QUANTUM SINGULAR COMPLETE INTEGRABILITY

THIERRY PAUL AND LAURENT STOLOVITCH

Abstract. We consider some perturbations of a family of pairwise commuting linear quantum Hamiltonians on the torus with possibly dense pure point spectra. We prove that the Rayleigh-Schrödinger perturbation series converge near each unperturbed eigenvalue under the form of a convergent quantum Birkhoff normal form. Moreover the family is jointly diagonalised by a common unitary operator explicitly constructed by a Newton type algorithm. This leads to the fact that the spectra of the family remain pure point. The results are uniform in the Planck constant near $\hbar = 0$. The unperturbed frequencies satisfy a small divisors condition and we explicitly estimate how this condition can be released when the family tends to the unperturbed one.

Contents

1. Introduction

Perturbation theory belongs to the history of quantum mechanics, and even to its pre-history, as it was used before the works of Heisenberg and Schrödinger 1925/1926. The goal at that time was to understand what should be the Bohr-Sommerfeld quantum conditions for systems nearly integrable [MB], by quantizing the perturbation series provided by celestial mechanics [HP]. After (or rather during its establishment) the functional analysis point of view was settled for quantum mechanics, the "modern" perturbation theory took

Research of L. Stolovitch was supported by ANR grant "ANR-10-BLAN 0102" for the project DynPDE .

place, mostly by using the Neumann expansion of the perturbed resolvent, providing efficient and rigorous ways of establishing the validity of the Rayleigh-Schrödinger expansion and leading to great success of this method, in particular the convergence under a simple argument of size of the perturbation in the topology of operators on Hilbert spaces [TK], and Borel summability for (some) unbounded perturbations [GG, BS]. On the other hand, by relying on the comparison between the size of the perturbation and the distance between consecutive unperturbed eigenvalues, the method has two inconveniences: it remains local in the spectrum in the (usual in dimension larger than one) case of spectra accumulating at infinity and is even inefficient in the case of dense point unperturbed spectra which can be the case in the present article.

In the present article, we consider some commuting families of operators on $L^2(\mathbb{T}^d)$ close to a commuting family of unperturbed Hamiltonians whose spectra are pure point and might be dense. As already emphasized, standard (Neumann series expansion) perturbation theory does not apply in this context. Nevertheless, we prove that the pure point property is preserved and moreover, we show that the perturbed spectra are analytic functions of the unperturbed ones. All these results are obtained using a method inspired by classical local dynamics, namely the analysis of quantum Birkhoff forms. Let us first recall some known fact of (classical) Brikhoff normal forms.

In the framework of (classical) local dynamics, Rüssman proved in [Ru] (see also [Bru]) the remarkable result which says that, when the Birkhoff normal form (BNF), at any order, depends only on the unperturbed Hamiltonian, then it converges provided that the small divisors of the unperturbed Hamiltonian do not accumulate the origin too fast (we refer to [Ar2] for an introduction to this subject). This leads to the integrability of the perturbed system. On the other hand, Vey proved two theorems about the holomorphic normalization of families of $l-1$ (resp. l) of commuting germs of holomorphic vector fields, volume preserving (resp. Hamiltonian) in a neighborhood of the origin of $\mathbb{C}^l$ (resp. $\mathbb{C}^{2l}$) (and vanishing at the origin) with diagonal and independent 1-jets [JV1, JV2].

These results were extended by one of us in [LS1, LS2], in the framework of general local dynamics of a families of of $1 \leqslant m \leqslant l$ commuting germs of holomorphic vector fields near a fixed point. It is proved that under an assumption on the formal (Poincaré) normal form of the family and and under a generalized Brjuno type condition of the family of linear parts, there exists an holomorphic transformation of the family to a normal form. This fills up therefore the gap between Rüssman-Brjuno and the complete integrability of Vey. In these directions, we should also mention works by H. Ito [It] and N.-T. Zung [Zu] in the analytic case and H. Eliasson [El] in the smooth case, and Kuksin-Perelman [KP] for a specific infinite dimensional version.

In [GP] one of us (the other) gave with S. Graffi a quantum version of the Rüssmann theorem in the framework of perturbation theory of the quantization of linear vector fields on the torus $\mathbb{T}^l$. Moreover, in this setting, it is possible to read on the original perturbation if the Rüssman condition is satisfied and the results are uniform in the Planck constant belonging to $[0,1]$. The method seats in the framework of Lie method perturbation theory initiated in classical mechanics in [De, Ho] and uses the quantum setting established in [BGP].

The goal of the present paper is to provide a full spectral resolution for certain families of commuting quantum Hamiltonians, not treatable by standard methods due to possible spectral accumulation, through the convergence of quantum normal Birkhoff forms and underlying unitary transformations. These families generalize the quantum version of Rüsmman theorem treated in [GP], to the quantum version of "singular complete integrability" treated in [LS1]. The methods use the quantum version of the Lie perturbative algorithm together with a newton type scheme in order to overcome the difficulty created by small divisors.

Let $m \leqslant l \in \mathbb{N}^*$. For $\omega = (\omega_i)_{i=1\dots m}$ with $\omega_i = (\omega_i^j)_{j=1\dots l} \in \mathbb{R}^l$, let us denote by $L_\omega = (L_{\omega_i})_{i=1\dots m}$, the operator valued vector of components

$$L_{\omega_i} = -i\hbar\omega_i.\nabla_x = -i\hbar\sum_{j=1}^{l}\omega_i^j\frac{\partial}{\partial x_j},\ i = 1\dots m$$

on $L^2(\mathbb{T}^l)$.

We define the operator valued vector $H = (H_i)_{i=1\dots m}$ by

$$H = L_\omega + V, \tag{1.1}$$

where V is a bounded operator valued vector on $L^2(\mathbb{T}^l)$ whose action is defined after a function $\mathcal{V}:\ (x,\xi,\hbar) \in T^*\mathbb{T}^l \times [0,1] \mapsto \mathcal{V}(x,\xi,\hbar) \in \mathbb{R}^m$ by the formula (Weyl quantization)

$$(Vf)(x) = \int_{\mathbb{R}^l\times\mathbb{R}^l}\mathcal{V}((x+y)/2,\xi,\hbar)e^{i\frac{\xi(x-y)}{\hbar}}f(y)\frac{dyd\xi}{(2\pi\hbar)^l}, \tag{1.2}$$

where $f(\cdot)$ and $\mathcal{V}((x+\cdot)/2)$ are considered as periodic functions on $\mathbb{R}^l$.

We make the following assumptions.

Main assumptions

(A1) There exist $\gamma > 0, \tau \geqslant l$ such that the family of frequencies vectors ω fulfills the **generalized Brjuno condition**

$$\sum_{l=1}^{\infty}\frac{\log\mathcal{M}_{2^k}}{2^k} < +\infty \text{ where } \mathcal{M}_M := \min_{1\leqslant i\leqslant m}\max_{0\neq|q|\leqslant M}|\langle\omega_i,q\rangle|^{-1}. \tag{1.3}$$

We will sometimes impose to ω tu fulfill the strongest **collective Diophantine condition**

$$\forall q \in \mathbb{Z}^l,\ q \neq 0,\ \min_{1 \leqslant i \leqslant m} |\langle \omega_i, q\rangle|^{-1} \leqslant \gamma |q|^\tau. \tag{1.4}$$

Remark : usually, $\frac{1}{\mathcal{M}_M}$ is denoted by ω_M in the literature [Bru, LS1]

(A2) $\mathcal{V}$ takes the form, for some $\mathcal{V}'$: $(\Xi, x, \hbar) \in \mathbb{R}^m \times \mathbb{T}^l \times [0,1] \mapsto \mathcal{V}'(\Xi, x, \hbar) \in \mathbb{R}^m$, analytic in (Ξ, x) and kth times differentiable in $\hbar$,

$$\mathcal{V}(x, \xi, \hbar) = \mathcal{V}'(\omega_1.\xi, \dots, \omega_m.\xi, x, \hbar), \tag{1.5}$$

(A3) The family H satisfies

$$[H_i, H_j] = 0, \quad 1 \leqslant i,j \leqslant m,\ 0 \leqslant \hbar \leqslant 1. \tag{1.6}$$

Moreover we will suppose that the vectors $\omega_j,\ j = 1\dots m$ are independent over $\mathbb{R}$. The case where they are not being easily reducible to the case of the $n < m$ independent ones of the family, the $m-n$ other being linear combination of the preceding.

We define

$$\underline{\omega} := \sum_{j=1}^m |\omega_j| = \sum_{j=1}^m \left(\sum_{i=1}^l (\omega_j^i)^2 \right)^{1/2} \tag{1.7}$$

Let us define for $\rho > 0,\ k \in \{0\} \cup \mathbb{N}$ and $\mathcal{V}'$: $(\Xi, x, \hbar) \in \mathbb{R}^m \times \mathbb{T}^l \times [0,1] \mapsto \mathcal{V}'(\Xi, x, \hbar) \in \mathbb{R}^m$

$$\|\mathcal{V}'\|_{\rho,\underline{\omega},k} = \sum_{j=1}^m \sum_{r=0}^k \|\partial_\hbar^r \widehat{\widehat{\mathcal{V}_j'}}\|_{L^1_{\rho,\underline{\omega},r}(\mathbb{R}^m \times \mathbb{Z}^l) \otimes L^\infty([0,1])} \text{ and } \|\nabla \mathcal{V}'\|_{\rho,\underline{\omega},k} = \max_{i=1\dots l} \sum_{j=1}^m \sum_{r=0}^k \|\partial_\hbar^r \partial_{\Xi_j} \mathcal{V}_i'\|_{\rho,\underline{\omega},k},$$

where $\widehat{\widehat{\cdot}}$ denotes the Fourier transform on $\mathcal{S}(\mathbb{R}^m \times \mathbb{T}^l)$ and $L^1_{\rho,\underline{\omega},k}(\mathbb{R}^m \times \mathbb{Z}^l)$ is the L^1 space equipped with the weighted norm $\sum_{q \in \mathbb{Z}^l} \int_{\mathbb{R}^m} |f(p,q)|(1 + |\omega \cdot p| + |q|)^{\frac{r}{2}} e^{\rho(\underline{\omega}|p|+|q|)} dp$,

Let us remark that $\|\mathcal{V}'\|_{\rho,\underline{\omega},k} < \infty$ implies that $\mathcal{V}'$ is analytic in a complex strip $\Im x < \rho, \Im \xi < \rho\underline{\omega}$ and kth time differentiable in $\hbar \in [0,1]$.

We will denote by $\overline{\mathcal{V}'}(\Xi) := \frac{1}{(2\pi)^l} \int_{\mathbb{T}^l} \mathcal{V}'(\Xi, x) dx$.

Our main result reads (see Theorems 26, 27 and 36 for more precise and explicit statements):

Theorem 1. *Let $k \in \mathbb{N} \cup \{0\}$ and $\rho > 0$ be fixed. Let H satisfy the Main Assumption above and $\|\mathcal{V}'\|_{\rho,\underline{\omega},k}, \|\nabla \overline{\mathcal{V}'}\|_{\rho,\underline{\omega},k}$ be small enough.*

Then there exists a family of vector-valued functions $\mathcal{B}_\infty^\hbar(\cdot)$, $\partial_\hbar^j \mathcal{B}_\infty^\hbar(\cdot)$ being holomorphic in $\{|\Im z_i| < \frac{\rho}{2}, i = 1\dots m\}$ uniformly with respect to $\hbar \in [0,1]$ and $0 \leqslant j \leqslant k$, such that the family H is jointly unitary conjugated to $\mathcal{B}_\infty^\hbar(L_\omega)$ and therefore the spectrum of each H_i is pure point and equals the set $\{(\mathcal{B}_\infty^\hbar)_i(\omega \cdot n),\ n \in \mathbb{Z}^l\}$.

Here we denoted $\omega \cdot n = (<\omega_i, n>)_{i=1\dots m}$.

Our results being uniform in $\hbar$ we get as a partial bi-product of the preceding result the following global version of [LS1]:

Theorem 2. *Let $\rho > 0$ be fixed. Let $\mathcal{H}$ be a family of $m \leqslant l$ Poisson commuting classical Hamiltonians $(\mathcal{H}_i)_{i=1\dots m}$ on $T^*\mathbb{T}^l$ of the form $\mathcal{H} = \mathcal{H}^0 + \mathcal{V}$, $\mathcal{H}^0(x,\xi) = \omega.\xi$, ω and $\mathcal{V}$ satisfying assumption (A1) and $\mathcal{V}$ on the form $\mathcal{V}(x,\xi) = \mathcal{V}'(\omega_1.\xi,\dots,\omega_m.\xi,x)$. Let finally $\|\mathcal{V}'\|_{\rho,\underline{\omega}}, \|\nabla\overline{\mathcal{V}'}\|_{\rho,\underline{\omega},0}$ be small enough (here we consider $\mathcal{V}'$ as a function constant in $\hbar$).*

Then $\mathcal{H}$ is (globally) symplectomorphically and holomorphically conjugated to $\mathcal{B}^0_\infty(\mathcal{H}^0)$.

Once again let us mention that our results are much more explicit and precise as expressed in Theorems 26, 27 and 36 and Corollary 32.

Moreover it appears in the proofs that the statement in Theorem 1, as well as in Theorems 26, 27 and 36 and Corollary 32, is valid for fixed value of the Planck constant $\hbar$ under the Main Assumption lowed down by restricting (1.6) to $\hbar$ fixed. More precisely under the Main Assumption with (A3) restricted to, e.g., $\hbar = 1$, the Theorem 1 is still valid by putting in the statement $k = 0$ and $\hbar = 1$. Let us mention also that, as in the original formulations in [Ru]-[LS1], one easily sees that condition (A2) can be replaced by the fact that the quantum Birkhoff normal form (see section 2 below for the precise definition) at each order is a function of $(\omega_1.\xi,\dots,\omega_m.\xi)$ only.

Finally let us emphasize the two extreme cases, that is $m = l$ and $m = 1$.

Corollary 1 (Quantum Vey theorem)**.** *Assume that the $\omega_j \in \mathbb{R}^l$, $j = 1,\dots,l$, are independent over $\mathbb{R}$. Assume that the $H_i = L_{\omega_i} + V_i$, $i = 1,\dots,l$ are pairwise commuting. Let the perturbation V_i be the quantization of any small enough analytic function $\mathcal{V}_i$. Then the family H is jointly unitary conjugated to $\mathcal{B}^\hbar_\infty(L_\omega)$ as defined in theorem 1.*

We emphasized that this last result do not require neither a small divisors condition nor a condition on the perturbation.

Corollary 2 (consolidated Graffi-Paul theorem)**.** *Assume that $\omega \in \mathbb{R}^l$ satisfies Brjuno condition ($m = 1$). Assume that $H = L_\omega + V$, where the perturbation V is small enough and $\mathcal{V}(\xi,x) = \mathcal{V}'(\omega.\xi,x)$. Then H is unitary conjugated to $\mathcal{B}^\hbar_\infty(L_\omega)$ as defined in theorem 1.*

The main difference between this last result and the main result of [GP] is the small divisors condition used (a Siegel type condition with constraints). See Section 10 for the proofs.

Le us finally mention a by-product of our resul, a kind of inverse result, obtained thanks to the fact that we carefully took care of the precise estimations and constants all a long the proofs. This result is motivated by the remark that, though a small divisors condition is necessary to obtain the perturbed integrability (and Brjuno condition is sufficient), such a condition should disappear when the perturbation vanishes, as the Hamiltonian H^0 is always integrable, whatever the frequencies ω are. Our last result quantifies this remark.

Let us define, for ω satisfying (1.4) and $\alpha < 2\log 2$,

$$B_\alpha(\gamma,\tau) = 2\log\left[2^\tau\gamma(\frac{\tau}{e\alpha})^\tau\right]$$

(note that $B_\alpha(\gamma,\tau)\to\infty$ as γ and/or $\tau\to\infty$).
The next Theorem shows that, in the Diophantine case, the small divisors condition can be released as $B_\alpha(\gamma,\tau)$ diverging logarithmically as the perturbation vanishes.

Theorem 3. *Let $k\in\mathbb{N}\cup\{0\}$ and $\rho>0$ be fixed. Let ω and $\mathcal{V}$ satisfy (A1) (Diophantine case), (A2) and (A3), and let $0<\underline{\omega}_-\leqslant\underline{\omega}\leqslant\underline{\omega}_+<\infty$ and $\|\mathcal{V}'\|_{\rho,\underline{\omega}_+,k}$, $\|\nabla\overline{\mathcal{V}'}\|_{\rho,\underline{\omega}_+,k}$ be small enough (depending only on k).*

Then there exist a constant $C_{\underline{\omega}_-}$ such that the conclusions of Theorems 1 hold as soon as, for some $\alpha<\rho/2$, $\alpha<2\log 2$,

$$B_\alpha(\gamma,\tau)<\frac{1}{3}\log\left(\frac{1}{\|V\|_{\rho,\underline{\omega}_+,k}}\right)+C_{\underline{\omega}_-}.$$

See Corollary 37 for details and the Remark after on the case of the Brjuno condition. Let us remark that an equivalent result for Theorem 2 is straightforwardly obtainable.

Let us finish this section by mentioning three comments and remarks concerning our results.

First of all, as mentioned earlier, no hypothesis on the minimal distance between two consecutive unperturbed eigenvalues is required in our article. More, the spectra of our unperturbed operators L_{ω_i} might be dense for all value of $\hbar$ (actually in the Diophantine case for $m=1$, $l>1$ they are) so the Neumann series expansion is not possible. For $m>1$ the non degeneracy of the unperturbed eigenvalues is not even insured by the arithmetical property of ω because it relies on the minimum over $i\leqslant m$ of the inverse of the small denominator of the vector ω_i. In fact, for a resonant ω_j the operator H_j will have an eigenvalue with infinite degeneracy, so the projection of the perturbation V_j on the corresponding and infinite dimensional eigenspace, which leads to the first order perturbation correction to the unperturbed eigenvalue, might have continuous spectrum. Nevertheless our results show that the perturbed spectra are analytic functions of the spectra of the L_{ω_i}'s.

Secondly, because of the fact that non degeneracy of some of the unperturbed spectra is not even guaranteed by our assumptions, the standard argument on existence of a common eigenbasis of commuting operators with simple spectra cannot be involved here. This existence is a bi-product of our results.

Finally let us mention that, as it was the case in [GP], though our hypothesis on the perturbations are restrictive, our results, compared with the usual construction of quasi-modes [Ra, CdV, PU, Po1, Po2], have the property of being global in the spectra (full diagonalization), and exact (no smoothing or $O(\hbar^\infty)$ remainder), together of course with sharing the property of being uniform in the Planck constant.

Let us point out that this paper has been written in order to be self-contained

Notations

Function valued vectors in $\mathbb{R}^n$ will be denoted in general in calligraphic style, and operator valued vectors by capital letters, e.g. $V=(V_l)_{l=1\dots m}$ or $\mathcal{V}=(\mathcal{V}_l)_{l=1\dots m}$.

For $i, j \in \mathbb{Z}^n$ we will denote by $\cdot_{ij}$ or $\cdot_{,ij}$ when $\cdot$ has already an index, the matrix element of an (vector) operator in the basis $\{e_j,\ e_j(x) = e^{ij.x}/(2\pi)^{\frac{l}{2}}, \theta \in \mathbb{T}^l\}$, namely

$$V_{ij} = (V_{l,ij})_{l=1\dots m} = ((e_i, V_l e_j)_{L^2(\mathbb{T}^n)})_{l=1\dots m},$$

and by $\overline{V}$ the diagonal part of V:

$$\overline{V}_{ij} = V_{ii}\delta_{ij},$$

together with

$$\overline{\mathcal{V}} = (2\pi)^{-l}\int_{\mathbb{T}^l}\mathcal{V}dx.$$

We will denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^m$ (or $\mathbb{C}^m$), $|Z|^2 = \sum_{i=1}^{m}|Z_i|^2$, and by $\|\cdot\|_{L^2(\mathbb{T}^l)\to L^2(\mathbb{T}^l)}$ the operator norm on the Hilbert space $L^2(\mathbb{T}^l)$.

Finally for $\omega = (\omega_i \in \mathbb{R}^l)_{i=1\dots m}$ and $\xi \in \mathbb{R}^l,\ p \in \mathbb{R}^m,\ q \in \mathbb{Z}^l$ we will denote

$$\omega \cdot \xi = (< \omega_i, \xi >_{\mathbb{R}^l})_{i=1\dots m} \in \mathbb{R}^m, \tag{1.8}$$

$$p.\omega = \left(\sum_{i=1}^{m} p_i\omega_i^j\right)_{j=1\dots l} \in \mathbb{R}^l \tag{1.9}$$

and

$$p.\omega.q = \sum_{i=1}^{m}\sum_{j=1}^{l} p_i\omega_i^j q_j = \langle p\cdot\omega, q\rangle_{\mathbb{R}^l}. \tag{1.10}$$

2. Strategy of the proofs

The general idea in proving Theorem 1 will be to construct a Newton-type iteration procedure consisting in constructing a family of unitary operators U_r such that (norms will be defined later)

$$U_r^{-1}(\mathcal{B}_r^\hbar(L_\omega) + V_r)U_r = \mathcal{B}_{r+1}^\hbar(L_\omega) + V_{r+1}, \tag{2.1}$$

with $\|V_{r+1}\|_{r+1} \leqslant D_{r+1}\|V_r\|_r^2$ and $\mathcal{B}_0^\hbar(L_\omega) = L_\omega,\ V_0 = V$.

U_r will be chosen of the form

$$U_r = e^{i\frac{W_r}{\hbar}},\ W_r \text{ self-adjoint}. \tag{2.2}$$

It is easy to realize that (2.2) implies (2.1) if W_r satisfies the (approximate) cohomological equation

$$\frac{1}{i\hbar}[\mathcal{B}_r^\hbar(L_\omega), W_r] + V_r = \mathcal{D}_{r+1}(L_\omega) + O(\|V_r\|_r^2), \tag{2.3}$$

or equivalently

$$\frac{1}{i\hbar}[\mathcal{B}_r^\hbar(L_\omega), W_r] + V_r^{co} = \mathcal{D}_{r+1}(L_\omega) + O(\|V_r\|_r^2), \tag{2.4}$$

for any V_r^{co} such that $\|V_r^{co} - V_r\|_r = O(\|V_r\|_r^2)$.

We will solve for each r the equation (2.4) where V_r^{co} will be obtained by a suitable "cut-off" in order to have to solve (2.4) with only small denominators of finite order (see Brjuno condition (1.3)).

In fact we will see in Section 3 that we can find a (scalar) solution of the (vector) equation (2.4) satisfying

$$\frac{1}{i\hbar}[\mathcal{B}_r^\hbar(L_\omega), W_r] + V_r^{co} = \mathcal{B}_{r+1}^\hbar(L_\omega) + R_r, \tag{2.5}$$

where $\|R_r\|_{k+1} = O(\|V_r\|_r^2)$. To do this we will remark that since the components of $\mathcal{B}_r^\hbar(L_\omega)+ V_r$ commute with each other (since the ones of $L_\omega + V$ do) we have that

$$[(\mathcal{B}_r^\hbar(L_\omega))_l, (V_r)_{l'}] - [(\mathcal{B}_r^\hbar(L_\omega))_{l'}, (V_r)_l] = [(V_r)_{l'}, (V_r)_l] = O(V_r^2) \tag{2.6}$$

which is an almost compatibility condition (see Section 3 for details).

Summarizing, the solution W_r of (2.4) will provide a unitary operator U_r such that (2.1) will hold with $\mathcal{B}_{r+1}^\hbar = \mathcal{B}_r^\hbar + \mathcal{D}_{r+1}$ and V_{k+1} being the sum of three terms:

- $V_{r+1}^1 = U_r^{-1}(\mathcal{B}_r^\hbar(L_\omega) + V_r)U_r - (\mathcal{B}_r^\hbar(L_\omega) + V_r) - \frac{1}{i\hbar}[\mathcal{B}_r^\hbar(L_\omega), W_r]$
- $V_{r+1}^2 = V_r - V_r^{co}$
- $V_{r+1}^3 = R_r$

The choice of the family of norms $\|\cdot\|_r$ will be made in order to have that

$$\|V_{r+1}\|_{r+1} = \|V_{r+1}^1 + V_{r+1}^2 + V_{r+1}^3\|_{r+1} \leqslant D_{r+1}\|V_r\|_r^2$$

with D_r satisfying

$$\prod_{r=1}^{R} D_r^{2^{R-r}} \leqslant C^{2^R}.$$

Hence, we have

$$\|V_{R+1}\|_{R+1} \leqslant (C\|V_0\|_0)^{2^R},$$

so that $\|V_{R+1}\|_{R+1} \to 0$ as $R \to \infty$ if $\|V_0 = V\|_0 < C^{-1}$ and $\|\cdot\|_\infty$ exists.

3. The cohomological equation: the formal construction

In this section we want to show how it is possible to construct the solution of the equation

$$\frac{1}{i\hbar}[\mathcal{B}^\hbar(L_\omega), W] + V = \mathcal{D}(L_\omega) + O(V^2), \tag{3.1}$$

where we denote by L_ω, $\omega = (\omega_i \in \mathbb{R}^l)_{i=1\dots m}$, the operator valued vector of components (with a slight abuse of notation) $L_{\omega_i} = -i\hbar\omega_i.\nabla_x$, $i = 1\dots m$ on $L^2(\mathbb{T}^l)$ and V is a "cut-off"ed.

$$V_{ij} = 0 \text{ for } |i-j| > M.$$

We will present the strategy only in the case of the Brjuno condition, the Diophantine case being very close.

