *Let $n\geq 3$. Let $f:\mathbb{R}\to\mathbb{R}$ be Schwartz class. We have for any $\kappa\neq 0$,*

$$\lim_{\substack{\lambda\to 0\\ \lambda>0}}\int_0^{+\infty}\frac{r^{n-1}f(r)}{r^2-\kappa^2+\lambda^2+2i\kappa\lambda}\,\mathrm{d}r = \mathrm{P.V.}\int_0^{+\infty}\frac{r^{n-1}f(r)}{r^2-\kappa^2}\,\mathrm{d}r - \mathrm{sgn}(\kappa)\frac{i\pi}{2}\kappa^{n-2}f(|\kappa|).$$

We postpone the proof of this claim at the end of the section. We apply this formula with $f(r)=|\hat\sigma_2(r)|^2$, $\lambda=\alpha|k|/c$ and $\kappa=\beta|k|/c$ in order to obtain

$$\begin{aligned}&\lim_{\substack{\alpha\to 0\\ \alpha>0}}\mathscr{L}p_c\big((\alpha+i\beta)|k|\big)\\ &\quad=\frac{1}{c^2}\frac{|\mathbb{S}^{n-1}|}{(2\pi)^n}\left(\mathrm{P.V.}\int_0^{+\infty}\frac{r^{n-1}|\widehat{\sigma}_2(r)|^2}{r^2-\frac{\beta^2|k|^2}{c^2}}\,\mathrm{d}r-\mathrm{sgn}(\beta)\frac{i\pi}{2}\Big(\frac{\beta|k|}{c}\Big)^{n-2}\left|\widehat{\sigma}_2\Big(\frac{|\beta k|}{c}\Big)\right|^2\right).\end{aligned}$$

We point out that Lemma 5.2 cannot be applied with $\beta=0$, nevertheless the previous formula makes sense even when $\beta=0$: in this case a direct application of the dominated convergence theorem allows us to obtain

$$\lim_{\substack{\alpha\to 0\\ \alpha>0}}\mathscr{L}p_c(\alpha|k|) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{|\widehat{\sigma}_2(\zeta)|^2}{c^2|\zeta|^2}\,\mathrm{d}\zeta = \frac{\kappa}{c^2}.$$

which is consistent with the general formula.

Therefore, we obtain the following expression for $\mathscr{L}\mathscr{K}(i\beta|k|,k)$ which identifies the real and imaginary parts

$$\mathscr{L}\mathscr{K}(i\beta|k|,k) = \frac{\rho_0}{c^2}\frac{|\mathbb{S}^{n-1}|}{(2\pi)^n}\,|\widehat{\sigma}_1(k)|^2\left(\mathscr{R}(\beta|k|,k)+i\mathscr{I}(\beta|k|,k)\right),$$

where

$$\begin{aligned}\mathscr{R}(\beta|k|,k) = &-\left(\mathrm{P.V.}\int_0^{+\infty}\frac{r^{n-1}|\widehat{\sigma}_2(r)|^2}{r^2-\frac{\beta^2|k|^2}{c^2}}\,\mathrm{d}r\right)\left(\mathrm{P.V.}\int_{\mathbb{R}}\frac{\mu'_{k/|k|}(r)}{r+\beta}\,\mathrm{d}r\right)\\ &-\mathrm{sgn}(\beta)\frac{\pi^2}{2}\Big(\frac{\beta|k|}{c}\Big)^{n-2}\left|\widehat{\sigma}_2\Big(\frac{|\beta k|}{c}\Big)\right|^2\mu'_{k/|k|}(-\beta),\end{aligned}$$

and

$$\begin{aligned}\mathscr{I}(\beta|k|,k) = &-\pi\,\mu'_{k/|k|}(-\beta)\left(\mathrm{P.V.}\int_0^{+\infty}\frac{r^{n-1}|\widehat{\sigma}_2(r)|^2}{r^2-\frac{\beta^2|k|^2}{c^2}}\,\mathrm{d}r\right)\\ &+\mathrm{sgn}(\beta)\frac{\pi}{2}\Big(\frac{\beta|k|}{c}\Big)^{n-2}\left|\widehat{\sigma}_2\Big(\frac{|\beta k|}{c}\Big)\right|^2\left(\mathrm{P.V.}\int_{\mathbb{R}}\frac{\mu'_{k/|k|}(r)}{r+\beta}\,\mathrm{d}r\right).\end{aligned}$$

