Incompressible Navier-Stokes equations

Jean-Yves Chemin
Laboratoire J.-L. Lions, Case 187
Université Pierre et Marie Curie, 75 230 Paris Cedex 05, France
chemin@ann.jussieu.fr

February 3, 2012

Introduction

This text is notes of a two lectures given at Saint Étienne de Tinée near Nice during the winter 2012. The purpose is to give a self contained introduction to recent results about global smooth solution for the tridimensionnal incompressible Navier-Stokes equations in the whole space $\mathbb{R}^3$. A different version with some additionnal chapter will be published as Lectures Notes of the Beijing Academy of Sciences.The notes are organized as follows:

In the first part, we first present the now classical theory of globall wellposedness for small initial data in the framework of Kato's method. The understanding of what "small" means in the core of the subject. Then, we present a recent result where global wellposedness is obtained without the hypothesis of smallness on the initial data, but for the relative smallness of the first iterate The partcicular structure of the Navier-Stokes equation is pointed out. In this part, we insists also on the key role of oscillations in the stabilization of the system.

In the second part, we investigate the case of initial data which vary slowly in one direction. This is in some sense the dual case of fast oscillating data. The first case is the so called "well prepared" case. Here, the initial data converges to a bidimensionnal divergence free vector field. The variation is respect to the vertical variable is slow enough, then the solution exists globally.

The second case is the so called "ill prepared case". Here, the horizontal part is a divergence of size 1. Because of the divergence free condition, the size of the vertical component of the vector died is the inverse of the speed of variation in the vertical direction, this very large. A rescaling in the vertical variable leads to a problem which looks like an ill posed. Using a global Cauchy- Kovalevska method, which is explain is a model case, we can proved global existence a smooth solution of this initial data which are very large. This method consists in the control of the decay of the radius of analyticity of the solution.

Acknowlegments

I want to thank very much Gérard Iooss and Laurent Stolovitch for inviting me to give this lectures at Saint Etienne de Tinée and for the organization of this very nice and very interesting meeting.

Contents

Chapter 1

Global wellposedness and Kato theory.

1.1 Weak solutions and Kato's approach

Let us define the concept of (weak) solutions of the incompressible Navier-Stokes system. Let us first recall what the incompressible Navier Stokes system is. We consider as unknown the speed $u = (u^1, u^2, u^3)$ a time dependant divergence free vector field on $\mathbb{R}^3$ and the pressure p. We consider the system

$$(NS) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \Delta u = -\nabla p + f \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0. \end{cases}$$

The notion of C^2 solution (i.e. classical solution) is not efficient because singularity can appear here and also we can be interested in rough initial data. This has been pointed out by C. Ossen (see [51] and [52]) that another concept of solution must be used. This has been formalized by J. Leray in 1934 in his seminal work [45]. Let us define the notion of weak solution (that we shall denote simply solution in all that follows).

Definition 1.1.1 *A time-dependent vector field u with components in $L^2_{loc}([0,T] \times \mathbb{R}^d)$ is a weak solution (simply a solution in this paper) of (NS) if for any smooth compactly supported divergence free vector field Ψ,*

$$\langle u(t,\cdot), \Psi(t,\cdot)\rangle = \langle u_0, \Psi(0,\cdot)\rangle + \sum_{j=1}^{3} \int_0^t \int_{\mathbb{R}^3} u^j(t',x)\big(\partial_t \Psi^j(t',x) + \Delta\Psi(t',x)\big) dt' dx$$
$$+ \sum_{j,k} \int_0^t \int_{\mathbb{R}^3} (u^j u^k)(t',x) \partial_j^k \Psi(t',x)\, dt' dx.$$

This definition is too weak in the sense there is not enough constraints on the solution. In particular it ignores the fundamental concept of energy. J. Leray introduced in his seminal paper [45] the concept of turbulent solution (we shall not use in the notes)

Definition 1.1.2 *A turbulent solution of (NS) is a divergence free vector field u which is a weak solution, has component is $L^\infty_T(L^2) \cap L^2_T(H^1)$ and satisfies in addition the energy* *inequality*

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(t')\|_{L^2}^2 dt' \leq \frac{1}{2}\|u_0\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^3} f(t',x)\cdot u(t',x) dt' dx. \tag{1.1}$$

Remark For a turbulent solution, Definition 1.1.1 of a weak solution becomes

$$\int_{\mathbb{R}^d} u(t,x)\cdot\Psi(t,x)\,dx = \int_{\mathbb{R}^d} u_0(x)\cdot\Psi(0,x)\,dx + \int_0^t \langle f(t'),\Psi(t')\rangle dt'$$
$$- \int_0^t\int_{\mathbb{R}^d}\Big(\nabla u : \nabla\Psi - u\otimes u : \nabla\Psi - u\cdot\partial_t\Psi\Big)(t',x)\,dx\,dt'.$$

In [45], J. Leray proved the following theorem.

Theorem 1.1.1 *Let u_0 be a divergence free vector field in $L^2(\mathbb{R}^d)$. Then a turbulent solution u exists on $\mathbb{R}^+\times\mathbb{R}^3$.*

The proof of this theorem relies on compactness methods, We shall not develop this notion here. We are going to focuse on results of existence (and thus uniqueness) that can to prove by a fixed point theorem. In order to do so, let us follow Kato's approach. Let us define the biinear operator $B(u,v)$ as the solution of the evolution Stokes problem

$$\left\{\begin{array}{c} \partial_t B^j - \Delta B^j = \dfrac{1}{2}\displaystyle\sum_{k=1}^3 \partial_k(u^j v^k + v^j u^k) - \partial_j p \\ \operatorname{div} B = 0 \quad \text{and} \quad B_{|t=0} = 0. \end{array}\right. \tag{1.2}$$

In weak formulation, this writes

$$\begin{aligned} \langle u(t,\cdot),\Psi(t,\cdot)\rangle = \sum_{j=1}^3 \int_0^t\int_{\mathbb{R}^3} u^j(t',x)\big(\partial_t\Psi^j(t',x) + \Delta\Psi(t',x)\big)dt'dx \\ + \sum_{j,k}\int_0^t\int_{\mathbb{R}^3}\partial_k(u^jv^k + v^ju^k)(t',x)\partial_j\Psi^k(t',x)\,dt'dx. \end{aligned} \tag{1.3}$$

It is obvious that u a solution of (NS) if and only if u satisfies

$$u = e^{t\Delta}u_0 + B(u,u).$$

Solving globally (NS) is equivalent to find a fixed point for the map

$$u \longmapsto e^{t\Delta}u_0 + B(u,u).$$

Now let us assume that we have a Banach space X of functions locally in L^2 on $\mathbb{R}^+\times\mathbb{R}^3$ such that B is a bilinar map from $X\times X$ into X. Then Picard's fixed point theorem implies the existence of a unique solution. Such a space X will be called "adapted".

Let us remark there is a strong constrain on X due to the scaling property. If u is a solution of (NS) on $[0,T]\times\mathbb{R}^3$, then for any positive λ, the vector field $u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x)$ is also a solution of (NS) on $[0,\lambda^{-2}T]\times\mathbb{R}^3$. Thus, if X adapted, it must be scaling invariant (and also translation invariant) in the sense that

$$\forall\lambda>0\,,\ \forall\overrightarrow{a}\in\mathbb{R}^3\,,\ u\in X \Longleftrightarrow u(\lambda(\cdot-\overrightarrow{a}))\in X \quad\text{and}\quad \|u\|_X \sim \|u(\lambda(\cdot-\overrightarrow{a}))\|_X.$$

Let us give a first example of an adapted spaces: the space $L^4(\mathbb{R}^+;\dot H^1)$. Let us define an operator which will be of some use later one.

Definition 1.1.3 *We denote by L_0 the operator defined by the fact that $L_0 f$ is the solution of*

$$\left\{\begin{array}{c}\partial_t L_0 f - \Delta L_0 f = f - \nabla p \\ \operatorname{div} L_0 f = 0 \quad \text{and} \quad L_0 f_{|t=0} = 0.\end{array}\right.$$

The key lemma is the following.

Lemma 1.1.1 *The operator L_0 maps continously $L^2(R^3; \dot{H}^{-\frac{1}{2}})$ into $L^4(\mathbb{R}^+; \dot{H}^1)$.*

Proof. In Fourier space, we can write that

$$\mathcal{F} L_0 f(t,\xi) = \int_0^t e^{-(t-t')|\xi|^2}\, \mathbb{P}(\xi)\widehat{f}(t',\xi)dt'$$

where $\mathbb{P}(\xi)$ is the orthogonal projection in $\mathbb{R}^3$ orthogonal of ξ. Thus, we get

$$|\xi||\mathcal{F} L_0 f(t,\xi)| \leq \int_0^t e^{-(t-t')|\xi|^2} |\xi|^{\frac{3}{2}} \theta(t',\xi) \|f(t',\cdot)\|_{\dot{H}^{-\frac{1}{2}}} dt'$$

with $\|\theta(t',\cdot)\|_{L^2} = 1$ for any t' of $\mathbb{R}^+$. Taking the L^2 in ξ norm in the above inequality gives

$$\|L_0 f(t,\cdot)\|_{\dot{H}1} \lesssim \int_0^t \frac{1}{(t-t')^{\frac{3}{4}}} \|f(t',\cdot)\|_{\dot{H}^{-\frac{1}{2}}} dt'.$$

Then Hardy-Littlewood-Sobolev inequality allows to conclude the proof. □

As a corollary, we get

Corollary 1.1.1 *The operator B maps $L^4(\mathbb{R}^+, \dot{H}^1) \times L^4(\mathbb{R}^+, \dot{H}^1)$ into $L^4(\mathbb{R}^+, \dot{H}^1)$.*

Proof. Let us obseve that, thank to diverngence free condition, we have

$$\frac{1}{2}\sum_{k=1}^{3} \partial_k (u^j v^k + v^j u^k) = \frac{1}{2}(u\cdot\nabla v + v\cdot\nabla u).$$

Thank you the Sobolev embeddings (and its dual), we have

$$\begin{aligned}\|u\cdot\nabla v + v\cdot\nabla u((t',\cdot)\|_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|u\cdot\nabla v + v\cdot\nabla u((t',\cdot)\|_{L^{\frac{3}{2}}} \\ &\lesssim \|\nabla u(t')\|_{L^2}\|\nabla(t')\|_{L^2}.\end{aligned}$$

Thus

$$\|u\cdot\nabla v + v\cdot\nabla u\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac{1}{2}})} \lesssim \|u\|_{L^4(\mathbb{R}^+;\dot{H}^1)}\|v\|_{L^4(\mathbb{R}^+;\dot{H}^1)}.$$

□

As we have

$$\|e^{t\Delta}u_0\|_{L^4(\mathbb{R}^+;\dot{H}^1)} \lesssim \|u_0\|_{\dot{H}^{\frac{1}{2}}}$$

we get that if $\|u_0\|_{\dot{H}^{\frac{1}{2}}}$ is small enough, then a unique global solution exists in $L^4(\mathbb{R}^+; \dot{H}^1)$. This is Fujita-Kato theorem.

1.2 The Kato theory is the L^p framework

Let us first define operatros that will be of some use later on.

Definition 1.2.1 *For $j \in \{1,2,3\}$, we denote by L_j the operator defined by the fact that $L_j f$ is the solution of*

$$\begin{cases} \partial_t L_j f - \Delta L_j f = \partial_j f - \nabla p \\ \operatorname{div} L_j f = 0 \quad \text{and} \quad L_j f_{|t=0} = 0. \end{cases}$$

Let us introduce spaces adpated to theses operators.

Definition 1.2.2 *For p in $[1,\infty]$ and for positive σ, we denote by K_p^σ the space of functions on $\mathbb{R}^+ \times \mathbb{R}^3$ such that*

$$\|u\|_{K_p^\sigma} \stackrel{def}{=} \sup_{t>0} t^{\frac{\sigma}{2}} \|u(t)\|_{L^p} < \infty.$$

Let us remark that when $\sigma = 1 - \dfrac{3}{p}$, the space K_p^σ is scaling invariant. THe key proposition is the following

Proposition 1.2.1 *For $p > 3$ the operator L_j maps continuously $K_{\frac{p}{2}}^{2\left(1-\frac{3}{p}\right)}$ into $K_p^{1-\frac{3}{p}}$.*

Proof. It relies on the explicite computation of the operator L_j. We have the following lemma.

Lemma 1.2.1 *Let σ be an homogeneous function of degree $\alpha > -d$. Then a constant C exists such that, for any positive t,*

$$\left|\mathcal{F}^{-1}\big(\sigma(\xi)e^{-t|\xi|^2}\big)(x)\right| \leq \frac{C}{(\sqrt{t} + |x|)^{\alpha+d}} \cdot$$

Proof. By changing variable in the integral

$$I(x) = \int_{\mathbb{R}^d} e^{i(x|\xi)} \sigma(\xi) e^{-t|\xi|^2} d\xi$$

it is enough to prove the estimate in the case when$t = 1$. As $\alpha > -d$, it is obvious that the function is bounded. Let us write that

$$I(x) = \int_{\mathbb{R}^d} e^{i(x|\xi)} \chi(\lambda^{-1}\xi)\sigma(\xi)e^{-t|\xi|^2} d\xi + \int_{\mathbb{R}^d} e^{(ix|\xi)} \big(1 - \chi(\lambda^{-1}\xi)\big)\sigma(\xi)e^{-t|\xi|^2} d\xi.$$

where χ is a smooth cut-off function. We have

$$\left|\int_{\mathbb{R}^d} e^{i(x|\xi)} \chi(\lambda^{-1}\xi)\sigma(\xi)e^{-t|\xi|^2} d\xi\right| \lesssim \lambda^{\alpha+d}.$$

Using that $|x|^{2N} e^{i(x|\xi)} = (-\Delta_\xi)^N e^{i(x|\xi)}$, we infer that, for small λ,

$$|x|^{2N} \left|\int_{\mathbb{R}^d} e^{i(x|\xi)} \big(1 - \chi(\lambda^{-1}\xi)\big)\sigma(\xi)e^{-t|\xi|^2} d\xi\right| \lesssim \lambda^{\alpha+d-2N}.$$

Choosing $\lambda|x| = 1$ gives the result. □

As a consequence, we get

$$L_j f^k(t,x) = \sum_\ell \int_0^t \Gamma^k_{j,\ell}(t-t',\cdot)\star f^\ell(t',\cdot)dt' \tag{1.4}$$

where the functions $\Gamma^k_{j,\ell}$ satisfy $\Gamma^k_{j,\ell}(\tau,z)| \lesssim \dfrac{1}{(\sqrt{\tau}+|z|)^4}\cdot$

Proof of Proposition 1.2.1 Young inequality ans definition of the spaces K^σ_p gives

$$\begin{aligned}\|L_j f(t,\cdot)\|_{L^p} &\leq \int_0^t \|\Gamma(t-t',\cdot)\|_{L^{p'}}\|f(t',\cdot)\|_{L^{\frac{p}{2}}}dt'\\ &\lesssim \|f\|_{K_{\frac{p}{2}}^{2\left(1-\frac{3}{p}\right)}}\int_0^t \frac{1}{(t-t')^{\frac12+\frac{3}{2p}}t'^{1-\frac3p}}dt'\\ &\lesssim \frac{1}{\sqrt{t}^{1-\frac3p}}\|f\|_{K_{\frac{p}{2}}^{2\left(1-\frac{3}{p}\right)}}.\end{aligned}$$

This proves the proposition. □

As $K_p^{1-\frac3p}\cdot K_p^{1-\frac3p}\subset K_{\frac{p}{2}}^{2\left(1-\frac{3}{p}\right)}$, we proves that B maps continuously $K_p^{1-\frac3p}\times K_p^{1-\frac3p}$ into $K_p^{1-\frac3p}$. Thus, we have the following theorem

Theorem 1.2.1 *If $\|e^{t\Delta}u_0\|_{K_p^{1-\frac3p}}$ is small enough, then there is a unique global solution of (NS) in $K_p^{1-\frac3p}$.*

Now we have to understand what is the space of distribution u_0 such that $e^{t\Delta}u_0$ belongs to $K_p^{1-\frac3p}$. This is the family of Besov spaces.