Let us recall also that equation (3.1) is in fact a system of m equations and that it might seem surprising at the first glance that the same W solves (3.1) for all $\ell = 1\dots m$.

3.1. **First order.** At the first order the cohomological equation is

$$\frac{[L_{\omega_\ell}, W]}{i\hbar} + V_\ell = \mathcal{D}_\ell(L_\omega), \quad l = 1 \dots m \tag{3.2}$$

solved on the eigenbasis of any L_{ω_ℓ} by $\mathcal{D}_\ell(L_\omega) = diag(V_\ell)$ and

$$W_{ij} = -\frac{(V_\ell - \mathcal{D}_\ell)_{ij}}{i\omega_\ell \cdot (i-j)}. \tag{3.3}$$

In (3.3) we will picked up, for every ij such that $|i-j| \leqslant M$, an index $\ell = \ell_{i-j}$ which minimize the quantity

$$|\langle \omega_{\ell_q}, q\rangle|^{-1} := \min_{1\leqslant i\leqslant m} |\langle \omega_i, q\rangle|^{-1} \leqslant \mathcal{M}_M. \tag{3.4}$$

We define W by

$$W_{ij} = -\frac{(V_{\ell_{i-j}})_{ij}}{i\omega_{\ell_{i-j}} \cdot (i-j)}, \quad i-j \neq 0 \tag{3.5}$$

Since $[H_\ell, H_{\ell'}] = 0$, then we have that $[L_{\ell'}V_\ell] + [V_{\ell'}, L_\ell] = -[V_\ell, V_{\ell'}]$. Therefore, evaluating the operators on e_j and taking the scalar product with e_i, leads to

$$\omega_{\ell'} \cdot (i-j)(V_\ell)_{ij} = \omega_\ell \cdot (i-j)(V_{\ell'})_{ij} - ([V_\ell, V_{\ell'}])_{ij} \tag{3.6}$$

that is

$$\frac{(V_\ell)_{ij}}{\omega_\ell \cdot (i-j)} = \frac{(V_{\ell'})_{ij}}{\omega_{\ell'} \cdot (i-j)} - \frac{([V_\ell, V_{\ell'}])_{ij}}{\omega_\ell \cdot (i-j)\omega_{\ell'} \cdot (i-j)}$$

(note that when $\omega_{\ell'} \cdot (i-j) = 0$ on has $(V_{\ell'})_{ij} = \frac{-([V_{\ell_{i-j}}, V_{\ell'}])_{ij}}{\omega_{\ell_{i-j}} \cdot (i-j)}$).

Let us remark that, though $[V_\ell, V_{\ell'}]$ is quadratic in V, it has the same cut-off property as V, namely $([V_\ell, V_{\ell'}])_{ij} = 0$ if $|i-j| > M$ as seen clearly by (3.6).

This means that W defined by (3.5) satisfies

$$\frac{[L_\omega, W]}{i\hbar} + V = \mathcal{D}(L_\omega) + \widehat{V},$$

where

$$(\widehat{V}_\ell)_{ij} = \frac{([V_\ell, V_{\ell_{i-j}}])_{ij}}{i\hbar\omega_{\ell_{i-j}} \cdot (i-j)}. \tag{3.7}$$

Note that this construction is different from the one used in [LS1].

3.2. **Higher orders.** The cohomological equation at order r will follow the same way, at the exception that L_ω has to be replaced by $\mathcal{B}_r^\hbar(L_\omega)$.

The corresponding cohomological equation is therefore of the form

$$\frac{[\mathcal{B}_r^\hbar(L_\omega), W_r]}{i\hbar} + V_r = O((V_r)^2), \tag{3.8}$$

equivalent to

$$\frac{\mathcal{B}^\hbar_r(\hbar\omega\cdot i)-\mathcal{B}^\hbar_r(\hbar\omega\cdot j)}{i\hbar}(W_r)_{ij}+(V_r)_{ij}=O((V_r)^2). \tag{3.9}$$

Lemma 4. *For $\mathcal{B}^\hbar_r$ close enough to the identity there exists a $m\times m$ matrix $A^r(i,j)$ such that*

$$\frac{\mathcal{B}^\hbar_r(\hbar\omega\cdot i)-\mathcal{B}^\hbar_r(\hbar\omega\cdot j)}{i\hbar}=\left(I+A^r(i,j)\right)\omega.(i-j), \tag{3.10}$$

where I is the $m\times m$ identity matrix and $\omega.(i-j)=(\omega_l.(i-j))_{l=1\dots m}$. Moreover

$$\|A^r(i,j)\|_{\mathbb{C}^m\to\mathbb{C}^m}\leqslant\|\nabla(\mathcal{B}^\hbar_r-\mathcal{B}^\hbar_0)\|_{(\mathbb{C}^m\to\mathbb{C}^m)\otimes L^\infty(\mathbb{R}^m)}\leqslant\max_{j=1\dots m}\sum_{i=1}^m\|\nabla_j(\mathcal{B}^\hbar_r-\mathcal{B}^\hbar_0)_i\|_{L^\infty(\mathbb{R}^m)}. \tag{3.11}$$

Proof. We have

$$\begin{aligned}\frac{\mathcal{B}^\hbar_r(\hbar\omega\cdot i)-\mathcal{B}^\hbar_r(\hbar\omega\cdot j)}{i\hbar} &= \omega.(i-j)+\int_0^1\partial_t\left[(\mathcal{B}^\hbar_r-\mathcal{B}^\hbar_0)(t\hbar\omega.i+(1-t)\hbar\omega.j)\right]\frac{dt}{\hbar}\\ &= \omega.(i-j)+\int_0^1\left[\nabla(\mathcal{B}^\hbar_r-\mathcal{B}^\hbar_0)(t\hbar\omega.i+(1-t)\hbar\omega.j)\right]\cdot[\omega.(i-j)]dt\end{aligned}$$

so $A^r(i,j)=\int_0^1\nabla(\mathcal{B}^\hbar_r-\mathcal{B}^\hbar_0)(t\hbar\omega.i+(1-t)\hbar\omega.j)dt$ and the first part of (3.11) follows. The second part is a standard estimate of the operator norm. □

Plugging (7.5) in (3.9) we get that W must solve

$$\omega.(i-j)W_{ij}=\left(I+A^r(i,j)\right)^{-1}\left[-(V_r)_{ij}+O((V_r)^2)\right], \tag{3.12}$$

and we are reduced to the first order case with $V_r\to\widetilde{V}^r$ where

$$\widetilde{V}^r_{ij}:=\left(I+A^r(i,j)\right)^{-1}(V_r)_{ij}. \tag{3.13}$$

3.3. **Toward estimating.** We will first have to estimate $\widetilde{V}^r$: this will be done out of its matrix coefficients given by (3.13) by the method developed in Section 5.1. We will estimate $\left(I+A^r(i,j)\right)^{-1}\widetilde{V}^r$ in section 6 by using the formula $\left(I+A^r(i,j)\right)^{-1}=\sum\limits_{k=0}^\infty(-A^r(i,j))^k$ and a bound of the norm of $(-A^r(i,j))^k\widetilde{V}^r$ of the form $|C|^k$ times the norm of $\widetilde{V}^r$ leading to a bound of $\left(I+A^r(i,j)\right)^{-1}\widetilde{V}^r$ of the form $\frac{1}{1-|C|}$ times the norm of $\widetilde{V}^r$, by summation of the geometric series $\sum\limits_{k=0}^\infty C^k$, possible at the condition that $|C|<1$.

We will then have to estimate W defined through

$$W_{ij}=-\frac{(\widetilde{V}^r_{\ell_{i-j}})_{ij}}{i\omega_{\ell_{i-j}}\cdot(i-j)},\quad i-j\neq 0 \tag{3.14}$$

with again $(\widetilde{V}^r_{\ell_{i-j}})_{ij}=0$ for $|i-j|>M$. We get

$$|W_{ij}|\leqslant\mathcal{M}_M|(\widetilde{V}_{\ell_{i-j}})_{ij}|,$$

and we will get an estimate of W, $\|W\|\leqslant\mathcal{M}_M\|\widetilde{V}^r\|$, for a norm $\|\cdot\|$ to be specified later.

Finally we will have to estimate

$$(\widehat{V}_l^r)_{ij} = \frac{([\widetilde{V}_l^r, \widetilde{V}_{\ell_{i-j}}^r])_{ij}}{i\hbar\omega_{\ell_{i-j}} \cdot (i-j)}. \tag{3.15}$$

We will get immediately $\|\widehat{V}_l^r\| \leqslant \mathcal{M}_M\|P\|$, $P_{ij} = \frac{([\widetilde{V}_l^r, \widetilde{V}_{\ell_{i-j}}^r])_{ij}}{i\hbar}$ and the estimate of the commutator will be done by the method developed in Section 5.

In the two next sections we will define the norms and the Weyl quantization procedure used in order to precise the results of this section,

4. Norms

Let m, l be positive integers. For $\mathcal{F} \in C^\infty(\mathbb{R}^m \times \mathbb{T}^l \times [0,1]; \mathbb{C})$ we will use the following normalization for the Fourier transform.

Definition 5 (Fourier transforms)**.** Let $p \in \mathbb{R}^m$ and $q \in \mathbb{Z}^l$

$$\widehat{\mathcal{F}}(p,x,\hbar) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} \mathcal{F}(\xi,x,\hbar)e^{-i\langle p,\xi\rangle}\,d\xi \tag{4.1}$$

$$\widetilde{\mathcal{F}}(\xi,q;\hbar) = \frac{1}{(2\pi)^l}\int_{\mathbb{T}^l} \mathcal{F}(\xi,x;\hbar)e^{-i\langle q,x\rangle}\,dx \tag{4.2}$$

$$\widehat{\widetilde{\mathcal{F}}}(p,q,\hbar) = \frac{1}{(2\pi)^{m+l}}\int_{\mathbb{R}^m\times\mathbb{T}^l} \mathcal{F}(\xi,x,\hbar)e^{-i\langle p,\xi\rangle - i\langle q,x\rangle}d\xi dx \tag{4.3}$$

$$= \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} \widetilde{\mathcal{F}}(\xi,q,\hbar)e^{-i\langle p,\xi\rangle}d\xi \tag{4.4}$$

$$= \frac{1}{(2\pi)^l}\int_{\mathbb{T}^l} \widehat{\mathcal{F}}(p,x,\hbar)e^{-i\langle q,x\rangle}dx \tag{4.5}$$

Note that

$$\mathcal{F}(\xi,x,\hbar) = \int_{\mathbb{R}^m} \widehat{\mathcal{F}}(p,x,\hbar)e^{i\langle p,\xi\rangle}\,dp \tag{4.6}$$

$$= \sum_{q\in\mathbb{Z}^l} \widetilde{\mathcal{F}}(\xi,q;\hbar)e^{i\langle q,x\rangle} \tag{4.7}$$

$$= \sum_{q\in\mathbb{Z}^l}\int_{\mathbb{R}^m} \widehat{\widetilde{\mathcal{F}}}(p,q,\hbar)e^{i\langle p,\xi\rangle + i\langle q,x\rangle}\,dp \tag{4.8}$$

Set now for $k \in \mathbb{N} \cup \{0\}$ and $p \cdot \omega = (\sum_{j=1\dots m} p_j.\omega_j^i)_{i=1\dots l}$:

$$\mu_k(p,q) := (1 + |p\cdot\omega|^2 + |q|^2)^{\frac{k}{2}} \tag{4.9}$$

(note that $\mu_r(p-p', q-q') \leqslant 2^{\frac{k}{2}}\mu_r(p,q)\mu_r(p',q')$ because $|x-x'|^2 \leqslant 2(|x|^2 + |x'|^2)$ and that $|p\cdot\omega| \to \infty$ as $|p| \to \infty$ because the vectors $(\omega^i)_{1=1\dots l}$ are independent over $\mathbb{R}$).

Definition 6 (Norms I)**.** *For $\rho > 0$, $\mathcal{F} \in C^\infty(\mathbb{R}^m \times \mathbb{T}^l \times [0,1]; \mathbb{C})$ we introduce the weighted norms*

$$\|\mathcal{F}\|^{\dagger}_{\rho} = \|\mathcal{F}\|^{\dagger}_{\rho,\underline{\omega}} := \max_{\hbar\in[0,1]} \int_{\mathbb{R}^m} \sum_{q\in\mathbb{Z}^l} |\widehat{\widehat{\mathcal{F}}}(p,q,\hbar)|\, e^{\rho(\underline{\omega}|p|+|q|)}\, dp. \tag{4.10}$$

$$\|\mathcal{F}\|^{\dagger}_{\rho,\underline{\omega},k} = \|\mathcal{F}\|^{\dagger}_{\rho,\underline{\omega},k} := \max_{\hbar\in[0,1]} \sum_{j=0}^{k} \int_{\mathbb{R}^m} \sum_{q\in\mathbb{Z}^l} \mu_{k-j}(p,q)\partial^j_{\hbar}|\widehat{\widehat{\mathcal{F}}}(p,q,\hbar)|\, e^{\rho(\underline{\omega}|p|+|q|)}\, dp. \tag{4.11}$$

Note that $\underline{\omega}$ is given by (1.7) and $\|\cdot\|^{\dagger}_{\rho;0} = \|\cdot\|^{\dagger}_{\sigma}$.

Definition 7 (Norms II)**.** *Let $\mathcal{O}_\omega$ be the set of functions $\mathcal{F}:\mathbb{R}^l\times\mathbb{T}^l\times[0,1]\to\mathbb{C}$ such that $\mathcal{F}(\xi,x;\hbar) = \mathcal{F}'(\omega\cdot\xi,x,\hbar)$ for some $\mathcal{F}':\ \mathbb{R}^m\times\mathbb{T}^l\times[0,1]\to\mathbb{C}$. Define, for $\mathcal{F}\in\mathcal{O}_\omega$:*

$$\|\mathcal{F}\|_{\rho,\underline{\omega},k} := \|\mathcal{F}'\|^{\dagger}_{\rho,\underline{\omega},k}. \tag{4.12}$$

We will also need the following definition for $\mathcal{F}\in\mathcal{O}_\omega$:

$$\|\mathcal{F}\|^{\hbar}_{\rho,\underline{\omega},k} := \sum_{j=0}^{k} \int_{\mathbb{R}^m} \sum_{q\in\mathbb{Z}^l} \mu_{k-j}(p,q)\partial^j_{\hbar}|\widehat{\widehat{\mathcal{F}'}}(p,q,\hbar)|\, e^{\rho(\underline{\omega}|p|+|q|)}\, dp. \tag{4.13}$$

Let us note that, obviously, $\|\cdot\|^{\hbar}_{\rho,\underline{\omega},k} \leqslant \|\cdot\|_{\rho,\underline{\omega},k}$.

We will need an extension of the previous definition to the vector case. Consider now $\mathcal{F}\in C^\infty(\mathbb{R}^m\times\mathbb{T}^l\times[0,1];\mathbb{C}^m)$ and $\mathcal{G}\in C^\infty(\mathbb{R}^m\times[0,1];\mathbb{C}^m)$. The definition of the Fourier transform is defined as usual, component by component.

Definition 8. [Norms III] *Let $\mathcal{F} = (\mathcal{F}_i)_{i=1\dots m}\in C^\infty(\mathbb{R}^m\times\mathbb{T}^l\times[0,1];\mathbb{C}^m)$. We define*

(1)
$$\|\mathcal{F}\|^{\dagger}_{\rho,\underline{\omega},k} = \sum_{i=1}^{m} \|\mathcal{F}_i\|^{\dagger}_{\rho,\underline{\omega},k} \tag{4.14}$$

(2) *Let*
$$\mathcal{O}^m_\omega = \left\{\mathcal{F} = (\mathcal{F}_i)_{i=1\dots m} : \mathbb{R}^m\times\mathbb{T}^l\times[0,1]\to\mathbb{C}^m/\ \mathcal{F}_i\in\mathcal{O}_\omega, i=1\dots m\right\} \tag{4.15}$$

Let $\mathcal{F}\in\mathcal{O}^m_\omega$. We define:

$$\|\mathcal{F}\|_{\rho,\underline{\omega},k} = \sum_{i=1}^{m} \|\mathcal{F}_i\|_{\rho,\underline{\omega},k} \tag{4.16}$$

Let
$$\mathcal{O}^{m\times m}_\omega = \left\{\mathcal{F} = (\mathcal{F}_{ij})_{i,j=1\dots m} : \mathbb{R}^m\times\mathbb{T}^l\times[0,1]\to\mathbb{C}^m/\ \mathcal{F}_{ij}\in\mathcal{O}_\omega, i,j=1\dots m\right\} \tag{4.17}$$

Let $\mathcal{F}\in\mathcal{O}^{m\times m}_\omega$. We define:

$$\|\mathcal{F}\|_{\rho,\underline{\omega},k} = \sup_{i=1\dots m} \sum_{j=1\dots m} \|\mathcal{F}_{ij}\|_{\rho,\underline{\omega},k}. \tag{4.18}$$

(3) *Finally we denote F the Weyl quantization of $\mathcal{F}$ recalled in Section 5 and*

$$\|F\|_{\rho,\underline{\omega},k} = \|\mathcal{F}\|_{\rho,\underline{\omega},k} \tag{4.19}$$

$$\begin{aligned}
\mathcal{J}_k^{\dagger}(\rho,\underline{\omega}) &= \{\mathcal{F} \,|\, \|\mathcal{F}\|^{\dagger}_{\rho,\underline{\omega},k} < \infty\}, &(4.20)\\
J_k^{\dagger}(\rho,\underline{\omega}) &= \{F \,|\, \mathcal{F} \in \mathcal{J}_k^{\dagger}(\rho,\underline{\omega})\}, &(4.21)\\
\mathcal{J}_k(\rho,\underline{\omega}) &= \{\mathcal{F} \in \mathcal{O}_\omega \,|\, \|\mathcal{F}\|_{\rho,\underline{\omega},k} < \infty\}, &(4.22)\\
J_k(\rho,\underline{\omega}) &= \{F \,|\, \mathcal{F} \in \mathcal{J}_k(\rho,\underline{\omega})\}. &(4.23)\\
\mathcal{J}_k^{\hbar}(\rho,\underline{\omega}) &= \{\mathcal{F} \in \mathcal{O}_\omega \,|\, \|\mathcal{F}\|^{\hbar}_{\rho,\underline{\omega},k} < \infty\}, &(4.24)\\
J_k^{\hbar}(\rho,\underline{\omega}) &= \{F \,|\, \mathcal{F} \in \mathcal{J}_k^{\hbar}(\rho,\underline{\omega})\}. &(4.25)\\
\mathcal{J}_k^{m}(\rho,\underline{\omega}) &= \{\mathcal{F} \in \mathcal{O}_\omega^m \,|\, \|\mathcal{F}\|_{\rho,\underline{\omega},k} < \infty\}, &(4.26)\\
J_k^{m}(\rho,\underline{\omega}) &= \{F \,|\, \mathcal{F} \in \mathcal{J}_k^{m}(\rho,\underline{\omega})\}. &(4.27)\\
\mathcal{J}_k^{m\times m}(\rho,\underline{\omega}) &= \{\mathcal{F} \in \mathcal{O}_\omega^{m\times m} \,|\, \|\mathcal{F}\|_{\rho,\underline{\omega},k} < \infty\}, &(4.28)\\
J_k^{m\times m}(\rho,\underline{\omega}) &= \{F \,|\, \mathcal{F} \in \mathcal{J}_k^{m\times m}(\rho,\underline{\omega})\} &(4.29)
\end{aligned}$$

and $\mathcal{J}^{@}(\rho,\underline{\omega}) = \mathcal{J}^{@}_{k=0}(\rho,\underline{\omega}),\ J^{@}(\rho,\underline{\omega}) = J^{@}_{k=0}(\rho,\underline{\omega})\ \forall @ \in \{\dagger, m, m\times m\}$.
When there will be no confusion we will forget about the subscript $\underline{\omega}$ in the label of the norms and also denote by $\mathcal{J}_k^{@}(\rho) = \mathcal{J}_k^{@}(\rho,\underline{\omega})$.

5. Weyl quantization and first estimates

We express the definitions and results of this section in case of scalar operators and symbols. The extension to the vector case is trivial component by component. The reader only interested by explicit expression can skip the beginning of the next paragraph and go directly to Definition 5.4.

5.1. **Weyl quantization, matrix elements and first estimates.** In this section we recall briefly the definition of the Weyl quantization of $T^*\mathbb{T}^l$. The reader is referred to [GP] for more details (see also e.g. [Fo]).

Let us recall that the Heisenberg group over $T^*\mathbb{T}^l \times \mathbb{R}$, denoted by $\mathbb{H}_l(\mathbb{R}^l \times \mathbb{Z}^l \times \mathbb{R})$, is (the subgroup of the standard Heisenberg group $\mathbb{H}_l(\mathbb{R}^l \times \mathbb{R}^l \times \mathbb{R})$) topologically equivalent to $\mathbb{R}^l \times \mathbb{Z}^l \times \mathbb{R}$ with group law $(u,t)\cdot(v,s) = (u+v, t+s+\frac{1}{2}\Omega(u,v))$. Here $u := (p,q), p \in \mathbb{R}^l, q \in \mathbb{Z}^l$, $t \in \mathbb{R}$ and $\Omega(u,v)$ is the canonical 2−form on $\mathbb{R}^l \times \mathbb{Z}^l$: $\Omega(u,v) := \langle u_1, v_2\rangle - \langle v_1, u_2\rangle$.

The unitary representations of $\mathbb{H}_l(\mathbb{R}^l \times \mathbb{Z}^l \times \mathbb{R})$ in $L^2(\mathbb{T}^l)$ are defined for any $\hbar \neq 0$ as follows

$$(U_\hbar(p,q,t)f)(x) := e^{i\hbar t + i\langle q,x\rangle + \hbar\langle p.q\rangle/2} f(x+\hbar p) \tag{5.1}$$

Consider now a family of smooth phase-space functions indexed by $\hbar$, $\mathcal{A}(\xi,x,\hbar) : \mathbb{R}^l \times \mathbb{T}^l \times [0,1] \to \mathbb{C}$, written under its Fourier representation

$$\mathcal{A}(\xi,x,\hbar) = \int_{\mathbb{R}^l} \sum_{q\in\mathbb{Z}^l} \widehat{\widetilde{\mathcal{A}}}(p,q;\hbar) e^{i(\langle p.\xi\rangle + \langle q,x\rangle)}\, dp \tag{5.2}$$

Definition 9 (Weyl quantization I). *By analogy with the usual Weyl quantization on $T^*\mathbb{R}^l$[Fo], the (Weyl) quantization of $\mathcal{A}$ is the operator $A(\hbar)$ defined as*

$$A(\hbar) := (2\pi)^l \int_{\mathbb{R}^l} \sum_{q\in\mathbb{Z}^l} \widehat{\widetilde{\mathcal{A}}}(p,q;\hbar) U_\hbar(p,q,0)\,dp \tag{5.3}$$

(note that the factor $(2\pi)^l$ in (5.3) is due to the (convenient for us) normalization of the Fourier transform in Definition 5).

It is a straightforward computation to show that, considering $f \in L^2(\mathbb{T}^l)$ and $\mathcal{V}((x+\cdot)/2)$ as periodic functions on $\mathbb{R}^l$, we get the equivalent definition

Definition 10 (Weyl quantization II).

$$(A(\hbar)f)(x) := \int_{\mathbb{R}^l_\xi\times\mathbb{R}^l_y} \mathcal{A}((x+y)/2,\xi,\hbar) e^{i\frac{\xi(x-y)}{\hbar}} f(y) \frac{d\xi dy}{(2\pi\hbar)^l} \tag{5.4}$$

Remark 11. The expression (10) is exactly the same as the definition of Weyl quantization on $T^*\mathbb{R}^l$ except the fact that f is periodic. Note that $A(\hbar)f$ is periodic thanks to the fact that $\mathcal{A}(x,\xi,\hbar)$ is periodic:
$\int \mathcal{A}((x+2\pi+y)/2,\xi)e^{i\frac{\xi(x+2\pi-y)}{\hbar}} f(y)\frac{d\xi dy}{\hbar^l} = \int \mathcal{A}((x+2\pi+y+2\pi)/2,\xi)e^{i\frac{\xi(x-y)}{\hbar}} f(y+2\pi)\frac{d\xi dy}{\hbar^l} = \int \mathcal{A}((x+y)/2+2\pi,\xi)e^{i\frac{\xi(x-y)}{\hbar}} f(y)\frac{d\xi dy}{\hbar^l} = \int \mathcal{A}((x+y)/2,\xi)e^{i\frac{\xi(x-y)}{\hbar}} f(y)\frac{d\xi dy}{\hbar^l} = (A(\hbar))f(x)$.

The first results concerning this definition are contained in the following Proposition.