It leads to the *Penrose stability criterion*, hereafter denoted **(P)**:

If

$$\frac{\mathrm{sgn}(\beta)}{2}\Big(\frac{\beta|k|}{c}\Big)^{n-2}\Big|\widehat{\sigma}_2\Big(\frac{|\beta k|}{c}\Big)\Big|^2\left(\mathrm{P.V.}\int_{\mathbb{R}}\frac{\mu'_{k/|k|}(r)}{r+\beta}\,\mathrm{d}r\right) = \mu'_{k/|k|}(-\beta)\left(\mathrm{P.V.}\int_0^{+\infty}\frac{r^{n-1}|\widehat{\sigma}_2(r)|^2}{r^2-\frac{\beta^2|k|^2}{c^2}}\,\mathrm{d}r\right),$$

then

$$-\frac{\rho_0}{c^2}\frac{|\mathbb{S}^{n-1}|}{(2\pi)^n}|\widehat{\sigma}_1(k)|^2\left\{\left(\mathrm{P.V.}\int_0^{+\infty}\frac{r^{n-1}|\widehat{\sigma}_2(r)|^2}{r^2-\frac{\beta^2|k|^2}{c^2}}\,\mathrm{d}r\right)\left(\mathrm{P.V.}\int_{\mathbb{R}}\frac{\mu'_{k/|k|}(r)}{r+\beta}\,\mathrm{d}r\right) + \mathrm{sgn}(\beta)\frac{\pi^2}{2}\Big(\frac{\beta|k|}{c}\Big)^{n-2}\Big|\widehat{\sigma}_2\Big(\frac{|\beta k|}{c}\Big)\Big|^2\mu'_{k/|k|}(-\beta)\right\} < 1.$$

When $\mathbb{X}^d = \mathbb{R}^d$, the Penrose criterion **(P)** has to be completed with the following criterion (hereafter denoted **(P')**): for all $\omega \in \mathbb{S}^d$

$$\text{if } \mu'_\omega(-\beta) = 0 \text{ then } -\frac{\rho_0\kappa}{c^2}|\widehat{\sigma}_1(0)|^2\left(\mathrm{P.V.}\int_{\mathbb{R}}\frac{\mu'_\omega(r)}{r+\beta}\,\mathrm{d}r\right) < 1,$$

We conclude that, when **(P)** (resp. **(P)** and **(P')**) is satisfied, then (**L**) holds. This criterion is much more involved than the Penrose criterion for the Vlasov equation, because the memory term p_c completely changes the evaluation of the symbol $\mathscr{L}\mathscr{K}$ and does not keep a simple separation between the real and imaginary parts.

Remark 5.3 *Let us rescale the problem as in [9]: roughly speaking, it amounts to replace the wave equation by*

$$\partial^2_{tt}\psi - c^2\Delta_z\psi = -c^2\sigma_2\ \sigma_1\star\rho.$$

Letting c run to $+\infty$, the problem looks like the Vlasov equation where the self-consistent potential is defined by the convolution $-\kappa\sigma_1\star\sigma_1\star\rho$. According to [27], the stability criterion for this limiting problem reads

$$\textit{if } \mu'_{k/|k|}(-\beta) = 0,\ \textit{then } -\rho_0\kappa\,|\widehat{\sigma_1}(k)|^2\left(\mathrm{P.V.}\int_{\mathbb{R}}\frac{\mu'_{k/|k|}(r)}{r+\beta}\,\mathrm{d}r\right) < 1,$$

which corresponds to the limit $c \to +\infty$ in the rescaled version of **(P)** *(note that in this scaling the symbol $\mathscr{L}\mathscr{K}$ is muitliplied by c^2). In particular, mind the minus sign in front of the coefficient $\rho_0|\widehat{\sigma_1}(k)|^2$: it makes the situation very similar to those of the* attractive *Vlasov-system.*

We finish this section with the proof of the Plemelj like formula that we used in order to compute the Laplace transform of p_c.

