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The Kraichnan–Kazantsev Dynamo

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We investigate the dynamo effect generated by an incompressible, helicity-free flow drawn from the Kraichnan statistical ensemble. The quantum formalism introduced by Kazantsev [A. P. Kazantsev, *Sov. Phys. JETP* **26**, 1031–1034 (1968)] is shown to yield the growth rate and the spatial structure of the magnetic field. Their dependences on the magnetic Reynolds number and the Prandtl number are analyzed. The growth rate is found to be controlled by the largest between the diffusive and the viscous characteristic times. The same holds for the magnetic field correlation length and the corresponding spatial scales.

KEY WORDS: Turbulent transport; Magnetohydrodynamics; Dynamo effect; Kraichnan statistical ensemble.

1. INTRODUCTION

Magnetic fields generated by turbulent motion of conductive fluids are relevant to many astrophysical applications.[1] Two competing mechanisms are at stake: magnetic field's amplification by the gradients of the advecting flow and magnetic energy's dissipation due to the finite resistivity of the fluid. Which one prevails, depends on the specific properties of the flow and does not bear a general answer. There are however some specific models where a complete analysis can be carried out and those will be the subject of interest of this work.

The evolution of an initially given magnetic field $\mathbf{B}(\mathbf{r}, 0)$ in an incompressible flow of a conductive fluid is determined by the following equations[2]

$$\begin{cases} \partial_t \mathbf{B} + (\mathbf{v} \cdot \nabla)\, \mathbf{B} = (\mathbf{B} \cdot \nabla)\, \mathbf{v} + \kappa \nabla^2 \mathbf{B} \\ \nabla \cdot \mathbf{B} = 0 \end{cases} \tag{1}$$

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where $\mathbf{v}(\mathbf{r}, t)$ denotes the velocity field. The magnetic diffusivity κ, assumed to be uniform and constant, is proportional to the inverse of the electric conductivity of the fluid.

The dimensionless number which expresses the viscosity-to-diffusivity ratio is the Prandtl number $Pr = \nu/\kappa$, where ν indicates the viscosity of the fluid.

In Eqs. (1) the term $(\mathbf{v}\cdot\nabla)\,\mathbf{B}$ is a purely advective contribution that preserves the magnetic energy. The stretching term $(\mathbf{B}\cdot\nabla)\,\mathbf{v}$ acts either as an energy source or as a sink depending on the local properties of the flow. Finally, the diffusive term $\kappa\nabla^2\mathbf{B}$ is responsible for the small-scale ohmic dissipation and balances the inertial terms at the diffusive scale r_d.

The relative importance of the two contributions on the right-hand side of (1) is given by the magnetic Reynolds number $R_m = UL/\kappa$, where L denotes the integral scale of the flow and U the characteristic velocity at such scale. The number R_m can be regarded as a dimensionless measure of the fluid conductivity. For $R_m \to 0$ the diffusion dominates and the magnetic energy density (proportional to B^2) always decays to zero in time. In the opposite limit, $R_m \to \infty$, the diffusion term is relevant only at very small scales and the magnetic field is almost frozen in the fluid. At high magnetic Reynolds numbers we can expect that the flow be able to enhance the magnetic field, producing a consequent growth in time of B^2. The last process is called dynamo effect, referring to the energy transfer from the velocity field to the magnetic one.

The field $\mathbf{B}$ acts on the velocity via the Lorentz force, which yields a term proportional to $(\mathbf{B}\cdot\nabla)\,\mathbf{B}$ in the Navier–Stokes equations. Generally speaking, it would be necessary to take into account such feedback action on $\mathbf{v}$. However, since we are interested in understanding the initial generation of the magnetic field, we can assume for the initial conditions $B^2 \ll v^2$ and neglect the Lorentz force contribution. In this kinematic approach $\mathbf{v}$ is a prescribed velocity field and the evolution equations (1) are totally uncoupled from Navier–Stokes equations. Given the initial condition $\mathbf{B}(\mathbf{r}, 0)$ and appropriate boundary conditions, Eqs. (1) completely determine the magnetic field evolution.

To prescribe the velocity, we use the Kraichnan statistical ensemble,[(3)] where $\mathbf{v}$ is taken Gaussian, homogeneous, isotropic and δ-correlated in time. The motivation is that it allows an analytical solution of the dynamo problem.

The flow is assumed to be characterized by two scales: the integral scale L and the viscous scale η, determined by the balance between dissipation and transport in Navier–Stokes equations. The velocity is supposed to be smooth up to the viscous scale η and to scale as $r^{\xi/2}$ $(0 \leqslant \xi \leqslant 2)$ in the inertial range $\eta \ll r \ll L$. The parameter $\xi/2$ is the Hölder exponent of the velocity and can be thought of as a measure of the field roughness: for $\xi = 2$ the velocity is smooth in space, while the case $\xi = 0$ corresponds to a diffusive field.

The magnetic Reynolds number and the Prandtl number are related to the relative importance of the scales involved in the physical problem by the relations: $R_m \simeq L/r_d$ and $Pr \simeq (\eta/r_d)^{\xi}$.

It is well known that magnetic dynamo can emerge for a helical flow due to the α-effect.[1,5] Here we will restrict to a parity invariant statistical ensemble so that the α-effect is ruled out. (For recent results on helical Kraichnan velocity fields see, e.g., ref. 6).

The analysis of kinematic dynamo for a Kraichnan velocity field is made easier by a simple quantum mechanics formulation, first introduced by Kazantsev.[4] The δ-correlation in time of the flow allows for the single time correlation function for the magnetic field $\langle B_i(\mathbf{x}, t)\, B_i(\mathbf{x}+\mathbf{r}, t)\rangle$ to be expressed in terms of a function that satisfies a one-dimensional Schrödinger-like equation. The problem of the dynamo effect can thus be mapped into that of studying the bound states of a quantum particle in a given potential that only depends on the velocity correlation function. In particular, the ground state energy E_0 will turn out to be the asymptotic magnetic field rate-of-growth.