Proposition 12. *Let $A(\hbar)$ be defined by the expression* (5.4)*. Then:*

(1) $\forall \rho > 0, \forall\, k \geqslant 0$ *we have:*

$$\|A(\hbar)\|_{L^2(\mathbb{T}^l)\to L^2(\mathbb{T}^l)} \leqslant \|\mathcal{A}\|_{\rho,k} \tag{5.5}$$

and, if $\mathcal{A}(\xi,x,\hbar) = \mathcal{A}'(\omega\cdot\xi,x;\hbar)$

$$\|A(\hbar)\|_{L^2(\mathbb{T}^l)\to L^2(\mathbb{T}^l)} \leqslant \|\mathcal{A}'\|_{\rho,k}. \tag{5.6}$$

(2) *Let, for $n\in\mathbb{Z}^l$, $e_n(x) = \frac{e^{inx}}{(2\pi)^l}$. Then for all m,n in $\mathbb{Z}^l$,*

$$\langle e_m, A(\hbar)e_n\rangle_{L^2(\mathbb{T}^l)} = \widetilde{A}((m+n)\hbar/2, m-n,\hbar) \tag{5.7}$$

(3) *Reciprocally, let $A(\hbar)$ be an operator whose matrix elements satisfy* (5.7) *for some $\mathcal{A}$ belonging to $\mathcal{J}^@$, $@\in\ \{\dagger, m, m\times m\}$. Then $A(\hbar)$ is the Weyl quantization of $\mathcal{A}$.*

Proof. (5.7) is obtained by a simple computation. It also implies that

$$\|A(\hbar)e_m\|^2_{L^2(\mathbb{R}^l)} = \sum_{q\in\mathbb{Z}^l} |\widetilde{\mathcal{A}}(\hbar(m+q)/2, m-q,\hbar)|^2 \leqslant \sup_{\xi\in\mathbb{R}^l}\sum_{q\in\mathbb{Z}^l} |\widetilde{\mathcal{A}}(\xi,q,\hbar)|^2.$$

So that

$$\|A(\hbar)\sum_{\mathbb{Z}^l} c_m e_m\|^2_{L^2(\mathbb{R}^l)} \leqslant \sum_{\mathbb{Z}^l}|c_m|^2 \sup_{\xi\in\mathbb{R}^l}\sum_{q\in\mathbb{Z}^l}|\widetilde{\mathcal{A}}(\xi,q,\hbar)|^2 \leqslant (\sum_{\mathbb{Z}^l}|c_m|^2)\left(\sum_{q\in\mathbb{Z}^l}\sup_{\xi\in\mathbb{R}^l}|\widetilde{\mathcal{A}}(\xi,q,\hbar)|\right)^2.$$

And therefore, since by (4.6)-(4.7)-(4.8) $\widetilde{\mathcal{A}}(\xi,q,\hbar)=\int_{\mathbb{R}^l}\widehat{\widetilde{\mathcal{A}}}(p,q,\hbar)e^{i<\xi,p>}dp$ so that $|\widetilde{\mathcal{A}}(\xi,q,\hbar)| \leqslant \int_{\mathbb{R}^l}|\widehat{\widetilde{\mathcal{A}}}(p,q,\hbar)|dp$,

$$\|A(\hbar)\|_{L^2(\mathbb{T}^l)\to L^2(\mathbb{T}^l)} \leqslant \sum_{q\in\mathbb{Z}^l}\sup_{\xi\in\mathbb{R}^l}|\widetilde{\mathcal{A}}(\xi,q,\hbar)| \leqslant \int_{\mathbb{R}^l}\sum_{q\in\mathbb{Z}^l}|\widehat{\widetilde{\mathcal{A}}}(p,q,\hbar)|dp \leqslant \|\mathcal{A}\|_{\rho,k}, \forall\rho>0, k\geqslant 0. \tag{5.8}$$

In the case $\mathcal{A}(\xi,x,\hbar)=\mathcal{A}'(\omega\cdot\xi,x;\hbar)$ we get, $\forall\rho>0, k\geqslant 0$:

$$\|A(\hbar)\|_{L^2(\mathbb{T}^l)\to L^2(\mathbb{T}^l)} \leqslant \sum_{q\in\mathbb{Z}^l}\sup_{\xi\in\mathbb{R}^l}|\widetilde{\mathcal{A}}(\xi,q,\hbar)| = \sum_{q\in\mathbb{Z}^l}\sup_{Y\in\mathbb{R}^m}|\widetilde{\mathcal{A}}'(Y,q,\hbar)| \leqslant \int_{\mathbb{R}^m}\sum_{q\in\mathbb{Z}^l}|\widehat{\mathcal{A}}'(p,q,\hbar)|dp \leqslant \|\mathcal{A}'\|_{\rho,k}.$$

(3) is obvious. □

5.2. **Fundamental estimates.** This section contains the fundamental estimates which will be the blocks of the estimates needed in the proofs of our main results. These primary estimates are contained in the following Proposition.

Proposition 13. *We have:*

(1) *For $F,G\in J_k^1(\rho)$, $FG\in J_k^1(\rho)$ and fulfills the estimate*

$$\|FG\|_{\rho,k}\leqslant (k+1)8^k\|F\|_{\rho,k}\cdot\|G\|_{\rho,k} \tag{5.9}$$

(2) *There exists a positive constant C' such that for $F\in J_k^m(\rho)$ and for $G\in J_k^1(\rho)$, we have, $\forall\delta_1>0,\delta\geqslant 0,\rho>\delta+\delta_1$,*

$$\left\|\frac{[F,G]}{i\hbar}\right\|_{\rho-\delta-\delta_1,k}\leqslant\frac{2(k+1)8^k}{e^2\delta_1(\delta+\delta_1)}\|F\|_{\rho,k}\|G\|_{\rho-\delta,k}, \tag{5.10}$$

$$\frac{1}{d!}\|\underbrace{[G,\dots[G}_{d\ times},F]\cdots]/(i\hbar)^d\|_{\rho-\delta,k}\leqslant\frac{1}{2\pi}\left(\frac{2(1+k)8^k}{\delta^2}\right)^d\|F\|_{\rho,k}\|G\|^d_{\rho,k}, \tag{5.11}$$

and

$$\left\|\frac{[L_\omega,G]}{i\hbar}\right\|_{\rho-\delta}\leqslant\frac{\omega}{e\delta}\|G\|_{\rho,k} \tag{5.12}$$

(3) *For $\mathcal{F},\mathcal{G}\in\mathcal{J}_k^1(\rho)$, $\mathcal{F}\mathcal{G}\in\mathcal{J}_k^1(\rho)$ and*

$$\|\mathcal{F}\mathcal{G}\|_{\rho,k}\leqslant (k+1)4^k\|\mathcal{F}\|_{\rho,k}\cdot\|\mathcal{G}\|_{\rho,k}. \tag{5.13}$$

(4) *Let $V=(V_l)_{l=1\dots m}\in J_k^m(\rho)$ and let W be defined by $\langle e_m,We_n\rangle=\frac{\langle e_m,V_{l_{m-n}}e_n\rangle}{\omega_{\ell_{m-n}}\cdot(m-n)}$ where, $\forall m,n\in\mathbb{Z}$, $\omega_{\ell_{m-n}}$ is such that $|\langle\omega_{\ell_{m-n}},m-n\rangle|^{-1}:=\min_{1\leqslant i\leqslant m}|\langle\omega_i,m-n\rangle|^{-1}$. Then*

$$\|W\|_{\rho-d,k}\leqslant\gamma\frac{\tau^\tau}{(ed)^\tau}\|V\|_{\rho,k} \tag{5.14}$$

in the Diophantine case and (obviously)

$$\|W\|_{\rho,k}\leqslant\mathcal{M}_M\|V\|_{\rho,k} \tag{5.15}$$

in the case of the Brjuno condition.

(5) *Let finally* $V=(V_l)_{l=1\dots m}\in J_k^m(\rho)$ *and let* P *be defined by* $(P_l)_{ij}=\frac{([V_l,V_{\ell_{i-j}}])_{ij}}{i\hbar}$ *for any choice of* $(i,j)\to\ell_{i-j}$*. Then* $P=(P_l)_{l=1\dots m}\in J_k^m(\rho-\delta),\forall\delta_1\geqslant 0,\delta>0,\rho>\delta+\delta_1$ *and*

$$\|P\|_{\rho-\delta-\delta_1,k}\leqslant\frac{2(k+1)8^k}{e^2\delta_1(\delta+\delta_1)}\|V\|_{\rho,k}\|V\|_{\rho-\delta,k} \tag{5.16}$$

(6) *Moreover let* $\mathcal{F}:\xi\in\mathbb{R}^m\mapsto\mathcal{F}(\xi)\in\mathbb{R}^m$ *be in* $\mathcal{J}_k^m(\rho)$*. Let us define* $\nabla\mathcal{F}$ *the matrix* $((\nabla\mathcal{F})_{ij})_{i,j=1\dots m}$ *with*

$$(\nabla\mathcal{F})_{ij}:=\partial_{\xi_i}\mathcal{F}_j. \tag{5.17}$$

Then, for all $\delta>0$, $\nabla\mathcal{F}\in\mathcal{J}_k^{m\times m}(\rho-\delta)$ *and*

$$\|\nabla\mathcal{F}\|_{\rho-\delta,k}\leqslant\frac{1}{e\delta}\|\mathcal{F}\|_{\rho,k}. \tag{5.18}$$

Let us remark that, as the proof will show, Proposition 13 remains valid when the norm $\|\cdot\|_{\rho,k}$ is replaced by the norm $\|\cdot\|^{\hbar}_{\rho,k}$

Proof. Items **(1)** and **(2)** are simple extension to the multidimensional case of the corresponding results for $m=1$ proven in [GP]. For sake of completeness we give here an alternative proof in the case $m=1$. The proof will use the three elementary inequalities,

$$\begin{aligned}
\mu_k(p+p',q+q') &\leqslant 2^{\frac{k}{2}}\mu_k(p,q)\mu_k(p',q') && (5.19)\\
|(p.\omega.q'-p'\omega.q)/2|^k &\leqslant \mu_k(p,q)\mu_k(p',q') && (5.20)\\
\left|\partial_\hbar^k\frac{\sin x\hbar}{\hbar}\right| &\leqslant |x|^{k+1} && (5.21)\\
|p\cdot\omega\cdot q| &\leqslant \underline{\omega}\max_{j-1\dots m}|p_j||q|\leqslant\underline{\omega}|p||q| && (5.22)
\end{aligned}$$

where we have used the notation (1.10) and the definition (1.7).

(in order to prove (5.19), (5.20), (5.21) and (5.22) just use $|X+X'|^2\leqslant 2(|X|^2+|X'|^2)$ for all $X,X'\in\mathbb{R}^{2l}$, $|(p.\omega.q'-p'\omega.q)/2|^2\leqslant(|p\cdot\omega|^2+|q|^2)(|p\cdot\omega'|^2+|q'|^2)$, $\frac{\sin x\hbar}{\hbar}=\int_0^x\cos(s\hbar)ds$ and $|p\cdot\omega\cdot q|\leqslant\sum_{j=1}^m|p_j||\sum_{i=1}^l\omega_j^iq_i|=\sum_{j=1}^m|p_j||\omega_j\cdot q|\leqslant\sum_{j=1}^m|p_j||\omega_j||q|$ by Cauchy-Schwarz, respectively).

We start with (5.9). Since $F,G\in J^1(\rho)$ we know that there exist two functions $\mathcal{F}',\mathcal{G}'$ such that the symbols of F,G are $\mathcal{F}(\xi,x)=\mathcal{F}'(\omega.\xi,x)$, $\mathcal{G}(\xi,x)=\mathcal{G}'(\omega.\xi,x)$. By (5.7) we have that

$$
\begin{aligned}
(FG)_{mn} &= \sum_{q'\in\mathbb{Z}^l} F_{mq'}G_{q'n} \\
&= \sum_{q'\in\mathbb{Z}^l} \widetilde{\mathcal{F}}\left(\frac{m+q'}{2}\hbar, m-q'\right)\widetilde{\mathcal{G}}\left(\frac{q'+n}{2}\hbar, q'-n\right) \\
&= \sum_{q'\in\mathbb{Z}^l} \widetilde{\mathcal{F}}\left(\frac{m+n+q'}{2}\hbar, m-n-q'\right)\widetilde{\mathcal{G}}\left(\frac{q'+2n}{2}\hbar, q'\right) \\
&= \sum_{q'\in\mathbb{Z}^l} \widetilde{\mathcal{F}'}\left(\omega\cdot\frac{m+n+q'}{2}\hbar, m-n-q'\right)\widetilde{\mathcal{G}'}\left(\omega\cdot\frac{q'+2n}{2}\hbar, q'\right) \\
&= \sum_{q'\in\mathbb{Z}^l} \widetilde{\mathcal{F}'}\left(\omega\cdot\frac{m+n+q'}{2}\hbar, m-n-q'\right)\widetilde{\mathcal{G}'}\left(\omega\cdot\frac{q'+m+n-(m-n)}{2}\hbar, q'\right) \qquad (5.23)
\end{aligned}
$$

Calling $\mathcal{P}$ the symbol of FG we have that, by (5.7) again, $(FG)_{mn} = \widetilde{\mathcal{P}}(\xi, q)$ with $\xi = \frac{m+n}{2}\hbar$ and $q = m-n$. Therefore

$$
\widetilde{\mathcal{P}}(\xi, q) = \sum_{q'\in\mathbb{Z}^l} \widetilde{\mathcal{F}'}\left(\omega.\xi + \omega\cdot\frac{q'}{2}\hbar, q-q'\right)\widetilde{\mathcal{G}'}\left(\omega.\xi + \omega\cdot\frac{q'-q}{2}\hbar, q'\right), \qquad (5.24)
$$

so we see that $\mathcal{P}(\xi,\cdot)$ depends only on $\omega.\xi$: $\mathcal{P}(\xi, x) = \mathcal{P}'(\omega.\xi, x)$. Moreover, since by (4.3) $\widehat{\widetilde{\mathcal{P}'}}(p,\cdot) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\widetilde{\mathcal{P}'}(\Xi,\cdot)e^{-i<\Xi,p>}d\Xi$ we get easily by simple changes of integration variables and the fact that the Fourier transform of a product is a convolution,

$$
\widehat{\widetilde{\mathcal{P}'}}(p,q) = \int_{\mathbb{R}^m}\sum_{q'\in\mathbb{Z}^l}\left(\widehat{\widetilde{\mathcal{F}'}}(p-p', q-q')e^{i\frac{\hbar}{2}(p-p').\omega.q'}\right)\left(\widehat{\widetilde{\mathcal{G}'}}(p', q')e^{i\frac{\hbar}{2}p'.\omega.(q'-q))}\right)dp'. \qquad (5.25)
$$

Therefore $\|FG\|_{\rho,k}$ is equal to the maximum over $\hbar\in[0,1]$ of

$$
\sum_{\gamma=0}^{k}\int_{\mathbb{R}^{2m}}\sum_{(q,q')\in\mathbb{Z}^{2l}}\mu_{k-\gamma}(p,q)|\partial_\hbar^\gamma\left[\widehat{\widetilde{\mathcal{F}'}}(p-p', q-q')e^{i\frac{\hbar}{2}((p-p').\omega.q'-p'.\omega.(q-q'))}\widehat{\widetilde{\mathcal{G}'}}(p', q')\right]|e^{\rho(\underline{\omega}|p|+|q|)}dpdp'. \qquad (5.26)
$$

Writing, by (5.20), that

$$
\begin{aligned}
&|\partial_\hbar^\gamma\left[\widehat{\widetilde{\mathcal{F}'}}(p-p',q-q')e^{i\frac{\hbar}{2}((p-p').\omega.q'-p'.\omega.(q-q'))}\widehat{\widetilde{\mathcal{G}'}}(p',q')\right]| \\
\leqslant\ & \sum_{\mu=0}^{\gamma}\binom{\gamma}{\mu}\sum_{\nu=0}^{\gamma-\mu}\binom{\gamma-\mu}{\nu}|\partial_\hbar^{\gamma-\mu-\nu}\widehat{\widetilde{\mathcal{F}'}}(p-p',q-q')||\partial_\hbar^{\nu}e^{i\frac{\hbar}{2}((p-p').\omega.q'-p'.\omega.(q-q'))}||\partial_\hbar^{\mu}\widehat{\widetilde{\mathcal{G}'}}(p',q')| \\
\leqslant\ & \sum_{\mu=0}^{\gamma}\binom{\gamma}{\mu}\sum_{\nu=0}^{\gamma-\mu}\binom{\gamma-\mu}{\nu}|\partial_\hbar^{\gamma-\mu-\nu}\widehat{\widetilde{\mathcal{F}'}}(p-p',q-q')||((p-p').\omega.q'-p'.\omega.(q-q'))/2|^{\nu}|\partial_\hbar^{\mu}\widehat{\widetilde{\mathcal{G}'}}(p',q')| \\
=:\ & \mathbb{P}(\mathcal{F}',\mathcal{G}') \qquad (5.27)\\
\leqslant\ & \sum_{\mu=0}^{\gamma}\binom{\gamma}{\mu}\sum_{\nu=0}^{\gamma-\mu}\binom{\gamma-\mu}{\nu}\mu_\nu(p-p',q-q')\mu_\nu(p',q')|\partial_\hbar^{\gamma-\mu-\nu}\widehat{\widetilde{\mathcal{F}'}}(p-p',q-q')||\partial_\hbar^{\mu}\widehat{\widetilde{\mathcal{G}'}}(p',q')| \\
& (\text{changing } \mu\to\gamma',\ \nu\to\nu':=\gamma-\gamma'-\nu) \\
\leqslant\ & \sum_{\gamma'=0}^{\gamma}\binom{\gamma}{\gamma'}\sum_{\nu'=0}^{\gamma-\gamma'}\binom{\gamma-\gamma'}{\nu}\mu_{\gamma-\gamma'-\nu'}(p-p',q-q')\mu_{\gamma-\gamma'-\nu'}(p',q')|\partial_\hbar^{\nu'}\widehat{\widetilde{\mathcal{F}'}}(p-p',q-q')||\partial_\hbar^{\gamma'}\widehat{\widetilde{\mathcal{G}'}}(p',q')| \\
& (\text{since } \binom{m}{n}\leqslant 2^m, \gamma\leqslant k, \gamma-\gamma'\leqslant k) \\
\leqslant\ & \sum_{\gamma'=0}^{k}2^k\sum_{\nu'=0}^{k}2^k\mu_{\gamma-\gamma'-\nu'}(p-p',q-q')\mu_{\gamma-\gamma'-\nu'}(p',q')|\partial_\hbar^{\nu'}\widehat{\widetilde{\mathcal{F}'}}(p-p',q-q')||\partial_\hbar^{\gamma'}\widehat{\widetilde{\mathcal{G}'}}(p',q')| \\
& \qquad (5.28)
\end{aligned}
$$

using (5.19) under the form

$$\mu_k(p,q)\leqslant 2^{\frac{k}{2}}\mu_k(p-p',q-q')\mu_k(p',q')$$

together with the fact that $\mu_k(p,q)$ is increasing in k and $\mu_k\mu_{k'}=\mu_{k+k'}$.

We find that

$$
\begin{aligned}
&\mu_{k-\gamma}(p,q)\mathbb{P}(\mathcal{F}',\mathcal{G}') \\
\leqslant\ & 4^k2^{\frac{k}{2}}\sum_{\gamma'=0}^{k}\sum_{\nu'=0}^{k}\mu_{k-\gamma+\gamma-\gamma'-\nu'}(p-p',q-q')\mu_{k-\gamma+\gamma-\gamma'-\nu'}(p',q')|\partial_\hbar^{\nu'}\widehat{\widetilde{\mathcal{F}'}}(p-p',q-q')||\partial_\hbar^{\gamma'}\widehat{\widetilde{\mathcal{G}'}}(p',q')| \\
& (\text{replacing } 2^{\frac{k}{2}} \text{ by } 2^k \text{ to avoid heavy notations and since } k-\gamma'-\nu'\leqslant k-\gamma',\ k-\nu') \\
\leqslant\ & 8^k\sum_{\gamma'=0}^{k}\sum_{\nu'=0}^{k}\mu_{k-\nu'}(p-p',q-q')\mu_{k-\gamma'}(p',q')|\partial_\hbar^{\nu'}\widehat{\widetilde{\mathcal{F}'}}(p-p',q-q')||\partial_\hbar^{\gamma'}\widehat{\widetilde{\mathcal{G}'}}(p',q')|. \qquad (5.29)
\end{aligned}
$$

Note that γ disappeared from (5.29) so the $\sum_{\gamma=0}^{k}$ in (5.26) gives a fractor $(1+k)$. We get that $(1+k)^{-1}8^{-k}\|FG\|_{\rho,k}$ is majored by the maximum over $\hbar\in[0,1]$ (note the change $\nu'\to\gamma$)

$$\sum_{q,q'\in\mathbb{Z}^l}\int_{\mathbb{R}^{2m}}\sum_{\gamma,\gamma'=0}^{k}\mu_{k-\gamma}(p,q)|\partial_\hbar^\gamma\widehat{\widetilde{\mathcal{F}'}}(p,q)|\mu_{k-\gamma'}(p',q')|\partial_\hbar^{\gamma'}\widehat{\widetilde{\mathcal{G}'}}(p',q')|e^{\rho(\underline{\omega}|p|+\underline{\omega}|p'|+|q|+|q'|)}dpdp' \tag{5.30}$$

which is equal to

$$\|\mathcal{F}'\|_{\rho,k}\|\mathcal{G}'\|_{\rho,k}.$$

The proof of (5.10) follows the same lines, except that it is easy to see that, in (5.25), $e^{i\frac{\hbar}{2}((p-p').\omega.q'-p'.\omega.(q-q'))}$ has to be replaced by $2\sin\left(\frac{\hbar}{2}((p-p').\omega.q'-p'.\omega.(q-q'))\right)$, since (5.23) becomes

$$\begin{aligned}\left(\frac{[F,G]}{i\hbar}\right)_{mn} &= \sum_{q'\in\mathbb{Z}^l}\frac{F_{mq'}G_{q'n}-G_{mq'}F_{q'n}}{i\hbar}\\ &= \frac{1}{i\hbar}\sum_{q'\in\mathbb{Z}^l}\left[\widetilde{\mathcal{F}}\left(\frac{m+n+q'}{2}\hbar,m-n-q'\right)\widetilde{\mathcal{G}}\left(\frac{m+n+q'-(m-n)}{2}\hbar,q'\right)\right.\\ &\quad\left.-\widetilde{\mathcal{G}}\left(\frac{m+n+q'}{2}\hbar,m-n-q'\right)\widetilde{\mathcal{G}}\left(\frac{m+n+q'-(m-n)}{2}\hbar,q'\right)\right]\end{aligned}\tag{5.31}$$

It generates in (5.26) the replacement of $|((p-p').\omega.q'-p'.\omega.(q-q'))/2|^\nu$by the term

$$2|((p-p').\omega.q'-p'.\omega.(q-q'))/2|^{\nu+1}\leqslant\mu_\nu(p-p',q-q')\mu_\nu(p',q')(|p\cdot\omega\cdot q'-p'\cdot\omega\cdot q|)$$

thanks to (5.20), and we get by a discussion verbatim the same than the one contained in equations (5.27)-(5.30) that

$$\|[F,G]/i\hbar\|_{\rho-\delta-\delta_1,k}\leqslant(1+k)8^k\sum_{q,q'\in\mathbb{Z}^l}\int_{\mathbb{R}^{2m}}\sum_{\gamma,\gamma'=0}^{k}\mu_{k-\gamma}(p,q)\mu_{k-\gamma'}(p',q')\mathbb{Q}dpdp', \tag{5.32}$$

where thanks to (5.22),

$$\begin{aligned}\mathbb{Q} &= |\partial_\hbar^\gamma\widehat{\widetilde{\mathcal{F}'}}(p,q)|(|p\cdot\omega\cdot q'|+|p'\cdot\omega q|)|\partial_\hbar^{\gamma'}\widehat{\widetilde{\mathcal{G}'}}(p',q')|e^{(\rho-\delta-\delta_1)(\underline{\omega}|p|+|q|+\underline{\omega}|p'|+|q'|)}\\ &\leqslant \left[|\partial_\hbar^\gamma\widehat{\widetilde{\mathcal{F}'}}(p,q)||\partial_\hbar^{\gamma'}\widehat{\widetilde{\mathcal{G}'}}(p',q')|e^{\rho(\underline{\omega}|p|+|q|)+(\rho-\delta)(\underline{\omega}|p'|+|q'|)}\right](\underline{\omega}|p||q'|+\underline{\omega}|p'||q|)e^{-(\delta+\delta_1)(\underline{\omega}|p|+|q|)-\delta_1(\underline{\omega}|p'|+|q'|))}\\ &\leqslant \frac{2}{e^2\delta_1(\delta+\delta_1)}|\partial_\hbar^\gamma\widehat{\widetilde{\mathcal{F}'}}(p,q)||\partial_\hbar^{\gamma'}\widehat{\widetilde{\mathcal{G}'}}(p',q')|e^{\rho(\underline{\omega}|p|+|q|)+(\rho-\delta)(\underline{\omega}|p'|+|q'|)}\end{aligned}\tag{5.33}$$

because ($e^{-x}\leqslant 1, x\geqslant 0$ and)

$$\sup_{x\in\mathbb{R}^+}xe^{-\alpha x}=\frac{1}{e\alpha}. \tag{5.34}$$

(5.10) follows immediatly from (5.32).