Proof of Lemma 5.2. Let us denote by $I(\lambda)$ the quantity under consideration and $f(r)=g(r^2)$; with the change of variable $u=r^2$ we get

$$I(\lambda)=\frac{1}{2}\int_0^{+\infty}\frac{\gamma(u)}{u-\kappa^2+\lambda^2+2i\kappa\lambda}\,\mathrm{d}u,$$

where $\gamma(u)=u^{n/2-1}g(u)$. We adapt the computations that lead to Plemelj's formula. It is crucial to remark that

$$\gamma'\in L^p((0,\infty))\text{ for some }1<p<2. \tag{51}$$

(At worst, $\gamma'(u)$ has the same singularity as $1/\sqrt{u}$ as $u\to 0$.) We start with

$$I(\lambda)=\frac{1}{2}\int_0^{+\infty}\frac{\gamma(u)}{(u-\kappa^2+\lambda^2)^2+4\kappa^2\lambda^2}(u-\kappa^2+\lambda^2)\,\mathrm{d}u$$
$$-\frac{2i\kappa\lambda}{2}\int_0^{+\infty}\frac{\gamma(u)}{(u-\kappa^2+\lambda^2)^2+4\kappa^2\lambda^2}\,\mathrm{d}u.$$

Setting $v=u-\kappa^2+\lambda^2$, and $w=v/(2|\kappa|\lambda)$, the second term recasts as

$$-\frac{i}{2}\frac{\kappa}{|\kappa|}\int_{-\kappa^2+\lambda^2}^{+\infty}\frac{\gamma(v+\kappa^2-\lambda^2)}{\left(\frac{v}{2|\kappa|\lambda}\right)^2+1}\frac{\mathrm{d}v}{2|\kappa|\lambda}=-\mathrm{sgn}(\kappa)\frac{i}{2}\int_{-\frac{1}{2}\left(\frac{\lambda}{|\kappa|}-\frac{|\kappa|}{\lambda}\right)}^{+\infty}\frac{\gamma(2|\kappa|\lambda w+\kappa^2-\lambda^2)}{w^2+1}\,\mathrm{d}w$$

which tends to $-i\,\mathrm{sgn}(\kappa)\pi\,\gamma(\kappa^2)/2$ as $\lambda\to 0$. Similarly, we consider

$$J(\lambda)=\int_{-\kappa^2+\lambda^2}^{+\infty}\frac{v}{v^2+4\kappa^2\lambda^2}\,\gamma(v+\kappa^2-\lambda^2)\,\mathrm{d}v.$$

Since λ is intended to tend to 0, we can consider $\kappa^2\gg\lambda^2>0$ Given $0<\delta<\kappa^2-\lambda^2$, we split into 2 parts

$$J(\lambda)=\int_{|v|>\delta}...\,\mathrm{d}v+\int_{-\delta}^{+\delta}...\,\mathrm{d}v=J^\delta(\lambda)+J_\delta(\lambda).$$

First, we show that $J_\delta(\lambda)$ tends to 0 as $\delta\to 0$, uniformly with respect to λ. Indeed, since $v\mapsto v/(v^2+\lambda^2)$ is odd and thanks to (51), we have

$$|J_\delta(\lambda)|=\left|\int_{-\delta}^{+\delta}\frac{v}{v^2+4\kappa^2\lambda^2}\left[\gamma(v+\kappa^2-\lambda^2)-\gamma(\kappa^2-\lambda^2)\right]\,\mathrm{d}v\right|$$
$$\leq\|\gamma'\|_{L^p}\int_{-\delta}^{+\delta}\frac{1}{|v|^{1/p}}\,\mathrm{d}v\ \xrightarrow[\delta\to 0]{}\ 0.$$

By dominated convergence, we get (owing to the fast decay at infinity of γ')

$$\lim_{\lambda\to 0}J^\delta(\lambda)=\int_{|v|>\delta}\mathbf{1}_{v\geq-\kappa^2}\frac{\gamma(v+\kappa^2)}{v}\,\mathrm{d}v$$
$$=\int_{-\kappa^2}^{-\delta}\frac{\gamma(v+\kappa^2)-\gamma(\kappa^2)}{v}\,\mathrm{d}v+\int_{\delta}^{\kappa^2}\frac{\gamma(v+\kappa^2)-\gamma(\kappa^2)}{v}\,\mathrm{d}v+\int_{\kappa^2}^{+\infty}\frac{\gamma(v+\kappa^2)}{v}\,\mathrm{d}v.$$

The same reasoning shows that this quantity admits a limit as δ goes 0, that we write with the shorthand notation

$$\lim_{\delta\to 0}\lim_{\lambda\to 0} J^\delta(\lambda) = \text{P.V.} \int_{-\kappa^2}^{\infty} \frac{\gamma(v+\kappa^2)}{v}\,\mathrm{d}v.$$

■

5.3 Stable and unstable states

The criterion **(P)** is a bit ugly and not that practical. Nevertheless, some relevant information can be extracted from the formula, showing again the similarity with the attractive Vlasov-Poisson equation.