In ref. 4 Kazantsev finally restricted himself to the limiting case of $R_m \to \infty$ and $Pr \to 0$. He proved that dynamo can take place only for a velocity scaling exponent in the range $1 \leqslant \xi \leqslant 2$ and he provided a numerical evaluation of E_0 vs ξ for $1.25 < \xi < 2$. The rate-of-growth for a smooth field ($\xi = 2$) was theoretically estimated on the ground of quantum mechanical considerations.

The Kazantsev quantum model was extended by Ruzmaĭkin and Sokolov[7] to a more realistic velocity field. In particular, the main consequence of finite magnetic Reynolds numbers was found to be the existence of a threshold value $R_m^{(cr)}$ for the appearance of dynamo. Further, the magnetic field was shown to be concentrated at small scales (of the order of $R_m^{-1/2}$) and to be always anticorrelated at large scales. The results of ref. 7 were later generalized by Novikov *et al.*[8] to consider an inertial scaling behavior with scaling exponent $\xi = 2/3$.

In more recent years the Kraichnan–Kazantsev dynamo problem was exactly solved in the special case of a smooth turbulent velocity field. Gruzinov *et al.*[9] found the formula which determines the rate-of-growth for a d-dimensional flow and generalized it to a non-δ-correlated flow. The exact analysis of moments and multipoint correlation functions of the magnetic field was carried out by Chertkov *et al.*[10] by means of a Lagrangian approach. They also obtained the expression for the rate-of-growth of the $2n$th moment of the field $\mathbf{B}$ in terms of the Lyapunov exponents of the turbulent flow.

Finally, the last contribution in solving the Kazantsev model is due to Schekochihin *et al.*,[11] which describe the case of a d-dimensional Kraichnan velocity field with a generic degree of compressibility in the limiting case of very large Prandtl numbers (of interest in astrophysical applications).

The aim of this paper is to give a comprehensive description of the Kraichnan–Kazantsev model, zeroing in on the dependence of the dynamo effect on the dimensionless numbers R_m and Pr. We have been motivated by the observation that the literature has provided many results valid in different limiting cases, while an unified treatment of the problem as a function of R_m and Pr still lacked.

We find that, while the magnetic Reynolds number determines the presence of dynamo, the Prandtl number influences the magnetic field correlation length and its rate-of-growth. More precisely, the correlation length of the field $\mathbf{B}$ is shown to be the largest between r_d and η, and its rate-of-growth is proportional to the largest between the diffusive and the viscous characteristic time-scale.

At the end of the paper we also point out a non-monotonic dependence of the rate-of-growth on the Prandtl number. This results might be of relevance to physical applications.

Many of the results appearing in this paper are completed by numerical computations based on the variation-iteration method described in Appendix A. This algorithm has the advantage of not relying on any assumption on the functional form of the solution of the Schrödinger-like equation. That was the case for previous numerical computations and therefore the selected behavior for the wave function was not always the right one.

The rest of paper is organized as follows. In Section 2 we define more precisely the Kraichnan model and, following Kazantsev,[4] we describe the quantum formalism mentioned above. In particular we derive the Schrödinger equation which is at the core of the quantum approach. In Section 3, we revisit the case of infinite magnetic Reynolds number and zero Prandtl number. Starting from these results, we then study how the dynamo effect is influenced by R_m and Pr. Section 4 is devoted to conclusions.

2. THE KRAICHNAN–KAZANTSEV MODEL

In this section we recall in detail the quantum formalism introduced by Kazantsev in ref. 4. The random velocity field is assumed to be incompressible, Gaussian, homogeneous, isotropic, parity invariant, and δ-correlated in time. Under these hypotheses it is completely defined by its correlation matrix

$$\begin{aligned}\langle v_i(\mathbf{x}, t)\, v_j(\mathbf{x}', t)\rangle &= \delta(t-t')\, \mathscr{D}_{ij}(\mathbf{r}) \\ &= \delta(t-t')[\mathscr{D}_{ij}(0)-S_{ij}(\mathbf{r})] \qquad (\mathbf{r}=\mathbf{x}-\mathbf{x}') \qquad (2)\end{aligned}$$

where $S_{ij}(\mathbf{r})$ denotes the structure function of the field $\mathbf{v}$.

The δ-correlation in time of $\mathbf{v}$ is an essential property in order to write a closed equation for the magnetic field correlation function, which (under a suitable transformation) reduces to a Schrödinger-like equation.

We impose homogeneous and isotropic initial conditions for $\mathbf{B}$. Therefore, on account of the translational and rotational invariance of Eqs. (1), the magnetic field maintains homogeneous and isotropic statistics at every time t. Its correlation tensor has thus the form (see, e.g., ref. 14)

$$\langle B_i(\mathbf{x}, t)\, B_j(\mathbf{x}', t)\rangle = G_1(r, t)\, \delta_{ij} + G_2(r, t)\, \frac{r_i r_j}{r^2} \tag{3}$$

Because of the solenoidality condition $\nabla \cdot \mathbf{B} = 0$, the functions G_1 and G_2 are related by the following differential equation

$$\frac{\partial G_1}{\partial r} = -\frac{1}{r^2}\frac{\partial}{\partial r}(G_2 r^2) \tag{4}$$

The covariance of $\mathbf{B}$ is then completely described by a single scalar function, e.g., its trace $H(r, t) = 3G_1(r, t) + G_2(r, t)$. Obviously, the dynamo effect will correspond to an unbounded growth in time of $H(r, t)$.

The correlation function $H(r, t)$ can be transformed into another function $\Psi(r, t)$ that solves the imaginary time Schrödinger equation

$$-\frac{\partial \Psi}{\partial t} + \left[\frac{1}{m(r)}\frac{\partial^2}{\partial r^2} - U(r)\right]\Psi = 0 \tag{5}$$

in which the mass and the potential depend on r only through $S_{ii}(r)$. (For the details see Appendix B and ref. 4).