1.3 Besov spaces of negative index

In this section, we interpret Theorem 1.2.1 in term of Besov spaces. Let us introduce the following spaces.

Definition 1.3.1 *Let s be be in $]0,\infty]$ and (p,r) in $[1,\infty]^2$. We define the space $B^{-s}_{p,r}$ as the space of tempered distribution u such that*

$$\|u\|_{B^{-s}_{p,r}} \stackrel{def}{=} \Big\|\|t^s e^{t\Delta}u\|_{L^p}\Big\|_{L^r(\mathbb{R}^+,\frac{dt}{t})}<\infty.$$

Let us point out that as, for $q\geq p$, $e^{t\Delta}u = e^{\frac{t}{2}\Delta}e^{\frac{t}{2}\Delta}u$ and

$$\|e^{\tau\Delta}\|_{\mathcal{L}(L^p;L^q)} \leq C\tau^{-\frac{d}{2}\left(\frac1p-\frac1q\right)}. \tag{1.5}$$

Thus we get that

$$\|u\|_{\dot{B}_q^{-s+d\left(\frac1p-\frac1q\right)}} \leq C\|u\|_{\dot{B}^{-s}_p}. \tag{1.6}$$

Homogeneous functions of negative degree will also belong to some Besov spaces. We have the following proposition.

Proposition 1.3.1 *Let p be in $]1,\infty]$ and α in $]d/p,d[$. Then a homogeneous function f of degree α of $\mathbb{R}^d\setminus\{0\}$ which is bounded on the sphere of $\mathbb{R}^d$ belongs to $\dot{B}_p^{-\alpha+\frac{d}{p}}$.*

Proof. We have, by changing of variable and because homogeneïty of f that

$$(e^{t\Delta}f)(x)=t^{-\frac{\alpha}{2}}(e^{\Delta}f)\Big(\frac{x}{\sqrt{t}}\Big)\cdot$$

As f is bounded on the sphere of $\mathbb{R}^d$, we have

$$|f(x)|\leq C|x|^{-\alpha}$$

As α belongs to $]d/p,d[$, the function f belongs to L^1+L^p. Thus $e^{\Delta}f$ belongs to L^p. After a change of variable, we get the result. □

From this, we can infer the following theorem about so called "self similar solutions" of (NS).

Corollary 1.3.1 *Let u_0 be a smooth divergence free vector field on $\mathbb{R}^3\setminus\{0\}$ homogeneous of degree -1. Then, if u_0 is small enough in $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}$, then there exists a unique solution of (NS) which is self-similar in the sense that it satisfies*

$$u(t,x)=\frac{1}{\sqrt{t}}U\Big(\frac{x}{\sqrt{t}}\Big)\quad \textit{with}\quad U(x)=u(1,x).$$

Proof. Using the scaling invariance of the Navier-Stokes equation, we have that, for any positive λ, $\lambda u(\lambda^2t,\lambda x)$ is the global solution with initial data $\lambda u_0(\lambda x)$ which is equal $u_0(x)$ because of the homogeneity. Thus, for any positive λ, we have

$$u(t,x)=\lambda u(\lambda^2t,\lambda x)$$

Choosing $\lambda=(\sqrt{t})^{-1}$ gives the result. □

Proposition 1.3.2 *Let ϕ be a function of $\mathcal{S}(\mathbb{R}^d)$, p in $]1,\infty[$ and s in $\Big]0,\dfrac{d}{p'}\Big[$. For (ε,Λ) in $]0,1]\times[1,\infty[$, let us define*

$$\phi_{\varepsilon,\Lambda}(x)\stackrel{\text{def}}{=}e^{i\frac{x_1}{\varepsilon}}\phi(x_1,\Lambda x_2,x_3).$$

Then, we have

$$\|\phi_{\varepsilon,\Lambda}\|_{\dot{B}^{-s}_{p,1}}\leq C_\phi\varepsilon^s\Lambda^{-\frac{1}{p}}\quad \textit{and}\quad \|\phi_{\varepsilon,\Lambda}\|_{\dot{B}^{-\frac{d}{p'}}_{p,\infty}}\leq C_\phi\varepsilon^s\Lambda^{-\frac{1}{p}}.$$

Proof. Using (1.5), we get that

$$t^{\frac{s}{2}}\|e^{t\Delta}\phi_{\varepsilon,\Lambda}\|_{L^p}\leq C_\phi t^{\frac{s}{2}}\Lambda^{-\frac{1}{p}}.$$

Thus we infer

$$\int_0^{\varepsilon^2}t^{\frac{s}{2}}\|e^{t\Delta}\phi_{\varepsilon,\Lambda}\|_{L^p}\frac{dt}{t}\leq C_\phi\varepsilon^s\Lambda^{-\frac{1}{p}}. \tag{1.7}$$

Now, let us assume that $t^{\frac{1}{2}}\geq\varepsilon$. We can assume without loss of generality Using that $\omega=(1,0,\cdots,0)$. We have

$$(-i\varepsilon)^k e^{i\frac{x|\omega)}{\varepsilon}}=\partial_1^k e^{i\frac{x|\omega)}{\varepsilon}}.$$

We get after k integrations by parts and Leibnitz formula,

$$t^{\frac{s}{2}}e^{t\Delta}\phi_{\varepsilon,\Lambda} = (-i\varepsilon)^k t^{\frac{s}{2}}\sum_{\ell=0}^{k} C_k^\ell \frac{1}{t^{\frac{d}{2}+\frac{\ell}{2}}} f_\ell\Big(\frac{\cdot}{\sqrt{t}}\Big)\star\big((e^{\frac{i(x|\omega)}{\varepsilon}}\phi_{k-\ell}(x_1,\Lambda^{-1}x_2,x_3)\big)$$

where f_ℓ and $\phi_{k-\ell}$ are functions of $\mathcal{S}(\mathbb{R}^d)$. As Λ is assume to be greater than 1, convolution inequalities give

$$\|t^{\frac{s}{2}}e^{t\Delta}\phi_\varepsilon\|_{L^p} \leq C_{\phi,k}\Lambda^{-\frac{1}{p}}\sum_{\ell=0}^{k}\min\Big\{\Big(\frac{1}{\sqrt{t}}\Big)^{\ell-s},\Big(\frac{1}{\sqrt{t}}\Big)^{\ell-s+\frac{d}{p'}}\Big\}.$$

Let us assume that $k>s$. If $k\geq \ell > s$, we use

$$\varepsilon^k\int_{\varepsilon^{-2}} t^{-1+\frac{1}{2}(s-\ell)}dt \lesssim \varepsilon^s.$$

If $\ell\leq s$, then we use that

$$\varepsilon^k\int_{\varepsilon^{-2}} t^{-1+\frac{1}{2}\left(s-\ell-\frac{d}{p'}\right)}dt \lesssim \varepsilon^{k+s-\ell-\frac{d}{p'}}.$$

If k is large enough, we get the result. □

Now, let us prove a bound from below.

Proposition 1.3.3 *Let ϕ be a function of $\mathcal{S}(\mathbb{R}^3)$ and s in $]0,3[$. For (ε,Λ) in $]0,1]\times[1,\infty[$, let us define*

$$\phi_{\varepsilon,\Lambda}(x) \stackrel{\text{def}}{=} e^{i\frac{x_1}{\varepsilon}}\phi(x_1,\Lambda x_2,x_3).$$

Then, if $\Lambda\varepsilon$ is small enough, we have

$$\|\phi_{\varepsilon,\Lambda}\|_{\dot{B}^{-s}_{\infty,\infty}} \geq C_\phi\varepsilon^s.$$

Proof. Let us first observe that, as the space of smooth compactly supported functions is dense in $\mathcal{S}$ and the Fourier transform is continuous on $\mathcal{S}$. Thus, for any positive η, a function φ exists, the Fourier transform of which is smooth and compactly supported such that, denoting as before $\theta_{\varepsilon,\Lambda}(x) = e^{i\frac{x_3}{\varepsilon}}\theta(x_1,\Lambda x_2,x_3)$,

$$\|\phi_\varepsilon-\theta_\varepsilon\|_{\dot{B}^{-\sigma}_{\infty,\infty}} \leq \eta\varepsilon^\sigma \quad\text{and}\quad \|\phi-\theta\|_{L^\infty}\leq\eta. \tag{1.8}$$

As the support of the Fourier transform of θ is included in the ball $B(0,R)$ for some positive R, that of $\theta(x_1,\Lambda x_2,x_3)$ is included in the ball $B(0,R\Lambda)$. Then the support of $\mathcal{F}\theta_{\varepsilon,\Lambda}$ is included in the ball $B(\varepsilon^{-1}(1,0,0),\Lambda R)$ which can be written as

$$\frac{1}{\varepsilon}B((1,0,0),\Lambda\varepsilon R)$$

If $\lambda\varepsilon$ is small enough, we can assume that this set is included in $\varepsilon^{-1}\mathcal{C}$ where $\mathcal{C}$ denotes a fixed ring. By definition of $B^{-s}_{\infty,\infty}$, we have

$$\begin{aligned}\|\theta_{\varepsilon,\Lambda}\|_{\dot{B}^{-s}_{\infty,\infty}} &= \sup_{t>0} t^{\frac{s}{2}}\|e^{t\Delta}\theta_{\varepsilon,\Lambda}\|_{L^\infty}\\ &\geq C\varepsilon^s\|e^{\varepsilon^2\Delta}\theta_{\varepsilon,\Lambda}\|_{L^\infty}.\end{aligned}$$

For any function h such that the support of $\widehat{h}$ is included in $\varepsilon^{-1}\mathcal{C}$, we have

$$\|\mathcal{F}^{-1}(e^{\varepsilon^2|\xi|^2}h)\|_{L^\infty} \leq C\|h\|_{L^\infty}.$$

Applied with $h = e^{\varepsilon^2\Delta}\theta_{\varepsilon,\Lambda}$, this inequality gives

$$\|\theta_{\varepsilon,\Lambda}\|_{L^\infty} \leq C\|e^{\varepsilon^2\Delta}\theta_{\varepsilon,\Lambda}\|_{L^\infty} \quad \text{and thus} \quad \|\theta_{\varepsilon,\Lambda}\|_{\dot{B}^{-s}_{\infty,\infty}} \geq C^{-1}\varepsilon^s\|\theta_{\varepsilon,\Lambda}\|_{L^\infty} = C^{-1}\varepsilon^s\|g\|_{L^\infty}.$$

Now let us write that

$$\begin{aligned} \|\phi_\varepsilon\|_{\dot{B}^{-s}_{\infty,\infty}} &\geq \|\theta_{\varepsilon,\Lambda}\|_{\dot{B}^{-s}_{\infty,\infty}} - \eta\varepsilon^s \\ &\geq C^{-1}\varepsilon^s(\|\phi\|_{L^\infty} - 2\eta). \end{aligned}$$

Together with (1.8), this gives the proposition. □

Let us give another example of estimates of Besov norm of negative index in the case of slowly varying functions.

Proposition 1.3.4 *Let (f, g) be in $\mathcal{S}(\mathbb{R}^2)$ and $\mathcal{S}(\mathbb{R})$ respectively. Let us define*

$$h^\varepsilon(x_h, x_3) \stackrel{\text{def}}{=} f(x_h)g(\varepsilon x_3).$$

We have, if ε is small enough,

$$\|h^\varepsilon\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)} \geq \frac{1}{4}\|f\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^2)}\|g\|_{L^\infty(\mathbb{R})}.$$

Proof. By the definition of $\|\cdot\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)}$, we have to bound from below $\|e^{t\Delta}h^\varepsilon\|_{L^\infty(\mathbb{R}^3)}$. Let us write that

$$(e^{t\Delta}h^\varepsilon)(t, x) = (e^{t\Delta_h}f)(t, x_h)(e^{t\partial_3^2}g)(\varepsilon^2 t, \varepsilon x_3).$$

Let us consider a positive time t_0 such that

$$t_0^{\frac{1}{2}}\|e^{t_0\Delta_h}f\|_{L^\infty(\mathbb{R}^2)} \geq \frac{1}{2}\|f\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^2)}.$$

Then we have

$$\begin{aligned} t_0^{\frac{1}{2}}\|e^{t_0\Delta}h^\varepsilon\|_{L^\infty(\mathbb{R}^3)} &= t_0^{\frac{1}{2}}\|e^{t_0\Delta_h}f\|_{L^\infty(\mathbb{R}^2)}\|(e^{t_0\partial_3^2}g)(\varepsilon^2 t_0, \varepsilon\cdot)\|_{L^\infty(\mathbb{R})} \\ &\geq \frac{1}{2}\|f\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^2)}\|e^{\varepsilon^2 t_0\partial_3^2}g\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

As $\lim\limits_{\varepsilon\to 0} e^{\varepsilon^2 t_0\partial_3^2}g = g$ in $L^\infty(\mathbb{R})$, the proposition is proved. □

As a conclusion of this section about Besov space, let us prove the following property.

