EUROPHYSICS LETTERS 15 December 2002

*Europhys. Lett.*, **60** (6), pp. 820–826 (2002)

# Lorenz deterministic diffusion

R. Festa [1], A. Mazzino [2,1] and D. Vincenzi [3,1] (*)

[1] *INFM-Dipartimento di Fisica, Università di Genova - I-16146 Genova, Italy*

[2] *ISAC/CNR, Sezione di Lecce - Strada provinciale Lecce-Monteroni km 1.2 73100 Lecce, Italy*

[3] *CNRS, Observatoire de la Côte d'Azur - B.P. 4229, 06304 Nice Cedex 4, France*

(received 26 March 2002; accepted in final form 25 September 2002)

PACS. 05.45.-a – Nonlinear dynamics and nonlinear dynamical systems.
PACS. 05.60.-k – Transport processes.

**Abstract.** – The Lorenz 1963 dynamical system is known to reduce in the steady state to a one-dimensional motion of a classical particle subjected to viscous damping in a past history-dependent potential field. If the potential field is substituted by a periodic function of the position, the resulting system shows a rich dynamics where (standard) diffusive behaviours, ballistic motions and trapping take place by varying the model control parameters. This system permits to highlight the intimate relation between chaos and long-time deterministic diffusion.

*Introduction.* – Chaotic dynamical systems are known to exhibit typical random processes behaviour due to their strong sensitivity to initial conditions. Deterministic diffusion arises from the chaotic motion of systems whose dynamics is specified, and it should be distinguished from noise-induced diffusion where the evolution is governed by probabilistic laws. Diffusive (standard and anomalous) behaviours have been observed in periodic chaotic maps (see, *e.g.*, refs. [1, 2] and references therein) and in continuous-time dynamical systems [3, 4]. The analysis of deterministic diffusion is relevant for the study of non-equilibrium processes in statistical physics. The major aim is to understand the relationship between the deterministic microscopic dynamics of a system and its stochastic macroscopic description (think, for example, at the connection between Lyapunov exponents, Kolmogorov-Sinai entropy and macroscopic transport coefficients, firstly highlighted by Gaspard and Nicolis [5]).

In this brief communication we present a first analysis of a new model of one-dimensional deterministic diffusion, suggested by a classical-mechanics interpretation of the celebrated Lorenz 1963 system [6]. The steady-state chaotic dynamics of the Lorenz system can indeed be recasted as the one-dimensional motion of a classical particle subjected to viscous damping in a past history-dependent potential field (see refs. [7,8], and ref. [9] for an earlier preliminary analysis).

We shortly recall that the (scaled) Lorenz dynamical system is given by

$$\begin{cases} \dot{x} &= \sigma(y - x), \\ \dot{y} &= -y + x + (r-1)(1-z)x, \\ \dot{z} &= b(xy - z), \end{cases} \tag{1}$$

(*) E-mail: vincenzi@obs-nice.fr

with $r > 1$. For $1 < r < r_c$, where $r_c = \sigma(\sigma + b + 3)/(\sigma - b - 1)$, three fixed points exist: $(0, 0, 0)$ (unstable) and $(\pm 1, \pm 1, 1)$ (stable). For $r > r_c$, all fixed points are unstable, and the Lorenz system can exhibit either periodic or chaotic behaviour on a strange attractor set (see, *e.g.*, ref. [10] for a comprehensive exposition on the subject matter).

In the steady state, the system (1) can be reduced to a one-dimensional integro-differential equation for the $x$-coordinate [8, 9],

$$\ddot{x} + \eta\dot{x} + \left(x^2 - 1\right)x = -\alpha\left[x^2 - 1\right]_\beta x, \tag{2}$$

where $\alpha = (2\sigma/b) - 1$, $\beta = [2b/(r-1)]^{1/2}$, $\eta = (\sigma + 1)/[(r-1)b/2]^{1/2}$, and the time is scaled by a factor $[(r-1)b/2]^{1/2}$ with respect to the time coordinate in eqs. (1). The square brackets in eq. (2) indicate the *exponentially vanishing memory* which is defined, for any suitable time function $f(t)$, by

$$[f]_k(t) \equiv k \int_0^\infty \mathrm{d}s e^{-ks} f(t-s).$$

According to eq. (2), the Lorenz-system chaotic dynamics corresponds to a one-dimensional motion in a constant-in-time quartic potential $U(x) = (x^2 - 1)^2/4$. Even in the presence of friction ($\eta \neq 0$), the motion can be sustained by a time-dependent memory term which takes into account the system past evolution.