The proof of (5.12) follows also the same line and is obtained thanks to the remark (5.34): indeed since

$$\left(\frac{[L_\omega, W]}{i\hbar}\right)_{mn} = -i\omega\cdot(m-n)W_{mn},$$

we see, again by (5.7), that the symbol $\mathcal{Q}(\xi,x)$ of $[L_\omega,W]/i\hbar$ is given trough the formula

$$\widetilde{\mathcal{Q}}(\xi,q) = ([L_\omega,W]/i\hbar)_{mn} \text{ for } \xi = \frac{m+n}{2}\hbar \text{ and } q = m-n.$$

Therefore $\widetilde{\mathcal{Q}}(\xi,q) = (-i\omega\cdot q)\widetilde{\mathcal{W}}(\xi,q)$, so $\mathcal{Q}(\xi,x) = \mathcal{Q}'(\omega.\xi,x)$ with

$$\widetilde{\mathcal{Q}'}(\omega.\xi,q) = -i\omega.q\ \widetilde{\mathcal{W}'}(\omega.\xi,q).$$

We get immediatly

$$\widehat{\widetilde{\mathcal{Q}'}}(p,q)e^{(\rho-\delta)(\underline{\omega}|p|+|q|)} \leqslant \frac{\omega}{e\delta}\widehat{\widetilde{\mathcal{W}'}}(p,q)e^{\rho(\underline{\omega}|p|+|q|)}$$

and (5.12) follows.

(5.11) is easily obtained by iteration of (5.10) and the Stirling formula: consider the finite sequence of numbers $\delta_s = \frac{d-s}{d}\delta$. We have $\delta_0=\delta$, $\delta_d=0$ and $\delta_{s-1}-\delta_s=\frac{\delta}{d}$. Let us define $G_0 := F$ and $G_{s+1} := \frac{1}{i\hbar}[G,G_s]$, for $0\leqslant s\leqslant d-1$. According to (5.10), we have

$$\|G_s\|_{\rho-\delta_{d-s}} \leqslant \frac{c_k}{e^2\delta_{d-s}(\frac{\delta}{d})}\|G\|_\rho\|G_{s-1}\|_{\rho-\delta_{d-s+1}},$$

where

$$c_k := 2(k+1)8^k.$$

Hence, by induction, we obtain

$$\begin{aligned}
\frac{1}{d!}\|G_d\|_{\rho-\delta_0} &\leqslant \frac{c_k^{d-1}}{d!e^{2(d-1)}\delta_0\cdots\delta_{d-2}(\frac{\delta}{d})^{d-1}}\|G\|_\rho^{d-1}\|G_1\|_{\rho-\delta_{d-1}} \qquad (5.35)\\
&\leqslant \frac{c_k^d}{d!e^{2d}\delta_0\cdots\delta_{d-1}\delta_{d-1}(\frac{\delta}{d})^{d-1}}\|G\|_\rho^d\|F\|_\rho\\
&\leqslant \frac{c_k^d}{d!e^{2d}d!(\frac{\delta}{d})^d\delta_{d-1}(\frac{\delta}{d})^{d-1}}\|G\|_\rho^d\|F\|_\rho\\
&\leqslant \frac{1}{2\pi}\left(\frac{c_kd^2}{e^2\delta^2}\right)^d\frac{1}{(d-1)!d!}\|G\|_\rho^d\|F\|_\rho\\
&= \frac{1}{2\pi}\left(\frac{c_k}{\delta^2}\right)^d\left(\frac{\sqrt{2\pi d}d^de^{-d}}{d!}\right)^2\|G\|_\rho^d\|F\|_\rho\\
&\leqslant \frac{1}{2\pi}\left(\frac{c_k}{\delta^2}\right)^d\|G\|_\rho^d\|F\|_\rho
\end{aligned}$$

since $\frac{d!}{\sqrt{2\pi d}e^{-d}d^d} \geqslant 1$. This weel know inequality can be seen from Binet's second expression for the $\log\Gamma(z)$ [WW][p. 251] :

$$\log\left(\frac{n!}{\left(\frac{n}{e}\right)^n\sqrt{2\pi n}}\right) = 2\int_0^\infty \frac{\arctan(t/n)}{e^{2\pi t}-1}dt \geqslant 0.$$

Finally (5.13) is obtained by noticing that $\|\mathcal{F}\mathcal{G}\|_{\rho,k}$ has the same expression as $\|FG\|_{\rho,k}$ after removing the term $e^{i\frac{\hbar}{2}(p.\omega.q'-p'.\omega.q)}$ in (5.25).

To prove **(4)** it is enough to notice that by (5.7) the symbol of W satisfies $\widetilde{W}(\xi,q,\hbar) = \frac{\widetilde{V}_{\ell_q}(\xi,q,\hbar)}{\omega_{\ell_q}\cdot q}$, so that $\widehat{\widetilde{W}}(p,q,\hbar) = \frac{\widehat{\widetilde{V}}1_{\ell_q}(p,q,\hbar)}{\omega_{\ell_q}\cdot q}$ and therefore, for all $r\in\mathbb{N}$,

$$|\partial_\hbar^r\widehat{\widetilde{W}}(p,q,\hbar)| \leqslant \gamma|q|^\tau \sup_{l=1\dots m}|\partial_\hbar^r\widehat{\widetilde{V}}_l(p,q,\hbar)| \leqslant \gamma|q|^\tau\sum_{l=1}^m|\partial_\hbar^r\widehat{\widetilde{V}}_l(p,q,\hbar)|$$

out of which we deduce (5.14) by standard arguments ($x^\tau e^{-\delta x} \leqslant (\frac{\tau}{e\delta})^\tau,\ x>0$) in the Diophantine case, and

$$|\partial_\hbar^r\widehat{\widetilde{W}}(p,q,\hbar)| \leqslant \mathcal{M}_M \sup_{l=1\dots m}|\partial_\hbar^r\widehat{\widetilde{V}}_l(p,q,\hbar)| \leqslant \mathcal{M}_M\sum_{l=1}^m|\partial_\hbar^r\widehat{\widetilde{V}}_l(p,q,\hbar)|$$

from which (5.15) follows.

To prove (5.18) we just notice that $\widehat{\widetilde{\partial_{\xi_i}\mathcal{F}_j}}(p,q,\hbar) = p_i\widehat{\widetilde{\mathcal{F}}}_j(p,q,\hbar)$. So

$$|\widehat{\widetilde{\partial_{\xi_i}\mathcal{F}_j}}(p,q,\hbar)| \leqslant |p_i\widehat{\widetilde{\mathcal{F}}}_j(p,q,\hbar)| \leqslant |\widehat{\widetilde{\mathcal{F}}}_j(p,q,\hbar)||p|.$$

Therefore $|\partial_\hbar^r\widehat{\widetilde{\partial_{\xi_i}\mathcal{F}_j}}(p,q,\hbar)|e^{(\rho-\delta)|p|} \leqslant \frac{1}{e\delta}|\partial_\hbar^r\widehat{\widetilde{\partial_{\xi_i}\mathcal{F}_j}}(p,q,\hbar)|e^{\rho|p|}$ and (5.18) follows.

(5) is an easy extension of (5.10). Indeed we find immediately, by (5.7) and the fact that l_{i-j} depends only on $i-j$, that the Fourier transform of the symbol of P_l is $\widehat{\widetilde{\mathcal{P}}}(p,q,\hbar) = \widehat{\widetilde{\mathcal{X}}}_{l_q}(p,q,\hbar)$ where $\widehat{\widetilde{\mathcal{X}}}_{l_q}$ is the Fourier transform of the symbol of the operator $X_{l_q} = \frac{[V_l,V_{l_q}]}{i\hbar}$. Therefore $|\partial_\hbar^r\widehat{\widetilde{\mathcal{X}}}_{l_q}(p,q,\hbar)| \leqslant \max_{l=1\dots m}|\partial_\hbar^r\widehat{\widetilde{\mathcal{X}}}_l(p,q,\hbar)|, \forall r\geqslant 0, q\in\mathbb{Z}^l$. So $\|P_l\|_{\rho-\delta} \leqslant \max_{l'=1\dots m}\|[V_l,V_{l'}]/i\hbar\|_{\rho-\delta}$ and $\|P\|_{\rho-\delta} \leqslant \max_{l,l'=1\dots m}\|[V_l,V_{l'}]/i\hbar\|_{\rho-\delta} \leqslant \sum_{l,l'=1}^m\|[V_l,V_{l'}]/i\hbar\|_{\rho-\delta}$ and we conclude by using (5.10). □

6. Fundamental iterative estimates: Brjuno condition case

In all this section the norm subscripts $\underline{\omega}$ and k are omitted.

Let us recall from Sections 2 and 3 that we want to find W_r such that

$$e^{i\frac{W_r}{\hbar}}(H_r+V_r)e^{-i\frac{W_r}{\hbar}} = H_{r+1}+V_{r+1} \tag{6.1}$$

where $H_{r+1} = H_r + h_{r+1}$ and $H_r = \mathcal{B}_r(L_\omega),\ h_{r+1} = \overline{V_r} = \mathcal{D}_r(L_\omega)$ and, for $0<\delta<\rho<\infty$,

$$\|h_{r+1}\|_\rho = \|\overline{V_r}\|_\rho \leqslant \|V_r\|_\rho, \quad \|V_{r+1}\|_{\rho-\delta} \leqslant D_r\|V_r\|_\rho^2, \tag{6.2}$$

and that we look at W_r solving:

$$\frac{1}{i\hbar}[H_r, W_r] + V^{co,r} = \overline{V^{co,r}} + \widehat{V}^r \tag{6.3}$$

with

$$V^{co,r} = V_r - V^{M_r}.$$

V^{M_r} is given by

$$V^{M_r}_{ij} = (V_r)_{ij} \text{ if } |i-j| > M_r, \quad V^{M_r}_{ij} = 0 \text{ otherwise.} \tag{6.4}$$

(note that $\overline{V^{co,r}} = \overline{V^r}$).
$\widehat{V}^r = (\widehat{V}^r_l)_{l=1\dots m}$ is given by

$$(\widehat{V}^r_l)_{ij} = \frac{([\widetilde{V}^r_l, \widetilde{V}^r_{l(i-j)}])_{ij}}{i\hbar\omega_{l(i-j)} \cdot (i-j)}, \qquad \widetilde{V}^r_{ij} := (I + A^r(i,j))^{-1}\, V^{co}_{ij}, \qquad V^{co,r} = V_r - V^{M_r}, \tag{6.5}$$

where $A^r(i,j)$ is the matrix given by Lemma 4, that is:

$$\frac{\mathcal{B}_r(\hbar\omega \cdot i) - \mathcal{B}_r(\hbar\omega \cdot j)}{i\hbar} = (I + A^r(i,j))\, \omega.(i-j).$$

Let

$$Z_k = 2(k+1)8^k. \tag{6.6}$$

Let us denote ad_W the operator $H \mapsto [W, H]$. The l.h.s. of (6.1) is then:

$$H_r + V^r + \frac{1}{i\hbar}[H_r, W_r] + \sum_{j=1}^{\infty} \frac{1}{(-i\hbar)^j j!} \mathrm{ad}^j_{W_r}(V_r) + \sum_{j=2}^{\infty} \frac{1}{(-i\hbar)^j j!} \mathrm{ad}^j_{W_r}(H_r)$$

that is

$$H_r + \overline{V^{co,r}} + \widehat{V}^r + V^r - V^{co,r} + \sum_{j=1}^{\infty} \frac{1}{(-i\hbar)^j j!} \mathrm{ad}^j_{W_r}(V_r) + \sum_{j=2}^{\infty} \frac{1}{(-i\hbar)^j j!} \mathrm{ad}^j_{W_r}(H_r).$$

or

$$H_r + h_{r+1} + (V_r - V^{co,r}) + \widehat{V}_r + R_1 + R_2 \tag{6.7}$$

Let us set

$$V_{r+1} := (V_r - V^{co,r}) + \widehat{V}^r + R_1 + R_2. \tag{6.8}$$

We want to estimate V_{r+1}. We first prove the following proposition.

Proposition 14. *Let W be in $J_k(\rho)$ and $0 < \delta < \rho$. Then*

$$\|[H_r, W]/i\hbar\|_{\rho-\delta} \leqslant \frac{1}{e\delta}(\underline{\omega} + Z_k \|\nabla(\mathcal{B}^\hbar_r - \mathcal{B}^\hbar_0)\|_\rho)\|W\|_\rho. \tag{6.9}$$

and for $d \geqslant 2$,

$$\frac{1}{d!}\|[H_r, \underbrace{W], \dots}_{d\ times}]/(i\hbar)^d\|_{\rho-\delta} \leqslant \frac{\delta\underline{\omega}}{2\pi Z_k}(1 + Z_k\|\nabla(\mathcal{B}_r^\hbar - \mathcal{B}_0^\hbar)\|_\rho)\left(\frac{Z_k}{\delta_r^2}\right)^d \|W\|_\rho^d \tag{6.10}$$

Let now W_r *be the (scalar) solution of* (6.3). *Then, we have*

$$\|W_r\|_\rho \leqslant \frac{\mathcal{M}_{M_r}}{1 - Z_k\|D(\mathcal{B}_r^\hbar - \mathcal{B}_0^\hbar)\|_\rho}\|V_r\|_\rho, \tag{6.11}$$

and therefore for $d \geqslant 2$,

$$\frac{1}{d!}\|[H_r, \underbrace{W_r], \dots}_{d\ times}]/(i\hbar)^d\|_{\rho-\delta} \leqslant \underline{\omega}\frac{1 + Z_k\|D(\mathcal{B}_r^\hbar - \mathcal{B}_0^\hbar)\|_\rho}{2\pi Z_k/\delta}\left(\frac{Z_k\mathcal{M}_{M_r}/\delta_r^2}{1 - Z_k\|D(\mathcal{B}_r^\hbar - \mathcal{B}_0^\hbar)\|_\rho}\|V_r\|_\rho\right)^d \tag{6.12}$$

($\mathcal{M}_M$ *is defined in* (1.3) *and* $\|D\mathcal{B}\|_\rho$ *is meant for* $\max_{i=1\dots m}\sum_{j=1\dots m}\|\nabla_i\mathcal{B}_j\|_\rho$*).*

Proof. We first prove (6.9). Note that the proof is somehow close to the proof of Proposition 13, items (1) and (2).

Since $\mathcal{B}^0(L_\omega) = L_\omega$, (5.12) reads

$$\|[H_0, W_r]/i\hbar\|_{\rho-\delta} \leqslant \frac{\omega}{e\delta}\|W_r\|_\rho. \tag{6.13}$$

Note that $([H_r - H_0, W_r]/\hbar)_{ij} = \frac{\mathcal{G}^r(\omega.i\hbar)-\mathcal{G}^r(\omega.j\hbar)}{\hbar}W_{ij}$ where $\mathcal{G}^r(Y) = \mathcal{B}_r^\hbar(Y) - Y,\ Y \in \mathbb{R}^m$ (note that $\mathcal{G}^r$ has an explicit dependence in $\hbar$ that we omit to avoid heaviness of notations). Indeed, since each L_{ω_i} is self-adjoint on $L^2(\mathbb{T}^l)$, $\mathcal{B}_r^\hbar(L_\omega)$ can be defined by the spectral theorem. Hence, we have

$$\begin{aligned}
[\mathcal{B}_r^\hbar(L_\omega), W]_{ij} = (e_i, [\mathcal{B}_r^\hbar(L_\omega), W]e_j) &= (e_i, \mathcal{B}_r^\hbar(L_\omega)We_j - W\mathcal{B}_r^\hbar(L_\omega)e_i) \\
&= (\mathcal{B}_r^\hbar(\omega.i\hbar) - \mathcal{B}_r^\hbar(\omega.j\hbar))(e_i, We_j).
\end{aligned}$$

Using (5.13) we get that

$$\|[H_r - H_0, W_r]/i\hbar\|_{\rho-\delta} \leqslant \|X_r\|_{\rho-\delta}. \tag{6.14}$$

where X_r is defined through $(X_r)_{ij} = \frac{\mathcal{G}^r(\omega.i\hbar)-\mathcal{G}^r(\omega.j\hbar)}{\hbar}(W_r)_{ij}$.

In order to estimate the norm of X^r we need to express its symbol $\mathcal{X}_r$. This is done thanks to formula (5.7) and the fact that we know the matrix elements of X_r.

Expressing $(X_r)_{ij}$ as a function of $((i+j)\hbar/2, i-j)$ through $i, j = \frac{i+j}{2} \pm \frac{i-j}{2}$ and using (5.7) we get that

$$\widetilde{\mathcal{X}_r}(\xi, q, \hbar) = \frac{\mathcal{G}^r(\omega.\xi + \omega.q\hbar/2) - \mathcal{G}^r(\omega.\xi - \omega.q\hbar/2)}{\hbar}\widetilde{\mathcal{W}_r'}(\omega.\xi, q, \hbar) := \widetilde{\mathcal{X}_r'}(\omega.\xi, q, \hbar),$$

so that, using (remember that we denote $p.\omega.q = \sum_{j=1\dots m}\sum_{i=1\dots l} p_j\omega_j^i q_i$)

$$\int_{\mathbb{R}^m}(\mathcal{G}^r(\Xi + \omega.q\hbar/2) - \mathcal{G}^r(\Xi - \omega.q\hbar/2))e^{-i<\Xi,p>}dp = 2\sin[p.\omega.q\hbar/2]\int_{\mathbb{R}^m}\mathcal{G}^r(\Xi)e^{-i<\Xi,p>}dp,$$

$$\widehat{\widehat{\mathcal{X}'_r}}(p,q,\hbar)=\int_{\mathbb{R}^m}\widehat{\mathcal{G}^r_i}(p-p')\frac{\sin\left[(p-p').\omega.q\hbar/2\right]}{\hbar}\widehat{\widehat{\mathcal{W}'}}_r(p',q,\hbar)dp.$$

Therefore $\|X_r\|_{\rho-\delta}$ is equal to

$$\sum_{i=1}^m\sum_{q\in\mathbb{Z}^l}\int_{\mathbb{R}^{2m}}dpdp'|2\sum_{\gamma=1}^k\mu_{k-\gamma}(p,q)\partial_\hbar^\gamma\left[\widehat{\mathcal{G}^r_i}(p-p')\frac{\sin\left[(p-p').\omega.q\hbar/2\right]}{\hbar}\widehat{\widehat{\mathcal{W}}}_r(p',q,\hbar)\right]|e^{(\rho-\delta)(\underline{\omega}|p|+|q|)}$$

$$\leqslant\sum_{i=1}^m\sum_{q\in\mathbb{Z}^l}\int\sum_{\gamma=1}^k\mu_{k-\gamma}(p,q)\sum_{\mu=1}^{\gamma}\sum_{\nu=1}^{\gamma-\mu}\binom{\gamma}{\mu}\binom{\gamma-\mu}{\nu}\times$$

$$\times 2|\partial_\hbar^{\gamma-\mu-\nu}\widehat{\mathcal{G}^r_i}(p-p')||\partial_\hbar^\nu\frac{\sin\left((p-p').\omega.q\hbar/2\right)}{\hbar}||\partial_\hbar^\mu\widehat{\widehat{\mathcal{W}}}_r(p',q,\hbar)|e^{(\rho-\delta)(\underline{\omega}|p|+|q|)}dpdp' \quad (6.15)$$

$$(6.16)$$

Using now the inequalities (5.21) and (5.22), we get,

$$|\sum_{\gamma=1}^k\mu_{k-\gamma}(p,q)\sum_{\mu=1}^{\gamma}\sum_{\nu=1}^{\gamma-\mu}\binom{\gamma}{\mu}\binom{\gamma-\mu}{\nu}\partial_\hbar^{\gamma-\mu-\nu}\widehat{\mathcal{G}^r_i}(p-p')\partial_\hbar^\nu\frac{\sin\left((p-p').\omega.q\hbar\right)}{\hbar}\partial_\hbar^\mu\widehat{\widehat{\mathcal{W}}}_r(p',q,\hbar)|$$

$$\leqslant\underline{\omega}\max_{j=1\dots m}|p_j-p'_j||q|\sum_{\gamma=1}^k\mu_{k-\gamma}(p,q)\sum_{\mu=1}^{\gamma}\sum_{\nu=1}^{\gamma-\mu}\binom{\gamma}{\mu}\binom{\gamma-\mu}{\nu}|\partial_\hbar^{\gamma-\mu-\nu}\widehat{\mathcal{G}^r_i}(p-p')||(p-p').\omega.q|^\nu|\partial_\hbar^\mu\widehat{\widehat{\mathcal{W}}}_r(p',q,\hbar)|.$$

Therefore we notice (after changing $q\leftrightarrow q'$) that $\|X_r\|_{\rho-\delta}$ is majored by the maximum over $\hbar\in[0,1]$ of

$$\sum_{\gamma=0}^k\int_{\mathbb{R}^{2m}}\sum_{(q,q')\in\mathbb{Z}^{2l}}\mu_{k-\gamma}(p,q)\underline{\omega}\max_{j=1\dots m}|p_j-p'_j|\mathbb{P}(\underline{\mathcal{G}^r_i},\mathcal{W}_r)e^{(\rho-\delta)(\underline{\omega}|p|+|q|)}dpdp' \quad (6.17)$$

where $\mathbb{P}$ is defined in (5.27) and

$$\underline{\mathcal{G}^r_i}(\Xi,x)=\mathcal{G}^r_i(\Xi)\text{ so that }\widehat{\widehat{\underline{\mathcal{G}^r_i}}}(p,q)=\widehat{\mathcal{G}^r_i}(p)\delta_{q=0}.$$

Therefore we can verbatim use the argument contained between formulas (5.27)-(5.29) and we arrive, in analogy with (5.30), to the fact that $(1+k)^{-1}8^{-k}\|X_r\|_{\rho-\delta}$ is majored by the maximum over $\hbar\in[0,1]$ of

$$\sum_{q,q'\in\mathbb{Z}^l}\int_{\mathbb{R}^{2m}}\underline{\omega}\max_{j=1\dots m}|p_j||q'|\sum_{\gamma,\gamma'=0}^k\mu_{k-\gamma}(p,q)|\partial_\hbar^\gamma\widehat{\widehat{\underline{\mathcal{G}^r_i}}}(p,q)|\mu_{k-\gamma'}(p',q')|\partial_\hbar^{\gamma'}\widehat{\widehat{\mathcal{W}}}_r(p',q')|e^{(\rho-\delta)(\underline{\omega}|p|+\underline{\omega}|p'|+|q|+|q'|)}dpdp'$$

$$=\sum_{q'\in\mathbb{Z}^l}\int_{\mathbb{R}^{2m}}\underline{\omega}\max_{j=1\dots m}|p_j||q'|\sum_{\gamma,\gamma'=0}^k\mu_{k-\gamma}(p,0)|\partial_\hbar^\gamma\widehat{\mathcal{G}^r_i}(p)|\mu_{k-\gamma'}(p',q')|\partial_\hbar^{\gamma'}\widehat{\widehat{\mathcal{W}}}_r(p',q')|e^{(\rho-\delta)(\underline{\omega}|p|+\underline{\omega}|p'|+|q'|)}dpdp'$$

Since $|\widehat{\mathcal{G}}_i^r(p)||p_j| = |\widehat{\mathcal{G}}_i^r(p)p_j| = |\widehat{\nabla_j\mathcal{G}_i^r}(p)|$, we get that (use $\rho-\delta \leqslant \rho$ and again $|q'|e^{-\delta|q'|} \leqslant \frac{1}{e\delta}$)

$$\|X_r\|_{\rho-\delta} \leqslant \frac{(1+k)8^k\omega}{e\delta}\|\nabla G^r\|_\rho\|W_r\|_\rho \leqslant \frac{Z_k\omega}{e\delta}\|\nabla(\mathcal{B}_r^\hbar-\mathcal{B}_0^\hbar)\|_\rho\|W_r\|_\rho. \tag{6.18}$$

Here $\|\nabla(\mathcal{B}_r^\hbar-\mathcal{B}_0^\hbar)\|_\rho$ is understood in the sense of (5.17)-(4.18).

(6.9) follows form (6.13) and (6.14)-(6.18).

We will prove (6.10) by the same argument as in the proof of Proposition 13. Take (5.35) with $G_1 := \frac{1}{i\hbar}[W_r, H_r]$, $G_{s+1} = \frac{1}{i\hbar}[W_r, G_s]$ and $\gamma_s = \frac{d-s}{d}\delta$ for $1 \leqslant s \leqslant d-1$, $\gamma_0 = \delta$, $\gamma_{d-1} = \frac{\delta}{d}$.

We get

$$\begin{aligned}
\frac{1}{d!}\|G_d\|_{\rho-\gamma_0} &\leqslant \frac{Z_k^{d-1}}{d!e^{2(d-1)}\gamma_0\cdots\gamma_{d-2}(\frac{\delta}{d})^{d-1}}\|W_r\|_\rho^{d-1}\|G_1\|_{\rho-\gamma_{d-1}} \\
&\leqslant \frac{Z_k^{d-1}}{d!d!e^{2(d-1)}(\frac{\delta}{d})^{2d-2}}\|W_r\|_\rho^{d-1}\|G_1\|_{\rho-\delta/d} \\
&\leqslant \frac{1+Z_k\|\nabla(\mathcal{B}_r^\hbar-\mathcal{B}_0^\hbar)\|_\rho}{d!d!e^{2d-1}(\frac{\delta}{d})^{2d-1}}Z_k^{d-1}\|W_r\|_\rho^d \\
&\leqslant \frac{\delta}{2\pi d^2e^{-1}Z_k}(1+Z_k\|\nabla(\mathcal{B}_r^\hbar-\mathcal{B}_0^\hbar)\|_\rho)\left(\frac{d^de^{-d}\sqrt{2\pi d}}{d!}\right)^2\left(\frac{Z_k}{\delta^2}\right)^d\|W_r\|_\rho^d
\end{aligned}$$

and we get (6.10) by $\frac{d^de^{-d}\sqrt{2\pi d}}{d!} \leqslant 1$ and $d^2e^{-1} \geqslant 1$ if $d \geqslant 2$, and setting $\rho = \rho_r$, $\gamma_0 = \gamma_r$.