Proposition 1.3.5 *If σ is a homogeneous Fourier multiplier of order $m \geq 0$ and s a positive number such that $s + m > 0$. Then $\sigma(D)$ maps continuously $B^{-s}_{p,q}$ into $B^{-s-m}_{p,q}$.*

Proof. For the sake of simplicity, let us write it only for $m = 0$. as for any function v, we have

$$v = \int_0^\infty \tau^{\frac{s}{2}}(-\Delta)^{\frac{s}{2}+1}e^{\tau\Delta}d\tau$$

Using the semi group law on the heat flow and splitting $(-\Delta)^{\frac{s}{2}}$ into two parts, we infer that

$$t^{\frac{s}{2}}e^{t\Delta}\sigma(D)u = C_s t^{\frac{s}{2}}\sigma(D)(-\Delta)^{\frac{s}{4}}e^{\frac{t}{2}\Delta}\int_{\frac{t}{2}}^{\infty}\Big(\tau-\frac{t}{2}\Big)^{\frac{s}{2}}(-\Delta)^{\frac{s}{4}+1}e^{\frac{\tau}{2}\Delta}e^{\frac{\tau}{2}\Delta}u\,d\tau. \tag{1.9}$$

Using Lemma 1.2.1, we have

$$\|\sigma(D)(-\Delta)^{\frac{s}{4}}e^{\frac{t}{2}\Delta}a\|_{L^p}\lesssim t^{\frac{s}{4}}\|a\|_{L^p}\quad\text{and}\quad\|(-\Delta)^{\frac{s}{4}+1}e^{\frac{\tau}{2}\Delta}a\|_{L^p}\lesssim\tau^{\frac{s}{4}-1}\|a\|_{L^p}.$$

Plugging this into (1.9) gives

$$\|t^{\frac{s}{2}}e^{t\Delta}\sigma(D)u\|_{L^p}\lesssim t^{\frac{s}{4}}\int_{\frac{t}{2}}^{\infty}\tau^{\frac{s}{4}}\|e^{\frac{\tau}{2}\Delta}u\|_{L^p}\frac{d\tau}{\tau}\,\cdot$$

Using Hölder inequality with respet to the measure $\tau^{-1}d\tau$, we get

$$\left(\int_{\frac{t}{2}}^{\infty}\tau^{\frac{s}{4}+\frac{s}{8}}\|e^{\frac{\tau}{2}\Delta}u\|_{L^p}\tau^{-\frac{s}{8}}\frac{d\tau}{\tau}\right)^q\lesssim t^{-qs8}\int_{\frac{t}{2}}^{\infty}\tau^{\frac{3qs}{8}}\|e^{\frac{\tau}{2}\Delta}u\|_{L^p}^q\frac{d\tau}{\tau}$$

Thus we infer that

$$\begin{aligned}\int_{\mathbb{R}^+}t^{\frac{qs}{2}}\|e^{t\Delta}u\|_{L^p}^q\frac{dt}{t} &\lesssim \int_{\mathbb{R}^+}\left(\int_0^{2\tau}t^{\frac{qs}{8}}\frac{dt}{t}\right)\tau^{\frac{3qs}{8}}\|e^{\frac{\tau}{2}\Delta}u\|_{L^p}^q\frac{d\tau}{\tau}\\ &\lesssim \int_{\mathbb{R}^+}\tau^{\frac{qs}{2}}\|e^{\frac{\tau}{2}\Delta}u\|_{L^p}^q\frac{d\tau}{\tau}\,\cdot\end{aligned}$$

This concludes the proof of the proposition. □

1.4 The endpoint space for Picard's scheme

The following proposition guarantees that it is hopeless to go beyond the space $B^{-1}_{\infty,\infty}$.

Proposition 1.4.1 *Let E be a Banach space continuously embedded in the set $\mathcal{S}'(\mathbb{R}^3)$. Assume that, for any (λ,a) in $\mathbb{R}^+_\star\times\mathbb{R}^3$,*

$$\|f(\lambda(\cdot-a))\|_E=\lambda^{-1}\|f\|_E.$$

Then E is continuously embedded in $\dot{B}^{-1}_{\infty,\infty}$.

Proof. As B is continuously included in $\mathcal{S}'$, we have that $|\langle f,e^{-|\cdot|^2}\rangle|\leq C\|f\|_B$. Then by dilation and translation, we deduce that

$$\|f\|_{\dot{B}^{-1}_{\infty,\infty}}=\sup_{t>0}t^{\frac{1}{2}}\|e^{t\Delta}f\|_{L^\infty}\leq C\|f\|_B.$$

This proves the proposition. □

It turns out however that $\dot{B}^{-1}_{\infty,\infty}$ is too large a space. The mayine reason why is that if we want to solve the problem using an iterative scheme then we need that $e^{t\Delta}u_0$ belongs to $L^2_{loc}(\mathbb{R}^+\times\mathbb{R}^3)$ so that $B(e^{t\Delta}u_0,e^{t\Delta}u_0)$ makes sense. Taking into consideration the scaling and the translation invariance thus leads to the following definition.

Definition 1.4.1 *We denote by X_0 the space of tempered distributions u such that*

$$\|u\|_{X_0} \stackrel{def}{=} \|u\|_{\dot{B}^{-1}_{\infty,\infty}} + \sup_{\substack{x\in\mathbb{R}^3\\ R>0}} R^{-\frac{3}{2}}\Big(\int_{P(x,R)} |e^{t\Delta}u(y)|^2\,dy\,dt\Big)^{\frac{1}{2}} < \infty$$

where $P(x,R) = [0,R^2]\times B(x,R)$ and $B(x,R)$ denotes the ball of $\mathbb{R}^3$ of center x and radius R.

We denote by X be the space of functions f on $\mathbb{R}^+_\star \times \mathbb{R}^3$ such that

$$\|f\|_X \stackrel{def}{=} \sup_{t>0}\bigg(t^{\frac{1}{2}}\|f(t)\|_{L^\infty} + \sup_{\substack{x\in\mathbb{R}^3\\ R>0}} R^{-\frac{3}{2}}\Big(\int_{P(x,R)} |f(t,y)|^2\,dy\,dt\Big)^{\frac{1}{2}}\bigg) < \infty$$

We denote by Y the space of functions on $\mathbb{R}^+_\star\times\mathbb{R}^3$ such that

$$\|f\|_Y \stackrel{def}{=} \sup_{t>0} t\|f(t)\|_{L^\infty} + \sup_{\substack{x\in\mathbb{R}^3\\ R>0}} R^{-3}\int_{P(x,R)} |f(t,y)|\,dy\,dt < \infty.$$

Proposition 1.4.2 *For any p in $]3,\infty[$, the space $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ is continuously embedded in X_0.*

Proof. Let us notice that for any $x\in\mathbb{R}^3$ and $R>0$, we have

$$\int_0^{R^2}\int_{B(x,R)} \big|e^{t\Delta}u_0(y)\big|^2\,dy\,dt \leq \mu(B(x,R))^{1-\frac{2}{p}}\int_0^{R^2}\bigg(\int_{B(x,R)}\big|e^{t\Delta}u_0(y)\big|^p\,dy\bigg)^{\frac{2}{p}}dt.$$

By definition of the space $\dot{B}_p^{-1+\frac{3}{p}}$, we have

$$\int_0^{R^2}\int_{B(x,R)} \big|e^{t\Delta}u_0(y)\big|^2\,dy\,dt \leq \|u_0\|_{\dot{B}_p^{-1+\frac{3}{p}}}\,\mu(B(x,R))^{1-\frac{2}{p}}\int_0^{R^2} t^{-1+\frac{3}{p}}\,dt$$

which obviously entails the announced embedding. □

Proposition 1.4.3 *The space $\dot{B}^{-1}_{\infty,2}$ is included in X_0.*

Proof. As $\dot{B}^{-1}_{\infty,2}$ is included in $B^{-1}_{\infty,\infty}$, we have

$$\sup_{t>0} t^{\frac{1}{2}}\|e^{t\Delta}u\|_{L^\infty} \leq C\|u\|_{B^{-1}_{\infty,2}}. \tag{1.10}$$

Moreover, we have

$$\begin{aligned}\frac{1}{R^3}\int_0^{R^2}\int_{B(x,R)} |(e^{t\Delta}f)(y)|^2dydt &\leq \int_0^\infty \|e^{t\Delta}f\|^2_{L^\infty}dt\\ &\leq C\|u\|_{\dot{B}^{-1}_{\infty,2}}.\end{aligned}$$

Together with (1.10), this proves the proposition. □

As $X \cdot X \subset Y$, the fact that we have a global unique solution for small initial data in X_0 will follow from the following proposition.

Theorem 1.4.1 *Let us define the operators $(L_j)_{1\leq j\leq N}$ by*

$$\begin{cases} \partial_t L_j f - \Delta L_j f = \partial_j f - \nabla p \\ \operatorname{div} v = 0 \quad \text{and} \quad L_{j|t=0} = 0. \end{cases}$$

The operators L_j maps continuously Y into X.

Proof. Using Lemma 1.4 page 11, we get that

$$(L_j f)^k(t,x) = \sum_{\ell=1}^{3} \Gamma^k_{j,\ell}(t-t', x-y) f^\ell(t', y)\, dt'\, dy$$

with for all positive real number R,

$$|\Gamma^k_{j,\ell}(\tau,\zeta)| \leq \frac{C}{(\sqrt{\tau}+|\zeta|)^4} \leq C'\big(\Gamma^{(1)}_R(\tau,\zeta) + \Gamma^{(2)}_R(\tau,\zeta)\big)$$

with $\Gamma^{(1)}_R(\tau,\zeta) \stackrel{\text{def}}{=} \mathbf{1}_{|\zeta|\geq R}\dfrac{1}{|\zeta|^4}$ and $\Gamma^{(2)}_R(\tau,\zeta) \stackrel{\text{def}}{=} \mathbf{1}_{|\zeta|\leq R}\dfrac{1}{(\sqrt{\tau}+|\zeta|)^4}\cdot$

The operators of convolution with functions $\Gamma^{(1)}_R$ and $\Gamma^{(2)}_R$ may be bounded according to the following proposition.

Lemma 1.4.1 *There exists a constant C such that, for any $R > 0$,*

$$\|\Gamma^{(1)}_R \star f\|_{L^\infty([0,R^2]\times\mathbb{R}^3)} \leq \frac{C}{R}\|f\|_Y, \tag{1.11}$$

$$\|\Gamma^{(2)}_R \star f\|_{L^\infty([R^2,\infty[\times\mathbb{R}^3)} \leq \frac{C}{R}\|f\|_Y. \tag{1.12}$$

Proof. Let us decompose $\Gamma^{(1)}_R \star f(t,x)$ as a sum of integrals on annulus:

$$\begin{aligned} |\Gamma^{(1)}_R \star f(t,x)| &\leq \sum_{p=0}^{\infty} \int_0^t \int_{B(0,2^{p+1}R)\setminus B(0,2^pR)} \frac{1}{|y|^4}|f(t',x-y)|\, dy\, dt' \\ &\leq \frac{1}{R}\sum_{p=0}^{\infty} 2^{-p+3}(2^{p+1}R)^{-3} \int_0^t \int_{B(0,2^{p+1}R)} |f(t',x-y)|\, dy\, dt'. \end{aligned}$$

As p is nonnegative, we have for $t \leq R^2$,

$$\begin{aligned} |\Gamma^{(1)}_R \star f(t,x)| &\leq \frac{C}{R}\sum_{p=0}^{\infty} 2^{-p}(2^{p+1}R)^{-3} \int_{P(x,2^{p+1}R)} |f(t,z)|\, dt\, dz \\ &\leq \frac{C}{R}\sum_{p=0}^{\infty} 2^{-p} \sup_{R'>0} \frac{1}{R'^3} \int_{P(x,R')} |f(t,z)|\, dt\, dz. \end{aligned}$$

By definition of $\|\cdot\|_Y$, Inequality (1.11) is proved.

In order to prove the second inequality, let us observe that for all $x \in \mathbb{R}^3$ and $t \geq R^2$, we have

$$
\begin{aligned}
|(\Gamma_R^{(2)} \star f)(t,x)| &\leq \Gamma_R^{(21)}(t,x) + \Gamma_R^{(22)}(t,x) \quad \text{with} \\
\Gamma_R^{(21)}(t,x) &\stackrel{\text{def}}{=} \int_0^{\min(R^2,\frac{t}{2})} \int_{B(0,R)} \frac{1}{(\sqrt{t-t'}+|y|)^4} |f(t',x-y)| \, dy \, dt', \\
\Gamma_R^{(22)}(t,x) &\stackrel{\text{def}}{=} \int_{\min(R^2,\frac{t}{2})}^t \int_{B(0,R)} \frac{1}{(\sqrt{t-t'}+|y|)^4} |f(t',x-y)| \, dy \, dt'.
\end{aligned}
$$

For bounding $\Gamma_R^{(21)}(t,x)$, we use that $t \leq 2(t-t')$. We get

$$
\Gamma_R^{(21)}(t,x) \leq C\frac{R^3}{t^2}\biggl(\frac{1}{R^3}\int_0^{R^2}\int_{B(0,R)} |f(t',x-y)| \, dt' \, dy\biggr)
$$

so that, for any $t \geq R^2$ and x in $\mathbb{R}^3$,

$$
\Gamma_R^{(21)}(t,x) \leq \frac{C}{t^{\frac{1}{2}}}\|f\|_Y. \tag{1.13}
$$

In order to estimate $\Gamma_R^{(22)}$, let us use that $t \leq 2t'$ and that, for any $a > 0$,

$$
\int_{B(0,R)} \frac{dy}{(a+|y|)^4} \leq \frac{1}{a}\int_{\mathbb{R}^3} \frac{dz}{(1+|z|)^4}.
$$

This enables us to write that

$$
\begin{aligned}
\Gamma_R^{(22)}(t,x) &\leq \int_{\min(R^2,\frac{t}{2})}^t \int_{B(0,R)} \frac{1}{(\sqrt{t-t'}+|y|)^4} \|f(t',\cdot)\|_{L^\infty} \, dy \, dt', \\
&\leq C\|f\|_Y \biggl(\int_{t/2}^t \frac{1}{\sqrt{t-t'}} \frac{dt'}{t'} + \int_{R^2}^t \frac{\mu(B(0,R))}{t^2}\frac{dt'}{t'}\biggr), \\
&\leq C\|f\|_Y \biggl(\frac{1}{t^{\frac{1}{2}}} + \frac{1}{R^2}\frac{tR^3}{t^2}\biggr).
\end{aligned}
$$

As $R \leq \sqrt{t}$, this concludes the proof of the lemma. □

Proof of Lemma 1.4.1 (continued). Note that applying the above proposition with $R = \sqrt{t}$ yields

$$
\|(L_j f)(t,\cdot)\|_{L^\infty} \leq \frac{C}{t^{\frac{1}{2}}}\|f\|_Y. \tag{1.14}
$$

Hence, it suffices to estimate $\|L_j f\|_{L^2(P(x,R))}$ for an arbitrary $x \in \mathbb{R}^3$. Using translations and dilations, we can assume that $x = 0$ and $R = 1$. Let us write

$$
L_j f = L_j(\mathbf{1}_{^cB(0,2)} f) + L_j(\mathbf{1}_{B(0,2)} f).
$$

Observing that, for any $y \in B(0,1)$, we have

$$
|L_j(\mathbf{1}_{^cB(0,2)} f)(t,y)| \leq C\Gamma_1^{(1)} \star (\mathbf{1}_{^cB(0,2)}|f|)(t,y),
$$

and using Inequality (1.11), we get

$$
\|L_j(\mathbf{1}_{^cB(0,2)} f)\|_{L^\infty(P(0,1))} \leq C\|f\|_Y.
$$

As the volume of $P(0,1)$ is finite, we infer that

$$\|L_j(\mathbf{1}_{^cB(0,2)}f)\|_{L^2(P(0,1))} \leq C\|f\|_Y. \tag{1.15}$$

Now the proof of Lemma 1.4.1 is reduced to the proof the following Lemma.