Although eq. (2) has been deduced from the Lorenz system (1), it can be generalized to a wider class of equations showing similar dynamical properties [8]. Indeed, it can be usefully recast in the form

$$\ddot{x} + \eta\dot{x} + \left\{q(x) + \alpha[q(x)]_\beta\right\}\Phi'(x) = 0, \tag{3}$$

where the prime indicates the derivative with respect to $x$. Equation (2) is obtained for $\Phi(x) = x^2/2$ and $q(x) = x^2 - 1$. The generalized equation (3) can be regarded as the description of the motion of a unit mass particle subjected to a viscous force $-\eta\dot{x}$ and interacting with a potential field $\Phi(x)$ through a dynamically varying "charge" $q_t(x) = q(x) + \alpha[q(x)]_\beta$. This charge depends both on the instantaneous particle position $x(t)$ and on the past history $\{x(t-s) \mid 0 \leq s < \infty\}$. It is just the coupling of $[q(x)]_\beta$ with the fixed potential field $\Phi(x)$ the origin of an endogenous forcing term which can sustain the motion even in the presence of friction: the chaotic behaviour can actually arise from the synergy between this term and the viscosity.

Moreover, one can easily verify that eq. (3) corresponds to the *generalized Lorenz system*

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = -y + x + (r-1)(1-z)\Phi'(x), \\ \dot{z} = -bz + b\left[\dfrac{1}{2}q'(x)(y-x) + q(x) + 1\right]. \end{cases} \tag{4}$$

The specific Lorenz model can thus be viewed as singled out from a quite general class of dynamical systems which can exhibit chaotic behaviour, their common essential property being an exponentially vanishing memory effect together with a viscous damping.

In our previous paper [7,8] the main chaotic dynamical features of the original Lorenz system have been investigated through the analysis of the piecewise linear system corresponding to the choice $\Phi(x) = |x|$ and $q(x) = |x| - 1$.

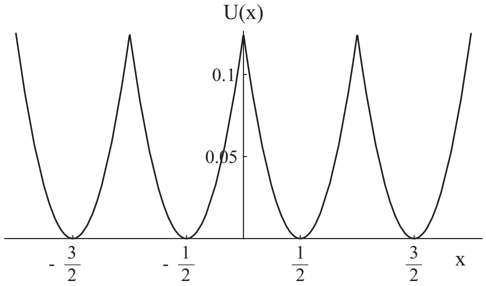


Fig. 1 – Periodic lattice of truncated parabolae. The system evolves in the periodic constant-in-time potential $U(x)$ subjected to a viscous damping and to a memory effect on the past motion.

*The Lorenz diffusion.* – If in eq. (3) one substitutes the quantities $q(x)$ and $\Phi'(x)$ with $x$-periodic functions, the chaotic jumps between the two infinite wells of the original quartic potential $U(x)$ correspond to chaotic jumps among near cells. The result is a deterministic diffusion in an infinite lattice, induced by a Lorenz-like chaotic dynamics. Equation (3) will be thus called the *Lorenz diffusion equation*. In order to use as far as possible analytical tools, we shall consider the unit wavelength periodic potential $U(x) = \frac{1}{2}\left(\{x\} - \frac{1}{2}\right)^2$ (corresponding to $q(x) = \{x\} - \frac{1}{2}$ and $\Phi(x) = \{x\}$), where $\{x\}$ indicates the fractionary part of $x$. This potential field obviously consists of a lattice of truncated parabolae (see fig. 1). By simple substitution one easily derives the equation

$$\ddot{x} + \eta\dot{x} + \{x\} - \frac{1}{2} + \alpha\left[\{x\} - \frac{1}{2}\right]_\beta = 0, \tag{5}$$

where $\eta$ denotes the friction coefficient, $\alpha$ is the memory amplitude and $\beta$ is related to the inertia whereby the system keeps memory of the past evolution.