To study the dynamo effect it is useful to put in evidence the time dependence of Ψ. As usual in quantum mechanics, we thus expand the "wave function" Ψ in terms of the "energy" eigenfunctions $\Psi(r, t) = \int \psi_E(r)\, e^{-Et} \varrho(E)\, dE$ [or $\Psi(r, t) = \sum_E \psi_E(r)\, e^{-Et}$ for discrete energy levels] and obtain the "stationary" equation

$$\frac{1}{m(r)}\frac{d^2\psi_E}{dr^2} + [E - U(r)]\, \psi_E = 0 \tag{6}$$

Referring to the meaning of Ψ, it is clear that an unbounded growth of the magnetic field corresponds to the existence of negative energies in Eq. (6). In particular, it is the sign of the ground state energy E_0 that determines the presence of dynamo and its value eventually represents the asymptotic growth rate of the magnetic field. Indeed, in this case it is the ground state $\psi_{E_0} e^{-E_0 t}$ that dominates the growth in time. (Recall that the negative energy levels of a Schrödinger equation are always discrete).

By looking at the variational expression for the eigenvalues derived from Eq. (6)

$$E = \frac{\int mU\psi_E^2 \, dr + \int (\psi'_E)^2 \, dr}{\int m\psi_E^2 \, dr} \tag{7}$$

one can easily conclude that the presence of dynamo effect is equivalent to the existence of bound states for a quantum particle of unit (r-independent) mass in the potential $V(r) = m(r)\, U(r)$.[12] Therefore, in order to state if dynamo can take place for a given velocity field, it is sufficient to study the properties of V.

Having summarized the quantum mechanics formalism for a magnetic field transported by a Kraichnan turbulent flow, in the next section we study the dynamo effect for a velocity correlation function that mimics the real physical situation. In particular, we numerically compute E_0 and describe the properties of the ground state eigenfunction as R_m and Pr are varied. From this analysis we are able to obtain information about the critical magnetic Reynolds number, the correlation length of the magnetic field, the asymptotic behaviors of its correlation function, and the characteristic time-scale of the magnetic field growth.

3. TURBULENT DYNAMO

We consider the realistic situation of a structure function $S_{ii}(r)$ that scales as r^2 for $r \ll \eta$ (as expected in the viscous range), as r^ξ $(0 \leqslant \xi \leqslant 2)$ in the inertial range $\eta \ll r \ll L$, and that tends to a constant value $\mathscr{D}_{ii}(0)$ for $r \gg L$.

The case $\xi = 0$ corresponds to the diffusive behavior, while the other limit $\xi = 2$ describes a velocity field that is smooth at all scales below the integral scale L. For the other values of ξ, the field $\mathbf{v}$ is only a Hölder continuous function of r with exponent $\xi/2$ (in the inertial range). The parameter ξ thus represents a measure of the field roughness.

An explicit expression for the velocity correlation tensor, which has the desired scaling properties, is, for example,

$$\mathscr{D}_{ij}(\mathbf{r}) = \int e^{i\mathbf{k}\cdot\mathbf{r}} \widehat{\mathscr{D}}_{ij}(\mathbf{k})\, d^3\mathbf{k} \tag{8}$$

with

$$\widehat{\mathscr{D}}_{ij}(\mathbf{k}) = D_0 \frac{e^{-\eta k}}{(k^2+L^{-2})^{(\xi+3)/2}} P_{ij}(\mathbf{k}) \tag{9}$$

The solenoidal projector $P_{ij}(\mathbf{k}) = (\delta_{ij} - k_i k_j / k^2)$ ensures the incompressibility of the velocity field.

In what follows we refer to Eq. (9) whenever we show numerical computations that exemplify our conclusions. However, it should be noted that our results are general: they depend only on the qualitative properties of $S_{ii}(r)$ and not on its explicit form.

3.1. Fully Developed Turbulent Dynamo

We first consider the limiting case of $R_m \to \infty$ and $Pr \to 0$. Under these conditions the diffusive scale r_d is in the inertial range and the presence of the cutoffs L and η is neglected: only the scaling behavior r^ξ $(0 \leqslant \xi \leqslant 2)$ is considered for the velocity structure function.

The general expression of $S_{ij}(r)$ for an homogeneous, isotropic, parity invariant, incompressible field that scales as r^ξ is[14]

$$\lim_{\substack{\eta \to 0 \\ L \to \infty}} S_{ij}(\mathbf{r}) = D_1 r^\xi \left[(2+\xi)\, \delta_{ij} - \xi \frac{r_i r_j}{r^2} \right] \tag{10}$$

where the coefficient D_1 has the dimensions of length$^{(2-\xi)}$/time.

In this limit the total energy $\mathscr{D}_{ii}(0)$ diverges with the infrared cutoff as L^ξ.

In order to analyze the existence of dynamo, let us turn to the quantum formulation described above. The potential V has the following asymptotic behaviors (see Appendix B and ref. 4 for the complete expression)

$$V(r) \sim \begin{cases} 2/r^2 & r \ll r_d \\ (2 - \frac{3}{2}\xi - \frac{3}{4}\xi^2)/r^2 & r_d \ll r \end{cases} \tag{11}$$

For sufficiently small ξ the potential is positive for all r, it does not generate bound states and therefore the dynamo can not take place. For larger ξ, V is repulsive up to $r \simeq r_d$ and becomes attractive at infinity (Fig. 2). A quantum mechanical analysis based on asymptotic behaviors (11) allows to establish that $\xi = 1$ is the exact threshold for the dynamo effect.[4, 12]

If $0 \leqslant \xi \leqslant 1$ the turbulent flow alone is unable to increase the magnetic field and B^2 finally decays in time. For those values of ξ, the presence of a forcing term in Eq. (1) is necessary to obtain a statistically stationary state.[12] A forcing term can represent boundary conditions on the field **B**, or it can be due to the presence of a large-scale mean magnetic field (see, e.g., ref. 13). This is, for example, the case of the solar corona, in which small-scale turbulent fluctuations and large-scale magnetic fields coexist.

From now on we restrict to the values $1 \leqslant \xi \leqslant 2$, for which the dynamo is present.

If Eq. (6) is rewritten in a rescaled form by means of the transformation $r \to r/r_d$, $r_d = (\kappa/D)^{1/\xi}$, it is easy to see that the eigenvalues of the energy must take the form

$$E = \epsilon(\xi)\, t_d^{-1} \tag{12}$$

where $\epsilon(\xi)$ depends only on the scaling exponent ξ and $t_d = r_d^2/\kappa$ is the characteristic time of magnetic diffusion.