In order to prove (6.11) we first estimate $\|\widetilde{V}^r\|_{\rho_r}$ defined by (6.5) where $A^r(i,j)$ is given by (7.5).

Lemma 15. *Let V' be defined by $V'_{ij} = A^r(i,j)V^{co}_{ij}$. Then*

$$\|V'\|_\rho \leqslant Z_k\|D(\mathcal{B}_r^\hbar-\mathcal{B}_0^\hbar)\|_\rho\|V^{co}\|_\rho. \tag{6.19}$$

Proof. The proof will actually be close to the one of (6.9). $A^r(i,j)\omega.(i-j) = \frac{\mathcal{G}^r(\omega.j\hbar)-\mathcal{G}(\omega-i\hbar)}{\hbar}$ so

$$A^r(i,j) = \int_0^1 \nabla\mathcal{G}^r((1-t)\omega.j\hbar + t\omega.i\hbar)dt.$$

Therefore

$$V' = \int_0^1 \sum_{n=1}^m \nabla_n\mathcal{G}^r((1-t)\omega.j\hbar + t\omega.i\hbar)V_n^{co}dt$$

so

$$\|V'\| \leqslant \sup_{0\leqslant t\leqslant 1}\sum_{n=1}^m \|\nabla_n\mathcal{G}^r((1-t)\omega.j\hbar + t\omega.i\hbar)V_n^{co}\|.$$

Let X_n^r be defined through

$$(X_n^r)_{ij} = \nabla_n\mathcal{G}^r((1-t)\omega.j\hbar + t\omega.i\hbar)(V_n^{co})_{ij} = \nabla_n\mathcal{G}^r(\omega\cdot\frac{i+j}{2}\hbar - (1-2t)(i-j)\frac{\hbar}{2})(V_n^{co})_{ij}$$

By the argument as before, using (5.7), we get that the symbol of X^r satisfies

$$\widetilde{\mathcal{X}_n^r}(\xi,q)=\nabla_n\mathcal{G}^r(\omega\cdot\xi-(1-2t)q\frac{\hbar}{2})\widetilde{\mathcal{V}_n^{co}}(\xi,q)=\widetilde{(\mathcal{X}_n^r)}'(\omega\cdot\xi,q):=\nabla_n\mathcal{G}^r(\omega\cdot\xi-(1-2t)q\frac{\hbar}{2})\widetilde{(\mathcal{V}_n^{co})}'(\omega\cdot\xi,q)$$

since $\mathcal{V}_n^{co}$ has the same structure as $\mathcal{V}$ so there exists $(\mathcal{V}_n^{co})'$ such that $\mathcal{V}_n^{co}(\xi,x)=(\mathcal{V}_n^{co})'(\omega\cdot\xi,x)$.

Taking now the Fourier transform of $\widetilde{(\mathcal{X}_n^r)}'(\Xi,q)$ with respect to Ξ one gets by translation-convolution

$$\widehat{\widetilde{(\mathcal{X}_n^r)}}'(p,q,\hbar)=\int_{\mathbb{R}^m}\widehat{\nabla_n\mathcal{G}^r}(p-p')e^{i(p-p').\omega.q(1-2t)\hbar/2}\widehat{\widetilde{\mathcal{V}_n^{co}}}(p',q,\hbar)dp'.$$

So, as before,

$$\begin{aligned}|\partial_\hbar^\gamma\widehat{\widetilde{(\mathcal{X}_n^r)}}'(p,q,\hbar)| &\leqslant \int\sum_{\mu=1}^{\gamma}\sum_{\nu=1}^{\gamma-\mu}|\partial_\hbar^{\gamma-\mu-\nu}\widehat{\nabla_n\mathcal{G}^r}(p-p')\partial_\hbar^\nu e^{i(p-p').\omega.q(1-2t)\hbar/2}\partial_\hbar^\mu\widehat{\widetilde{\mathcal{V}_n^{co}}}(p',q,\hbar)|\binom{\gamma}{\mu}\binom{\gamma-\mu}{\nu}dp'\\ &\leqslant \int_{\mathbb{R}^m}\sum_{\mu=1}^{\gamma}\sum_{\nu=1}^{\gamma-\mu}|\partial_\hbar^{\gamma-\mu-\nu}\widehat{\nabla_n\mathcal{G}^r}(p-p')(|p-p'||q|/2)^\nu\partial_\hbar^\mu\widehat{\widetilde{\mathcal{V}_n^{co}}}(p',q,\hbar)|\binom{\gamma}{\mu}\binom{\gamma-\mu}{\nu}dp'\end{aligned}$$

Following the same lines than in the proof of (6.9) we get that (remember that, by definition, $\|X^r\|_\rho:=\sum_{n=1}^m\|X_n^r\|_\rho,\ \|X_n^r\|_\rho=\|(\mathcal{X}_n^r)'\|_\rho$ by Definitions 8 and 7)

$$\|X^r\|_\rho\leqslant Z_k\sum_{i=1}^m\sum_{n=1}^m\|\nabla_n\mathcal{G}_i^r\|_\rho\|V_n^{co}\|_\rho\leqslant Z_k\max_n\|\nabla_n\mathcal{G}^r\|_\rho\sum_{n=1}^m\|V_n^{co}\|_\rho\leqslant Z_k\|\nabla\mathcal{G}^r\|_\rho\|V^{co}\|_\rho$$

and the Lemma is proved. □

Corollary 16. *Let V'' defined by $V''_{ij}=(1+A^r(i,j))^{-1}V_{ij}^{co}$. Then*

$$\|V''\|_\rho\leqslant\frac{1}{1-Z_k\|\nabla(\mathcal{B}_r^\hbar-\mathcal{B}_0^\hbar)\|_\rho}\|V^{co}\|_\rho. \tag{6.20}$$

(6.11) is now a consequence of (5.15) and the fact that $\|V^{co}\|_\rho\leqslant\|V\|_\rho$.
(6.12) is obtained by putting (6.11) in (6.10). The proposition is proved. □

We need finally the following obvious Lemma:

Lemma 17. *Define*

$$\mathcal{V}^M(x,\xi):=\sum_{|q|\geqslant M}\widetilde{\mathcal{V}}_q(\xi)e^{iqx}. \tag{6.21}$$

Then

$$\|\mathcal{V}^M\|_{\rho-\delta}\leqslant e^{-\delta M}\|\mathcal{V}\|_\rho \tag{6.22}$$

Corollary 18. *Let V^M be defined by*

$$\begin{aligned} V^M_{ij} &= V_{ij} \quad \text{when } |i-j| \geqslant M \\ &= 0 \quad \text{when } |i-j| < M. \end{aligned}$$

Then

$$\|V^M\|_{\rho-\delta} \leqslant e^{-\delta M}\|V\|_\rho. \tag{6.23}$$

Proof. Just notice that the symbol of V^M, $\mathcal{V}^M$, satisfies, by (5.7), $\widetilde{\mathcal{V}}^M(\xi, q, \hbar) = 0$ when $|q| \leqslant M$ and apply Lemma 17. ☐

Let us define, for a decreasing positive sequence $(\rho_r)_{r=0\dots\infty}$, $\rho_{r+1} = \rho_r - \delta_r$ to be specified later,

$$G_r = \|D(\mathcal{B}^\hbar_r - B^\hbar_0)\|_{\rho_r} = \max_{i=1\dots m} \sum_{j=1\dots m} \|\nabla_i(\mathcal{B}^\hbar_r - B^\hbar_0)_j\|_{\rho_r}. \tag{6.24}$$

We are now in position to derive the following fundamental estimates of the five terms in (6.7):

$$\|h_{r+1}\|_{\rho_r-\delta_r} \leqslant \|h_{r+1}\|_{\rho_r} = \|V^{co,r}\|_{\rho_r} = \|V_r\|_{\rho_r} \tag{6.25}$$

$$\|V_r - V^{co,r}\|_{\rho_r-\delta_r} = \|V^{M_r}\|_{\rho_r-\delta_r} \leqslant \left(\frac{e^{-M_r\delta_r}}{\|V^r\|_{\rho_r}}\right)\|V_r\|^2_{\rho_r} \tag{6.26}$$

$$\|\widehat{V}^r\|_{\rho_r-\delta_r} \leqslant \frac{Z_k\frac{\mathcal{M}_{M_r}}{\delta_r^2}}{(1-Z_kG_r)^2}\|V_r\|^2_{\rho_r} \tag{6.27}$$

$$\|R_1\|_{\rho_r-\delta_r} \leqslant \frac{Z_k\frac{\mathcal{M}_{M_r}}{\delta_r^2(1-Z_kG_r)}}{1-Z_k\frac{\mathcal{M}_{M_r}}{\delta_r^2(1-Z_kG_r)}\|V^r\|_{\rho_r}}\|V_r\|^2_{\rho_r} \tag{6.28}$$

$$\|R_2\|_{\rho_r-\delta_r} \leqslant \frac{Z_k\frac{\mathcal{M}^2_{M_r}\underline{\omega}(1+Z_kG_r)}{\delta_r^3(1-Z_kG_r)^2}}{1-Z_k\frac{\mathcal{M}_{M_r}}{\delta_r^2(1-Z_kG_r)}\|V^r\|_{\rho_r}}\|V_r\|^2_{\rho_r} \tag{6.29}$$

Indeed, (6.25) is obvious and (6.26) is nothing but Corollary 18.
(6.27) is derived by using Proposition 13, item (5) equation (5.16), Lemma 16 and equation (6.5). Note that, as pointed out before, $\widehat{V}^r$ is cut-offed as $V^{co,r}$ thanks to (3.6).
(6.28) is obtained through the definition $R_1 = \sum_{j=1}^\infty \frac{1}{(-i\hbar)^j j!}\mathrm{ad}^j_{W_r}(V_r)$, the fact that, by (5.11), $\|\frac{1}{(-i\hbar)^j j!}\mathrm{ad}^j_{W_r}(V_r)\|_{\rho_r-\delta_r} \leqslant (Z_k/\delta_r^2)^j\|V_r\|_{\rho_r}\|W_r\|^j_{\rho_r}$ and (6.11), so that

$$\|R_1\|_{\rho_r-\delta_r} \leqslant \sum_{j=1}^\infty (Z_k/\delta_r^2)^j\|V_r\|_{\rho_r}\left(\frac{\mathcal{M}_{M_r}}{1-Z_kG_r}\|V_r\|_{\rho_r}\right)^j.$$

(6.29) is proven by the definition $R_2 = \sum\limits_{j=2}^\infty \frac{1}{(-i\hbar)^j j!}\mathrm{ad}^j_{W_r}(H_r)$ and the fact that, by (6.12) we have that $\|\frac{1}{(-i\hbar)^j j!}\mathrm{ad}^j_{W_r}(H_r)\|_{\rho_r-\delta_r} \leqslant \underline{\omega}\frac{1+Z_kG_r}{Z_k/\delta_r}\left(\frac{Z_k\mathcal{M}_{M_r}/\delta_r^2}{1-Z_kG_r}\|V_r\|_{\rho_r}\right)^j$.

Collecting all the preceding estimates together with the definition (6.8) :

$$V_{r+1} := (V_r - V^{co,r}) + \widehat{V}^r + R_1 + R_2,$$

we obtain:

Proposition 19. *For* $r = 0, 1, \dots$ *, we have*

$$\|V_{r+1}\|_{\rho_r-\delta_r} \leqslant F_r\|V_r\|^2_{\rho_r} + e^{-\delta_r M_r}\|V_r\|_{\rho_r} \tag{6.30}$$

with

$$F_r = \frac{\mathcal{M}_{M_r} Z_k}{\delta_r^2(1 - Z_kG_r)^2}\left(1 + \frac{(1 - Z_kG_r) + \frac{\mathcal{M}_{M_r}}{\delta_r}\underline{\omega}(1 + Z_kG_r)}{1 - \frac{\mathcal{M}_{M_r}}{1-Z_kG_r}\frac{Z_k}{\delta_r^2}\|V_r\|_{\rho_r}}\right). \tag{6.31}$$

7. Fundamental iterative estimates: Diophantine condition case

In all this section also the norm subscripts $\underline{\omega}$ and k are omitted.

Let $0 < \delta < \rho$. Let us recall that we want to find W_r such that

$$e^{i\frac{W_r}{\hbar}}(H_r + V_r)e^{-i\frac{W_r}{\hbar}} = H_{r+1} + V_{r+1} \tag{7.1}$$

where $H_{r+1} = H_r + h_{r+1}$ and $H_r = \mathcal{B}^\hbar_r(L_\omega),\ h_{r+1} = \overline{V_r} = \mathcal{D}_r(L_\omega)$ and

$$\|h_{r+1}\|_\rho = \|\overline{V_r}\|_\rho \leqslant \|V_r\|_\rho, \qquad \|V_{r+1}\|_{\rho-\delta} \leqslant D_r\|V_r\|^2_\rho. \tag{7.2}$$

In the case where ω satisfies the Diophantine condition (1.4) we look at W_r solving:

$$\frac{1}{i\hbar}[H_r, W_r] = \overline{V_r} + \widehat{V}^r \tag{7.3}$$

with $\widehat{V}^r = (\widehat{V}^r_l)_{l=1\dots m}$ given by

$$(\widehat{V}^r_l)_{ij} = \frac{([\widetilde{V}^r_l, \widetilde{V}^r_{l(i-j)}])_{ij}}{i\hbar\omega_{l(i-j)}\cdot(i-j)}, \qquad \widetilde{V}^r_{ij} := (I + A^r_\varepsilon(i,j))^{-1}(V_r)_{ij}. \tag{7.4}$$

Here $A^r(i,j)$ is the matrix given by Lemma 4, that is:

$$\frac{\mathcal{B}^\hbar_r(\hbar\omega\cdot i) - \mathcal{B}_r(\hbar\omega\cdot j)}{i\hbar} = (I + A^r(i,j))\,\omega.(i-j), \tag{7.5}$$

The l.h.s. of (7.1) is:

$$H_r + V_r + \frac{1}{i\hbar}[H_r, W_r] + \sum_{j=1}^\infty \frac{1}{(-i\hbar)^j j!}\mathrm{ad}^j_{W_r}(V_r) + \sum_{j=2}^\infty \frac{1}{(-i\hbar)^j j!}\mathrm{ad}^j_{W_r}(H_r)$$

that is

$$H_r + \overline{V^r} + \widehat{V}^r + \sum_{j=1}^\infty \frac{1}{(-i\hbar)^j j!}\mathrm{ad}^j_{W_r}(V_r) + \sum_{j=2}^\infty \frac{1}{(-i\hbar)^j j!}\mathrm{ad}^j_{W_r}(H_r).$$

or

$$H_r + h_{r+1} + \widehat{V}^r + R_1 + R_2 \tag{7.6}$$

Proposition 20. *Let W_r the (scalar) solution of* (7.3)*.Then, for* $d \geqslant 2,\ 0<\delta<\rho<\infty$,

$$\frac{1}{d!}\|[H_r,\underbrace{W_r],\dots}_{d\ times}]/(i\hbar)^d\|_{\rho-\delta} \leqslant \frac{\delta\underline{\omega}}{2\pi Z_k}(1+Z_k\|\nabla(\mathcal{B}_r^\hbar-\mathcal{B}_0^\hbar)\|_\rho)\left(\frac{Z_k}{\delta^2}\right)^d\|W_r\|^d_{\rho-\delta} \tag{7.7}$$

$$\|W_r\|_{\rho-\delta} \leqslant \frac{2^\tau\gamma(\frac{\tau}{e\delta})^\tau}{1-Z_k\|D(\mathcal{B}_r^\hbar-\mathcal{B}_0^\hbar)\|_\rho}\|V_r\|_\rho, \tag{7.8}$$

so

$$\frac{1}{d!}\|[H_r,\underbrace{W_r],\dots}_{d\ times}]/(i\hbar)^d\|_{\rho-\delta} \leqslant \underline{\omega}\frac{1+Z_k\|D(\mathcal{B}_r^\hbar-\mathcal{B}_0^\hbar)\|_\rho}{2\pi Z_k/\delta}\left(\frac{2^\tau\gamma(\frac{\tau}{e\delta})^\tau}{1-Z_k\|D(\mathcal{B}_r^\hbar-\mathcal{B}_0^\hbar)\|_\rho}\|V_r\|_\rho\right)^d \tag{7.9}$$

(let us recall that $\mathcal{M}_M$ is defined in (1.3) *and $\|D\mathcal{B}\|_\rho$ is meant for $\max\limits_{j=1\dots m}\sum\limits_{i=1\dots m}\|\nabla_i\mathcal{B}_j\|_\rho$).*

Proof. The proof of Proposition 20 is the same than the one of Proposition 14 done in details in Section 6. The only minor difference is the discussion of the small denominators and is adaptable without pain. We omit the details here. □

Using notation (6.24), Proposition 13, last item (5.10), and Proposition 14 we can derive the following fundamental estimates of the four terms in (7.6)

$$\|h_{r+1}\|_{\rho_r-\delta_r} \leqslant \|h_{r+1}\|_{\rho_r} = \|V_r\|_{\rho_r}$$

$$\|\widehat{V}^r\|_{\rho_r-\delta_r} \leqslant \frac{Z_k\frac{2^{2+\tau}\gamma(\frac{\tau}{e\delta_r})^\tau}{\delta_r^2}}{(1-Z_kG_r)^2}\|V_r\|^2_{\rho_r}$$

$$\|R_1\|_{\rho_r-\delta_r} \leqslant \frac{Z_k\frac{\gamma(\frac{\tau}{e\delta_r})^\tau}{\delta_r^2(1-Z_kG_r)}}{1-Z_k\frac{\gamma(\frac{\tau}{e\delta_r})^\tau}{\delta_r^2(1-Z_kG_r)}\|V^r\|_{\rho_r}}\|V_r\|^2_{\rho_r}$$

$$\|R_2\|_{\rho_r-\delta_r} \leqslant \frac{Z_k\frac{(\gamma(\frac{\tau}{e\delta_r})^\tau)^2\underline{\omega}(1+Z_kG_r)}{\delta_r^3(1-Z_kG_r)^2}}{1-Z_k\frac{\gamma(\frac{\tau}{e\delta_r})^\tau}{\delta_r^2(1-Z_kG_r)}\|V^r\|_{\rho_r}}\|V_r\|^2_{\rho_r}$$

Collecting all the preceding results we get:

Proposition 21. *For* $r=0,1,\dots,$ $\|V_{r+1}\|_{\rho_r-\delta_r} \leqslant F'_r\|V_r\|^2_{\rho_r}$ *with*

$$F'_r = \frac{\gamma(\frac{\tau}{e\delta_r})^\tau Z_k}{\delta_r^2(1-Z_kG_r)^2}\left(2^{2+\tau}+\frac{(1-Z_kG_r)+\frac{\gamma}{\delta_r}(\frac{\tau}{e\delta_r})^\tau\underline{\omega}(1+Z_kG_r)}{1-\frac{\gamma(\frac{\tau}{e\delta_r})^\tau}{1-Z_kG_r}\frac{Z_k}{\delta_r^2}\|V_r\|_{\rho_r}}\right) \tag{7.10}$$

8. Strategy of the KAM iteration

In the case of the Diophantine condition, the strategy consists in finding a sequences δ_r such that, with F'_r given by (7.10),

$$\sum_{r=1}^{\infty} \delta_r = \delta < \infty \quad \text{and} \quad \prod_{i=1}^{r} D_i^{2^{r-i}} \leqslant R^{2^r},\ R > 0. \tag{8.1}$$

Indeed when (8.1) is satisfied and thanks to Proposition 21, the series $\sum_{r=1}^{\infty} V_r$, and therefore $\sum_{r=1}^{\infty} h_r = \sum_{r=1}^{\infty} \overline{V_r}$ are easily shown to be convergent in $J_k(\rho-\delta,\underline{\omega})$ for $\rho > \delta$, at the condition that

$$\|V\|_{\rho,\underline{\omega},k} < R.$$

This last sum is the quantum Birkhoff normal form $\mathcal{B}^{\hbar}_{\infty}$ of the perturbation. Estimates on the solution of the cohomological equations provide also the existence of a limit unitary operator conjugating the original Hamiltonian to its normal form.

The case of the Brjuno condition follows the same way, except that one has also to find a sequence of numbers M_r so that (6.31) holds. The main difference comes from the extra linear and non quadratic term in Proposition 19. This difficulty is overcome by deriving out of $\|V\|_{\rho,\underline{\omega},k}$ a sequence of quantities with a quadratic growth as in (7.10). This leads to an extra condition for the convergence of the iteration, condition involving only the arithmetical properties of ω and which can be removed by a scaling argument.

These ideas will be implemented in the following section which is organized as follows. We first prove the convergence of the KAM iteration in the Brjuno case with a restriction on ω (Theorem 26), restriction released in Theorem 27 thanks to the scaling argument already mentioned. This proves and precises **Theorem 1**. We then prove the corresponding classical version (Corollary 32, global Hamiltonian version of the singular integrability of [LS1], leading to **Theorem 2** precised. We end the section by more refined results under Diophantine condition on ω, Theorem 36, leading to the criterion contained in **Theorem 3**.

9. Proof of the convergence of the KAM iteration

In this section the norm subscripts $\underline{\omega}$ and k might be committed in the body of the proofs. They are nevertheless reestablished in the main statements.

Proposition 22. *Let us fix $0 < C < \eta < 1$, $\rho > 0$ and let us choose*

$$\rho_0 = \rho,\ \delta_r = \alpha 2^{-r},\ 0 < \alpha \leqslant \log 2, \quad \textit{and} \quad M_r = 2^r. \tag{9.1}$$

For $E \geqslant E_0$ defined below by (9.13) let us suppose:

$$\sum_{r=0}^{\infty} \left[\frac{|\log \mathcal{M}_{M_r}|}{2^{r-1}} - 3\frac{\log \delta_r}{2^r} + \frac{\log Z_k E}{2^r} \right] = C_k < \infty \tag{9.2}$$

and, for $1 \leqslant r \leqslant l$,

$$Z_k G_r < \eta - C/r, \tag{9.3}$$

$$\frac{\delta_r^3}{\delta_{r+1}^3} e^{(\delta_{r+1} M_{r+1} - 2\delta_r M_r)} > 2 \tag{9.4}$$

and

$$\frac{\mathcal{M}_{M_r} Z_k}{\delta_r^2} \|V_r\|_{\rho_r} < \frac{1}{2}(1 - \eta + C/r), \tag{9.5}$$

together with

$$Z_k G_0 = 0, \tag{9.6}$$

$$\frac{\delta_0^3}{\delta_1^3} e^{(\delta_1 M_1 - 2\delta_0 M_0)} > 2 \tag{9.7}$$

and

$$\frac{\mathcal{M}_{M_0} Z_k}{\delta_0^2} \|V\|_{\rho_0} < \frac{1}{2}. \tag{9.8}$$

Then, for $r \geqslant 0$

$$\|V_{r+1}\|_{\rho_{r+1}} \leqslant (D_k)^{2^{r+1}}. \tag{9.9}$$

where

$$D_k := e^{C_k}(\|V\|_\rho + \frac{e^{-\alpha}\alpha^3}{2\mathcal{M}_1^2 Z_k E}). \tag{9.10}$$

Note that $G_0 = 0$ and that, taking (9.3) for $r = 1$ we get:

$$1 > \eta > Z_k \|\nabla \overline{\mathcal{V}'}\|_\rho \text{ and } C < \eta - Z_k \|\nabla \overline{\mathcal{V}'}\|_\rho. \tag{9.11}$$

Therefore we will impose the condition

$$\|\nabla \overline{\mathcal{V}'}\|_\rho < \frac{\eta - C}{Z_k} \tag{9.12}$$

Proof. We first prove the two following Lemmas.

Lemma 23. *Under the hypothesis* (6.30), (9.3) *and* (9.5), *and* $\eta < 1$, *we have that, if*

$$E \geqslant \frac{3\alpha + (1+\eta)\underline{\omega}\mathcal{M}_1(\omega)}{(1-\eta)^2 \mathcal{M}_1(\omega)} =: E_0 \tag{9.13}$$

then

$$\|V_{r+1}\|_{\rho_{r+1}} \leqslant d_r \|V_r\|_{\rho_r}^2 + e^{-\delta_r M_r} \|V_r\|_{\rho_r} \qquad \text{with } d_r = \frac{\mathcal{M}_{M_r}^2 Z_k E}{\delta_r^3}.$$

The proof is immediate by noticing that, under proposition 19, (9.3) and (9.5), (6.31) gives that, for $r = 1, \dots,$

$$F_r \leqslant \frac{\mathcal{M}_{M_r} Z_k}{\delta_r^2 (1 - \eta + C/r)^2} \left(3 + 2\frac{\mathcal{M}_{M_r}}{\delta_r} \underline{\omega}(1 + \eta - C/r)\right)$$

so, for $r = 0, 1, \dots$

$$F_r \leqslant \frac{\mathcal{M}_{M_r} Z_k}{\delta_r^2 (1 - \eta)^2} \left(3 + 2\frac{\mathcal{M}_{M_r}}{\delta_r} \underline{\omega}(1 + \eta)\right).$$

The case $r \geqslant 1$ is obtained out of the preceding inequality, and the case $r = 0$ comes form the fact that $Z_k G_0 = 0 \leqslant \eta$.