Inside each potential cell, eq. (5) can be recasted in a third-order linear differential form. Indeed, by applying the operator $(\mathrm{d}/\mathrm{d}t + \beta)$ to each side of eq. (5), one obtains (for $x \neq n$)

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3} + (\beta + \eta)\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + (1 + \beta\eta)\frac{\mathrm{d}x}{\mathrm{d}t} + \beta(1 + \alpha)\left(x - n - \frac{1}{2}\right) = 0. \tag{6}$$

It is worth observing that the nonlinearity of the original model is simply reduced to a change of sign of the forcing term $\beta(1+\alpha)(\{x\}-\frac{1}{2})$ when $x$ crosses the cell boundaries (in our case the integer values). As we will see, chaotic dynamics essentially results from the unpredictability of the crossing times. Note that $\eta$ and $\beta$ play a symmetrical role in the dynamics: the solution of eq. (6) is indeed left invariant if one changes $\beta$ with $\eta$, while keeping $\beta(1 + \alpha)$ constant.

Partial solutions of the third-order nonlinear differential equation can be easily calculated inside each open interval $(n, n + 1)$. To obtain a global solution, such partial solutions should be matched at $x = n$ by assuming that the position $x$, the velocity $\dot{x}$ and the memory $[|x|]_\beta$ are continuous, whereas the acceleration $\ddot{x}$ turns out to be undefined. However, it is easily shown that each pair of acceleration values "immediately" before and after the crossing times are related by $\ddot{x}^{(+)} - \ddot{x}^{(-)} = \mathrm{sgn}(\dot{x})$.

The fixed points of eq. (5) are, of course, $x = n+\frac{1}{2}$, $n \in \mathbb{Z}$, and their local stability depends on the roots of the characteristic polynomial associated to eq. (6). All the fixed points are unstable when $\alpha$ is larger than the critical value $\alpha_\mathrm{c} = \beta^{-1}(1 + \beta\eta)(\beta + \eta) - 1$. In this case there are one real negative root $(-\lambda_0 < 0)$ and a complex conjugate pair of roots with positive

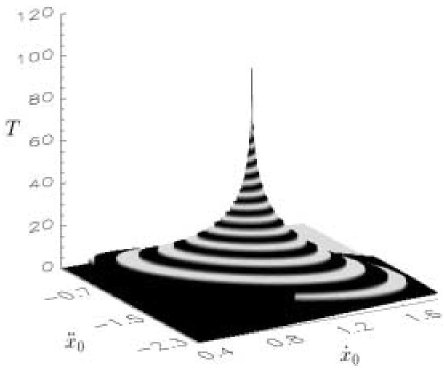

Fig. 2 – Dependence of the exiting time from the elementary cell on the initial conditions $\dot{x}_0$, $\ddot{x}_0$ for $\alpha = 6.50$, $\beta = 0.19$, $\eta = 0.78$. The graph has been obtained via numerical solution of eq. (8). Backward ($x = n$) and forward ($x = n + 1$) exiting cases have been distinguished by different colors. The structure of the graph can be guessed through the analysis of the contour lines of the function $T = T(\dot{x}_0, \ddot{x}_0)$ (see ref. [8] for further details).

real part $\lambda_\pm = \lambda \pm i\omega$ ($\lambda > 0$). For $\alpha > \alpha_c$, the partial solution in the generic open interval $(n, n + 1)$ can be finally written in the explicit form

$$x(t) = e^{\lambda t}(C_1 \cos(\omega t) + C_2 \sin(\omega t)) + C_3 e^{-\lambda_0 t} + n + 1/2, \tag{7}$$

where the constants $C_1$, $C_2$, $C_3$ are linearly related to the (cell-by-cell) entering conditions. The motion inside each cell consists of an amplified oscillation around a central point which translates towards the center of the cell. A change of cell yields a discontinuous variation of the acceleration $\ddot{x}$ and, consequently, of the coefficients $C_1$, $C_2$, $C_3$.