We have already noted that the ground state eigenfunction dominates the evolution in time and that E_0 is the asymptotic magnetic growth rate. We numerically compute $\epsilon_0(\xi)$ as a function of ξ by the variation-iteration method described in Appendix A. The quantity ϵ_0 grows with ξ as shown in Fig. 1. When ξ tends to one, ϵ_0 approaches zero and the bound states disappear. In the other limit, ϵ_0 reaches the value $15/2$ according to the known theoretical predictions.[4, 9, 10]

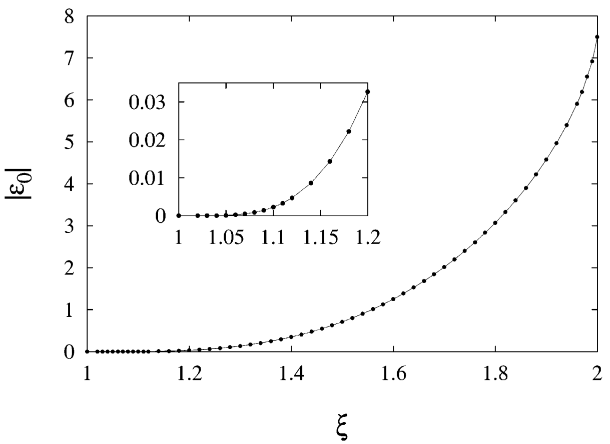

Fig. 1. The dependence of the magnetic growth rate $\epsilon_0 = E_0 t_d^{-1}$ on the scaling exponent ξ in the limit of infinite R_m and zero Pr, as computed by the variation-iteration method described in Appendix A.

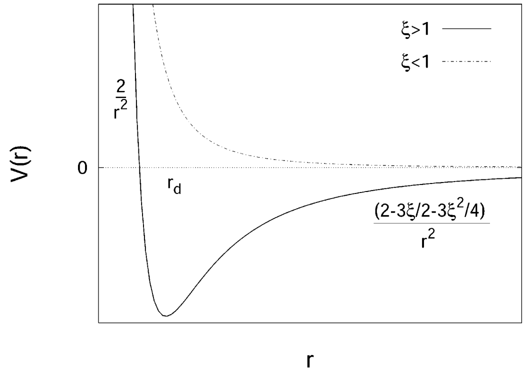

Fig. 2. The shape of the quantum potential V in the limit $R_m \to \infty$ and $Pr \to 0$ for $\xi > 1$ (dynamo effect) and for $\xi < 1$ (no dynamo effect).

An estimation for ϵ_0 vs ξ already appears in ref. 4, but there the results are limited to the values $1.25 < \xi < 2$. Moreover, the numerical computations in that paper are performed by a variational method based on the particular guess $r^2 e^{-\beta r}$ for the eigenfunction ψ_{E_0}. This ansatz is correct for $r \ll r_d$, but it fails to capture the right behavior for $r \gg r_d$. Indeed, if we insert the asymptotic behaviors (11) in Eq. (6), we find that (for $1 < \xi < 2$) $\psi_E(r)$ shows a stretched exponential decay with characteristic scale r_d and stretching exponent $(2-\xi)/2$ (Fig. 3). The variation-iteration method we used (see Appendix A) presents the big advantage of not requiring an explicit form for ψ_{E_0}. The algorithm provides as results both the eigenvalue and the corresponding eigenfunction.

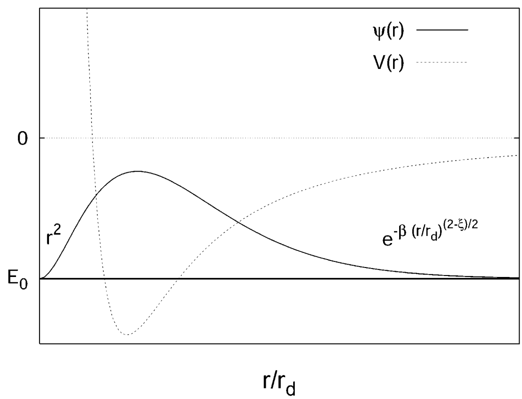

Fig. 3. The asymptotic behaviours of the "stationary wave function" ψ_{E_0} in the limit of infinite R_m and zero Pr. The maximum at $r \simeq r_d$ determines the magnetic field correlation length.

From the expressions of $\psi_{E_0}(r)$ we can recover the behavior of $H(r,\cdot)$ (see the definition (B2) in Appendix B). We have that, for $r \ll r_d$, the magnetic field correlation function is approximately constant, while, if $1<\xi<2$, $H(r,\cdot)$ decays for $r \gg r_d$ as a stretched exponential with characteristic scale r_d

$$H(r,\cdot) \propto -e^{-\beta\,(r/r_d)^{(2-\xi)/2}} \qquad (r_d \ll r \ll L) \tag{13}$$

where the prefactor

$$\beta = \frac{\sqrt{2\,|\epsilon_0(\xi)|}}{2-\xi} \tag{14}$$

depends on the growth rate $\epsilon_0(\xi)$. We can thus conclude that, for $1<\xi<2$, the magnetic field has a spatial distribution characterized by structures whose scales are of order r_d.

Observe that, as pointed out in refs. 7 and 8, the magnetic field has always anticorrelated tails in the large scale, due to the solenoidality condition. Formally, this is a consequence of the exponential decay of $\psi_{E_0}(r)$, which implies $\int_0^\infty H(\rho,\cdot)\,\rho^2\,d\rho = 0$ (see Eq. (B2) and ref. 7).

The cases $\xi=2$ and $\xi=1$ have to be treated separately. Indeed, the asymptotic properties cannot be deduced directly from Eq. (6).