Therefore E must be $\geqslant \frac{3+\underline{\omega}(1+\eta)\frac{\mathcal{M}_{M_r}}{\delta_r}(\omega)}{(1-\eta)^2\frac{\mathcal{M}_{M_r}}{\delta_r}(\omega)} \leqslant \frac{3+\underline{\omega}(1+\eta)\mathcal{M}_1(\omega)/\alpha}{(1-\eta)^2\mathcal{M}_1(\omega)/\alpha} = \frac{3\alpha+\underline{\omega}(1+\eta)\mathcal{M}_1(\omega)}{(1-\eta)^2\mathcal{M}_1(\omega)}$ since $\mathcal{M}_{M_r}$is increasing with M_r and the Lemma is proved.

Lemma 24. *Let* $\widetilde{V}_r = \|V_r\|_{\rho_r} + \frac{e^{-\delta_r M_r}}{2d_r}$ *where* $d_r = \frac{\mathcal{M}^2_{M_r} Z_k E}{\delta_r^3}$, V_r *satisfy* (6.31) *and* $V_0 := V, \rho_o = \rho$. *Then*

$$\widetilde{V}_{r+1} \leqslant d_r \widetilde{V}_r^2. \tag{9.14}$$

The proof reduces to completing the square in Proposition 19 and noticing that $\frac{e^{-2\delta_r M_r}}{4d_r} - \frac{e^{-\delta_{r+1}M_{r+1}}}{2d_{r+1}} > 0$ by (9.4), since $\mathcal{M}_{M_{r+1}} \geqslant \mathcal{M}_{M_r}$. The Lemma has for consequence the fact the

$$\widetilde{V}_{r+1} \leqslant \prod_{s=0}^{r} d_s^{2^{r-s}} \widetilde{V}_0^{2^r} \leqslant (e^{C_k}\widetilde{V}_0)^{2^r}. \tag{9.15}$$

This concludes the proof of Proposition 22 since $\|V_r\|_{\rho_r} \leqslant \widetilde{V}_r$. □

Proposition 25. *Let* $\|V_r\|_{\rho_r} \leqslant (D_k)^{2^r}$ *with* $D_k < e^{-P}$ *and* $D_k < M$, M *and* P *defined below by* (9.23) *and* (9.20). *Then* (9.3), (9.4) *and* (9.5) *hold.*

Note that $\sum\limits_{r=0}^{\infty} \delta_r = 2\alpha$.

Proof. (9.4):
it is trivial to show that (9.4) is satisfied when $\alpha \leqslant 2\log 2$.
(9.5):
(9.5)-(9.8) are equivalent to

$$\frac{1}{2^r}\log \mathcal{M}_{M_r} - \frac{\log \delta_r^2}{2^r} + \frac{\log Z_k}{2^r} + \log D_k < \frac{1}{2^r}\log\frac{1}{2}(1-\eta+\frac{C}{r}) \tag{9.16}$$

and

$$\log \mathcal{M}_1 - \log \delta_0^2 + \log Z_k + \log D_k < \log\frac{1}{2} \tag{9.17}$$

which is implied by

$$\log D_k \quad < \quad -\sum_{r=0}^{\infty} \frac{|\log \mathcal{M}_{M_r}|}{2^r} + \inf_{r\geqslant 0}\frac{\log \delta_r^2 - \log Z_k}{2^r} - \Delta \tag{9.18}$$

$$< \quad -\sum_{r=0}^{\infty} \frac{|\log \mathcal{M}_{M_r}|}{2^r} - \log Z_k + 2\log\alpha - \frac{2}{e} - \Delta \tag{9.19}$$

which is implied by

$$\log D_k < -\sum_{r=0}^{\infty} \frac{|\log \mathcal{M}_{M_r}|}{2^r} - \log Z_k - \frac{2}{e} - \Delta := -P \tag{9.20}$$

where

$$\Delta = -\inf_{r\geqslant 1}\left\{\frac{1}{2^r}\log\frac{1}{2}(1-\eta+\frac{C}{r}), \log\frac{1}{2}\right\} < \infty \text{ for } \eta < 1. \tag{9.21}$$

Note that $\Delta > 0$, $P > 0$.

(9.3):

remember that $\mathcal{B}_{r+1} = \mathcal{B}_r + \overline{V}_{r+1}$, and $\|\mathcal{B}_{r+1}\|_{\rho_{r+1}} \leqslant \|\mathcal{B}_r\|_{\rho_{r+1}} + \|\overline{\mathcal{V}}'_{r+1}\|_{\rho_{r+1}} \leqslant \|\mathcal{B}_r\|_{\rho_r} + \|\overline{\mathcal{V}}'_{r+1}\|_{\rho_r}$. So $\|D\mathcal{B}_{r+1}\|_{\rho_{r+1}} \leqslant \|D\mathcal{B}_r\|_{\rho_{r+1}} + \|D\overline{\mathcal{V}}'_{r+1}\|_{\rho_{r+1}}$.

Moreover one has, by (5.18), $\|D\overline{\mathcal{V}}'_{r+1}\|_{\rho_{r+1}-\frac{\delta_r}{2}} \leqslant \frac{\|\overline{\mathcal{V}}'r+1\|_{\rho_{r+1}}}{\frac{\delta_r}{2}e} \leqslant 2\frac{\|V_{r+1}\|_{\rho_{r+1}}}{\delta_r e} \leqslant 2\frac{(D_k)^{2^{r+1}}}{\delta_r e}$ out of which we conclude that

$Z_kG_r < \eta - C/r \Longrightarrow Z_kG_{r+1} < \eta - C/(r+1), \forall r \geqslant 1$, if

$$2Z_k\frac{D_k^{2^{r+1}}}{\alpha 2^{-r}e} < \frac{C}{r} - \frac{C}{r+1} = \frac{C}{r(r+1)} \tag{9.22}$$

which is implied by $D_k < M$ for

$$M = \inf_{r\geqslant 1}\left(\frac{\alpha 2^{-r}eC}{2Z_kr(r+1)}\right)^{2^{-(r+1)}} = \left(\frac{\alpha eC}{8Z_k}\right)^{1/4} < 1 \tag{9.23}$$

since $\alpha \leqslant \log 2$, $Z_k \geqslant 8$ and $C \leqslant 1$. □

Proposition 25 together with Proposition 22 shows clearly that

$$(9.12) \text{ and } \left[D_k < e^{-P} \text{ and } D_k < M\right] \Longrightarrow \|V_r\|_{\rho_r} \leqslant (D_k)^{2^r} \tag{9.24}$$

where D_k is given by (9.10) i.e. $D_k := e^{C_k}\|V\|_\rho + e^{C_k}\frac{e^{-\delta_0 M_0}}{2d_0}$.Note that since $M < 1$ so is $D_k \leqslant M$ leading ot the superquadratic convergence of the sequence $(V_r)_{r=0,\dots}$. In order for D_k to satisfy the two conditions of the bracket in the l.h.s. of (9.24) the two terms in D_k will have to both satisfy the two conditions. This remark will be the key of the main theorem below.

Let us denote by $\omega_j^i,\ j = 1\dots m,\ i = 1\dots l$ be the ith component of the vector ω_j. Let us remark that

$$\mathcal{M}_1(\omega) = \min_{j=1\dots m}\frac{1}{\min\limits_{i=1\dots l}|\omega_j^i|} \text{ and } \frac{1}{\mathcal{M}_1(\omega)} = \max_{j=1\dots m}\min_{i=1\dots l}|\omega_j^i|. \tag{9.25}$$

Let us denote

$$B(\omega) := \sum_{r=0}^{\infty}\frac{|\log\mathcal{M}_{2^r}|}{2^r}. \tag{9.26}$$

We have that, by (9.2) and (9.20),

$$C_k(\omega) = 2B(\omega) - 6\log\alpha + 6\log 2 + 2\log(Z_kE). \tag{9.27}$$

and

$$P(\omega) = B(\omega) + \log Z_k + \frac{2}{e} + \Delta. \tag{9.28}$$

Theorem 26. *[Brjuno case] Let $\alpha, \rho, \eta,$ and C be strictly positive constants satisfying*

$$\alpha < 2\log 2, \quad \rho > 2\alpha, \quad 0 < C < \eta < 1. \tag{9.29}$$

Let us define, for $\Delta = -\inf_{r\geqslant 1} \frac{1}{2^r}\log\frac{1}{2}(1-\eta+\frac{C}{r})$ and $M = \left(\frac{\alpha e C}{8Z_k}\right)^{\frac{1}{4}}$,

$$R_k(\omega) = \frac{(1-\eta)^4\mathcal{M}_1(\omega)^2}{(3\alpha+(1+\eta)\underline{\omega}\mathcal{M}_1(\omega))^2}\frac{\alpha^6 e^{-2B(\omega)}}{2^6 Z_k^2}\min\left\{\frac{e^{-B(\omega)-\Delta}}{2^{1/e}Z_k}, M\right\}. \tag{9.30}$$

Let us suppose that, in addition to Assumptions (A1), (A2) Brjuno case and (A3), ω satisfies

$$\frac{3\alpha+(1+\eta)\underline{\omega}\mathcal{M}_1(\omega)}{2e^{\alpha}(1-\eta)^2\mathcal{M}_1(\omega)^3} \leqslant \frac{\alpha^3 e^{-2B(\omega)}}{2^6 Z_k}\min\left\{\frac{e^{-B(\omega)-\Delta}}{2^{1/e}Z_k}, M\right\}. \tag{9.31}$$

and the perturbation V satisfies

$$\|V\|_{\rho,\underline{\omega},k} < R_k(\omega), \ \|\nabla\overline{\mathcal{V}'}\|_{\rho,\underline{\omega},k} < \frac{\eta-C}{Z_k}. \tag{9.32}$$

Then the BNF as constructed in section 2 converges in the space $\mathcal{J}_k^\dagger(\rho-2\alpha,\underline{\omega})$ to $\mathcal{B}_\infty^\hbar$ and

$$\|\mathcal{B}_\infty^\hbar - \mathcal{B}_0^\hbar\|_{\rho-2\alpha,\underline{\omega},k} = O(\|V\|^2_{\rho,\underline{\omega},k}) \text{ as } \|V\|_{\rho,\underline{\omega},k} \to 0. \tag{9.33}$$

That is to say that there exists a (scalar) unitary operator U_∞ such that the family of operators $H = (H_i)_{i=1\dots m}, H_i = L_{\omega_i} + V_i$, satisfies, $\forall\hbar \in (0,1]$,

$$U_\infty^{-1} H U_\infty = \mathcal{B}_\infty^\hbar(L_\omega). \tag{9.34}$$

U_∞ is the limit as $r \to \infty$ of the sequence of operators $U_r = e^{i\frac{W_r}{\hbar}}\dots e^{i\frac{W_0}{\hbar}}$ constructed in Section 2 and

$$\|U_\infty - U_r\|_{L^2(\mathbb{T}^l)\to L^2(\mathbb{T}^l)} \leqslant \frac{A_r}{\hbar} = O\left(\frac{E^{2r}}{\hbar}\right) \text{ as } r\to\infty \text{ for some } E<1,$$

here A_r is defined by (9.53).

Moreover, $U_\infty - I \in J_0^\hbar(\rho-2\alpha,\underline{\omega})$ and

$$\|U_\infty - I\|^\hbar_{\rho-2\alpha,\underline{\omega},0} = O\left(\frac{\|V\|_{\rho,\underline{\omega},0}}{\hbar}\right) \text{ as } \|V\|_{\rho,\underline{\omega},0} \to 0, \tag{9.35}$$

and, for any operator X for which there exists $\overline{X}_{k,\rho}$ such that for all $W \in J_k(\rho,\underline{\omega})$,

$$\|[X,W]/i\hbar\|_{\rho-\delta,\underline{\omega},k} \leqslant \frac{Z_k}{\delta^2}\overline{X}_{k,\rho}\|W\|_{\rho,\underline{\omega},k}, \tag{9.36}$$

$U_\infty^{-1}XU_\infty - X \in J_k(\rho-2\alpha-\delta,\underline{\omega})$ and

$$\|U_\infty^{-1}XU_\infty - X\|_{\rho-2\alpha-\delta,\underline{\omega},k} \leqslant \frac{D}{\delta^2}\sup_{\rho-2\alpha\leqslant\rho'\leqslant\rho}\overline{X}_{k,\rho'} = O\left(\frac{\|V\|_{\rho,\underline{\omega},k}}{\delta^2}\sup_{\rho-2\alpha\leqslant\rho'\leqslant\rho}\overline{X}_{k,\rho'}\right) \tag{9.37}$$

where D is given by (9.57).

Note that the second condition in (9.32) “touches” only the average $\overline{V}$ and not the full perturbation V. It can also be replaced for any $\rho' > \rho$ by $\|\overline{V}\|_{\rho'} \leqslant \|V\|_{\rho'} \leqslant e\frac{\rho'-\rho}{Z_k}$ since $\|\nabla\overline{\mathcal{V}'}\|_{\rho-\delta} \leqslant \frac{\|\overline{V'}\|_\rho}{e\delta}$ for any $\delta > 0$.

Before we start the proof of theorem 26, let us show the way of overcoming the condition (9.31).
We first notice that multiplying the family $L_\omega + V$ by $\lambda > 0$ preserves of course integrability. Moreover $\lambda(L_\omega + V) = L_{\lambda\omega} + \lambda V$.
On the other side we see easily that:

$$B(\lambda\omega) = B(\omega) - 2\log\lambda,\ \mathcal{M}_1(\lambda\omega) = \lambda^{-1}\mathcal{M}_1(\omega) \text{ and therefore } \underline{\omega}\mathcal{M}_1 \text{ is invariant by scaling.} \tag{9.38}$$

Let us show that, for λ large enough, (9.31) will be satisfied for $\omega_\lambda := \lambda\omega$. More precisely, let us define

$$\begin{aligned}\mu &= \frac{\alpha + 2[(1-\eta)\alpha + (1+\eta)\underline{\omega}\mathcal{M}_1(\omega)]}{2e^\alpha(1-\eta)^2\mathcal{M}_1(\omega)^3}\frac{2^6 Z_k}{\alpha^3 e^{-2B(\omega)}}\\ \nu &= \frac{e^{-B(\omega)-\Delta}}{2^{1/e}Z_k}\end{aligned}$$

we easily see that the following number λ_0 is uniquely defined:

$$\lambda_0 = \lambda_0(\omega) := \inf\left\{\lambda > 0 \text{ such that } M\lambda - \mu \geqslant 0 \text{ and } \nu\lambda^3 - \mu \geqslant 0\right\} = \sup\left\{\frac{\mu}{M}, \left(\frac{\mu}{\nu}\right)^{\frac{1}{3}}\right\}. \tag{9.39}$$

Elementary algebra leads to
Lemma. $\forall\omega$, $\forall\lambda \geqslant \lambda_o(\omega)$, (9.31) is satisfied for $\omega_\lambda := \lambda\omega$.

Since the BNF of λH is the BNF of H multiplied by λ we get that the latter will exist and be convergent if $\lambda\|V\|_{\rho,\lambda\underline{\omega},k} \leqslant R_k(\lambda\omega)$ and $\lambda\|\nabla\overline{\mathcal{V}'}\|_{\rho,\lambda\underline{\omega},k} \leqslant \lambda\frac{\eta-C}{Z_k}$. we get the

Theorem 27. *Let $\alpha, \rho, \eta,$ and C be strictly positive constants satisfying*

$$\alpha < 2\log 2, \quad \rho > 2\alpha, \quad 0 < C < \eta < 1. \tag{9.40}$$

Let us define $\Delta = -\inf_{r\geqslant 1}\frac{1}{2^r}\log\frac{1}{2}(1-\eta+\frac{C}{r})$, $M = \left(\frac{\alpha e C}{8Z_k}\right)^{\frac{1}{4}} = \inf_{r\geqslant 1}\left(\frac{\alpha 2^{-r}eC}{2Z_k r(r+1)}\right)^{2^{-(r+1)}}$ *and, for* $\lambda \geqslant \lambda_0(\omega)$ *given by* (9.39),

$$R_{\lambda,k}(\omega) = \frac{R_k(\lambda\omega)}{\lambda} = \lambda\frac{(1-\eta)^4\mathcal{M}_1(\omega)^2}{(3\alpha + (1+\eta)\underline{\omega}\mathcal{M}_1(\omega))^2}\frac{\alpha^6 e^{-2B(\omega)}}{2^6 Z_k^2}\min\left\{\lambda^2\frac{e^{-B(\omega)-\Delta}}{2^{1/e}Z_k}, M\right\}. \tag{9.41}$$

Let us suppose that the general assumption (A1), (A2) Brjuno case and (A3) hold and

$$\|V\|_{\rho,\lambda\underline{\omega},k} < R_{\lambda,k}(\omega),\ \|\nabla\overline{\mathcal{V}'}\|_{\rho,\lambda\underline{\omega},k} \leqslant \frac{\eta - C}{Z_k}. \tag{9.42}$$

Then the BNF as constructed in section 2 converges in the space $\mathcal{J}_k^\dagger(\rho-2\alpha,\lambda\underline{\omega})$ *to* $\mathcal{B}_\infty^\hbar$ *and*

$$\|\mathcal{B}_\infty^\hbar-\mathcal{B}_0^\hbar\|_{\rho-2\alpha,\lambda\underline{\omega},k}=O(\|V\|^2_{\rho,\lambda\underline{\omega},k}). \tag{9.43}$$

That is to say that there exists a (scalar) unitary operator U_∞, $U_\infty-I\in J_k(\rho-2\alpha,\lambda\underline{\omega})$ *such that the family of operators* $H=(H_i)_{i=1\dots m}, H_i=L_{\omega_i}+V_i$, *satisfies,* $\forall\hbar\in(0,1]$,

$$U_\infty^{-1}HU_\infty=\mathcal{B}_\infty^\hbar(L_\omega). \tag{9.44}$$

U_∞ *is the limit as* $r\to\infty$ *of the sequence of operators* $U_r=e^{i\frac{W_r}{\hbar}}\dots e^{i\frac{W_0}{\hbar}}$ *constructed in Section 2 and*

$$\|U_\infty-U_r\|_{L^2(\mathbb{T}^l)\to L^2(\mathbb{T}^l)}\leqslant\frac{A_r}{\hbar}=O\left(\frac{E^{2r}}{\hbar}\right)\ \textit{as } r\to\infty \textit{ for some } E<1,$$

here A_r *is defined by* (9.53).

Moreover, $U_\infty-I\in J_0^\hbar(\rho-2\alpha,\lambda\underline{\omega})$ *and*

$$\|U_\infty-I\|^\hbar_{\rho-2\alpha,\lambda\underline{\omega},0}=O\left(\frac{\|V\|_{\rho,\lambda\underline{\omega},0}}{\hbar}\right)\ \textit{as } \|V\|_{\rho,\lambda\underline{\omega},0}\to 0, \tag{9.45}$$

and, for any operator X *for which there exists* $\overline{X}_{k,\rho,\lambda}$ *such that for all* $W\in J_k(\rho,\lambda\underline{\omega})$,

$$\|[X,W]/i\hbar\|_{\rho-\delta,\lambda\underline{\omega},k}\leqslant\frac{Z_k}{\delta^2}\overline{X}_{k,\rho,\lambda}\|W\|_{\rho,\lambda\underline{\omega},k}, \tag{9.46}$$

$U_\infty^{-1}XU_\infty-X\in J_k(\rho-2\alpha-\delta,\lambda\underline{\omega})$ *and*

$$\|U_\infty^{-1}XU_\infty-X\|_{\rho-2\alpha-\delta,\lambda\underline{\omega},k}\leqslant\frac{D}{\delta^2}\sup_{\rho-2\alpha\leqslant\rho'\leqslant\rho}\overline{X}_{k,\rho',\lambda}=O\left(\frac{\|V\|_{\rho,\lambda\underline{\omega},k}}{\delta^2}\sup_{\rho-2\alpha\leqslant\rho'\leqslant\rho}\overline{X}_{k,\rho',\lambda}\right) \tag{9.47}$$

where D *is given by* (9.57).

Remark 28. Note that, for λ large enough, $R_{\lambda,k}(\omega)=\lambda R_k(\omega)$. Therefore the radius of convergence increases by dilating ω. But this fact is compensated by the fact that the norm in the condition of convergence (9.42) (we take here $\overline{\mathcal{V}}=0$), $\|\mathcal{V}'\|_{\rho,\lambda\underline{\omega},0}<R_{\lambda,k}(\omega)$, increases at least as λ (an actually highly non sharp estimate as the Gaussian case shows clearly) when λ is large, as shown by the following lemma. Therefore the optimization on λ of (9.42) remains between bounded values of λ.

Lemma 29. *For* $\underline{\omega}'\geqslant\underline{\omega}$, $\|\mathcal{F}\|_{\rho,\underline{\omega}',k}-\|\mathcal{F}\|_{\rho,\underline{\omega}.k}\geqslant(\underline{\omega}'-\underline{\omega})\rho\|\nabla\mathcal{F}\|_{\rho,\underline{\omega},k}$.

The proof is an immediate consequence of

$$e^{\rho'X}-e^{\rho X}=e^{\rho X}(e^{(\rho'-\rho)X}-1)\geqslant e^{\rho X}(\rho'-\rho)X,\ X\geqslant 0.$$

Proof of Theorem 26. First notice that $\mathcal{B}_r^\hbar-\mathcal{B}_{r-1}^\hbar=\overline{V}_r=\widetilde{V}_r(\cdot,0,\hbar)$ so $\|\mathcal{B}_r^\hbar-\mathcal{B}_0^\hbar\|_{\rho_r}\leqslant\sum_1^r\|V_l\|_l$ which is convergent under (9.31) and (9.32). What is left is to show that the sequence of unitary operators $U_r:=e^{i\frac{W_r}{\hbar}}\dots e^{i\frac{W_1}{\hbar}}$ converges to a unitary operator on $L^2(\mathbb{T}^l)$. This is done by proving that the sequence U_r is Cauchy ($\hbar\in(0,1]$). For $p>n$ let us denote

$$E_{np}=e^{i\frac{W_{n+p}}{\hbar}}e^{i\frac{W_{n+p-1}}{\hbar}}\dots e^{i\frac{W_{n+1}}{\hbar}}-I, \tag{9.48}$$

so that $U_{n+p} - U_n = E_{np}U_n$. We have for all r,

$$e^{i\frac{W_r}{\hbar}} = I + T_r \text{ with } T_r = i\frac{W_r}{\hbar}\int_0^1 e^{it\frac{W_r}{\hbar}}dt. \tag{9.49}$$

Therefore

$$\hbar\|T_r\|_{L^2(\mathbb{T}^l)\to L^2(\mathbb{T}^l)} \leqslant \|W_r\|_{L^2(\mathbb{T}^l)\to L^2(\mathbb{T}^l)} \leqslant \|W_r\|_{\rho_r,k}. \tag{9.50}$$

By (6.11) we have also that

$$\begin{aligned}\|W_r\|_{\rho_r} = \|W_r\|_{\rho_r,k} &\leqslant \frac{\mathcal{M}_{M_r}}{1-\eta+C/r}\|V_r\|_{\rho_r,k} \leqslant \frac{\mathcal{M}_{M_r}}{1-\eta+C/r}D_k^{2^r} \quad l>0\\ \|W_0\|_\rho = \|W_0\|_{\rho,k} &\leqslant \mathcal{M}_1\|V\|_{\rho,k}\end{aligned} \tag{9.51}$$

Note that, by the Brjuno condition, we have for all r $\mathcal{M}_{M_r} \leqslant e^{B(\omega)2^r}$ and, by the condition on D_k insuring the convergence of the BNF, $D_k < e^{-B(\omega)}$, so that:

$$A := \sum_{r=1}^{\infty}\frac{\mathcal{M}_{M_r}}{1-\eta+C/r}D_k^{2^r} \leqslant \frac{(e^{B(\omega)}D_k)^{2^r}}{1-\eta+C/r} < \infty. \tag{9.52}$$

We also define, for $n \geqslant 1$,

$$A_n = \sum_{r=n}^{\infty}\frac{\mathcal{M}_{M_r}}{1-\eta+C/r}D_k^{2^r}. \tag{9.53}$$

Note that $A_n = O(E^{2^n})$ as $n\to\infty$ for $E = e^B D_k < 1$ by (9.24).