Lemma 1.4.2 *For any function $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ such that $f(t,\cdot)$ be supported in $B(0,2)$ for all $t \in [0,1]$, we have*

$$\|(L_jf)(t,\cdot)\|_{L^2([0,1]\times\mathbb{R}^3)} \leq C\|f\|_Y$$

Proof. Let us point out that, for any t, $L_jf(t) = \mathbb{P}\,\widetilde{L}_jf(t)$ where $\widetilde{L}_j$ is the solution of

$$\partial_t\widetilde{L}_jf - \Delta\widetilde{L}_jf = \partial_jf \quad \text{and} \quad \widetilde{L}_jf_{|t=0} = 0.$$

As $\mathbb{P}$ is an orthogonal projection in L^2, we get

$$\|L_jf(t,\cdot)\|_{L^2} \leq \|\widetilde{L}_jf(t,\cdot)\|_{L^2}. \tag{1.16}$$

Let us decompose f into low and high frequencies in the sense of the heat flow:

$$f = f^\flat + f^\sharp \quad \text{with} \quad f^\flat(t,\cdot) \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\widehat{\theta}(t^{\frac{1}{2}}\xi)\widehat{f}(t,\xi))$$

where θ denotes a function such that $\widehat{\theta}$ be compactly supported and with value 1 near the origin. Let us write that

$$\begin{aligned}
\|f^\sharp\|^2_{L^2([0,1];\dot{H}^{-1})} &= (2\pi)^{-3}\int_{[0,1]\times\mathbb{R}^3} \frac{|1-\widehat{\theta}(t^{\frac{1}{2}}\xi)|^2}{t|\xi|^2}\, t|\widehat{f}(t,\xi)|^2\,dt\,d\xi\\
&\leq C\int_{[0,1]\times\mathbb{R}^3} t\|f(t,\cdot)\|^2_{L^2}\,dt\\
&\leq C\|f\|_{L^1([0,1]\times\mathbb{R}^3)}\sup_{t>0} t\|f(t,\cdot)\|_{L^\infty}.
\end{aligned}$$

So using the energy estimate on the heat equation and (1.16), we end up with

$$\|L_jf^\sharp\|_{L^2([0,1]\times\mathbb{R}^3)} \leq C\|f\|_Y. \tag{1.17}$$

Now let us estimate $\|\widetilde{L}_jf^\flat\|_{L^2([0,1]\times\mathbb{R}^3)}$. Let us first observe that, by definition of $\widetilde{L}_j$ and $f^\flat$, we have

$$\begin{aligned}
\mathcal{F}\widetilde{L}_jf^\flat(t,\xi) &= i\xi_j\int_0^t e^{(-t-t')|\xi|^2}\widehat{f^\flat}(t',\xi)\,dt'\\
&= i\xi_je^{-t|\xi|^2}\int_0^t \mathcal{F}(\widetilde{f^\flat})(t',\xi)\,dt' \quad \text{with} \quad \mathcal{F}\widetilde{f^\flat}(t',\xi) \stackrel{\text{def}}{=} e^{t'|\xi|^2}\widehat{\theta}(t'|\xi|^2)\widehat{f}(t,\xi).
\end{aligned}$$

Let us notice that, by definition of θ, we have that

$$\widetilde{f^\flat}(t,\cdot) = t^{-\frac{3}{2}}\widetilde{\theta}\Big(\frac{\cdot}{\sqrt{t}}\Big)\star f(t,\cdot) \quad \text{with} \quad \widetilde{\theta} \in \mathcal{S}(\mathbb{R}^3). \tag{1.18}$$

Thus, using (1.16), we get

$$\sum_{j=1}^3 \|L_jf^\flat\|^2_{L^2([0,1]\times\mathbb{R}^3)} \leq \mathcal{N}(f) \quad \text{with} \quad \mathcal{N}(f) \stackrel{\text{def}}{=} \int_0^1 \Big\|\nabla e^{t\Delta}\int_0^t \widetilde{f^\flat}(t')\,dt'\Big\|^2_{L^2}\,dt.$$

By symmetry, we can write

$$\begin{aligned} \mathcal{N}(f) &= 2\int_A \big(\nabla e^{t\Delta}\widetilde{f}^\flat(t'')\big|\nabla e^{t\Delta}\widetilde{f}^\flat(t')\big)_{L^2}\,dt''\,dt'\,dt \quad \text{with} \\ A &\stackrel{\text{def}}{=} \{(t'',t',t)\in[0,1]^3\ /\ t''\leq t'\leq t\}. \end{aligned}$$

By integration by parts and because $e^{t\Delta}$ is self-adjoint on L^2, we get

$$\big(\nabla e^{t\Delta}\widetilde{f}^\flat(t'')\big|\nabla e^{t\Delta}\widetilde{f}^\flat(t')\big)_{L^2} = -\langle \Delta e^{2t\Delta}\widetilde{f}^\flat(t''),\widetilde{f}^\flat(t')\rangle.$$

For any positive t' and t'' such that $t''\leq t'$, the function $\widetilde{f}^\flat(t',\cdot)$ and $\widetilde{f}^\flat(t'',\cdot)$ have Fourier transform with compact support in a ball of radius $C/\sqrt{t''}$. In this space (denoted in $\mathcal{F}L^2_{t''}$), we have, in the sense of $\mathcal{L}(\mathcal{F}L^2_{t''})$,

$$2\Delta e^{t\Delta} = -\frac{d}{dt}e^{2t\Delta}\,.$$

We infer that

$$\forall (t'',t')\in]0,1]^2\,,\quad \big(\nabla e^{t\Delta}\widetilde{f}^\flat(t'')\big|\nabla e^{t\Delta}\widetilde{f}^\flat(t')\big)_{L^2} = -\frac{1}{2}\frac{d}{dt}\big(e^{2t\Delta}\widetilde{f}^\flat(t'')|\widetilde{f}^\flat(t')\big)_{L^2}.$$

By integration, we deduce that

$$\begin{aligned} \int_{t'}^1 \big(\nabla e^{t\Delta}\widetilde{f}^\flat(t'')\big|\nabla e^{t\Delta}\widetilde{f}^\flat(t')\big)_{L^2}dt &= -\int_{t'}^1 \frac{d}{dt}\big(e^{2t\Delta}\widetilde{f}^\flat(t'')\big|\widetilde{f}^\flat(t')\big)_{L^2}dt \\ &= \big((e^{2t'\Delta}-e^{2\Delta})\widetilde{f}^\flat(t'')\big|\widetilde{f}^\flat(t')\big)_{L^2} \end{aligned}$$

Thanks to Fubini' theorem, we deduce that

$$\mathcal{N}(f) = \int_0^1 \Big((e^{2t'\Delta}-e^{2\Delta})\int_0^{t'}\widetilde{f}^\flat(t'')\,dt''\Big|\widetilde{f}^\flat(t')\Big)_{L^2}dt'$$

By definition of the L^2 inner product, we infer that

$$\mathcal{N}(f)\leq \|\widetilde{f}^\flat\|_{L^1([0,1]\times\mathbb{R}^3)}\sup_{t'\in[0,1]}\Big\|(e^{2t'\Delta}-e^{2\Delta})\int_0^{t'}\widetilde{f}^\flat(t'')\,dt''\Big\|_{L^\infty}.$$

Using (1.18), we infer that

$$\mathcal{N}(f)\leq \|f\|_{L^1([0,1]\times\mathbb{R}^3)}\sup_{t'\in[0,1]}\Big\|(e^{2t'\Delta}-e^{2\Delta})\int_0^{t'}\widetilde{f}^\flat(t'')\,dt''\Big\|_{L^\infty}. \tag{1.19}$$

First of all, let us notice that, using (1.18) and the fact that operator $e^{2\Delta}$ maps $L^1(\mathbb{R}^3)$ in $L^\infty(\mathbb{R}^3)$, we have

$$\Big\|e^{2\Delta}\int_0^{t'}\widetilde{f}^\flat(t'')\,dt''\Big\|_{L^\infty}\leq C\|f^\flat\|_{L^1([0,1]\times\mathbb{R}^3)}.$$

Thanks to (1.18), we have, as f is supported in the ball $B(0,2)$,

$$\forall t\in[0,1]\,,\,,\ \|f^\flat(t,\cdot)\|_{L^1(\mathbb{R}^3)}\leq C\|f\|_Y. \tag{1.20}$$

Thus, we get

$$\Big\|e^{2\Delta}\int_0^{t'}\widetilde{f}^\flat(t'')\,dt''\Big\|_{L^\infty}\leq C\|f\|_Y. \tag{1.21}$$

Let us admit for a while the following two lemmas.

Lemma 1.4.3 *Let θ in $\mathcal{S}(\mathbb{R}^3)$. Let us define*

$$\widetilde{f}_\theta(t,\cdot) \stackrel{\rm def}{=} t^{-\frac{3}{2}}\theta(t^{-\frac{1}{2}}\cdot)\star f(t,\cdot)$$

We have

$$\|\widetilde{f}_\theta\|_Y \leq C\|f\|_Y.$$

Lemma 1.4.4 *Let us define*

$$Ef(t) = e^{t\Delta}\int_0^t f(t')dt'.$$

Then a constant C exists such that for any function f in Y and supported in $[0,1]\times B(0,2)$, we have

$$\|Ef\|_{L^\infty([0,1]\times\mathbb{R}^3)} \leq C\|f\|_Y$$

Conclusion of the proof of Theorem 1.4.1 These two lemmas imply that

$$\sup_{t'\in[0,1]}\Big\|e^{2t'\Delta}\int_0^{t'}\widetilde{f}^\flat(t'')\,dt''\Big\|_{L^\infty} \leq C\|f\|_Y.$$

Inequalities (1.14), (1.17) and (1.21) allows to conclude the proof of the theorem. □

Proof of Lemma 1.4.3 Let us first obverse that, for any t, we have

$$\|\widetilde{f}(t,\cdot)\|_{L^\infty} \leq \|\theta\|_{L^1}\|f(t,\cdot)\|_{L^\infty}.$$

Thus we have

$$t\|\widetilde{f}(t,\cdot)\|_{L^\infty} \leq \|\theta\|_{L^1}t\|f(t,\cdot)\|_{L^\infty}. \tag{1.22}$$

Now, let us write that, for any x in the ball of center 0 and radius R, we have

$$\begin{aligned}|\widetilde{f}_\theta(t,x)| \quad &\leq \quad t^{-\frac{3}{2}}\int_{\mathbb{R}^3}\Big|\theta\Big(\frac{x-y}{\sqrt{t}}\Big)\Big|\mathbf{1}_{B(0,2R)}(y)\,|f(t,y)|\,dy \\ &\qquad\qquad + Ct^{-\frac{3}{2}}\int_{\mathbb{R}^3}\frac{1}{\Big(1+\frac{|x-y|}{\sqrt{t}}\Big)^4}\frac{t}{R^2}|f(t,y)|\,dy \\ &\leq \quad t^{-\frac{3}{2}}\Big(|\theta(t^{-\frac{1}{2}}\cdot)|\star\mathbf{1}_{B(0,2R)}|f(t,\cdot)|\Big)(x) + \frac{C}{R^2}\sup_{t>0}t\|f(t,\cdot)\|_{L^\infty}.\end{aligned}$$

Thus, we infer

$$\frac{1}{R^3}\|\widetilde{f}_\theta\|_{L^1(P(0,R))} \leq \frac{C}{R^3}\int_{P(0,R)}|f(t,y)|\,dt\,dy + C\sup_{t>0}t\|f(t,\cdot)\|_{L^\infty}.$$

This proves Lemma 1.4.3. □

Proof of Lemma 1.4.4 Because of translation invariance, it is enough to bound

$$I_f(t) \stackrel{\rm def}{=} \Big(e^{2t\Delta}\int_0^t f(t',\cdot)dt'\Big)(0).$$

We decompose the space $\mathbb{R}^3$ as a disjoint union of cubes of center $n\sqrt{t}$ and radius $\frac{1}{2}\sqrt{t}$ where $n=(n_1,n_2,n_3)$ is the generic point of $\mathbb{Z}^3$. This gives

$$I_f(t) = \sum_{n\in\mathbb{Z}^3}\int_{[0,T]\times B(n\sqrt{t},\frac{1}{2}\sqrt{t})}\frac{1}{8\pi t^{\frac{3}{2}}}e^{-\frac{|y|^2}{8t}}f(t',y)dt'dy.$$

As y belongs to $B(n\sqrt{t},\frac{1}{2}\sqrt{t})$, we have

$$\frac{|y|}{\sqrt{t}}\geq |n|-\frac{1}{2}\,\cdot$$

Thus we get

$$\begin{aligned}\|I_f(t)| &\lesssim \sum_{n\in\mathbb{Z}^3} e^{-\frac{|n|^2}{8}}\frac{1}{t^{\frac{3}{2}}}\int_{P(n\sqrt{t},\frac{1}{2}\sqrt{t})}|f(t',y)|dt'dy\\ &\lesssim \|f\|_Y\sum_{n\in\mathbb{Z}^3} e^{-\frac{|n|^2}{8}}\\ &\lesssim \|f\|_Y.\end{aligned}$$

This concludes the proof of Lemma 1.4.4. □

Of course, this proves the following theorem

Theorem 1.4.2 *If $e^{t\Delta}u_0$ is small enough then there is a unique global solution to NS) associated with u_0.*

1.5 An abstract non linear smallness condition

Let us define a space which is a good space for being an external force that Theorem 1.4.2.

Definition 1.5.1 *We shall denote by E the space of functions f in $L^1(\mathbb{R}^+;\dot B^{-1}_{\infty,2})$ such that $\sup\limits_{t>0} t\|f(t)\|_{B^{-1}_{\infty,\infty}}$ is finite equipped with the norm*

$$\|f\|_E\stackrel{def}{=}\|f\|_{L^1(\mathbb{R}^+;\dot B^{-1}_{\infty,1})}+\sup_{t>0} t\|f(t)\|_{B^{-1}_{\infty,\infty}}.$$

Theorem 1.5.1 *There is a constant C_0 such that the following result holds. Let u_0 be a divergence free vector field in $B^{-1}_{\infty,2}$. Let us assume that*

$$\big\|e^{t\Delta}u_0\cdot\nabla e^{t\Delta}u_0\big\|_E\leq C_0^{-1}\exp\big(-C_0U_0^4\big)\quad\textit{with}\quad U_0\stackrel{def}{=}\Big(\|u_0\|^2_{\dot B^{-1}_{\infty,2}}+\|u_0\|^4_{\dot B^{-1}_{\infty,4}}\Big)^{\frac{1}{4}}\qquad(1.23)$$

Let us define

$$U(t)=\Big(\|e^{t\Delta}u_0\|^2_{L^\infty}+t\|e^{t\Delta}u_0\|^4_{L^\infty}\Big)^{\frac{1}{4}}\quad\textit{and}\quad v_\lambda(t,x)\stackrel{def}{=}\exp\Big(-\lambda\int_0^t U(t')dt'\Big)v(t,x).$$

Then there is a unique global solution to (NS) such that

$$\big\|\big(u(t)-e^{t\Delta}u_0\big)_\lambda\big\|_X\leq C_0^{-1}\exp(-CU_0^4).$$

The proof of this theorem is the purpose of all this section.