The smooth case is solved by Chertkov *et al.* in ref. 10 by a Lagrangian approach that relates the growth rate to the Lyapunov exponents. There is a big difference between the situation of a smooth velocity field and one that is just Hölder continuous. In the former case the correlation function is found to depend on the spatial coordinate as $H(r,t) \propto r^{-5/2}$ (equivalent to $\psi_{E_0}(r) \propto r^{1/2}$), which implies the presence of structures with at least one dimension of inertial range size. Actually the magnetic field in the smooth case has been shown to be characterized by strip-like objects.

The case $\xi=1$ can be solved exactly. Indeed, the appropriate ground state eigenfunction of Eq. (6) is (recall that $\xi=1$ is the threshold for dynamo and hence $E_0=0$)

$$\psi_0(x) = C\,\frac{\sqrt{1+x}\,(-2x+(2+x)\ln(1+x))}{x}, \qquad (x=r/r_d) \tag{15}$$

where the constant C is related to the value of $H(0,\cdot)$ by the relation $C = 3\sqrt{\kappa}\,r_d^2 H(0,\cdot)$.

If we neglect logarithmic corrections, the asymptotic behavior of ψ_0 for $r \gg r_d$ is $\psi_0(r) \propto r^{1/2}$, which yields again $H(r,\cdot) \propto r^{-5/2}$ for $r \gg r_d$.

The results we have outlined in this section will be useful in the following to describe the general case where the velocity energy spectrum has an infrared and an ultraviolet cutoff. Indeed, we will study a structure function that for $r \ll \eta$ scales as r^2 and so takes the $\xi = 2$ behavior, while for $r \gg L$ tends to a constant value like in the diffusive case $\xi = 0$.

3.2. Finite Reynolds Effect

Let us analyze the situation of finite R_m (and zero Prandtl number). The principal fact is that a large-scale cutoff L appears for velocity field correlations. The diffusive scale r_d is again within the inertial range of the velocity fluctuations, and the presence of the viscous cutoff can be neglected. The velocity structure function therefore scales as r^ξ for $r \ll L$ and tends to $\mathscr{D}_{ii}(0)$ for $r \gg L$.

The potential V behaves as in the previous case for $r \ll L$, while it takes the $\xi = 0$ behavior for $r \gg L$

$$V(r) \sim \begin{cases} 2/r^2 & r \ll r_d \\ (2 - \frac{3}{2}\xi - \frac{3}{4}\xi^2)/r^2 & r_d \ll r \ll L \\ 2/r^2 & L \ll r \end{cases} \tag{16}$$

The main consequence of a finite R_m is that V is repulsive also at large scales. It is thus clear that, for sufficiently high R_m, a potential well is present at scales of order r_d. On the contrary, if R_m is too small, the well can be absent or anyway not deep enough to generate bound states[7, 8] (see Fig. 4). Therefore, for sufficiently small R_m, the dynamo does not take place, even for $1 < \xi < 2$.

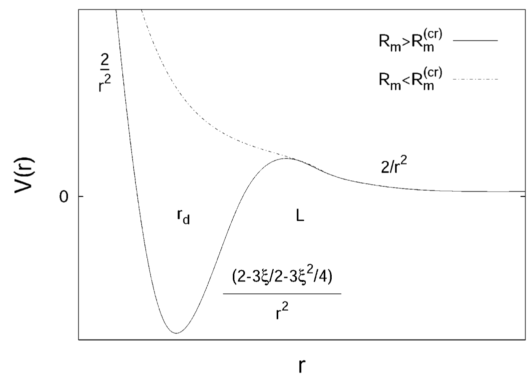

Fig. 4. A qualitative picture of the quantum potential shape for R_m respectively above and below the critical value $R_m^{(cr)}$ $(1 < \xi < 2)$.

The effect of a large-scale cutoff on the velocity energy spectrum is thus the presence of a critical Reynolds number $R_m^{(cr)}$. For R_m smaller than that value the potential V has not bound states or equivalently, on account of our quantum mechanic interpretation, the velocity field is unable to favor the magnetic field growth and the ohmic dissipation eventually prevails on stretching.

The dependence of the dimensionless rate-of-growth $|E_0|\, t_d^{-1}$ on R_m is shown in Fig. 5 for $R_m > R_m^{(cr)}$ in the case of the scaling $\xi = 4/3$. (Observe that, as a consequence of the velocity field δ-correlation in time, the value $\xi = 4/3$ corresponds to the Kolmogorov scaling). It should be noted that, for $R_m \gg R_m^{(cr)}$, E_0 takes the inertial range behavior $E_0 \simeq \epsilon_0(\xi)\, t_d^{-1}$. A numerical estimation of $R_m^{(cr)}$ was already given in ref. 8 for a different velocity structure function by a reverse iterations method. Once more we note that such numerical algorithm is based on a particular choice for the wave function, which does not take into account the stretched exponential decay typical of the inertial range.

We can again deduce from Eq. (6) some properties of the function $H(r,\cdot)$. The correlation length of the magnetic field is again of order r_d and, at $r \gg L$, $H(r,\cdot)$ shows an exponential decay

$$H(r,\cdot) \propto -e^{-\gamma(r/L)} \qquad (L \ll r) \tag{17}$$

with $\gamma = E_0[L^2/(2\bar{\kappa})]^{-1}$, $\bar{\kappa} = \kappa + \mathscr{D}_{ii}(0)/6$. (The negative exponential decay of the wave function at scales much larger than L was already pointed out in refs. 7 and 8).

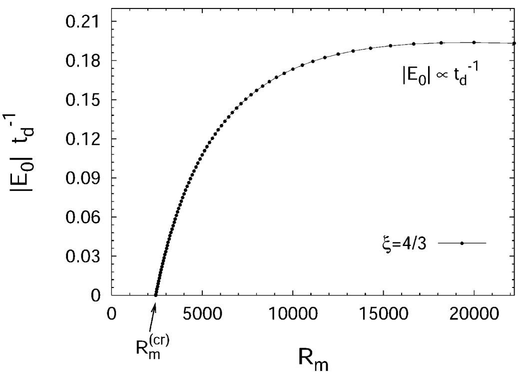

Fig. 5. The dependence of the magnetic growth rate on the magnetic Reynolds number for $Pr \to 0$ and $\xi = 4/3$. The numerical computation is performed using expression (9) for the correlation tensor of the magnetic field.