By (9.49) we get that

$$\begin{aligned}E_{np} &= e^{i\frac{W_{n+p}}{\hbar}}E_{np-1} - I + e^{i\frac{W_{n+p}}{\hbar}}\\ &= e^{i\frac{W_{n+p}}{\hbar}}E_{np-1} + T_{n+p}\\ &= e^{i\frac{W_{n+p}}{\hbar}}e^{i\frac{W_{n+p-2}}{\hbar}}E_{np-1} + e^{i\frac{W_{n+p}}{\hbar}}T_{n+p-1} + T_{n+p}.\end{aligned} \tag{9.54}$$

By iteration we find easily that

$$E_{np} = \sum_{k=2}^{p} e^{i\frac{W_{n+p}}{\hbar}}\dots e^{i\frac{W_{n+p-k+1}}{\hbar}}T_{n+p-k} + e^{i\frac{W_{n+p}}{\hbar}}T_{n+p-1} + T_{n+p}$$

and, by unitarity of $e^{i\frac{W_r}{\hbar}}$ and (9.50),

$$\begin{aligned}\|E_{np}\|_{L^2(\mathbb{T}^l)\to L^2(\mathbb{T}^l)} &\leqslant \sum_{k=0}^{p}\|T_{n+k}\|_{L^2(\mathbb{T}^l)\to L^2(\mathbb{T}^l)} \leqslant \sum_{k=0}^{p}\|T_{n+k}\|_{\rho_{n+k}} \leqslant \sum_{k=0}^{p}\frac{\|W_{n+k}\|_{\rho_{n+k}}}{\hbar}\\ &\leqslant \sum_{k=0}^{\infty}\frac{\|W_{n+k}\|_{\rho_{n+k}}}{\hbar} \leqslant \frac{A_n}{\hbar} \to 0 \text{ as } n\to\infty \text{ since } A<\infty.\end{aligned}$$

So $\|E_{np}\|_{L^2(\mathbb{T}^l)\to L^2(\mathbb{T}^l)} \to 0$ as $n\to\infty$ and so does $U_{n+p} - U_n = E_{np}U_n$ by unitarity of U_n, and U_n converges to U_∞ in the operator topology. Moreover we get as a by-product of the preceding estimate that

$$\|U_\infty - U_r\|_{L^2(\mathbb{T}^l)\to L^2(\mathbb{T}^l)} \leqslant \frac{A_r}{\hbar}.$$

Since U_∞ is a perturbation of the identity which doesn't belong to any $J(\rho)$, $\rho > 0$, there is no hope to estimate $\|U_\infty\|_{\rho,\underline{\omega},k}$. Nevertheless, and somehow more interesting, we will estimate $U_\infty - I$ in the $\|\cdot\|_{\rho-2\alpha,\underline{\omega},0}$ topology. In the sequel of this proof we will denote $\|\cdot\|_\rho := \|\cdot\|_{\rho,\underline{\omega},0}$ and use the fact that $\|\cdot\|_{\rho_r} \geqslant \|\cdot\|_{\rho-2\alpha}$, $\forall r \in \mathbb{N}$.

We first remark that, for $r \geqslant 0$, since $\|\cdot\|^\hbar_\rho \leqslant \|\cdot\|_\rho$,

$$\|T_r\|^\hbar_{\rho-2\alpha} = \|e^{i\frac{W_r}{\hbar}} - I\|^\hbar_{\rho-2\alpha} \leqslant \|e^{i\frac{W_r}{\hbar}} - I\|^\hbar_{\rho_r} \leqslant \sum_{j=1}^{\infty} \frac{(Z_0)^{j-1}\|\frac{W_r}{\hbar}\|^j_{\rho_r}}{j!} = \frac{e^{\frac{Z_0\|W_r\|_{\rho_r}}{\hbar}} - 1}{Z_0}$$

We first remark also that

$$(I + T_{r+1})U_r = U_{r+1}.$$

Therefore, denoting $P_r = U_r - I$,

$$P_{r+1} = (I + T_{r+1})P_r + T_{r+1}$$

so

$$\|P_{r+1}\|_\rho \leqslant \|P_r\|(Z_0\|T_{r+1}\|_\rho + 1) + \|T_{r+1}\|_\rho = (\|P_r\|_\rho + \frac{1}{Z_0})\|(Z_0\|T_{r+1}\|_\rho + 1) - \frac{1}{Z_0}$$

so $\|P_{r+1}\|_\rho + \frac{1}{Z_0} \leqslant (\|P_r\|_\rho + \frac{1}{Z_0})\|(Z_0\|T_{r+1}\|_\rho + 1)$ and

$$\begin{aligned}\|P_{r+1}\|^\hbar_{\rho-2\alpha} + \frac{1}{Z_0} \leqslant (\|P_0\|^\hbar_\rho + \frac{1}{Z_0}\prod_{j=1}^{r+1}\|(Z_0\|T_j\|^\hbar_{\rho_j} + 1) &\leqslant (\|P_0\|^\hbar_\rho + \frac{1}{Z_0})\prod_{j=1}^{r+1} e^{\frac{\|W_j\|_{\rho_j}}{\hbar}} \\ &= e^{\sum_{j=1}^{r+1}\frac{\|W_j\|_{\rho_j}}{\hbar}}(\|P_0\|^\hbar_\rho + \frac{1}{Z_0}) \\ &\leqslant e^{\frac{A}{\hbar}}(\|P_0\|^\hbar_\rho + \frac{1}{Z_0})\end{aligned}$$

Therefore

$$\|U_\infty - I\|^\hbar_{\rho-2\alpha} = \|P_\infty\|^\hbar_{\rho-2\alpha} \leqslant e^{\frac{A}{\hbar}}\left(\frac{\mathcal{M}_1\|V\|_\rho}{\hbar} + \frac{1}{Z_0}\right) - \frac{1}{Z_0}$$

Let us note that, by construction, $A = O(\frac{D_k^2}{1-\eta})$ and that D_k depends on η through (9.10).

Lemma 30. *$\exists\eta = \eta(\|V\|_\rho)$ such that*

$$\frac{D_k^2}{1-\eta} = O(\|V\|_\rho) \text{ as } \|V\|_\rho \to 0.$$

Proof. By looking at the expression of the radius of convergence which tends to 0 as $\eta \to 1$ we see that as $V \to 0$ one can take values of $\eta \to 1$ which makes the second term in the definition of D_k of order $\|V\|_\rho$ and the ratio $\frac{D_k}{1-\eta}$ of order $\|V\|_\rho$. □

By application of the Lemma we find that

$$\|U_\infty - I\|^\hbar_{\rho-2\alpha} = \|P_\infty\| \leqslant e^{\frac{A}{\hbar}}\left(\frac{\mathcal{M}_1\|V\|_\rho}{\hbar} + \frac{1}{Z_0}\right) - \frac{1}{Z_0} = O\left(\frac{\|V\|_\rho}{\hbar}\right).$$

which gives (9.45).

In order to prove (9.47) we first denote $V_r = e^{i\frac{W_r}{\hbar}}$.

We have, actually for any operator X, that $V_r X V_r^{-1} - X = \sum_{j=1}^{\infty} \frac{1}{j!}\mathrm{ad}^j_{W_r}(X)$.

Let us suppose now that $\exists \overline{X}_{\rho,k}$ such that for all $W \in J_k(\rho)$

$$\|[X,W]/i\hbar\|_{\rho-\delta} \leqslant \frac{Z_k}{\delta^2}\overline{X}_{k,\rho}\|W\|_\rho. \tag{9.55}$$

(e.g. $\overline{X}_{k,\rho} = \|X\|_{k,\rho}$).

Using (5.11) we get (since $2\pi l > 1$) we get

$$\|V_r F V_r^{-1}\|_{\rho-\delta} \leqslant \frac{\|F\|_\rho}{1-\frac{Z_k}{\delta^2}\|W_r\|_\rho}, \tag{9.56}$$

and also (let us recall again that $\rho_{r+1} = \rho_r - \delta_r,\ \rho_0 = \rho$)

$$\begin{aligned}\|\frac{1}{j!\hbar^j}\mathrm{ad}^j_{W_r}(X)\|_{\rho_r-\delta_r-\delta=\rho_{r+1}-\delta} &\leqslant \left(\frac{Z_k}{\delta_r^2}\right)^{j-1}\|[X,W_r]/i\hbar\|_{\rho_r-\delta}\|W_l\|^{j-1}_{\rho_r-\delta}\\ &\leqslant \left(\frac{Z_0}{\delta_r^2}\right)^{j-1}\|[X,W_r]/i\hbar\|_{\rho_r-\delta}\|W_l\|^{j-1}_{\rho_r},\end{aligned}$$

out of which we get

$$\|V_0XV_0^{-1} - X\|_{\rho_1-\delta} \leqslant \frac{\|[X,W_0]/i\hbar\|_{\rho_0-\delta}}{1-\frac{Z_k}{\delta_0^2}\|W_0\|_\rho}.$$

by $V_1V_0XV_0^{-1}V_1^{-1} - V_1XV_1^{-1} = V_1(V_0XV_0^{-1}-X)V_1^{-1}$ and (9.56)

$$\begin{aligned}\|V_1V_0XV_0^{-1}V_1^{-1} - V_1XV_1^{-1}\|_{\rho_2-\delta} &\leqslant \frac{\|V_0XV_0^{-1}-X\|_{\rho_1-\delta}}{1-\frac{Z_k}{\delta_1^2}\|W_1\|_{\rho_1}}\\ &\leqslant \frac{\|[X,W_0]/i\hbar\|_{\rho_0-\delta}}{(1-\frac{Z_k}{\delta_0^2}\|W_0\|_\rho)(1-\frac{Z_k}{\delta_1^2}\|W_1\|_{\rho_1})}\\ &\text{and by iteration}\\ \|U_rXU_r^{-1} - U_rV_0^{-1}XV_0U_r^{-1}\|_{\rho_{r+1}-\delta} &\leqslant \frac{\|[X,W_0]/i\hbar\|_{\rho_0-\delta}}{(1-\frac{Z_k}{\delta_0^2}\|W_0\|_\rho)\dots(1-\frac{Z_k}{\delta_r^2}\|W_r\|_{\rho_r})}\\ &\text{and by } X\to V_0^{-1}XV_0\\ \|U_rV_0^{-1}XV_0U_r^{-1} - U_rV_1^{-1}V_0^{-1}XV_0V_1U_r^{-1}\|_{\rho_{r+1}-\delta} &\leqslant \frac{\|[X,W_1]/i\hbar\|_{\rho_1-\delta}}{(1-\frac{Z_k}{\delta_1^2}\|W_1\|_\rho)\dots(1-\frac{Z_k}{\delta_r^2}\|W_r\|_{\rho_r})}\\ &\ \ \vdots\\ \|V_rXV_r^{-1} - X\|_{\rho_{r+1}-\delta} &\leqslant \frac{\|[X,W_r]/i\hbar\|_{\rho_r-\delta}}{1-\frac{Z_k}{\delta_r^2}\|W_r\|_{\rho_r}}.\end{aligned}$$

So that by summing the telescopic sequence we get

$$\begin{aligned}
\|U_rXU_r^{-1}-X\|_{\rho_{r+1}-\delta} &\leqslant \sum_{s=0}^{r}\|[X,W_s]/i\hbar\|_{\rho_s-\delta}\prod_{j=s}^{r}\frac{1}{1-\frac{Z_k}{\delta_j^2}\|W_j\|_{\rho_j}}\\
&\leqslant \sum_{s=0}^{r}\frac{\overline{X}_{k,\rho_s}}{\delta^2}Z_k\|W_s\|_{\rho_s}e^{-\sum_{j=s}^{r}\log\left(1-\frac{Z_k}{\delta_j^2}\|W_j\|_{\rho_j}\right)}\\
\text{and since } (1-a)(1+2a)\geqslant 1 \quad &\text{if} \quad 0<a\leqslant 1/2\\
&\leqslant \sum_{s=0}^{r}\frac{\overline{X}_{k,\rho_s}}{\delta^2}Z_k\|W_s\|_{\rho_s}e^{\sum_{j=s}^{r}\log\left(1+2\frac{Z_k}{\delta_j^2}\|W_j\|_{\rho_j}\right)}\\
&\leqslant \sum_{s=0}^{r}\frac{\overline{X}_{k,\rho_s}}{\delta^2}Z_k\|W_s\|_{\rho_s}e^{2\sum_{j=s}^{r}\frac{Z_k}{\delta_j^2}\|W_j\|_{\rho_j}}\\
&\leqslant \sum_{s=0}^{r}\frac{\overline{X}_{k,\rho_s}}{\delta^2}Z_k\|W_s\|_{\rho_s}e^{2\sum_{j=0}^{\infty}\frac{Z_k}{\delta_j^2}\|W_j\|_{\rho_j}}\\
&\leqslant \frac{\sup_{\rho-2\alpha\leqslant\rho'\leqslant\rho}\overline{X}_{k,\rho'}}{\delta^2}Z_k\sum_{s=0}^{r}\|W_s\|_{\rho_s}e^{2B}\\
&\leqslant \frac{\sup_{\rho-2\alpha\leqslant\rho'\leqslant\rho}\overline{X}_{k,\rho'}}{\delta^2}D
\end{aligned}$$

with $B=\frac{Z_k}{\alpha^2}\mathcal{M}_1\|V\|_\rho+\sum_{j=1}^{\infty}\frac{Z_k2^j\mathcal{M}_{M_j}}{\alpha^2(1-\eta+C/j)}D_k^{2^j}<\infty$ and

$$D=Z_k(\mathcal{M}_1\|V\|_\rho+A)e^{2B}=O(\|V\|_\rho) \tag{9.57}$$

by Lemma 30.

Therefore we get, by letting $r\to\infty$ so that $\rho_r\to\rho-2\alpha$,,

$$\|U_\infty^{-1}XU_\infty-X\|_{\rho-2\alpha-\delta}\leqslant\frac{D}{\delta^2}\sup_{\rho-2\alpha\leqslant\rho'\leqslant\rho}\overline{X}_{k,\rho'}. \tag{9.58}$$

$$\infty>\|U_\infty^{-1}XU_\infty-X\|_{\rho-2\alpha-\delta}=O\left(\frac{\|V\|_\rho}{\delta^2}\sup_{\rho-2\alpha\leqslant\rho'\leqslant\rho}\overline{X}_{k,\rho'}\right) \tag{9.59}$$

The theorem is proved. □

Remark 31. [Diophantine case] In the Diophantine case one immediately sees that

$$B(\omega)\leqslant 2\log\left[\gamma 2^\tau\right] \tag{9.60}$$

Moreover one easily sees that $R_k(\omega)$ and $R_{\lambda,k}(\omega)$, together with $\lambda_0(\omega)$, are decreasing functions of $B(\omega)$. Therefore $R_k(\omega)\geqslant R_k^{Dio}(\omega)$ and $R_{\lambda,k}(\omega)\geqslant R_{\lambda,k}^{Dio}(\omega)$ where $R_k^{Dio}(\omega)$ and $R_{\lambda,k}^{Dio}(\omega)$ are obtain by replacing $B(\omega)$ by $2\log\left[\gamma 2^\tau\right]$ in the r.h.s. of (9.30) and (9.41). It follows that Theorem 26 (resp. Theorem 27) is valid with $R_k^{dio}(\omega)$ in place of $R_k(\omega)$ (resp. $R_{\lambda,k}^{dio}(\omega)$ in place of $R_{\lambda,k}(\omega)$).

Since all the estimates are uniform in $\hbar$, the methods of the present paper allow to prove the following result

Corollary 32. *Let $\mathcal{H}$ a family of $m \leqslant l$ classical Hamiltonians $(\mathcal{H}_i)_{i=1\ldots m}$ on $T^*(\mathbb{T}^l)$ of the form $\mathcal{H}(x,\xi) = \ \omega\cdot\xi + \mathcal{V}(x,\xi) = \mathcal{H}^0(\omega\cdot\xi) + \mathcal{V}'(\omega\cdot\xi, x)$. Then, under the hypothesis on ω of Theorem 27 (resp. Theorem 26) and the conditions*

$$\begin{array}{rcl} \{\mathcal{H}_i, \mathcal{H}_j\} &=& 0 \quad 1 \leqslant i,j \leqslant m \\ \|\mathcal{V}\|_\rho &<& \overline{R}_{\lambda,0}(\omega) \quad (resp. \ < R_0(\omega)) \\ \|\nabla\overline{\mathcal{V}'}\|_\rho &<& \dfrac{\eta - C}{Z_0} \end{array}$$

$\mathcal{H}$ is (globally) symplectomorphically and holomorphically conjugated to $\mathcal{B}^0_\infty(\omega.\xi)$: for all $\delta > 0$,
there exist a symplectomorphism Φ_∞^{-1} such that

$$\mathcal{H}\circ\Phi_\infty^{-1} = \mathcal{B}^0_\infty(\mathcal{H}_0).$$

Moreover, $\Phi_\infty^{-1} - I \in J(\rho - 2\alpha - \delta)$ (in particular Φ_∞^{-1} is holomorphic) and for any positive $\delta < \rho$,

$$\|\Phi_\infty^{-1} - I\|_{\rho-2\alpha-\delta} \leqslant \frac{\mathcal{D}}{\delta}\|\mathcal{V}\|_\rho \tag{9.61}$$

where $\mathcal{D}$ is given by (9.68) *below.*

Finally, for any function $\mathcal{X}$ satisfying (9.66)*, we have*

$$\|\mathcal{X}\circ\Phi_\infty^{-1} - \mathcal{X}\|_{\rho-2\alpha-\delta} \leqslant \frac{\mathcal{D}}{\delta^2}\sup_{\rho-2\alpha\leqslant\rho'\leqslant\rho}\overline{\mathcal{X}}_{0,\rho'} = O\left(\frac{\|V\|_\rho}{\delta^2}\sup_{\rho-2\alpha\leqslant\rho'\leqslant\rho}\overline{\mathcal{X}}_{0,\rho'}\right) \tag{9.62}$$

where $\overline{\mathcal{X}}_{0,\rho'}$ is defined in (9.66).

Proof. Once again the function $\mathcal{B}^\hbar_\infty$ is by construction uniform in $\hbar \in [0,1]$ so it has a limit $\mathcal{B}^0_\infty$ as $\hbar \to 0$. It is easy to get convinced that the construction of $\mathcal{B}^0_\infty$ is the same as the one of $\mathcal{B}^\hbar_\infty$ after the substitution (we use capital letters for operators and calligraphic ones for their symbols at $\hbar = 0$):

$$\begin{array}{rcl} AB &\longrightarrow& \mathcal{A}\times\mathcal{B} \\ \dfrac{[A,B]}{i\hbar} &\longrightarrow& \{\mathcal{A},\mathcal{B}\} \\ e^{i\frac{W}{\hbar}} &\longrightarrow& e^{\mathcal{L}_{\mathcal{W}}} \\ e^{i\frac{W_1}{\hbar}}e^{i\frac{W_2}{\hbar}} &\longrightarrow& e^{\mathcal{L}_{\mathcal{W}_1}}\circ e^{\mathcal{L}_{\mathcal{W}_2}} \\ e^{i\frac{W}{\hbar}}Ae^{-i\frac{W}{\hbar}} &\longrightarrow& \mathcal{A}\circ e^{\mathcal{L}_{\mathcal{W}}}. \end{array}$$

Here $\times$ is the usual function multiplication, $\{.,.\}$ denotes the Poisson bracket and $e^{\mathcal{L}_{\mathcal{W}}}$ the Hamiltonian flow at time 1 of Hamiltonian $\mathcal{W}$ (Lie exponential).

What is left is to prove the convergence of the sequence of flows $e^{\mathcal{L}_{\mathcal{W}_r}}\dots e^{\mathcal{L}_{\mathcal{W}_1}} := \Phi^r$ as $r\to\infty$. This is done by the same Cauchy argument than in the proof of Theorem 26.

For $\Phi : T^*\mathbb{T}^l \to T^*\mathbb{T}^l$ we denote $\|\Phi\|_\rho = \sum_{i=1}^{2l}\|\Phi_i\|_\rho$ where Φ_i are the components of Φ and we define $\mathcal{E}_{np}$ by

$$\mathcal{E}_{np} = e^{\mathcal{L}_{\mathcal{W}_{n+p}}}\circ e^{\mathcal{L}_{\mathcal{W}_{n+p-1}}}\circ\dots\circ e^{\mathcal{L}_{\mathcal{W}_{n+1}}} - I_{T^*\mathbb{T}^l\to T^*\mathbb{T}^l},$$

so that $\Phi^{n+p}-\Phi^n = \mathcal{E}_{np}\circ\Phi^n$.

We will need the following

Lemma 33. *Let $\mathcal{F}(z,\theta)$ be analytic in $\{|\Im z|, |\Im\theta|\leqslant\rho\}$. Then,*

$$\|\mathcal{F}\|_{L^\infty(|\Im z|,|\Im\theta|\leqslant\rho)} \leqslant \|\mathcal{F}\|_\rho. \tag{9.63}$$

Proof. As in section 5 write

$$|\mathcal{F}(z,\theta)| = |\sum_q\int\widehat{\widehat{\mathcal{F}}}(p,q)e^{i<p,\xi>+i<q,x>}dp| \leqslant \sum_q\int|\widehat{\widehat{\mathcal{F}}}(p,q)|e^{\rho(|p|+|q|)}dp = \|F\|_\rho.$$

□

We will denote

$$\|\cdot\|_\rho^\infty = \|\cdot\|_{L^\infty(|\Im z|,|\Im\theta|\leqslant\rho)}.$$

Proposition 34. *Let $H_\rho = \{(z,\theta).\ |\Im z|\leqslant\rho$ and $|\Im\theta|\leqslant\rho\}$.*
Under the hypothesis of Theorems 26 and 27, Φ^r is analytic $H_\rho\to H_{\rho_r}$.

Remember that $\rho_r = \rho - \sum_{j=0}^{r-1}\delta_r,\ \delta_r = \alpha 2^{-r}$.

Proof. Let us first remark that the "rule" $\frac{[A,B]}{i\hbar}\longrightarrow\{\mathcal{A},\mathcal{B}\}$ is in fact (and of course) a Lemma.

Lemma 35. *Let $F\in J^m(\rho),\ \mathcal{G}\in J^1(\rho)$. Then $\frac{[F,G]}{i\hbar}$ is the Weyl quantization of a function $\sigma_\hbar$ on $T^*\mathbb{T}^l$ and*

$$\lim_{\hbar\to 0}\sigma_\hbar = \sigma_0 = \{\mathcal{F},\mathcal{G}\}.$$

The proof is an easy exercise which consists (again) in computing the symbol of $\frac{[F,G]}{i\hbar}$ through its matrix elements using Proposition 12, after expressing these matrix elements out of the ones of F,G, themselves expressed through the symbols $\mathcal{F},\mathcal{G}$ of F,G thanks of formula (5.7). The limit $\hbar\to 0$ leads naturally to the Poisson bracket. Since these techniques have been extensively used through the present article, we omit the details.