1.5.1 Main steps of the proof

Let us start by remarking that in the case when u_0 is small then there is nothing to be proved, so in the following we shall suppose that $\|u_0\|_{\dot{B}^{-1}_{\infty,2}}$ is not small, say $\|u_0\|_{\dot{B}^{-1}_{\infty,2}} \geq 1$.

We search for the solution u under the form

$$u_L + R \quad \text{where} \quad u_L(t) \stackrel{\text{def}}{=} e^{t\Delta}u_0.$$

Then R is the solution of

$$(NS') \qquad R = B(u_L, u_L) + 2B(u_L, R) + B(R, R).$$

To prove the global existence of u, we are reduced to prove that (MNS) has a global solution. We use the following easy lemma, the proof of which is omitted.

Lemma 1.5.1 *Let X be a Banach space, let L be a continuous linear map from X to X, and let B be a bilinear map from $X \times X$ to X. Let us define*

$$\|L\|_{\mathcal{L}(X)} \stackrel{\text{def}}{=} \sup_{\|x\|=1} \|Lx\| \quad \text{and} \quad \|B\|_{\mathcal{B}(X)} \stackrel{\text{def}}{=} \sup_{\|x\|=\|y\|=1} \|B(x,y)\|.$$

If $\|L\|_{\mathcal{L}(X)} < 1$, then for any x_0 in X such that

$$\|x_0\|_X < \frac{(1 - \|L\|_{\mathcal{L}(X)})^2}{4\|B\|_{\mathcal{B}(X)}},$$

the equation

$$x = x_0 + Lx + B(x,x)$$

has a unique solution in the ball of center 0 and radius $\dfrac{1 - \|L\|_{\mathcal{L}(X)}}{2\|B\|_{\mathcal{B}(X)}}$.

Let us introduce the functional space for which we shall apply the above lemma. We define the quantity

$$U(t) \stackrel{\text{def}}{=} \|u_L(t)\|^2_{L^\infty} + t\|u_L(t)\|^4_{L^\infty},$$

which satisfies, tby definition of Besov spaces,

$$\begin{aligned} \int_0^\infty U(t)dt &\leq C\|u_0\|^2_{\dot{B}^{-1}_{\infty,2}} + C\|u_0\|^4_{\dot{B}^{-1}_{\infty,4}} \\ &\leq C\|u_0\|^4_{\dot{B}^{-1}_{\infty,2}} \end{aligned} \tag{1.24}$$

recalling that we have supposed that $\|u_0\|_{\dot{B}^{-1}_{\infty,2}} \geq 1$ to simplify the proof.

For all $\lambda \geq 0$, let us denote by X_λ the set of functions on $\mathbb{R}^+ \times \mathbb{R}^3$ such that

$$\|v\|_\lambda \stackrel{\text{def}}{=} \sup_{t>0} \Big(t^{\frac{1}{2}} \|v_\lambda(t)\|_{L^\infty} + \sup_{\substack{x\in\mathbb{R}^3 \\ R>0}} R^{-\frac{3}{2}} \Big(\int_{P(x,R)} |v_\lambda(t,y)|^2 dy \Big)^{\frac{1}{2}} \Big) < \infty, \tag{1.25}$$

where

$$v_\lambda(t,x) \stackrel{\text{def}}{=} v(t,x) \exp\Big(-\lambda \int_0^t U(t')dt'\Big)$$

while $P(x,R)=[0,R^2]\times B(x,R)$ and $B(x,R)$ denotes the ball of $\mathbb{R}^3$ of center x and radius R. Let us point out that, in the case when $\lambda=0$, this is exactly the space of Definition 1.4.1 page 16. For any non negative λ and that for any $\lambda\geq 0$ we have due to (1.24),

$$\|v\|_\lambda \leq \|v\|_X \leq C\|v\|_\lambda \exp\Big(C\lambda\|u_0\|^4_{\dot B^{-1}_{\infty,2}}\Big). \tag{1.26}$$

As B maps continously $X\times X$ into X, we infer that

$$\|B(v,w)\|_\lambda \leq C\|v\|_\lambda\|w\|_\lambda \exp\big(C\lambda\|u_0\|^4_{\dot B^{-1}_{\infty,2}}\big). \tag{1.27}$$

Theorem 1.5.1 follows from the following two lemmas we admit for a while.

Lemma 1.5.2 *There is a constant $C>0$ such that the following holds. For any non negative λ, for any $t\geq 0$ and any $f\in E$, we have*

$$\Big\|\int_0^t e^{(t-t')\Delta}f(t')dt'\Big\|_\lambda \leq C\|f\|_E.$$

Lemma 1.5.3 *Let $u_0\in \dot B^{-1}_{\infty,2}$ be given, and define $u_L(t)=e^{t\Delta}u_0$. There is a constant $C>0$ such that the following holds. For any $\lambda\geq 1$, for any $t\geq 0$ and any $v\in X_\lambda$, we have*

$$\|B(u_L,v)(t)\|_\lambda \leq \frac{C}{\lambda^{\frac14}}\|v\|_\lambda.$$

Conclusion of the proof of Theorem 1.5.1 Let us apply Lemma 1.5.1 to Equation (NS') satisfied by R, in a space X_λ. We choose λ so that according to Lemma 1.5.3,

$$\|B(u_L,\cdot)(t)\|_{\mathcal{L}(X_\lambda)} \leq \frac14\cdot$$

Then according to Lemma 1.5.1, there is a unique solution R to (NS') in X_λ as soon as $B(u_L,u_L)$ satisfies

$$\|B(u_L,u_L)\|_{X_\lambda} \leq \frac{1}{16\|B\|_{\mathcal{B}(X_\lambda)}}\cdot$$

But (1.27) guarantees that

$$\|B\|_{\mathcal{B}(X_\lambda)} \leq \exp\Big(\lambda\|u_0\|^4_{\dot B^{-1}_{\infty,2}}\Big),$$

So, if

$$\|B(u_L,u_L)\|_{X_\lambda} \leq C^{-1}\exp\Big(-\lambda\|u_0\|^4_{\dot B^{-1}_{\infty,2}}\Big),$$

by Lemma 1.5.1, there is a unique solution of (MNS). The above condition is exactly re is precisely condition (1.23) of Theorem 1.5.1, so under assumption (1.23), there is a unique small (in the sense of $\|\cdot\|_\lambda$) solution R to (NS').

Proof of Lemma 1.5.2 Thanks to (1.26), it is enough to prove Lemma 1.5.2 for $\lambda=0$.

Let us start by proving that $L_0f \stackrel{\text{def}}{=} \displaystyle\int_0^t e^{(t-t')\Delta}f(t')dt'$ belongs to $L^2(\mathbb{R}^+;L^\infty)$. Let us observe that

$$\|L_0f(t,\cdot)\|_{L^\infty} \leq \int_0^t \|e^{(t-t')\Delta}f(t')\|_{L^\infty}dt'.$$

Let us observe that, after a change of variable

$$\|L_0 f\|_{L^2(\mathbb{R}^+;L^\infty)} \leq \sup_{\|\varphi\|_{L^2(\mathbb{R}^+)}\leq 1} \int_{\mathbb{R}^2_+} \|e^{\tau\Delta} f(t')\|_{L^\infty} \varphi(t'+\tau) d\tau\, dt'\,.$$

By Cauchy-Schwarz inequality and by definition of the norm on $B^{-1}_{\infty,\infty}$, we get

$$\begin{aligned}\|L_0 f\|_{L^2(\mathbb{R}^+;L^\infty)} &\leq \int_0^\infty \Big(\int_0^\infty \|e^{\tau\Delta} f(t')\|_{L^\infty}\Big)^{\frac{1}{2}} dt' \\ &\leq \|f\|_{L^1(\mathbb{R}^+;B^{-1}_{\infty,2})}.\end{aligned}$$

In order to estimate $t^{\frac{1}{2}} f(t)\|_{L^\infty}$, let us write that

$$\begin{aligned} t^{\frac{1}{2}} L_0 f(t,\cdot)\|_{L^\infty} &\leq \sqrt{2}\int_0^{\frac{t}{2}} (t-t')^{\frac{1}{2}} \big\|e^{(t-t')\Delta} f(t')\|_{L^\infty} dt' \\ &\qquad + \sqrt{2}\int_{\frac{t}{2}}^t \frac{1}{\sqrt{(t-t')t'}} t'(t-t')^{\frac{1}{2}} \big\|e^{(t-t')\Delta} f(t')\|_{L^\infty} dt' \\ &\leq \sqrt{2}\int_0^{\frac{t}{2}} \|f(t')\|_{B^{-1}_{\infty,2}} dt' + \sup_{t>0} \|f(t)\|_{B^{-1}_{\infty,\infty}} \int_{\frac{t}{2}}^t \frac{1}{\sqrt{(t-t')t'}}\, dt' \\ &\lesssim \|f\|_{B^{-1}_{\infty,2}} + \sup_{t>0} \|f(t)\|_{B^{-1}_{\infty,\infty}}. \end{aligned}$$

This proves Lemma 1.5.2. □

Proof of Lemma 1.5.3 From Proposition 1.4 page 11, we have

$$\begin{aligned} B(v,w)(t,x) &= \int_0^t \int_{\mathbb{R}^3} k(t-t',y) v(t',x-y) w(t',x-y) dy dt' \\ &= k\star(vw)(t,x) \quad \text{with} \quad |k(\tau,\zeta)| \leq \frac{C}{(\sqrt{\tau}+|\zeta|)^4}\,. \end{aligned}$$

The proof relies now mainly on the following proposition.

Proposition 1.5.1 *Let $u_0 \in \dot{B}^{-1}_{\infty,2}$ be given, and define $u_L(t) = e^{t\Delta} u_0$. There is a constant C such that the following holds. Consider, for any positive R and for $(\tau,\zeta) \in \mathbb{R}^+ \times \mathbb{R}^3$, the following functions:*

$$K^{(1)}_R(\tau,\zeta) \stackrel{\text{def}}{=} \mathbf{1}_{|\zeta|\geq R} \frac{1}{|\zeta|^4} \quad \textit{and} \quad K^{(2)}_R(\tau,\zeta) \stackrel{\text{def}}{=} \mathbf{1}_{|\zeta|\leq R} \frac{1}{(\sqrt{\tau}+|\zeta|)^4}.$$

Then for any $\lambda \geq 1$ and any $R > 0$,

$$\Big\| e^{-\lambda\int_0^t U(t')dt'} K^{(1)}_R \star (u_L v)\Big\|_{L^\infty([0,R^2]\times\mathbb{R}^3)} \leq \frac{C}{\lambda^{\frac{1}{2}} R} \|v\|_\lambda. \tag{1.28}$$

Moreover, for any $\lambda \geq 1$ and any $R > 0$,

$$\Big\| e^{-\lambda\int_0^t U(t')dt'} K^{(2)}_R \star (u_L v)\Big\|_{L^\infty([R^2,2R^2]\times\mathbb{R}^3)} \leq \frac{C}{\lambda^{\frac{1}{4}} R} \|v\|_\lambda. \tag{1.29}$$

Proof. Let us write that

$$
\begin{aligned}
V_\lambda^{(1)}(t,x) &\stackrel{\text{def}}{=} e^{-\lambda\int_0^t U(t')dt'}|K_R^{(1)}\star(u_L v)(t,x)|\\
&\leq \int_0^t\int_{{}^cB(0,R)} \frac{1}{|y|^4} e^{-\lambda\int_{t'}^t U(t'')dt''}\|u_L(t',\cdot)\|_{L^\infty}|v_\lambda(t',x-y)|dt'dy.
\end{aligned}
$$

By the Cauchy-Schwarz inequality and by definition of U, we infer that

$$
\begin{aligned}
V_\lambda^{(1)}(t,x) &\leq \biggl(\int_0^t\int_{{}^cB(0,R)} \frac{1}{|y|^4} e^{-2\lambda\int_{t'}^t U(t'')dt''}\|u_L(t',\cdot)\|_{L^\infty}^2 dt'dy\biggr)^{\frac12}\\
&\qquad\qquad\times\biggl(\int_0^t\int_{{}^cB(0,R)} \frac{1}{|y|^4}|v_\lambda(t',x-y)|^2dt'dy\biggr)^{\frac12}\\
&\leq \Bigl(\frac{C}{\lambda R}\Bigr)^{\frac12}\biggl(\int_0^t\int_{{}^cB(0,R)} \frac{1}{|y|^4}|v_\lambda(t',x-y)|^2dt'dy\biggr)^{\frac12}. \qquad (1.30)
\end{aligned}
$$

Now let us decompose the integral on the right on rings; this gives

$$
\begin{aligned}
\int_0^t\int_{{}^cB(0,R)} \frac{1}{|y|^4}|v_\lambda(t',x-y)|^2dt'dy &= \sum_{p=0}^\infty \int_0^t\int_{B(0,2^{p+1}R)\setminus B(0,2^pR)} \frac{1}{|y|^4}|v_\lambda(t',x-y)|^2dt'dy\\
&\leq \frac1R\sum_{p=0}^\infty 2^{-p+3}(2^{p+1}R)^{-3}\\
&\qquad\times\int_0^t\int_{B(0,2^{p+1}R)}|v_\lambda(t,x-y)|^2dtdy.
\end{aligned}
$$

As $t\leq R^2$ and p is non negative, we have

$$
\begin{aligned}
\int_0^t\int_{{}^cB(0,R)} \frac{1}{|y|^4}|v_\lambda(t',x-y)|^2dt'dy &\leq \frac CR\sum_{p=0}^\infty 2^{-p}(2^{p+1}R)^{-3}\int_{P(x,2^{p+1}R)}|v_\lambda(t,z)|^2dtdz\\
&\leq \frac CR\sum_{p=0}^\infty 2^{-p}\sup_{R'>0}\frac{1}{R'^3}\int_{P(x,R')}|v_\lambda(t,z)|^2dtdz.
\end{aligned}
$$

By definition of $\|\cdot\|_\lambda$, we infer that

$$
\int_0^t\int_{{}^cB(0,R)} \frac{1}{|y|^4}|v_\lambda(t',x-y)|^2dt'dy \leq \frac CR\|v\|_\lambda^2.
$$

Then, using (1.30), we conclude the proof of (1.28).