3.3. Nonzero Prandtl Effect

Finally, we consider the situation of nonzero Prandtl number (at infinite Reynolds number). This is equivalent to look at the influence of the viscous scale on the dynamo effect.

If $Pr < 1$, the diffusive scale r_d is in the inertial range, while, if $Pr > 1$, it lies within the viscous range. The structure function $S_{ii}(r)$ scales as r^2 for $r \ll \eta$ and as r^ξ for $r \gg \eta$.

From the previous considerations we can expect for the potential V the same asymptotic behaviors for $r \to \infty$ as in the case of $Pr = 0$. Therefore, if $R_m \to \infty$, the Prandtl number does not affect the presence of dynamo. (Note however that, if the magnetic Reynolds number is finite, a critical Prandtl number exists[8, 11]). What is sensitive to Pr is the correlation length of the magnetic field, that approximately corresponds to the scale at which the function ψ_{E_0} begin its exponential-like decay. When $Pr < 1$, the potential has nearly the same shape as in the case $Pr = 0$

$$V(r) \sim \begin{cases} 2/r^2 & r \ll r_d \\ (2 - \frac{3}{2}\xi - \frac{3}{4}\xi^2)/r^2 & r_d \ll r \end{cases} \tag{18}$$

and the correlation length is of order r_d.

On the contrary, when $Pr > 1$, the potential well is modified by an attractive $\xi = 2$ contribution

$$V(r) \sim \begin{cases} 2/r^2 & r \ll r_d \\ -4/r^2 & r_d \ll r \ll \eta \\ (2 - \frac{3}{2}\xi - \frac{3}{4}\xi^2)/r^2 & \eta \ll r \end{cases} \tag{19}$$

For these Pr the function $\psi_{E_0}(r)$ grows as r^2 for $r \ll r_d$, as $r^{1/2}$ in the range $r_d \ll r \ll \eta$ and has a stretched exponential decay for $\eta \ll r$. We can thus conclude that, when $Pr > 1$, the magnetic field correlation length is of order η.

In consequence, the correlation length of $\mathbf{B}$ is always the largest between the diffusive scale r_d and the viscous scale η, their ratio being controlled by the Prandtl number $Pr \simeq (\eta/r_d)^\xi$.

On account of what we have just seen, we expect that for $Pr \ll 1$ the ground state energy will be proportional to the diffusive time: $E_0 \simeq \epsilon_0(\xi)\, t_d^{-1}$. In the other limit, $Pr \gg 1$, we can predict an approximate expression for E_0 by a simple scaling argument. Indeed, for large Pr the potential V behaves like in the case $\xi = 2$ and we can expect $E_0 \propto D_1$ (see ref. 10 for the discussion of the smooth case). Knowing that $S_{ii}(r) \propto r^2$ for $r \ll \eta$ and $S_{ii}(r) \propto D_1 r^\xi$ in the inertial range, we can match the previous behaviors to obtain $D_1 \propto \eta^{\xi-2}$. Finally, we recall that from dimensional arguments we

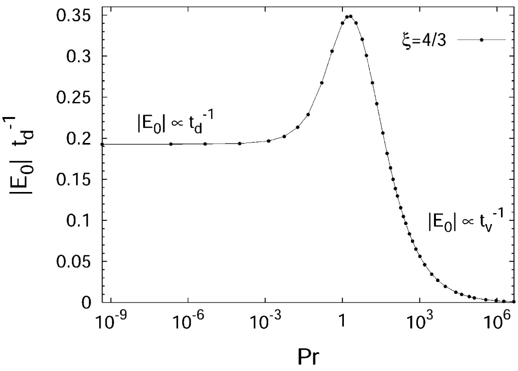

Fig. 6. The dependence of the magnetic growth rate on the Prandtl number for $\xi = 4/3$ and in the limit $R_m \to \infty$. The numerical computation is performed using expression (9) for the correlation tensor of the magnetic field.

have $\eta \simeq (\nu/D_1)^{1/\xi}$. Summarizing the previous considerations, it is easily seen that, for $Pr \gg 1$, the relation $E_0 \propto t_\nu^{-1}$ holds (the time $t_\nu = \eta^2/\nu$ is the characteristic one for the velocity diffusion).

The Prandtl number $Pr \simeq (t_\nu/t_d)^{\xi/(\xi-2)}$ thus influences also the magnetic field rate-of-growth: in the presence of dynamo, B^2 increases with a characteristic time-scale determined by the largest between the viscous and the diffusive time (see Fig. 6).

To conclude this section, we discuss a result that emerges from numerical computations: the magnetic rate-of-growth has a non-monotonic dependence on the Prandtl number and it reaches a maximum for $Pr \simeq 1$ (Fig. 6). We can explain this behavior referring once more to the Kazantsev quantum formalism. For $Pr < 1$ the $\xi = 2$ behavior is practically absent in the potential V, while, when Pr approaches the value 1, the scale η begins to come into play yielding a strongly attractive $-4/r^2$ contribution at scales $r_d \ll r \ll \eta$. The $\xi = 2$ potential is more attractive than that of $\xi < 2$ and the ground state energy increases in absolute value. Then, as Pr becomes larger, $|E_0|$ decreases as explained above. In other words as long as the viscous behavior affects only the potential shape around r_d, its only effect is to make the well deeper and so to favor the dynamo. When viscosity becomes very large, the level of velocity fluctuations lowers significantly, inducing eventually the depletion of the rate-of-growth.

4. CONCLUSIONS

We have presented a unified treatment of the kinematic dynamo problem in the framework of the Kraichnan–Kazantsev model. Much attention has been paid to highlighting the influence of the magnetic Reynolds number and of the Prandtl number on the dynamo effect. As already noted, the previous analysis depends only on the qualitative properties of the velocity structure function. We thus expect that our conclusions hold for a generic turbulent flow with the same statistical symmetries and be relevant for real applications.