Let us, by a slight abuse of notation, define again $\mathrm{ad}_{\mathcal{W}}$ the operator $\mathcal{F}\mapsto\{\mathcal{W},\mathcal{F}\}$. Being uniform in $\hbar$, the formula (5.11) taken with $k=0$ leads, for any $\mathcal{F}\in\mathcal{J}^m(\rho_r)$, to

$$\frac{1}{j!}\|\mathrm{ad}^j_{\mathcal{W}_r}(\mathcal{F})\|_{\rho_r-\delta_r}\leqslant\left(\frac{Z_0}{\delta_r^2}\right)^j\|\mathcal{F}\|_{\rho_r}\|\mathcal{W}_r\|^j_{\rho_r}$$

Let us denote $\varphi_r = e^{\mathcal{L}_{\mathcal{W}_r}}$ and $\mathrm{ad}_{\mathcal{W}_r}(\mathcal{F}) := \{\mathcal{W}_r, \mathcal{F}\}$. Since $\mathcal{F} \circ \varphi_r^{-1} = \sum_{j=0}^{\infty} \frac{1}{j!}\mathrm{ad}^j_{\mathcal{W}_r}(\mathcal{F})$ we get

$$\|\mathcal{F} \circ \varphi_r^{-1}\|_{\rho_{r+1}} \leqslant \frac{\|\mathcal{F}\|_{\rho_r}}{1 - \frac{Z_0}{\delta_r^2}\|\mathcal{W}_r\|_{\rho_r}}.$$

Therefore, under the hypothesis of Theorem 27 (resp. Theorem 26), $\mathcal{F} \circ \varphi_r^{-1}$ is analytic in $H_{\rho_{r+1}}$ for all $\mathcal{F}$ analytic in H_{ρ_r} and so φ_r^{-1} maps analytically $H_{\rho_{r+1}}$ to H_{ρ_r} and so φ_r maps analytically H_{ρ_r} to $H_{\rho_{r+1}}$. Writing $\Phi^r = \varphi_r \circ \varphi_{r-1} \circ \cdots \circ \varphi_1$ gives the result. □

Let us write now for all r in $\mathbb{N}$,

$$\varphi_r = I + \mathcal{T}_r,$$

with, as for (5.18),

$$\|\mathcal{T}_r\|^{\infty}_{\rho-2\alpha} \leqslant \|\mathcal{T}_r\|^{\infty}_{\rho_r-\delta_r} = \sum_{i=1}^{2l} \|(\mathcal{T}_r)_i\|^{\infty}_{\rho_r-\delta_r} \leqslant \|\nabla\mathcal{W}_r\|^{\infty}_{\rho_r-\delta_r} := \max_i \sum_j \|(\nabla_j\mathcal{W}_r)_i\|^{\infty}_{\rho_r-\delta_r} \leqslant \frac{\|\mathcal{W}_r\|^{\infty}_{\rho_r}}{e\delta_r},$$

and so

$$\|\nabla\mathcal{T}_r\|^{\infty}_{\rho_r-\delta_r} \leqslant \frac{\|\nabla\mathcal{W}_r\|^{\infty}_{\rho_r-\delta_r/2}}{e\delta_r/2} \leqslant 4\frac{\|\mathcal{W}_r\|^{\infty}_{\rho_r}}{e^2\delta_r^2} \leqslant \frac{\|\mathcal{W}_r\|^{\infty}_{\rho_r}}{\delta_r^2}.$$

In analogy with (9.54) we write

$$\mathcal{E}_{np} = \varphi_{n+p} \circ (\mathcal{E}_{np-1} + I) - I = \varphi_{n+p} \circ (\mathcal{E}_{np-1} + I) - \varphi_{n+p} + (\varphi_{n+p} - I)$$

so

$$\|\mathcal{E}_{np}\|^{\infty}_{\rho-2\alpha} \leqslant \|\nabla\varphi_{n+p}\|^{\infty}_{\rho-2\alpha}\|\mathcal{E}_{np-1}\|^{\infty}_{\rho-2\alpha} + \|\mathcal{T}_{n+p}\|^{\infty}_{\rho-2\alpha}$$

and, by induction,

$$\begin{aligned}
\|\mathcal{E}_{np}\|^{\infty}_{\rho-2\alpha} &\leqslant \sum_{k=0}^{p-1} \|\mathcal{T}_{n+k}\|^{\infty}_{\rho-2\alpha} \prod_{s=k}^{p-1}(\|\nabla\varphi_{n+s}\|^{\infty}_{\rho-2\alpha}) + \|\mathcal{T}_{n+p}\|^{\infty}_{\rho-2\alpha} \\
&\leqslant \sum_{k=0}^{p-1} \|\mathcal{T}_{n+k}\|^{\infty}_{\rho-2\alpha} \prod_{s=k}^{p-1}(1 + \|\nabla\mathcal{T}_{n+s}\|^{\infty}_{\rho-2\alpha}) + \|\mathcal{T}_{n+p}\|^{\infty}_{\rho-2\alpha} \\
&\leqslant \sum_{k=0}^{p} \|\mathcal{T}_{n+k}\|^{\infty}_{\rho-2\alpha} \prod_{s=k}^{\infty}(1 + \|\nabla\mathcal{T}_{n+s}\|^{\infty}_{\rho-2\alpha}) \\
&\leqslant \sum_{k=0}^{p} \frac{\|W_{n+k}\|_{\rho_{n+k}}}{\delta_{n+k}} e^{\sum_{s=0}^{\infty} \frac{\|W_s\|_{\rho_s}}{\delta_s^2}} \\
&\leqslant \mathcal{A}_n e^{\mathcal{A}} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Here we defined

$$\mathcal{A} := \sum_{j=1}^{\infty} \frac{\mathcal{M}_{M_l}}{\delta_l^2(1-\eta+C/l)} D_k^{2^l} < \infty, \tag{9.64}$$

$$\mathcal{A}_n = \sum_{j=n}^{\infty} \frac{\mathcal{M}_{M_l}}{\delta_l(1-\eta+C/l)} D_k^{2l} \to 0 \text{ as } n \to \infty \text{ since } \mathcal{A} < \infty \tag{9.65}$$

and used (9.51).

So Φ^r converges to Φ^∞ in the $L^\infty(H_\rho)$ topology.

The proof of (9.62) is exactly the one of (9.47) by using the dictionary expressed earlier. Since it is a by-product of the proof of (9.61) we repeat it here. We denote $\|\cdot\|_\rho := \|\cdot\|_{\rho,\underline{\omega}}$ and use the fact that $\|\cdot\|_{\rho_r} \geqslant \|\cdot\|_{\rho-2\alpha-\delta},\ \forall r \in \mathbb{N}$.

We have, actually for any operator $\mathcal{X}$, that $\mathcal{X}\circ\varphi_r^{-1} - \mathcal{X} = \sum_{j=1}^{\infty} \frac{1}{j!}\mathrm{ad}^j_{\mathcal{W}_r}(\mathcal{X})$.

Taking (9.55) at $k=0$ we have

$$\|\{\mathcal{X},\mathcal{W}\}\|_{\rho-\delta} \leqslant \frac{Z_0}{\delta^2}\overline{\mathcal{X}}_{0,\rho}\|\mathcal{W}\|_\rho \tag{9.66}$$

We have

$$\|\mathcal{F}\circ\varphi_r^{-1}\|_{\rho-\delta} \leqslant \frac{\|\mathcal{F}\|_\rho}{1-\frac{Z_0}{\delta^2}\|\mathcal{W}_r\|_\rho}, \tag{9.67}$$

and also (let us recall again that $\rho_{r+1} = \rho_r - \delta_r,\ \rho_0 = \rho$)

$$\begin{aligned}\|\frac{1}{j!\hbar^j}\mathrm{ad}^j_{\mathcal{W}_r}(\mathcal{X})\|_{\rho_r-\delta_r-\delta=\rho_{r+1}-\delta} &\leqslant \left(\frac{Z_0}{\delta_r^2}\right)^{j-1}\|\{\mathcal{X},\mathcal{W}_r\}/i\hbar\|_{\rho_r-\delta}\|\mathcal{W}_r\|^{j-1}_{\rho_r-\delta}\\ &\leqslant \left(\frac{Z_0}{\delta_r^2}\right)^{j-1}\|\{\mathcal{X},\mathcal{W}_r\}/i\hbar\|_{\rho_r-\delta}\|\mathcal{W}_r\|^{j-1}_{\rho_r},\end{aligned}$$

out of which we get

$$\|\mathcal{X}\circ\varphi_0^{-1}-\mathcal{X}\|_{\rho_1-\delta} \leqslant \frac{\|\{\mathcal{X},\mathcal{W}_0\}\|_{\rho_0-\delta}}{1-\frac{Z_0}{\delta_0^2}\|\mathcal{W}_0\|_\rho}.$$

by $\mathcal{X}\circ\varphi_0^{-1}\circ\varphi_1^{-1} - \mathcal{X}\circ\varphi_1^{-1} = (\mathcal{X}\circ\varphi_0^{-1}-\mathcal{X})\circ\varphi_1^{-1}$ and (9.67)

$$\begin{aligned}\|\mathcal{X}\circ\varphi_0^{-1}\circ\varphi_1^{-1}-\mathcal{X}\circ\varphi_1^{-1}\|_{\rho_2-\delta} &\leqslant \frac{\|\mathcal{X}\circ\varphi_0^{-1}-\mathcal{X}\|_{\rho_1-\delta}}{1-\frac{Z_0}{\delta_1^2}\|\mathcal{W}_1\|_{\rho_1}}\\ &\leqslant \frac{\|\{\mathcal{X},\mathcal{W}_0\}\|_{\rho_0-\delta}}{(1-\frac{Z_0}{\delta_0^2}\|\mathcal{W}_0\|_\rho)(1-\frac{Z_0}{\delta_1^2}\|\mathcal{W}_1\|_{\rho_1})}\end{aligned}$$

and by iteration

$$\|\mathcal{X}\circ\Phi_s^{-1}-\mathcal{X}\circ\varphi_0\circ\Phi_s^{-1}\|_{\rho_{s+1}-\delta} \leqslant \frac{\|\{\mathcal{X},\mathcal{W}_0\}\|_{\rho_0-\delta}}{(1-\frac{Z_0}{\delta_0^2}\|\mathcal{W}_0\|_\rho)\dots(1-\frac{Z_0}{\delta_s^2}\|\mathcal{W}_s\|_{\rho_s})}$$

By the same argument we get

$$\begin{aligned}
\|\mathcal{X}\circ\varphi_0\circ\Phi_s^{-1}-\mathcal{X}\circ\varphi_0\circ\varphi_1\circ\Phi_s^{-1}\|_{\rho_{s+1}-\delta} &\leqslant \frac{\|\{\mathcal{X},\mathcal{W}_1\}\|_{\rho_1-\delta}}{(1-\frac{Z_0}{\delta_1^2}\|\mathcal{W}_1\|_\rho)\dots(1-\frac{Z_0}{\delta_s^2}\|\mathcal{W}_s\|_{\rho_s})}\\
&\cdot\\
&\cdot\\
&\cdot\\
\|\mathcal{X}\circ\varphi_r^{-1}-\mathcal{X}\|_{\rho_{r+1}-\delta} &\leqslant \frac{\|\{\mathcal{X},\mathcal{W}_r\}\|_{\rho_r-\delta}}{1-\frac{Z_0}{\delta_r^2}\|\mathcal{W}_r\|_{\rho_r}}.\\
\text{so that by}&\text{ summing the telescopic sequence}\\
&\leqslant \sum_{s=0}^r\frac{\overline{X}_{k,\rho_s}}{\delta^2}Z_0\|W_s\|_{\rho_s}e^{-\sum\limits_{j=s}^r\log\left(1-\frac{Z_0}{\delta_j^2}\|W_j\|_{\rho_j}\right)}\\
\text{and since } (1-a)(1+2a)\geqslant 1 \text{ if}&\ 0<a\leqslant 1/2\\
\leqslant\sum_{s=0}^r\frac{\overline{X}_{k,\rho_s}}{\delta^2}Z_0\|W_s\|_{\rho_s}e^{\sum\limits_{j=s}^r\log\left(1+\frac{2Z_0}{\delta_j^2}\|W_j\|_{\rho_j}\right)} &\leqslant \sum_{s=0}^r\frac{\overline{X}_{k,\rho_s}}{\delta^2}Z_0\|W_s\|_{\rho_s}e^{2\sum\limits_{j=s}^r\frac{Z_0}{\delta_j^2}\|W_j\|_{\rho_j}}\\
\leqslant\sum_{s=0}^r\frac{\overline{X}_{k,\rho_s}}{\delta^2}Z_0\|W_s\|_{\rho_s}e^{2\sum\limits_{j=0}^\infty\frac{Z_0}{\delta_j^2}\|W_j\|_{\rho_j}} &\leqslant \frac{\sup\limits_{\rho-2\alpha\leqslant\rho'\leqslant\rho}\overline{X}_{k,\rho'}}{\delta^2}Z_0\sum_{s=0}^r\|W_s\|_{\rho_s}e^{2\mathcal{B}}\\
&\leqslant \frac{\sup\limits_{\rho-2\alpha\leqslant\rho'\leqslant\rho}\overline{X}_{k,\rho'}}{\delta^2}\mathcal{D}
\end{aligned}$$

with $\mathcal{B}=\frac{Z_0}{\alpha^2}\mathcal{M}_1\|V\|_\rho+\sum\limits_{j=1}^\infty\frac{Z_0 2^j\mathcal{M}_{M_j}}{\alpha^2(1-\eta+C/j)}D_k^{2j}<\infty$ and

$$\mathcal{D}=Z_0(\mathcal{M}_1\|V\|_\rho+A)e^{2\mathcal{B}}=O(\|V\|_\rho) \tag{9.68}$$

by Lemma 30.

Therefore we get, by letting $r\to\infty$ so that $\rho_r\to\rho-2\alpha$,

$$\|\mathcal{X}\circ\Phi_\infty^{-1}-\mathcal{X}\|_{\rho-2\alpha-\delta}\leqslant\frac{\mathcal{D}}{\delta^2}\sup_{\rho-2\alpha\leqslant\rho'\leqslant\rho}\overline{\mathcal{X}}_{k,\rho'}=O\left(\frac{\|V\|_\rho}{\delta^2}\sup_{\rho-2\alpha\leqslant\rho'\leqslant\rho}\overline{\mathcal{X}}_{k,\rho'}\right). \tag{9.69}$$

Let now $\mathcal{X}\in\{\xi_1,\dots,\xi_l,x_1,\dots,x_l\}$, $\{\mathcal{X},\mathcal{W}_1\}=\pm\partial_\Xi\mathcal{W}_1$ where Ξ is the conjugate quantity to $\mathcal{X}$. Therefore $\|\{\mathcal{X},\mathcal{W}_0\}\|_{\rho-\delta}\leqslant\|\nabla\mathcal{W}_0\|_{\rho-\delta}\leqslant\frac{1}{\delta}\|\mathcal{W}_0\|_\rho$. Therefore $\overline{\mathcal{X}}_{k,\rho}=1$ and $\|\mathcal{X}\circ\Phi_r^{-1}-\mathcal{X}\|_{\rho_{r+1}-\delta}\leqslant\frac{D}{\delta^2}$, $\forall\mathcal{X}\in\{\xi_1,\dots,\xi_l,x_1,\dots,x_l\}$ which means that

$$\|\Phi_r^{-1}-I\|_{\rho_{r+1}-\delta}\leqslant\frac{\mathcal{D}}{\delta^2} \tag{9.70}$$

In fact we just proved that $\Phi_\infty^{-1}=I+\widetilde{\Phi}$ with $\|\widetilde{\Phi}\|_{\rho_\infty-\delta=\rho-2\alpha-\delta}\leqslant\frac{\mathcal{D}}{\delta^2}$. Corollary 32 is proved. □

Though the Diophantime case is covered by the Theorem 26 (see Remark 31), we can also use directly Proposition 21 in order to low down the hypothesis of the Theorem.

In fact Proposition 21 shows that the same proof will be possible by only replacing $\mathcal{M}_{M_r}(\omega)$ by $\gamma(\frac{\tau}{e\delta_r})^\tau$ and (9.13) by

$$E \geqslant \frac{2^{2+\tau}\alpha + 2[\alpha + (1+\eta)\underline{\omega}\gamma(\frac{\tau}{e\alpha})^\tau]}{(1-\eta)^2\gamma(\frac{\tau}{e\alpha})^\tau} = E_1 \tag{9.71}$$

Indeed (7.10) is verbatim the same as (6.31) after replacing $\mathcal{M}_{M_r}(\omega)$ by $\gamma(\frac{\tau}{e\delta_r})^\tau$ and the first term in the parenthesis, namely 1, by $2^{2+\tau}$. Therefore the proof will be the same by replacing $B(\omega)$ by $B_\alpha(\gamma,\tau)$

$$B_\alpha(\gamma,\tau) = \sum_{r=0}^{\infty} \log(\gamma(\frac{\tau}{e\delta_r})^\tau)2^{-r} = 2\log\left[\gamma(\frac{\tau}{e\alpha})^\tau\right] + 2\tau\log 2 = 2\log\left[2^\tau\gamma(\frac{\tau}{e\alpha})^\tau\right]$$

and of course C_k and P by the corresponding expressions C'_k, P'.

The very last change will concern D_k which now will be $D_k = e^{C_k}\|V\|_\rho$ because the estimate of Proposition 21 reads now directly$\|V^{r+1}\|_{\rho_l-\delta_l} \leqslant F'_r\|V^r\|^2_{\rho_l}$: this will imply that in the Diophantine case there is no condition for ω similar to (9.31), and no condition $\alpha < 2\log 2$. We get:

Theorem 36. *[Diophantine case] Let α, ρ, η, C and E be strictly positive constants satisfying*

$$\rho > 2\alpha,\ 0 < C < \eta < 1. \tag{9.72}$$

Let us define, for $\Delta = -\inf_{r\geqslant 1}\frac{1}{2^r}\log\frac{1}{2}(1-\eta+\frac{C}{r})$ and $M = \left(\frac{\alpha e C}{8Z_k}\right)^{\frac{1}{4}}$,

$$R_k(\omega) = \left(\frac{(1-\eta)^2\gamma(\frac{\tau}{e\alpha})^\tau}{2^{2+\tau}\alpha + 2[\alpha + (1+\eta)\underline{\omega}\gamma(\frac{\tau}{e\alpha})^\tau]}\right)^2 \frac{\alpha^6}{2^6 Z_k^2(2^\tau\gamma(\frac{\tau}{e\alpha})^\tau)^4}\min\left\{\frac{(2^\tau\gamma(\frac{\tau}{e\alpha})^\tau)^{-2}e^{-\Delta}}{2^{1/e}Z_k}, M\right\}. \tag{9.73}$$

Then if

$$\|V\|_{\rho,\underline{\omega},k} < R_k(\omega),\ \|\nabla\overline{\mathcal{V}'}\|_{\rho,\underline{\omega},k} < \frac{\eta - C}{Z_k}, \tag{9.74}$$

the same conclusions as in Theorem 26 and Corollary 32 hold.

As mentioned in the introduction we can use Theorem 27 to estimate the rate of divergence of the Brjuno constant as the system remains integrable while the perturbation is vanishing.

Let us suppose that we let ω vary in a way such that $\underline{\omega}$ remain in a bounded set $[\underline{\omega}_-, \underline{\omega}_+]$ of $(0,+\infty)$. (9.73) tells us that, in order that Theorem 36 holds, ω can be taken as we want as soon as as γ and τ satisfy $\|V\|_{\rho,\underline{\omega},k} <$ r.h.s. of (9.73) and $\|\nabla\overline{\mathcal{V}}\|_{\rho,\underline{\omega},k} < \frac{\eta - C}{Z_k}$. It is easy to check that, since $B_\alpha(\gamma,\tau) := 2\log\left[2^\tau\gamma(\frac{\tau}{e\alpha})^\tau\right] \to \infty$ as γ, τ or both of them diverge, we have, for $B_\alpha(\gamma,\tau)$ large enough (in order that the min in (9.73) is reached by the first term and that $2^{2+\tau} < B_\alpha(\gamma,\tau)$),

$$R_k(\omega) \geqslant 2Ke^{-3B_\alpha(\gamma,\tau)}$$

with $K = \frac{(1-\eta)^4\alpha^6}{(\alpha+2(1+\eta)\underline{\omega}_-)2^6 2^{1/e}Z_k^3}$. Therefore for $\|\nabla\overline{\mathcal{V}'}\|_{\rho,\underline{\omega},k} < \frac{\eta-C}{Z_k}$ and $\|V\|_{\rho,\underline{\omega},k}$ small enough (namely $\|V\|_{\rho,\underline{\omega},k} \leqslant 2Ke^{-3B_\alpha^-}$ where B_α^- is the smallest value of $B_\alpha(\gamma,\tau)$ which makes the min in (9.73) reached by the first term and which is larger than $2^{2+\tau}$), we have

Corollary 37. *The conclusions of Theorem 36 hold as soon as*

$$B_\alpha(\gamma,\tau) < \frac{1}{3}\log\left(\frac{1}{\|V\|_{\rho,\underline{\omega}_+,k}}\right) + \frac{1}{2}\log 2K.$$

Remark. In the case of the Brjuno condition, Theorem 26, it happens that $\lambda_0(\omega) \sim C'e^{2B(\omega)}$ and $R_{\lambda_0(\omega)} \sim C$ as $B(\omega) \to \infty$ for some bounded constants C, C'. Therefore our condition of convergence takes the form $\|V\|_{\rho,C'e^{2B(\omega)}\underline{\omega},k} < C$. This leads to a sufficient condition on $B(\omega)$ depending on the way $V \to 0$. For example it is easy to check that, if $V \to 0$ as $V = \epsilon V_0$, $\epsilon \to 0$ and V_0 with a symbol $\mathcal{V}_0'$ whose Fourier transform in ξ is compactly supported, one gets a condition of the form $B(\omega) < D\log\log\frac{1}{\epsilon} + D'$ for some constants $D,\ D'$.

10. The case $m = l$

Lemma 38. *Let the vectors $\omega_j,\ j = 1\ldots m = l$, be independent over $\mathbb{R}$. Then*
1. *any V satisfies* (1.5)
2. $\inf_{q\in\mathbb{Z}^l}\max_{j=1\ldots l}|\omega_j.q| \geqslant 1$ *(there is no small denominator).*

Proof. Let $\Omega = (\Omega_{ij})_{i,j=1\ldots l}$ with $\Omega_{ij} := \omega_i^j$. Then Ω is invertible by the independence of the ω_js. This proves (1). Moreover one has immediately that $1 \leqslant |q| \leqslant l|\Omega^{-1}|\max_{j=1\ldots l}|\omega_j.q|$. □

Therefore the main assumption reduces to:

Main assumptions (extreme case)

$$\omega_j \in \mathbb{R}^l,\ j = 1\ldots l,\ \text{are independent over } \mathbb{R} \text{ and } [H_i, H_j] = 0,\ \forall 1 \leqslant i,j \leqslant l. \qquad (10.1)$$

References

[Ar2] V.I.Arnold *Geometric Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag (1985)

[BGP] D. Bambusi, S.Graffi, T.Paul *Normal Forms and Quantization Formulae*, Comm.Math.Phys. **207**, 173-195 (1999)

[MB] Max Born, *The mechanics of the atom*, Ungar Pub. Co. (1925).

[Bru] A.D. Brjuno. Analytical form of differential equations. *Trans. Mosc. Math. Soc*, 25,131-288(1971); 26,199-239(1972), 1971-1972.

[CdV] Y.Colin de Verdière, *Quasi-modes sur les varietes Riemanniennes*, Inventiones Mathematicae **43**, 129-159 (1981)

[De] A.Deprit, *Canonical transformations depending on a small parameter*, Cel.Mech.Dyn.Astr. **1**, 12-30 (1969)

[El] L. H. Eliasson, *Normal forms for Hamiltonian systems with Poisson commuting integrals—elliptic case*, Comment. Math. Helv., 65(1),4–35,1990

[FM] F.M.Fedoryuk, V.P.Maslov, *Semi-classical approximation in quantum mechanics* D.Reidel, (1981)

[Fo] G.Folland, *Harmonic Analysis in Phase Space*, Annals of Mathematics Studies **122**, Princeton University Press (1989)

[Ga] G.Gallavotti, *Criterion of Integrability for Perturbed Nonresonant Harmonic Oscillators, "Wick Ordering" of the Perturbations in Classical Mechanics and Invariance of the Frequency Spectrum*, Comm.Math.Phys. **87**, 365-383 (1982)

[GG] S. Graffi, V. Grecchi, and B. Simon, Borel summability: application to the anharmonic oscillator, Phys. Lett. B **32**, 631-634 (1970)

[GP] S. Graffi, T.Paul, *Convergence of a quantum normal form and an exact quantization formula*, Journ. Func. Analysis, **262**, 3340-3393 (2012).

[Ho] G.I.Hori, *Theory of general perturbations with unspecified canonical variables*, Publ. Astr. Soc. Japan **18**, 287-296 (1966)

[It] H. Ito. Convergence of birkhoff normal forms for integrable systems. *Comment. Math. Helv.*, 64:412–461, 1989.

[TK] T. Kato, *Perturbation theory of linear operators*, Springer (1988).

[KP] S. Kuksin and G. Perelman, *Vey theorem in infinite dimensions and its application to KdV*, Discrete and Continuous Dynamical Systems. Series A **27**, 1-24 (2010)

[PU] T. Paul and A. Uribe, *A construction of quasimodes using coherent states*, Annales I.H.P., Physique Théorique **59**, 357-381 (1993)

[HP] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, T. 2 Gauthier-Villars (1892).

[Po1] A.Popov, *Invariant Tori, Effective Stability, and Quasimodes with Exponentially Small Error Terms I - Birkhoff Normal Forms*, Ann.Henri Poincaré **1**, 223-248 (200)

[Po2] A.Popov, *Invariant Tori, Effective Stability, and Quasimodes with Exponentially Small Error Terms II - Quantum Birkhoff Normal Forms*, Ann.Henri Poincaré **1**, 223-248 (2000)

[Ru] H.Rüssmann, *Über die Normalform analytischer Hamiltonscher Differentialgleichungen in der Nähe einer gleichgewichtlösung*, Math.Annalen **169**, 55-72 (1967)

[Ra] J. V. Ralston, *On the construction of quasimodes associated with stable periodic orbits*, Commun. Math. Phys. 51, 219-242 (1976)

[BS] B. Simon, *Determination of eigenvalues by divergent perturbation series*, Adv. in Math. **7**, 240-253 (1971)

[LS1] L. Stolovitch, *Singular complete integrabilty*, Publ. Math. I.H.E.S., **91**, 133-210, (2000)

[LS] L.Stolovitch, *Progress in normal form theory*, Nonlinearity **22**, 7423-7450 (2009)

[LS2] L. Stolovitch. Normalisation holomorphe d'algèbres de type Cartan de champs de vecteurs holomorphes singuliers. *Ann. of Math.*, (161):589–612, 2005.

[JV1] J. Vey, *Sur certains systmes dynamiques sparables*, Am. Journal of Math. **100** 591614, (1978)

[JV2] J. Vey, *Algbres commutatives de champs de vecteurs isochores*, Bull. Soc. Math. France,**107** 423432, (1979)

[Zu] N.T.Zung, *Convergence Versus Integrability in Normal Form Theory*, Ann. of Math.**161**, 141-156 (2005)

[WW] E. T. Whittaker and G. N. Watson, *A course of modern analysis. An introduction to the general theory of infinite processes and of analytic functions: with an account of the principal transcendental functions*, Fourth edition, Cambridge University Press, 1927.

CNRS and Centre de Mathématiques Laurent Schwartz, École Polytechnique, 91128 Palaiseau Cedex France

E-mail address: paul@math.polytechnique.fr

CNRS and Laboratoire J.-A. Dieudonné Université de Nice - Sophia Antipolis Parc Valrose 06108 Nice Cedex 02 France

E-mail address: Laurent.STOLOVITCH@unice.fr