In order to prove the second inequality, let us observe that

$$
e^{-\lambda\int_0^t U(t')dt'}|(K_R^{(2)}\star(u_Lv))(t,x)|\leq \mathcal{K}_R^{(21)}(t,x)+\mathcal{K}_R^{(21)}(t,x)\quad\text{with}
$$

$$
\begin{aligned}
\mathcal{K}_R^{(21)}(t,x) &\stackrel{\text{def}}{=} \int_0^{\frac t2}\int_{B(0,R)}\frac{1}{(\sqrt{t-t'}+|y|)^4}e^{-\lambda\int_{t'}^t U(t'')dt''}\|u_L(t',\cdot)\|_{L^\infty}|v_\lambda(t',x-y)|dt'dy\,,\\
\mathcal{K}_R^{(22)}(t,x) &\stackrel{\text{def}}{=} \int_{\frac t2}^t\int_{B(0,R)}\frac{1}{(\sqrt{t-t'}+|y|)^4}e^{-\lambda\int_{t'}^t U(t'')dt''}\|u_L(t',\cdot)\|_{L^\infty}|v_\lambda(t',x-y)|dt'dy.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, as $t \in [R^2, 2R^2]$ and $t \leq 2(t-t')$, we infer that

$$\begin{aligned}
\mathcal{K}_R^{(21)}(t,x) &\leq \Big(\int_0^{\frac{t}{2}} \int_{B(0,R)} \frac{1}{(\sqrt{t-t'}+|y|)^8} e^{-2\lambda \int_{t'}^t U(t'')dt''} \|u_L(t',\cdot)\|_{L^\infty}^2 dt' dy\Big)^{\frac12} \\
&\qquad\qquad \times \Big(\int_0^{\frac{t}{2}} \int_{B(0,R)} |v_\lambda(t',x-y)|^2 dt' dy\Big)^{\frac12} \\
&\leq \frac{C}{\lambda^{\frac12}} \Big(\int_{B(0,R)} \frac{dy}{(R+|y|)^8}\Big)^{\frac12} \Big(\int_0^{\frac{t}{2}} \int_{B(0,R)} |v_\lambda(t',x-y)|^2 dt' dy\Big)^{\frac12} \\
&\leq \frac{C}{(t\lambda)^{\frac12}} R^{-\frac32} \Big(\int_0^{R^2} \int_{B(0,R)} |v_\lambda(t',x-y)|^2 dt' dy\Big)^{\frac12},
\end{aligned}$$

so that

$$K_R^{(21)}(t,x) \leq \frac{C}{(t\lambda)^{\frac12}} \|v\|_\lambda. \tag{1.31}$$

In order to estimate $\mathcal{K}_R^{(22)}$, let us write that

$$\begin{aligned}
\mathcal{K}_R^{(22)}(t,x) &\leq \int_{\frac{t}{2}}^t \int_{\mathbb{R}^3} \frac{1}{(\sqrt{t-t'}+|y|)^4} e^{-\lambda \int_{t'}^t U(t'')dt''} \|u_L(t',\cdot)\|_{L^\infty} \|v_\lambda(t',\cdot)\|_{L^\infty} dt' dy \\
&\leq C\|v\|_\lambda \int_{\frac{t}{2}}^t \frac{1}{\sqrt{t-t'}} e^{-\lambda \int_{t'}^t U(t'')dt''} \frac{\|u_L(t',\cdot)\|_{L^\infty}}{t'^{\frac12}} dt'.
\end{aligned}$$

By definition of U and using the fact that $t \leq 2t'$, Hölder's inequality implies that

$$\begin{aligned}
\mathcal{K}_R^{(22)}(t,x) &\leq \frac{C}{t^{\frac12}} \|v\|_\lambda \Big(\int_0^t e^{-4\lambda \int_{t'}^t U(t'')dt''} t' \|u_L(t',\cdot)\|_{L^\infty}^4 dt'\Big)^{\frac14} \\
&\leq \frac{C}{\lambda^{\frac14} t^{\frac12}} \|v\|_\lambda.
\end{aligned}$$

Together with (1.31), this concludes the proof of the proposition. □

From this proposition, we infer immediately the following corollary. This corollary proves directly one half of Lemma 1.5.3, as it gives a control of $B(u_L, v)$ in the first norm out of the two entering in the definition of X_λ.

Corollary 1.5.1 *Under the assumptions of Proposition 1.5.1, we have*

$$t^{\frac12} e^{-\lambda \int_0^t U(t')dt'} \|B(u_L,v)(t,\cdot)\|_{L^\infty} \leq \frac{C}{\lambda^{\frac14}} \|v\|_\lambda.$$

Proof. Let us write that

$$k \star (u_L v)(t,x) = k \star (u_L \mathbf{1}_{{}^c B(x,2\sqrt{t})} v)(t,x) + k \star (u_L \mathbf{1}_{B(x,2\sqrt{t})} v)(t,x).$$

From Proposition 1.5.1, we infer that

$$\begin{aligned}
e^{-\lambda \int_0^t U(t')dt'} |k \star (u_L \mathbf{1}_{{}^c B(x,2\sqrt{t})} v)(t,x)| &\leq e^{-\lambda \int_0^t U(t')dt'} K_{2\sqrt{t}}^{(1)} \star (|u_L \mathbf{1}_{B(x,2\sqrt{t})} v|)(t,x) \\
&\leq \frac{C}{(t\lambda)^{\frac12}} \|v\|_\lambda.
\end{aligned}$$

Moreover, thanks to Proposition 1.5.1, we have also

$$\begin{aligned} e^{-\lambda\int_0^t U(t')dt'}|k\star(u_L\mathbf{1}_{B(x,2\sqrt{t})}v)\,(t,x)| &\leq e^{-\lambda\int_0^t U(t')dt'}K^{(2)}_{2\sqrt{t}}\star(|u_L|\mathbf{1}_{B(x,2\sqrt{t})}|v|)\,(t,x)\\ &\leq \frac{C}{\lambda^{\frac14}t^{\frac12}}\|v\|_\lambda. \end{aligned}$$

This proves the corollary. □

Proof of Lemma 1.5.3 (continued) Let us estimate $\|k\star(u_Lv)\|_{L^2(P(x,R))}$, for an arbitrary x in $\mathbb{R}^3$. Let us write that

$$k\star(u_Lv) = k\star(u_L\mathbf{1}_{^cB(x,2R)}v) + k\star(u_L\mathbf{1}_{B(x,2R)}v).$$

Observing that, for any $y\in B(x,R)$, we have

$$|k\star(u_L\mathbf{1}_{^cB(x,2R)}v)(t,y)| \leq CK^{(1)}_R\star(|u_L|\mathbf{1}_{^cB(x,2R)}|v|)(t,y),$$

and using Inequality (1.11) of Proposition 1.5.1, we get

$$e^{-\lambda\int_0^t U(t')dt'}\|k\star(u_L\mathbf{1}_{^cB(x,2R)}v\|_{L^\infty(P(x,R))} \leq \frac{C}{\lambda^{\frac12}R}\|v\|_\lambda.$$

As the volume of $P(x,R)$ is proportional to R^5, we infer that

$$\|k\star(u_L\mathbf{1}_{^cB(x,2R)}v_\lambda)\|_{L^2(P(x,R))} \leq \frac{C}{\lambda^{\frac12}}R^{\frac32}\|v\|_\lambda. \tag{1.32}$$

Now let us prove the equivalent of Lemma 1.4.2.

Lemma 1.5.4 *For any T, we have*

$$\big\|\big(B(u_L,v)\big)_\lambda\big\|_{L^2([0,T\times\mathbb{R}^3)} \leq \frac{C}{\sqrt{\lambda}}\|v_\lambda\|_{L^2([0,T\times\mathbb{R}^3)}$$

Proof. Let us write that

$$|\mathcal{F}(B(u_L,v))_\lambda(t,\xi)| \leq \int_0^t \exp\Big(-(t-t')|\xi|^2-\lambda\int_{t'}^t U(t'')^4dt''\Big)\big|\mathcal{F}(u_L(t')v_\lambda(t'))(\xi)\big|dt'.$$

We can write that

$$\big|\mathcal{F}(u_L(t')v_\lambda(t'))(\xi)\big| \leq a(t',\xi)\|u_L(t')\|_{L^\infty}\|v_\lambda(t')\|_{L^2} \quad\text{with}\quad \int_{\mathbb{R}^3}a^2(t',\xi)d\xi = 1.$$

Thus using Cauchy-Schwarz inequality, we infer that

$$\begin{aligned} \int_0^T|\mathcal{F}(B(u_L,v))_\lambda(t,\xi)|^2dt &\leq \int_0^T\bigg(\int_0^t\exp\Big(-(t-t')|\xi|^2-\lambda\int_{t'}^tU(t'')^4dt''\Big)\\ &\qquad\qquad\times a(t',\xi)\|u_L(t')\|_{L^\infty}\|v_\lambda(t')\|_{L^2}dt'\bigg)^2dt\\ &\leq \int_0^T\bigg(\int_0^t\exp\Big(-2\lambda\int_{t'}^tU(t'')^4dt''\Big)\|u_L(t')\|^2_{L^\infty}dt'\bigg)\\ &\qquad\times\bigg(\int_0^t|\xi|^2\exp\big(-2(t-t')|\xi|^2\big)a^2(t',\xi)\|v_\lambda(t')\|^2_{L^2}dt'\bigg)dt\\ &\leq \frac{1}{2\lambda}\int_0^T\Big(\int_{t'}^T|\xi|^2\exp\big(-2(t-t')|\xi|^2\big)dt\Big)a^2(t',\xi)\|v_\lambda(t')\|^2_{L^2}dt'\\ &\leq \frac{1}{4\lambda}\int_0^T a^2(t',\xi)\|v_\lambda(t')\|^2_{L^2}dt' \end{aligned}$$

An integration with respect to ξ gives the result. □

Conclusion the proof of Lemma 1.5.3 Together with Corollary 1.5.1 and (1.32), this concludes the proof of Lemma 1.5.3 □

1.6 A particular case of large oscillating data

It is not obvious that Theorem 1.5.1 is not empty. Of course, the non linear smallness condition is satisfied in the case when $\|u_0\|_{\dot{B}^{-1}_{\infty,2}}$ is small. Let us first state the theorem that presents a class of large oscillating initial data satisfying hypotheses of Theorem 1.5.1.

Proposition 1.6.1 *Let φ be a function in $\mathcal{S}(\mathbb{R}^3)$. Let us consider $P = (\varepsilon, \Lambda)$ in $]0,1] \times [1, \infty[$ such that $\varepsilon\Lambda$ is small enough. Let us define*

$$\varphi_\varepsilon(x) = \cos\Big(\frac{x_3}{\varepsilon}\Big)\varphi(x_1, \Lambda x_2, x_3).$$

If $A^2\Lambda^{-\frac{1}{3}} \leq C_0^{-1}\exp\big(-C_0A^4\big)$, the divergence free vector field

$$u_0(x) = \frac{A}{\varepsilon\Lambda}(-\partial_2\varphi_P(x), -\partial_1\varphi_P(x), 0)$$

satisfies

$$\|u_0\|_{\dot{B}^{-1}_{\infty,2}} \geq c_0 A \tag{1.33}$$

and the hypotheses of Theorem 1.5.1.

Proof. The fact that (1.33) is satisfied is an obvious consequence of Proposition 1.3.3.

As $\mathbb{P}$ is a Fourier multiplier operator of order 0, we have

$$\big\|\mathbb{P}\big(e^{t\Delta}u_0 \cdot \nabla e^{t\Delta}u_0\big)\big\|_E \leq C\big\|e^{t\Delta}u_0 \cdot \nabla e^{t\Delta}u_0\big\|_E.$$

Let us observe that

$$\big(e^{t\Delta}u_0\big)^1\partial_1\big(e^{t\Delta}u_0\big)^1 + \big(e^{t\Delta}u_0\big)^2\partial_2\big(e^{t\Delta}u_0\big)^1 = \Big(\frac{A}{\varepsilon}\Big)^2 e^{t\Delta}f_P e^{t\Delta}g_P \quad \text{and} \tag{1.34}$$

$$\big(e^{t\Delta}u_0\big)^1\partial_1\big(e^{t\Delta}u_0\big)^2 + \big(e^{t\Delta}u_0\big)^2\partial_2\big(e^{t\Delta}u_0\big)^2 = \Big(\frac{A}{\varepsilon}\Big)^2 \frac{1}{\Lambda} e^{t\Delta}\widetilde{f}_P e^{t\Delta}\widetilde{g}_P \tag{1.35}$$

where f, g, $\widetilde{f}$ and $\widetilde{g}$ are functions in $\mathcal{S}(\mathbb{R}^3)$. Now, the main lemma is the following.

Lemma 1.6.1 *There is a constant C such that the following result holds. Let f and g be in $\dot{B}^{-1}_{\infty,2} \cap \dot{H}^{-1}$. Then we have*

$$\|(e^{t\Delta}fe^{t\Delta}g)\|_E \leq C\Big(\|f\|_{\dot{B}^{-1}_{\infty,2}}\|g\|_{\dot{B}^{-1}_{\infty,2}}\Big)^{\frac{2}{3}}\Big(\|f\|_{\dot{B}^{-1}_{2,2}}\|g\|_{\dot{B}^{-1}_{2,2}}\Big)^{\frac{1}{3}}$$

Proof. Let us first prove the following interpolation result.

Lemma 1.6.2 *A constant C exists such that, for a and b in $L^2 \cap L^\infty$, we have*

$$\|ab\|_{B^{-1}_{\infty,2}} \lesssim \big(\|a\|_{L^2}\|b\|_{L^2}\big)^{\frac{1}{3}}\big(\|a\|_{L^\infty}\|b\|_{L^\infty}\big)^{\frac{2}{3}}.$$

Proof. By definition of the $B^{-1}_{\infty,2}$ norm, we have, for any positive τ_0,

$$\begin{aligned}
\|ab\|_{B^{-1}_{\infty,2}} &= \int_{\tau_0}^{\infty} \|e^{\tau\Delta}(ab)\|^2_{L^\infty} d\tau + \int_0^{\tau_0} \|e^{\tau\Delta}(ab)\|^2_{L^\infty} d\tau \\
&\leq \|a\|_{L^\infty}\|b\|_{L^\infty} \int_{\tau_0}^{\infty} \frac{d\tau}{\tau^3} + \|a\|_{L^2}\|b\|_{L^2} \int_0^{\tau_0} d\tau \\
&\leq \frac{1}{\tau_0^2}\|a\|_{L^\infty}\|b\|_{L^\infty} + \tau_0\|a\|_{L^2}\|b\|_{L^2}.
\end{aligned}$$

Chossing $\tau_0 = \Big(\dfrac{\|a\|_{L^\infty}\|b\|_{L^\infty}}{\|a\|_{L^2}\|b\|_{L^2}}\Big)^{\frac{1}{3}}$ gives the result. □

Proof of Lemma 1.6.1 (continued) Applying the above lemma this $a = e^{t\Delta}f$ and $e^{t\Delta}g$, this gives by Hölder inequality and by the definition of Besov spaces,

$$\begin{aligned}
\int_0^\infty \|e^{t\Delta}f\varepsilon^{t\Delta}g\|^2_{B^{-1}_{\infty,2}} dt &\leq \int_0^\infty \big(\|e^{t\Delta}f\|_{L^2}e^{t\Delta}g\|_{L^2}\big)^{\frac{1}{3}} \\
&\qquad\qquad \times \big(\|e^{t\Delta}f\|_{L^\infty}e^{t\Delta}g\|_{L^\infty}\big)^{\frac{2}{3}} dt \\
&\leq \big(\|e^{t\Delta}f\|_{L^2(\mathbb{R}^+;L^2)}e^{t\Delta}g\||_{L^2(\mathbb{R}^+;L^2)}\big)^{\frac{1}{3}} \\
&\qquad\qquad \times \big(\|e^{t\Delta}f\|_{L^2(\mathbb{R}^+;L^\infty)}e^{t\Delta}g\||_{L^2(\mathbb{R}^+;L^\infty)}\big)^{\frac{2}{3}} \\
&\leq \big(f\|_{B^{-1}_{2,2}}e^{t\Delta}g\||_{B^{-1}_{2,2}}\big)^{\frac{1}{3}}\big(f\|_{B^{-1}_{\infty,2}}e^{t\Delta}g\||_{B^{-1}_{\infty,2}}\big)^{\frac{2}{3}}.
\end{aligned}$$