APPENDIX A. VARIATION-ITERATION METHOD

For the numerical analysis of Schrödinger equation (6) we make the transformation $y = a^{-r}$ $(a > 1)$ which maps $(0, \infty)$ on the finite interval $(0, 1)$. (The constant a should be chosen to properly resolve this interval). Equation (6) can thus be rewritten in the form

$$\mathscr{L}\psi = \lambda \mathscr{M}\psi \tag{A1}$$

where

$$\begin{aligned} \mathscr{L} &= -(\ln a)^2 \left(y \frac{d^2}{dy^2} + \frac{d}{dy} \right) + \frac{m(y)}{y} (U(y) - U_{\min}) \\ \mathscr{M} &= \frac{m(y)}{y}, \qquad \lambda = E - U_{\min} \end{aligned} \tag{A2}$$

and $U_{\min}$ denotes the minimum value of U. $\mathscr{L}$ and $\mathscr{M}$ are positive-definite self-adjoint operators defining a spectrum of eigenvalues λ bounded from below and which extends to infinity. Moreover, $\mathscr{L}$ is invertible on all functions twice differentiable on $(0, 1)$ and vanishing at the boundaries of the interval. Under these hypotheses the variation-iteration method described in ref. 15 provides a valuable tool to compute the lowest eigenvalue λ_0 of Eq. (A1) and the corresponding eigenfunction ψ_0. Indeed, let φ_0 be an initial trial function such that $\int_0^1 \psi_0 \mathscr{M} \varphi_0 \, dy \neq 0$ and define the nth iterate φ_n as

$$\varphi_n \equiv \mathscr{L}^{-1} \mathscr{M} \varphi_{n-1} = (\mathscr{L}^{-1} \mathscr{M})^n \varphi_0 \tag{A3}$$

Then, as n is increased, the sequence φ_n converges to the eigenfunction φ_0. The nth approximation to λ_0 is given by the following variational expression employing φ_n as trial function

$$\lambda_0^{(n)} = \frac{\int_0^1 \varphi_n \mathscr{L} \varphi_n \, dy}{\int_0^1 \varphi_n \mathscr{M} \varphi_n \, dy} \tag{A4}$$

The set $\lambda_0^{(n)}$ form a monotonic sequence of decreasing values, approaching λ_0 from the above. The advantage of the variation-iteration technique is that no expression is required *a priori* for the function ψ_0. We only have to choose any guess for initial function φ_0 and then improve the result by iterating the method for sufficiently large n. The convergence is more rapid the smaller is the ratio between λ_0 and the following eigenvalue.

Finally, for the numerical implementation of the method, we exploited the first order discrete expression of $\mathscr{L}$ preserving the boundary conditions on ψ. If (0, 1) is divided in intervals of length Δ and $y_i = i\Delta$, we have

$$\mathscr{L}_{ij} = \frac{m(y_i)}{y_i}(U(y_i) - U_{\min}) + \frac{(\ln a)^2}{2\Delta^2} \times \begin{cases} (-y_{i-1} - y_i) & \text{if} \quad i = j+1 \\ (y_{i-1} + 2y_i + y_{i+1}) & \text{if} \quad i = j \\ (-y_i - y_{i+1}) & \text{if} \quad i = j-1 \end{cases} \tag{A5}$$

APPENDIX B. THE SCHRÖDINGER EQUATION IN THE DYNAMO THEORY

In the present appendix we refer to the notation adopted in the body of the paper. So, the trace of the correlation tensor $\langle B_i(\mathbf{x}, t)\, B_j(\mathbf{x}+\mathbf{r}, t)\rangle$ will be denoted by $H(r, t)$.

As a consequence of the velocity δ-correlation in time, H satisfies a closed equation that, under a suitable transformation, takes on the form of a one-dimensional Schrödinger-like equation. In order to exploit this fact, let us denote $s(r) = S_{ii}(r)$ and define the following quantities

$$\bar{s}(r) = \frac{1}{r^3}\int_0^r \frac{s(\rho)}{2}\,\rho^2\, d\rho$$
$$\Lambda(r) = \kappa + \bar{s}(r), \qquad \Lambda_1(r) = \Lambda(r) + 3\kappa + \frac{s(r)}{2} \tag{B1}$$

Then, the function

$$\Psi(r, t) = \sqrt{\kappa}\, \exp\left(\int_0^r \frac{\Lambda_1(\rho)}{2\rho\Lambda(\rho)}\, d\rho\right) \frac{1}{r^3}\int_0^r H(\rho, t)\,\rho^2\, d\rho \tag{B2}$$

solves the imaginary time Schrödinger equation

$$-\frac{\partial \Psi}{\partial t} + \left[\frac{1}{m(r)}\frac{\partial^2}{\partial r^2} - U(r)\right]\Psi = 0 \tag{B3}$$

where

$$m = \frac{1}{2\Lambda}, \qquad U = -\frac{1}{r}\frac{ds}{dr} + \frac{1}{2r^2}\frac{\Lambda_1^2}{\Lambda} + \Lambda \frac{d}{dr}\left(\frac{\Lambda_1}{r\,\Lambda}\right) \tag{B4}$$

(See ref. 4 for the detailed derivation). If we expand Ψ in terms of the energy eigenfunctions $\Psi(r,t) = \int \psi_E(r)\, e^{-Et} \varrho(E)\, dE$, we get the stationary equation

$$\frac{1}{m(r)} \frac{d^2\psi_E}{dr^2} + [E - U(r)]\, \psi_E = 0 \tag{B5}$$

The dynamo effect corresponds to the presence of negative eigenvalues in Eq. (B5).

The correlation function $H(r,\cdot)$ must tend to a constant value as $r \to 0$ and decreases to zero as $r \to \infty$. From the definition (B2) we have therefore that Eq. (B5) must be solved with the boundary conditions that $\psi_E(r)$ vanishes as $r \to 0$ and increases as $r \to \infty$ slowly enough to guarantee that $H(r,\cdot)$ decreases to zero. In particular, if $s(r)$ tends to a constant as $r \to \infty$, $\psi_E(r)$ cannot increase more rapidly than r.