Proposition 5.4 *Let $\mathbb{X}^d = \mathbb{R}^d$ with $d \geq 3$. Let $\mathscr{M}$ be a spatially homogeneous and radially symmetric equilibrium. Then, there exists a threshold for the wave speed $c_0(\mathscr{M}, \sigma_1, \sigma_2) > 0$ such that for any $0 < c < c_0(\mathscr{M}, \sigma_1, \sigma_2)$, $\mathscr{M}$ in an unstable equilibrium state.*

Proof. We find k and β such that $\mathscr{L}\mathscr{K}(i\beta|k|, k) = 1$. To this end, we use the fact that $\mathscr{L}p_c(i\beta|k|)$ belongs to $\mathbb{R}$ for $\beta = 0$ and the radial symmetry of $\mathscr{M}$ which implies that $\mathscr{L}(|k|^2 t\widehat{M}(tk))(i\beta|k|, k)$ is real too when $\beta = 0$:

$$\mathscr{L}\mathscr{K}(0,k) = -\rho_0\,|\widehat{\sigma}_1(k)|^2 \left(\text{P.V.} \int_{\mathbb{R}} \frac{\mu'_{k/|k|}(r)}{r}\,\mathrm{d}r\right) \frac{\kappa}{c^2}. \tag{52}$$

Moreover, the symmetry of $\mathscr{M}$ (and the condition on the dimension d, see Remark 5.5 below) also ensures (except for $\mathscr{M} = 0$, but 0 is obviously a stable state)

$$-\left(\text{P.V.} \int_{\mathbb{R}} \frac{\mu'_{k/|k|}(r)}{r}\,\mathrm{d}r\right) > 0.$$

Now let us pick a vector k_0 such that $\widehat{\sigma}_1(k_0) \neq 0$. As far as c is small enough, we have $\mathscr{L}\mathscr{K}(0, k_0) > 1$. Next,

$$\mathscr{L}\mathscr{K}(0, \lambda k_0) \underset{\lambda\to+\infty}{\longrightarrow} 0$$

and the continuity of $\lambda \in \mathbb{R} \mapsto \widehat{\sigma}_1(\lambda k_0)$ (observe that $\lambda k_0/|\lambda k_0|$ does not depend on λ and thus only $\widehat{\sigma}_1$ depends on λ in the expression of $\mathscr{L}\mathscr{K}(0, \lambda k_0)$), allow us to exhibit a $\lambda_0 \in \mathbb{R}$ such that $\mathscr{L}\mathscr{K}(0, \lambda_0 k_0) = 1$. ■

Remark 5.5 *The condition $d \geq 3$ ensures that all marginals of a non negative radially symmetric function $\mathscr{M}$ are non increasing function of $|v|$, see [27, Remark 2.2], which yields*

$$-\left(\text{P.V.} \int_{\mathbb{R}} \frac{\mu'_{k/|k|}(r)}{r}\,\mathrm{d}r\right) \geq 0. \tag{53}$$

When $d = 1$ or $d = 2$ this does not hold in full generality. Nevertheless, Proposition 5.4 still holds provided (53) *is fulfilled.*

Remark 5.6 *When $\mathbb{X}^d = \mathbb{T}^d$, the same proof shows that, for any spatially homogeneous and radially symmetric equilibrium, we can find some wave speed c such that $\mathscr{M}$ is unstable. However, since $k \in \mathbb{Z}^d$, it is not clear that we can exhibit a non trivial* interval $[0, c_0(\mathscr{M})]$ *such that instability occurs.*