Suppose that, at a given time, say $t = 0$, the particle enters the $n$-th cell at its left boundary with positive velocity (the reverse case can be symmetrically analyzed). In this case $C_3 = -(C_1 + \frac{1}{2})$. The question is now on whether the particle leaves the cell either from the left side (*i.e.* $x = n$) or from the right side of the cell (*i.e.* $x = n + 1$). Once assigned the model parameters $\lambda_0$, $\lambda$ and $\omega$, the minimum positive time $T$ such that

$$\left| e^{\lambda T}\big(C_1 \cos(\omega T) + C_2 \sin(\omega T)\big) - \big(C_1 + 1/2\big) e^{-\lambda_0 T} \right| = 1/2 \tag{8}$$

depends, of course, on $C_1, C_2$, and therefore on the entering conditions $\dot{x}_0, \ddot{x}_0$. Unfortunately, the direct problem is transcendent. Moreover, as shown in fig. 2, its solution is strongly sensitive to the entering conditions. This fact is a direct consequence of the crossing time definition: $T$ is indeed determined by the intersection of an amplified oscillation and a decreasing exponential. A small change in the entering conditions may thus cause a discontinuous variation of the crossing time. This is the very origin of the system chaotic dynamics which suggests a stochastic treatment of the Lorenz diffusion equation.

Despite the fact that we have a three-dimensional space of parameters to investigate the model behaviours, the interesting region is actually a limited portion. This easily follows from the following simple considerations. For $\eta$ large enough in eq. (5), the motion rapidly stops in one of the lattice fixed points. In order to have non-trivial solutions, the viscous coefficient must be smaller than a maximum value which can be explicitly derived from the condition $\alpha > \alpha_c$: $\eta_{\max} = \{[(1 + \beta^2)^2 + 4\alpha\beta^2]^{1/2} - 1 - \beta^2\}/(2\beta)$. In the opposite limit, if $\eta$ is too small, the friction term is negligible with respect to the memory term, and the resulting motion is

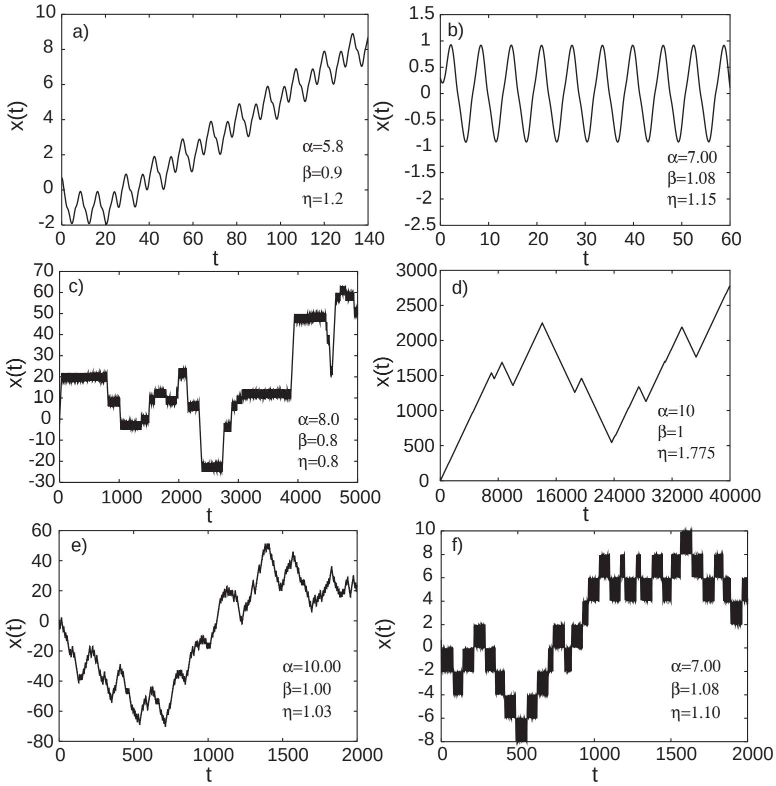


Fig. 3 – a) A typical ballistic motion with a more complex structure than a simple crossing of adjacent cells. Ballistic motion generally corresponds to a periodic behaviour in the cell crossing. b) Trapping in two cells. c)-f) Different kinds of trajectories generating diffusive motions.

"ballistic", *i.e.*, $\langle [x(t) - x(0)]^2 \rangle \sim t^2$ (1). Analogous considerations can be repeated for $\alpha$ and $\beta$. Inside this limited region of parameters the Lorenz diffusion equation generates a wide variety of behaviours: as we will see, the observed regimes are strongly sensitive to the control parameters, and, furthermore, this dependence is often in contrast with the intuitive meaning of $\alpha$, $\beta$, $\eta$.