Along the same lines, let us write that

$$\begin{aligned}
t\|e^{t\Delta}f\varepsilon^{t\Delta}g\|_{B^{-1}_{\infty,\infty}} &\lesssim t\|e^{t\Delta}f\varepsilon^{t\Delta}g\|_{B^{-1}_{\infty,\infty}} \\
&\lesssim \big(t^{\frac{1}{2}}\|e^{t\Delta}f\|_{L^2}t^{\frac{1}{2}}\|e^{t\Delta}g\|_{L^2}\big)^{\frac{1}{3}}\big(t^{\frac{1}{2}}\|e^{t\Delta}f\|_{L^\infty}t^{\frac{1}{2}}\|e^{t\Delta}g\|_{L^\infty}\big)^{\frac{2}{3}} \\
&\lesssim \big(f\|_{B^{-1}_{2,\infty}}e^{t\Delta}g\||_{B^{-1}_{2,\infty}}\big)^{\frac{1}{3}}\big(f\|_{B^{-1}_{\infty,\infty}}e^{t\Delta}g\||_{B^{-1}_{\infty,\infty}}\big)^{\frac{2}{3}} \\
&\lesssim \big(f\|_{B^{-1}_{2,2}}e^{t\Delta}g\||_{B^{-1}_{2,2}}\big)^{\frac{1}{3}}\big(f\|_{B^{-1}_{\infty,2}}e^{t\Delta}g\||_{B^{-1}_{\infty,2}}\big)^{\frac{2}{3}}.
\end{aligned}$$

This proves Lemma 1.6.1 □

Conclusion of the proof of Proposition 1.6.1 Then, using (1.34) and (1.35), we infer that

$$\begin{aligned}
\big\|\mathbb{P}\big(e^{t\Delta}u_0\cdot\nabla e^{t\Delta}u_0\big)\big\|_E &\leq \Big(\frac{A}{\varepsilon}\Big)^2\Big(\|f_P\|_{\dot B^{-1}_{\infty,2}}\|g_P\|_{\dot B^{-1}_{\infty,2}}\Big)^{\frac{2}{3}}\Big(\|f_P\|_{\dot B^{-1}_{2,2}}\|g_P\|_{\dot B^{-1}_{2,2}}\Big)^{\frac{1}{3}} \\
&\leq \Big(\frac{A}{\varepsilon}\Big)^2\varepsilon^{\frac{4}{3}}\big(\varepsilon^2\Lambda^{-1}\big)^{\frac{1}{3}} \\
&\leq A^2\Lambda^{-\frac{1}{3}}.
\end{aligned}$$

Thus if

$$\Lambda^{-\frac{1}{3}} \leq C_0^{-1} \leq A^{-2}C_0^{-1}\exp(-C_0A^4)$$

the hypotheses of Theorem 1.5.1 □

Remark Let us point out the importance of the algebraic structure of a non linear term. A term like

$$\big(e^{t\Delta}u_0\big)^1\partial_2\big(e^{t\Delta}u_0\big)^1$$

will produce terms we can bound only in E by $A^2\Lambda^{\frac{2}{3}}$.

1.7 The role of the special structure: a Navier-Stokes type equation which blows up

The incompressible Navier-Stokes system has three important features: the scaling invariance, the incompressibilty condition and the very special structure of the non linear term. This structure leads to energy estimate, but also appears in relations likes (1.34) and (1.35) which has been crucial for proving the global well posedness result of Theorem 1.6.1. The purpose of this section is to study a modifed system which has the first two properties (scaling invariance and divergence free condition) and with a different structure of the non linear term which will lead to blow up at finite time for some initial data that satisfies the hypotheses of Theorem 1.6.1,

$$\mathcal{E} \stackrel{\text{def}}{=} \bigl\{\xi \in \mathbb{R}^3\,,\ \xi_1\xi_2 < 0\,,\ \xi_2\xi_3 < 0 \text{ and } |\xi_1| < \min\{|\xi_2|, |\xi_3|\}\bigr\} \tag{1.36}$$

and the matrix $q(\xi)$

$$q(\xi) = \frac{\mathbf{1}_{\mathcal{E}}(\xi)}{|\xi|}\begin{pmatrix} \xi_2^2+\xi_3^2-\xi_1\xi_2-\xi_1\xi_3 & \xi_2^2+\xi_3^2-\xi_1\xi_2-\xi_1\xi_3 & \xi_2^2+\xi_3^2-\xi_1\xi_2-\xi_1\xi_3\\ \xi_1^2+\xi_3^2-\xi_1\xi_2-\xi_2\xi_3 & \xi_1^2+\xi_3^2-\xi_1\xi_2-\xi_2\xi_3 & \xi_1^2+\xi_3^2-\xi_1\xi_2-\xi_2\xi_3\\ \xi_1^2+\xi_2^2-\xi_1\xi_3-\xi_2\xi_3 & \xi_1^2+\xi_2^2-\xi_1\xi_3-\xi_2\xi_3 & \xi_1^2+\xi_2^2-\xi_1\xi_3-\xi_2\xi_3 \end{pmatrix}. \tag{1.37}$$

Let us observe that

$$q(\xi) = |\xi|\mathbf{1}_{\mathcal{E}}(\xi)\,\mathbb{P}(\xi)\begin{pmatrix} 1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}$$

if $\mathbb{P}(\xi)$ denotes the matrix of the Leray projection of divergence free vector field in the Fourier space. Let us consider the following modified incompressible Navier-Stokes system.

$$(MNS)\ \left\{\begin{array}{l} \partial_t u - \Delta u = Q(u,u)\\ \operatorname{div} u = 0\,,\ u_{|t=0} = u_0\end{array}\right. \quad \text{with}\quad \mathcal{F}(Q^j(u,u)) \stackrel{\text{def}}{=} \sum_{k=1}^{3} \mathbf{1}_{\mathcal{E}}(\xi) q_{j,k}(\xi)\mathcal{F}(u^j u^k).$$

The main point of this modified Navier-Stokes system is the following property which plays a key role in the proof of blow up for finite time.

Proposition 1.7.1 *The coefficients of the matrix $q(\xi)$ are non negative.*

Proof. Let us first notice that on $\mathcal{E}$, $\xi_1\xi_3$ is positive.The components of the first line may be written

$$\xi_2^2+\xi_3^2-\xi_1\xi_2-\xi_1\xi_3 = \xi_2^2-\xi_1\xi_2+\xi_3(\xi_3-\xi_1)$$

which is also positive since either ξ_1 and ξ_3 are both positive, in which case $\xi_3 > \xi_1$, or they are both negative in which case $\xi_3 < \xi_1$. Thus the first line of the above matrix is clearly made of positive scalars. The fact that the terms of the second line are non negative is obvious due to the sign condition imposed on the components of ξ.

Similarly one has

$$\xi_1^2+\xi_2^2-\xi_1\xi_3-\xi_2\xi_3 = \xi_1^2+\xi_2^2-\xi_3(\xi_1+\xi_2)$$

and either $\xi_1 < 0$, $\xi_2 > 0$, $\xi_3 < 0$ and $\xi_1+\xi_2 > 0$, or $\xi_1 > 0$, $\xi_2 < 0$, $\xi_3 > 0$ and $\xi_1+\xi_2 < 0$. So the third line is also made of positive real numbers. □

Theorem 1.7.1 *Let us consider an initial data u_0 such that, for any $j \in \{1,2,3\}$, the component $\widehat{u}_0^j$ is a non negative function. Let us assume that for some j_0, we have*

$$\widehat{u}_0^{j_0}(\xi) \geq \widehat{v}_0(\xi) \geq 0 \quad \textit{with} \quad \text{Supp } \widehat{v}_0 \subset \mathcal{C}_{r,R}. \tag{1.38}$$

where $\mathcal{C}_{r,R}$ is some ring of $\mathbb{R}^3$. Let us assume a positive real number m exists such that, for any ξ in the union of the iterated sum of Supp $\widehat{v}_0$,

$$q_{j_0,j_0}(\xi) \geq m|\xi|. \tag{1.39}$$

If the quantity

$$m\frac{r}{R}\|\widehat{v}_0\|_{L^1}$$

is large enough, then the unique solution to (MNS) associated with the initial date u_0 blows up for finite time.

Proof. As the positivity of the components of $\widehat{u}$ is preserved by the flow of (MSN), we have that

$$\begin{aligned}
\widehat{u}^{j_0}(\xi) &= e^{-t|\xi|^2}\widehat{u}_0^{j_0}(\xi) + \sum_{k=1}^{3}\int_0^t e^{-t-t')|\xi|^2} q_{j_0,k}(\xi)\big(\widehat{u}^{j_0}(t'\cdot)\star\widehat{u}^k(t',\cdot)\big)(\xi)dt' \\
&\geq e^{-t|\xi|^2}\widehat{u}_0^{j_0}(\xi) + \sum_{k=1}^{3}\int_0^t e^{-t-t')|\xi|^2} q_{j_0,j_0}(\xi)\big(\widehat{u}^{j_0}(t'\cdot)\star\widehat{u}^k(t',\cdot)\big)(\xi)dt'.
\end{aligned}$$

As $q_{j_0,j_0}(\xi)$ is non negative, we get that, for any t, $u^{j_0}(\xi) \geq \widehat{v}(\xi)$ where v is the solution of

$$\partial_t v - \Delta v = q_{j_0,j_0}(D)(v^2) \quad \text{with} \quad v_{|t=0} = v_0.$$

We give here a variation of the proof of [50]. Let us define the sequence $(t_k)_{k\in\mathbb{N}}$ by

$$t_0 = 0 \quad \text{and} \quad t_k \stackrel{\text{def}}{=} \frac{1}{R^2}\sum_{\ell=1}^{k} 2^{-2\ell}.$$

We use denote by $\underline{T}$ its limit (which is $4/(3R^2)$). Let us define the sequence $(v_0^{(\ell)})_{\ell\in\mathbb{N}}$ by

$$v_0^{(1)} \stackrel{\text{def}}{=} \widehat{v}_0 \quad \text{and} \quad v^{(\ell)} \stackrel{\text{def}}{=} v_0^{(1)} \star v_0^{(\ell-1)}.$$

Let us notice that

$$\text{Supp } v^{(\ell)} \subset \ell\mathcal{C}_{r,R} \quad \text{and} \quad \forall \xi \in \text{Supp } v_0^{(\ell)},\ q(\xi) \geq mr\ell.$$

Let us make the following induction hypothesis for some sequence $(A_k)_{k\in\mathbb{N}}$ which will be choosen later on:

$$(H_k) \quad \forall t \in [2\underline{T}, t_k]\,,\ \widehat{v}(t,\xi) \geq A_k v_0^{(2^k)}(\xi).$$

Using the hypothesis on the support of $\widehat{v}_0$, we have that, for any t in $[0, 2T]$,

$$\widehat{v}(t,\xi) \geq e^{-t\xi|^2}\widehat{v}_0(\xi) \geq e^{-2\underline{T}R^2}\widehat{v}_0(\xi).$$

Thus, if we choose $A_0 = e^{2\underline{T}R^2}$, (H_0) is satisfied. Now let us assume (H_k). As q_{j_0,j_0} and $\widehat{v}(t,\cdot)$ are non negative functions, we have, for any $t \geq t_{k+1}$,

$$\begin{aligned}
\widehat{v}(t,\xi) &= e^{-t|\xi|^2}\widehat{v}_0(\xi) + \int_0^t e^{-(t-t')|\xi|^2} q_{j_0,j_0}(\xi)\big(\widehat{v}(t',\cdot)\star\widehat{v}(t',\cdot)\big)(\xi)dt' \\
&\geq \Big(\int_t^{t_k} e^{-(t-t')|\xi|^2}dt'\Big)A_k^2 q_{j_0,j_0}(\xi)\big(v_0^{(2^k)}\star v_0^{(2^k)}\big)(\xi).
\end{aligned}$$

By definition of the sequence $(v_0^{(\ell)})_{\ell\in\mathbb{N}}$ we get using (1.39),

$$\widehat{v}(t,\xi) \geq \Big(\int_{t_k}^t e^{-(t-t')2^{2k+2}R^2}dt'\Big)mr2^k v_0^{(2^{k+1})}.$$

As $t \geq t_{k+1}$, we have $t - \dfrac{1}{2^{2k+2}R^2} \geq t_k$. Thus we get

$$\begin{aligned}
\widehat{v}(t,\xi) &\geq \Big(\int_{t-\frac{1}{2^{2k+2}R^2}}^t e^{-(t-t')2^{2k+2}R^2}dt'\Big)mr2^k v_0^{(2^{k+1})}(\xi) \\
&\geq 4e^{-1}\frac{r}{R^2}2^{-k}A_k^2 v_0^{(2^{k+1})}(\xi).
\end{aligned}$$

Choosing

$$A_{k+1} \stackrel{\text{def}}{=} m_0 2^{-k}A_k k^2 \quad \text{with} \quad m_0 \stackrel{\text{def}}{=} 4e^{-1}m\frac{r}{R^2}$$

gives (H_{k+1}). Let us compute A_k. By iteration, we find that

$$A_{k+1} = m_0^{2k+1}2^{-\sum_{\ell=0}^k \ell 2^{k-\ell}}A_0^{2^{k+1}}.$$

As $\displaystyle\sum_{\ell=0}^k \ell 2^{k-\ell} = 2^k$, we get that $A_{k+1} = 4\big(m_0 e^{-\frac{8}{3}}\big)^{2^{k+1}}$. As $\widehat{v}_0^{(\ell)}\|_{L^1} = \|\widehat{v}_0\|_{L^1}^\ell$, we infer that, for any t in $[\underline{T};2\underline{T}]$, we have

$$\widehat{v}(t,\xi) \geq 4\big(m_0 e^{-\frac{8}{3}}\|\widehat{v}_0\|_{L^1}\big)^{2^{k+1}}$$

Thus, if $m_0 e^{-\frac{8}{3}}\|\widehat{v}_0\|_{L^1}$ is large enough, then $\displaystyle\lim_{k\to\infty} A_k = +\infty$ and thus $\|\widehat{u}(t,\cdot)\|_{L^1}$ blows up for finite time and the theorem is proved. □

The purpose of this section is the proof of the following theorem.