To identify a threshold on c determining whether or not the stability criterion holds can be interpreted by means of Jeans' criterion, a standard criterion for the Vlasov-Poisson system, see [27, Proposition 2.1 & Remark 2.2]). To be more specific, let us consider a form function σ_1 defined on $\mathbb{R}^d$, the Fourier transform of which has a singularity at $\xi = 0$: typically $\widehat{\sigma}_1(k) = |k|^{-\alpha}$ for some $\alpha > 1$. Of course, such singular potential is beyond the analysis detailed in this paper; we only use this assumption to establish a parallel with the usual Jeans' criterion. Let $\sigma_1^{(L)}$ be the periodic potential defined on $\mathbb{T}_L^d = (\mathbb{R}/(2\pi L\,\mathbb{Z}))^d$ by

$$\sigma_1^{(L)}(x) = \sum_{k \in \mathbb{Z}^d} \sigma_1(x + 2\pi L\, k).$$

Observing that $\widehat{\sigma_1^{(L)}}(k) = \widehat{\sigma}_1(k/L)$, (52) becomes

$$\mathscr{L}\mathscr{K}(0,k) = -\rho_0 \frac{L^{2\alpha}}{|k|^{2\alpha}} \left(\text{P.V.} \int_{\mathbb{R}} \frac{\mu'_{k/|k|}(r)}{r}\, \mathrm{d}r \right) \frac{\kappa}{c^2},$$

where L has a role similar to $1/c$. In particular, for any spatially homogeneous equilibrium $\mathscr{M}$, there exists a critical length L_J beyond which the equilibrium can be unstable, this defines Jeans' length.

Remark 5.7 *Denoting $\mathscr{M} = \rho_0 M$, with M being normalized, we can equally say (with the same arguments) that, for any fixed wave speed c we can find a mass threshold $m_0(M, c, \sigma_1, \sigma_2) > 0$ such that for any $\rho_0 > m_0(M, c, \sigma_1, \sigma_2)$, $\mathscr{M}$ is unstable. Nevertheless we point out that, for c fixed, the mass ρ_0 of the profile $\mathscr{M}$ is not the unique quantity that governs the stability of $\mathscr{M}$, as indicated by the following claim*

Proposition 5.8 *Let $\mathscr{M}$ be a spatially homogeneous equilibrium. We can find two positive constants $C_1 = C_1(c, \sigma_1, \sigma_2)$ and $C_2 = C_2(c, \sigma_1, \sigma_2)$ such that*

$$\text{if, for any } \omega \in \mathbb{S}^d, \text{ we have } \int_0^{+\infty} u \left| \widehat{\mathscr{M}}(u\omega) \right| \mathrm{d}u \le C_1(c, \sigma_1, \sigma_2), \text{ then } \mathscr{M} \text{ is stable,}$$

$$\text{if there exists } \omega \in \mathbb{S}^d \text{ such that } \int_0^{+\infty} u \widehat{\mathscr{M}}(u\omega)\, \mathrm{d}u \ge C_2(c, \sigma_1, \sigma_2), \text{ then } \mathscr{M} \text{ is unstable}\ .$$

This statement can be interpreted as follows. For fixed c, σ_1 and σ_2 there always exist stable spatially homogeneous equilibria with an arbitrarily large mass (resp. kinetic energy), and there always exist unstable spatially homogeneous equilibria with an arbitrarily small mass (resp. kinetic energy). This comes from the fact that the constant C_1 and C_2 in Proposition 5.8 are left invariant by the rescaling $M \to M_\lambda(v) = \lambda^{d-2}\mathscr{M}(\lambda v)$, while the associated mass (resp. kinetic energy) is invariant for the scaling $M \to \lambda^d \mathscr{M}(\lambda v)$ (resp. $M \to \lambda^{d+2}\mathscr{M}(\lambda v)$). These findings are investigated on numerical grounds in [18].

Proof. The first part of the statement is a direct consequence of Proposition 3.10, which tells us that a given profile $\mathscr{M}$ is stable provided c is large enough. The second part of the statement is a direct consequence of Proposition 5.4 and it comes from the formula

$$\mathscr{L}(|k|^2 t\widehat{\mathscr{M}}(tk))(0,k) = \rho_0 \left(\text{P.V.} \int_{\mathbb{R}} \frac{\mu'_{k/|k|}(r)}{r} \, \mathrm{d}r \right) = \int_0^{+\infty} u\widehat{\mathscr{M}}(u\omega) \, \mathrm{d}u.$$

■

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