We consider now the explicit expression

$$\mathscr{D}_{ij}(\mathbf{r}) = \int e^{i\mathbf{k}\cdot\mathbf{r}} \hat{\mathscr{D}}_{ij}(\mathbf{k})\, d^3\mathbf{k} \tag{B6}$$

with

$$\hat{\mathscr{D}}_{ij}(\mathbf{k}) = D_0 \frac{e^{-\eta k}}{(k^2 + L^{-2})^{(\xi+3)/2}} P_{ij}(\mathbf{k}) \tag{B7}$$

The transverse projector $P_{ij}(\mathbf{k}) = (\delta_{ij} - k_i k_j / k^2)$ ensures the incompressibility of the velocity field.

In the limits $\eta \to 0$ and $L \to 0$, $S_{ij}(r)$ takes the form

$$\lim_{\substack{\eta \to 0 \\ L \to \infty}} S_{ij}(\mathbf{r}) = D_1 r^{\xi} \left[(2+\xi)\, \delta_{ij} - \xi \frac{r_i r_j}{r^2} \right] \tag{B8}$$

with

$$D_1 = \frac{4\pi \cos(\pi\xi/2)\, \Gamma(-1-\xi)}{\xi + 3} D_0 \tag{B9}$$

(The function Γ is the Euler function).

If we insert $s(r)=2(\xi+3)\,D_1 r^{\xi}$ in (B4), the transformation (B2) takes on the form

$$\Psi(r,t)=\frac{(\kappa+D_1 r^{\xi})^{1/2}}{r}\int_0^r H(\rho,t)\,\rho^2\,d\rho \tag{B10}$$

while its inverse reads

$$H(r,t)=\frac{(2\kappa-D_1 r^{\xi}(\xi-2))\,\Psi(r,t)+2r(\kappa+D_1 r^{\xi})\,\Psi'(r,t)}{2r^2(\kappa+D_1 r^{\xi})^{\frac{3}{2}}} \tag{B11}$$

For the mass and the potential we obtain the following expressions

$$m(r)=\frac{1}{2(\kappa+D_1 r^{\xi})} \tag{B12}$$

$$U(r)=\frac{4\kappa^2+A(\xi)\,\kappa D_1 r^{\xi}+B(\xi)\,D_1^2 r^{2\xi}}{r^2(\kappa+D_1 r^{\xi})} \tag{B13}$$

with $A(\xi)=(8-3\xi-\xi^2)$ and $B(\xi)=(4-3\xi-\frac{3}{2}\xi^2)$.

For the sake of completeness we write also the expressions of the trace $s(r)$, which we used to compute E_0 respectively in the case of finite R_m and in the case of nonzero Pr

$$\lim_{\eta\to 0} s(r)=\frac{4\pi D_0 L^{\xi}}{\Gamma\left(\frac{\alpha+\xi+1}{2}\right)}\times\left[\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma\left(\frac{\xi}{2}\right)-\sqrt{\pi}\,\frac{L}{r}\,G^{2\,1}_{1\,3}\left(\frac{r^2}{4L^2}\,\middle|\,\begin{matrix}1-\frac{\alpha}{2}\\ \frac{\xi+1}{2},\frac{1}{2},0\end{matrix}\right)\right] \tag{B14}$$

$$\lim_{L\to\infty} s(r)=8\pi D_0\eta^{\xi}\left(\Gamma(-\xi)+\frac{\eta}{r}\left(1+\frac{r^2}{\eta^2}\right)^{\frac{1+\xi}{2}}\Gamma(-1-\xi)\times\sin\left[(1+\xi)\arctan\left(\frac{r}{\eta}\right)\right]\right) \tag{B15}$$

(The function G denotes the G-Meijer's function of argument $r^2/(4L^2)$. See ref. 16 for the exact definition). The explicit expressions of the mass and the potential can be derived from (B4).

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REFERENCES

1. H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press, Cambridge, 1978).
2. E. N. Parker, *Cosmical Magnetic Fields* (Clarendon Press, Oxford, 1979).
3. R. H. Kraichnan, Small-scale structure of a scalar field convected by turbulence, *Phys. Fluids* **11**:945–953 (1968).
4. A. P. Kazantsev, Enhancement of a magnetic field by a conducting fluid, *Sov. Phys. JETP* **26**:1031–1034 (1968).
5. F. Krause and K.-H. Rädler, *Mean-field Magnetohydrodynamics and Dynamo Theory* (Pergamon Press, Frankfurt, 1980).
6. S. Boldyrev, *A Solvable Model for Nonlinear Mean Field Dynamo*, astro-ph/0105354 (2001).
7. A. A. Ruzmaĭkin and D. D. Sokolov, The magnetic field in mirror-invariant turbulence, *Sov. Astron. Lett.* **7**:388–390 (1981).
8. V. G. Novikov, A. A. Ruzmaĭkin, and D. D. Sokolov, Kinematic dynamo in a reflection-invariant random field, *Sov. Phys. JETP* **58**:527–532 (1983).
9. A. Gruzinov, S. Cowley, and R. Sudan, Small-scale-field dynamo, *Phys. Rev. Lett.* **77**:4342–4344 (1996).
10. M. Chertkov, G. Falkovich, I. Kolokolov, and M. Vergassola, Small-scale turbulent dynamo, *Phys. Rev. Lett.* **83**:4065–4068 (1999).
11. A. A. Schekochihin, S. A. Boldyrev, and R. M. Kulsrud, *Spectra and Growth Rates of Fluctuating Magnetic Fields in the Kinematic Dynamo Theory with Large Magnetic Prandtl Numbers*, astro-ph/0103333 (2001); submitted to *Astrophys. J.* (2001).
12. M. Vergassola, Anomalous scaling for passively advected magnetic fields, *Phys. Rev. E* **53**:R3021–R3024 (1996).
13. N. V. Antonov, A. Lanotte, and A. Mazzino, Persistence of small-scale anisotropies and anomalous scaling in a model of magnetohydrodynamics turbulence, *Phys. Rev. E* **61**:6586–6605 (2000).
14. A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics*, Vol. 2, J. Lumley, ed. (MIT Press, Cambridge, MA, 1975).
15. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw–Hill Book Company, Inc., United States of America, 1953).
16. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1965).