In fig. 3 a few numerical simulation of eq. (5) are shown, corresponding to different values of the parameters. These examples suggest that the variety of motion regimes ranges from "ballistic" ones to clearly "diffusive", and even "trapped" in one cell or in groups of nearby cells.

The analysis of the motion in the elementary cell shows that the system rapidly reaches a steady state. In the diffusive regimes the points corresponding to subsequent cell entering conditions $\dot{x}_0$, $\ddot{x}_0$ are quickly attracted on a particular locus of the plane (fig. 4). In the general situation, this attracting set does not define a univocal map between the entering velocity and acceleration. However, it is not difficult to see that, for large $T$, $\dot{x}_0$ and $\ddot{x}_0$ satisfy the piecewise

(1) Due to the deterministic nature of the system, averages should be intended over the initial conditions. In our numerical simulations we have typically chosen the initial conditions to be uniformly distributed over some real interval, and we have naturally assumed that the diffusion coefficient is independent of the choice of the initial ensemble.

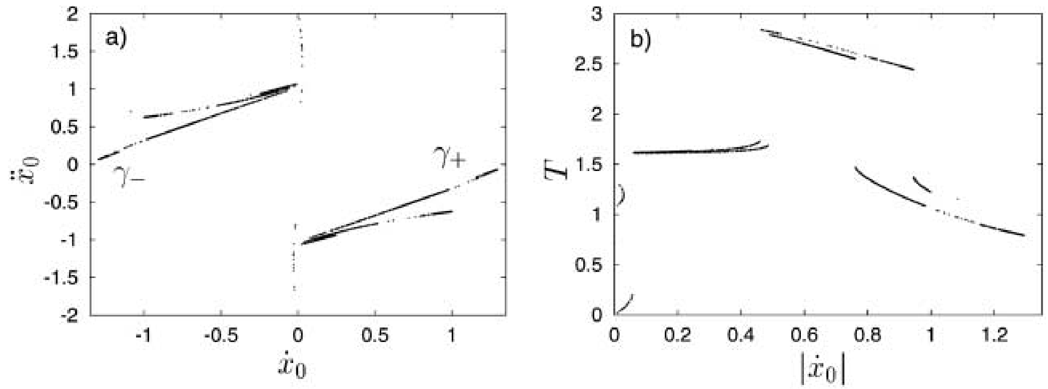


Fig. 4 – a) A typical attracting set for the $n$-th cell entering conditions $\dot{x}_0$, $\ddot{x}_0$. The corresponding trajectory is shown in fig. 3 e). One should note the presence of the straight lines defined by eq. (9), here denoted by $\gamma_+$ and $\gamma_-$. b) Attracting set in the plane $(T, \dot{x}_0)$. The step-like structure of the set derives from the superposition of the points in a) on a graph similar to that shown in fig. 2.

linear relation [8]

$$\ddot{x}_0 - 2\lambda\dot{x}_0 + \frac{1}{2}(2 - \lambda^2 - \omega^2)\mathrm{sgn}(\dot{x}_0) = 0\,. \tag{9}$$

If the diffusive regime admits large enough times of permanence, the couple of straight lines defined by eq. (9) therefore belongs to the entering-condition attracting set (fig. 4). Moreover, in the limiting case of very large $\lambda_0$, the attracting set exactly reduces to such lines, and in the steady state eq. (9) is satisfied by all entering conditions [8].

The macroscopic evolution of the system can be statistically analyzed by introducing the diffusion coefficient $D \equiv \lim_{t\to\infty}\langle[x(t) - x(0)]^2\rangle/(2t)$. Diffusive regimes are identified by a finite value of $D$. As shown in fig. 3, different values of the control parameters can lead to qualitatively very different diffusive motions. What is surprising is that the change of regime is very irregular and abrupt with $D$, and sometimes it does not reflect our intuition on the role played by the parameters in eq. (5). For instance, $D$ shows a nonmonotonic dependence on the viscosity $\eta$ (fig. 5), and it is possible to observe transitions from trapped (or diffusive) motion to ballistic ones even by increasing the viscosity itself (table I). It is worth recalling that similar complex behaviours in discrete nonlinear maps have been observed in refs. [11–13] and, more recently, in ref. [14].

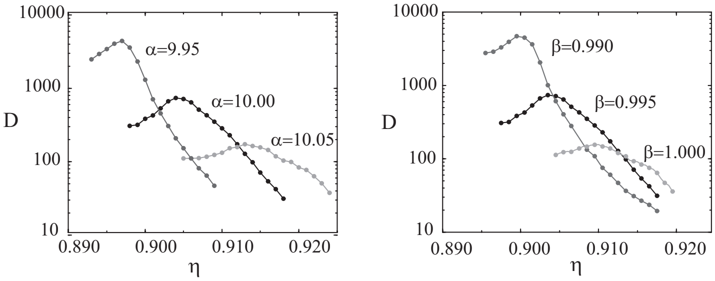


Fig. 5 – Dependence of the diffusion coefficient $D$ on the viscosity parameter $\eta$ for different values of $\alpha$ and $\beta$.

TABLE I – *Different long-time behaviours as a function of $\eta$, for fixed $\alpha$ and $\beta$. Note the nontrivial, highly structured dependence of the diffusion coefficient on the control parameter.*

| $\eta$ | Regime | $\eta$ | Regime |
|---|---|---|---|
| $< 0.850$ | ballistic | 1.425–1.725 | trapped |
| 0.850–0.875 | trapped | 1.725–1.975 | diffusive |
| 0.875–0.900 | ballistic | 1.975–2.050 | trapped |
| 0.900–1.050 | diffusive | 2.050–2.325 | diffusive |
| 1.050–1.175 | trapped | $2.325 <$ | trapped |
| 1.175–1.425 | diffusive | | |

Despite the somehow strange aspect of some system trajectories, it should be stressed that no anomalies (*i.e.* superdiffusive transport) have been encountered in the statistical analysis of diffusion, the distribution of long flies having always exponential tails.

To conclude, we have presented a deterministic system which can generate large-scale diffusive transport induced by a Lorenz-like microscopic chaotic dynamics. Two main problems should still be tackled. First, a more systematic analysis of the Lorenz diffusion equation could highlight interesting properties of the diffusion coefficient. Indeed, the very irregular dependence of $D$ on the control parameters might be indicative of a fractal structure of the diffusion, as already observed in other chaotic maps [1,14]. Second, it would be interesting to consider a two-dimensional extension of the Lorenz diffusion equation, mainly to investigate the possible emergence of anomalous diffusion induced by the increased spatial dimension.

* * *

The authors would like to thank R. KLAGES and M. LA CAMERA for very useful suggestions. DV was supported by grants of the University of Nice and of the University of Genova. This work has been partially supported by the INFM project GEPAIGG01 and Cofin 2001, prot. 2001023848.

REFERENCES

[1] KLAGES R. and DORFMAN J. R., *Phys. Rev. E*, **59** (1999) 5361.
[2] WEIBERT K., MAIN J. and WUNNER G., *Phys. Lett. A*, **292** (2001) 120.
[3] GASPARD P., *Phys. Rev. E*, **53** (1996) 4379.
[4] CASTIGLIONE P., MAZZINO A., MURATORE-GINANNESCHI P. and VULPIANI A., *Physica D*, **134** (1999) 75.
[5] GASPARD P. and NICOLIS G., *Phys. Rev. Lett.*, **65** (1990) 1693.
[6] LORENZ E. N., *J. Atmos. Sci.*, **20** (1963) 130.
[7] FESTA R., MAZZINO A. and VINCENZI D., *Europhys. Lett.*, **56** (2001) 47.
[8] FESTA R., MAZZINO A. and VINCENZI D., *Phys. Rev. E*, **65** (2002) 046205.
[9] TAKEYAMA K., *Progr. Theor. Phys.*, **60** (1978) 613.
[10] SPARROW C., *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors* (Springer-Verlag, New York) 1982.
[11] SCHELL M., FRASER S. and KAPRAL R., *Phys. Rev. A*, **26** (1982) 504.
[12] GROSSMANN S. and FUJISAKA H., *Phys. Rev. A*, **26** (1982) 1779.
[13] FUJISAKA H. and GROSSMANN S., *J. Phys. B Condens. Matter*, **48** (1982) 261.
[14] KORABEL N. and KLAGES R., arXiv:nlin.CD/0206027.