Theorem 1.7.2 *Let ϕ be a function in $\mathcal{S}(\mathbb{R}^3)$ such that its Fourier transform is non negative, even and has its support in the region $\mathcal{E}$. Let $P = (\varepsilon,\Lambda)$ be in $]0,1]\times[1,\infty[$ such that $\varepsilon\Lambda$ is small enough, and A a positive real number. Let us consider the initial data*

$$u_0(x) \stackrel{\text{def}}{=} A(\partial_2\varphi_P(x), -\partial_1\varphi_P(x), 0) \quad \textit{with} \quad \varphi_P(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon\Lambda}\cos\Big(\frac{x_3}{\varepsilon}\Big)(\partial_1\phi)(x_1,\Lambda x_2, x_3).$$

If

$$\Lambda^{\frac{4}{3}} \leq C_0^{-1}A^{-2}\exp(-C_0A^4) \quad \textit{and} \quad \frac{A}{\varepsilon} \quad \textit{large enough,}$$

then u_0 satisfies the hypothesis of Theorem 1.5.1 and the local solution to (MNS) blows up for finite time.

Proof. We have

$$u_0(x) = \frac{A}{\varepsilon}\cos\Big(\frac{x_3}{\varepsilon}\Big)\Big((\partial_1\partial_2\varphi)(x_1,\Lambda x_2,x_3), -\frac{1}{\Lambda}(\partial_1^2\varphi)(x_1,\Lambda x_2,x_3)\Big).$$

First of all, let us check that $\widehat{u}_0^j$ are non negative functions for $j \in \{1,2,3\}$ and that their support intersects the set $|\xi_j| \geq 1/2$. Indeed we have

$$\widehat{u}_0(\xi) = \frac{A}{2\varepsilon\Lambda^2}\bigg(\sum_{\pm}(-\xi_1\xi_2)\varphi\Big(\xi_1,\frac{\xi_2}{\Lambda},\xi_3 \pm \frac{1}{\varepsilon}\Big), \sum_{\pm} -\xi_1^2\varphi\Big(\xi_1,\frac{\xi_2}{\Lambda},\xi_3 \pm \frac{1}{\varepsilon}\Big)\bigg)$$

which gives the non negativity of the Fourier transform of the components of u_0. Then let us consider a point ω_0 such that

$$-\omega_0^1\omega_2\varphi(\omega_0) \geq 2c_0$$

and a real number ε_0 such that

$$\forall \xi \in B(\omega_0,\varepsilon_0), -\xi_1\xi_2\varphi(\xi) \geq c_0.$$

Let us define v_0 by

$$\widehat{v}_0(\xi \stackrel{\text{def}}{=} \frac{A}{2\varepsilon\Lambda^2} w_0\Big(\xi_1,\frac{\xi_2}{\Lambda},\xi_3 \pm \frac{1}{\varepsilon}\Big) \quad \text{with} \quad w_0(\eta) \stackrel{\text{def}}{=} \mathbf{1}_{B(\omega_0,\varepsilon_0)}(\eta)\eta_1\eta_2\widehat{\varphi}(\eta).$$

As we have

$$\begin{aligned} \|\widehat{v}_0\|_{L^1} &\geq c_0\frac{A}{\varepsilon\Lambda}\mu\Big\{\xi \in \mathbb{R}^3 \ / \ \Big(\xi_1,\frac{\xi_2}{\Lambda},\xi_3\Big) \in B(\omega_0,\varepsilon_0)\Big\} \\ &\geq \mu(B(\omega_0,\varepsilon_0)c_0\frac{A}{\varepsilon}, \end{aligned}$$

we infer that if A/ε is large enough, the hypotheses of Theorem 1.7.1 are satisfied and thus the theorem is proved. □

1.8 References and remarks

The Koch and Tataru theorem has been originally proved in [40]. The second and third section are adapted from [15]. Let us mention that in the framework of periodic boundary conditions, a type of non linear smallness condition has been introduced in [14]. The methods for proving blow up in section 1.7 has been introduced in [49] and the modified (MNS) appeared in [29].

Chapter 2

Slowly varying vector fields

Introduction

In this chapter, we continue to investigate the use of the structure of the incompressible 3D Navier-Stokes equation to construct large global solution. The idea is to work is a situation close to the 2D situation, i.e. in a situation which vector fields vary slowly in one direction, nameley are of the form

$$u_{0,\varepsilon}(x_h, x_3) = v_{0,\varepsilon}(x_h, \varepsilon x_3)$$

where x_h belongs to a two dimensionnal domain and x_3 belongs to $\mathbb{R}$ and where derivatives of $v_{0,\varepsilon}$ are bounded with respect to ε in say L^2 space. Of course, the divergence free condition will constrain the properties of the "profile" v. Indeed, let us write the profile

$$v_{0,\varepsilon} = (v^h_{0,\varepsilon}, v^3_{0,\varepsilon} = (v^1_{0,\varepsilon}, v^2_{0,\varepsilon}, v^3_{0,\varepsilon})$$

the divergence free condition on $u_{0,\varepsilon}$ implies that

$$\mathrm{div}_h\, v^h_{0,\varepsilon} + \varepsilon\partial_3 v^3_{0,\varepsilon} = 0.$$

Thus the form of the initial data is

$$\Big(v^h_{0,\varepsilon}(x_h, \varepsilon x_3), \frac{1}{\varepsilon} v^3_{0,\varepsilon}(x_h, \varepsilon x_3)\Big)$$

where the profile $v_{0,\varepsilon} = (v^h_{0,\varepsilon}, v^3_{0,\varepsilon}) = (v^1_{0,\varepsilon}, v^2_{0,\varepsilon}, v^3_{0,\varepsilon})$ is a smooth enough vector field, uniformely with respect to the parameter ε. Because of Proposition 1.3.4, such initial are very large.

For technical reason, it will be necesarry to work in the domain $\mathbb{T}^2 \times \mathbb{R}$.

In the firs section, we explain how the problem reduces, after a rescaling with respect to the vertical variable, to a resolution of a system with vanishing viscosity in the vertical variable and a modified gradient of the rescaled pressure. This system looks illposed.

In the second section, we present a global Cauchy-Kovalevska mehod in the model case

$$\begin{cases} \partial_t u + \gamma u + A(D)(u^2) = 0 \\ u_{|t=0} = u_0 \end{cases}$$

where γ is a positif real number and $A(D)$ a Fourier mulptiplier of order 0. We prove global a priori estimate for small analalytic data. We method used consists in defining an admissible rate of decay of the radius of analyticity.

In the third section, we introduce, in the case of the real problem the phase function which describe the rate of decay of the radius of analyticity in the vertical variable. The role of the horizontal and the vertical component in this definition is very different. Then, we reduce the proof of the global wellposedness problem to the proof of two propositions which describe how the control the dacay of the radius of analyticity.

The rest of the chapter is devoted to the proof of these two propositions.

2.1 Ill prepared data: the vertical rescaling

The theorem we want to in this case in the following.

Theorem 2.1.1 *Let a be a positive number. There are two positive numbers ε_0 and η such that for any divergence free vector field v_0 satisfying*

$$\|e^{a|D_3|}v_0\|_{H^4} \leq \eta,$$

then, for any positive ε smaller than ε_0, the initial data

$$u_{0,\varepsilon}(x) \stackrel{\text{def}}{=} \Big(v_0^h(x_h, \varepsilon x_3), \frac{1}{\varepsilon}v_0^3(x_h, \varepsilon x_3)\Big)$$

generates a global smooth solution of (NS) on $\mathbb{T}^2 \times \mathbb{R}$.

Let us notice that for the sake of simplicity, we assume that the profile v_0 does not depend on ε. The theorem is also true in the case when $v_{0,\varepsilon}$ is a family of profile such that, for all ε, $\|e^{a|D_3|}v_{0,\varepsilon}\|_{H^4} \leq \eta$.

We look for the solution under the form

$$u_\varepsilon(t,x) \stackrel{\text{def}}{=} \Big(v_\varepsilon^h(t, x_h, \varepsilon x_3), \frac{1}{\varepsilon}v_\varepsilon^3(t, x_h, \varepsilon x_3)\Big).$$

Let us notice that

$$\Delta u_\varepsilon(t, x_h, x_3) = (\Delta_\varepsilon v_\varepsilon)(x_h, \varepsilon x_3) \quad \text{with} \quad \Delta_\varepsilon \stackrel{\text{def}}{=} \partial_1^2 + \partial_2^2 + \varepsilon^2\partial_3^2 = \Delta_h + \varepsilon^2\partial_3^2.$$

Moreover,

$$\begin{aligned} u_\varepsilon \cdot \nabla(f(x_h, \varepsilon x_3) &= \big(v^h \cdot \nabla_h f\big)(x_h, \varepsilon x_3) + \frac{1}{\varepsilon}v^3(x_h, \varepsilon x_3)\partial_3(f(x_h, \varepsilon x_3)) \\ &= (v \cdot \nabla f)(x_h, \varepsilon x_3). \end{aligned}$$

This leads to the following rescaled Navier-Stokes system.

$$(RNS_\varepsilon) \left\{ \begin{array}{c} \partial_t v^h - \Delta_\varepsilon v^h + v \cdot \nabla v^h = -\nabla^h q \\ \partial_t v^3 - \Delta_\varepsilon v^3 + v \cdot \nabla v^3 = -\varepsilon^2 \partial_3 q \\ \operatorname{div} v = 0 \\ v_{|t=0} = v_0 \end{array} \right.$$

Here, we already notice the importance of the structure of the non linear term which prevents from terms of size ε^{-1} which would appear with terms like $u^3\partial_h$.

Let us notice that the system (RNS_ε) is far away from the following system

$$(ANS_\varepsilon)\quad \begin{cases} \partial_t v^h - \Delta_\varepsilon v^h + v\cdot\nabla v^h = -\nabla^h p \\ \partial_t v^3 - \Delta_\varepsilon v^3 + v\cdot\nabla v^3 = -\partial_3 q \\ \operatorname{div} v = 0 \\ v_{|t=0} = v_0 \end{cases}$$

For this system, it is proved in [12], that if

$$\|v_0\|_{L^2}\|\partial_3 v_0\|_{L^2} \leq c_0,$$

(with c_0 independent of ε) then the system (ANS_ε) is globally wellposed. The fact that the about system satisfied the L^2 energy estimate is crucial. Because of the term $\varepsilon^2\partial_3 q$, the system (RNS_ε) does not safisty any conservation of energy.

Asthere is no boundary, the rescaled pressure q can be computed with the formula

$$\Delta_\varepsilon q = \sum_{j,k} \partial_j v^k \partial_k v^j = \sum_{j,k} \partial_j\partial_k (v^j v^k). \tag{2.1}$$

It turns out that when ε goes to 0, Δ_ε^{-1} looks like Δ_h^{-1}. In the case of $\mathbb{R}^3$, for low horizontal frequencies, an expression of the type $\Delta_h^{-1}(ab)$ cannot be estimated in L^2 in general. This is the reason why we work in $\mathbb{T}^2\times\mathbb{R}$. In this domain, the problem of low horizontal frequencies reduces to the problem of the horizontal average that we denote by

$$(Mf)(x_3) \stackrel{\text{def}}{=} \overline{f}(x_3) \stackrel{\text{def}}{=} \int_{\mathbb{T}^2} f(x_h, x_3)dx_h.$$

Let us also define $M^\perp f \stackrel{\text{def}}{=} (\operatorname{Id} - M)f$. Notice that, because the vector field v is divergence free, we have $\overline{v}^3 \equiv 0$. The system (RNS_ε) can be rewritten in the following form.

$$(RNS_\varepsilon)\quad \begin{cases} \partial_t w^h - \Delta_\varepsilon w^h + M^\perp\big(v\cdot\nabla w^h + w^3\partial_3\overline{v}^h\big) = -\nabla^h q \\ \partial_t w^3 - \Delta_\varepsilon w^3 + M^\perp(v\cdot\nabla w^3) = -\varepsilon^2\partial_3 M^\perp q \\ \partial_t\overline{v}^h - \varepsilon^2\partial_3^2\overline{v}^h = -\partial_3 M(w^3 w^h) \\ \operatorname{div}(\overline{v}+w) = 0 \\ (\overline{v}, w)_{|t=0} = (\overline{v}_0, w_0). \end{cases}$$

The problem to solve this sytem is that there is no obvious way to compensate the loss of one vertical derivative which appears in the equation on w_h and $\overline{v}$ and also, but more hidden, in the pressure term. The method we use is inspired by the one introduced in [9] and can be understood as a global Cauchy-Kowalewski result. This is the reason why the hypothesis of analyticity in the vertical variable is required in our theorem.

Let us denote by $\mathcal{B}$ the unit ball of $\mathbb{R}^3$ and by $\mathcal{C}$ the annulus of small radius 1 and large radius 2. For non negative j, let us denote by L^2_j the space $\mathcal{F}L^2((\mathbb{Z}^2\times\mathbb{R})\cap 2^j\mathcal{C})$ and by L^2_{-1} the space $\mathcal{F}L^2((\mathbb{Z}^2\times\mathbb{R})\cap\mathcal{B})$ respectively equipped with the (semi) norms

$$\|u\|^2_{L^2_j} \stackrel{\text{def}}{=} (2\pi)^{-d}\int_{2^j\mathcal{C}} |\widehat{u}(\xi)|^2 d\xi \quad\text{and}\quad \|u\|^2_{L^2_{-1}} \stackrel{\text{def}}{=} (2\pi)^{-d}\int_{\mathcal{B}} |\widehat{u}(\xi)|^2 d\xi.$$

Let us now recall the definition of inhomogeneous Besov spaces modeled on L^2.

Definition 2.1.1 *Let s be a nonnegative real number. The space B^s is the subspace of L^2 such that*

$$\|u\|_{B^s} \stackrel{\text{def}}{=} \big\|\big(2^{js}\|u\|_{L^2_j}\big)_j\big\|_{\ell^1} < \infty.$$

We note that $u \in B^s$ is equivalent to writing $\|u\|_{L^2_j} \leq Cc_j 2^{-js}\|u\|_{B^s}$ where (c_j) is a non negative series which belongs to the sphere of ℓ^1. Let us notice that $B^{\frac{3}{2}}$ is included in $\mathcal{F}(L^1)$ and thus in the space of continuous bounded functions. Moreover, if we substitute ℓ^2 to ℓ^1 in the above definition, we recover the classical Sobolev space H^s.

The theorem we actually prove is the following.

Theorem 2.1.2 *Let a be a positive number. There are two positive numbers ε_0 and η such that for any divergence free vector field v_0 satisfying*

$$\|e^{a|D_3|}v_0\|_{B^{\frac{7}{2}}} \leq \eta,$$

then, for any positive ε smaller than ε_0, the initial data

$$u_{0,\varepsilon}(x) \stackrel{\text{def}}{=} \Big(v_0^h(x_h, \varepsilon x_3), \frac{1}{\varepsilon}v_0^3(x_h, \varepsilon x_3)\Big)$$

generates a global smooth solution of (NS) on $\mathbb{T}^2 \times \mathbb{R}$.

Before entering in the technicalities of the proof, let us have a discussion about the hypothesis on analyticity of the intial data. It is known to be somehing "unphysical" like for instance in the problem of boundary layer of vanishing viscosity (see the Prantl' problem). Here, for any ε, a positive real number ρ_ε exists such that if

$$\|u_0 - u_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}}} < \rho_\varepsilon$$

then u_0 generates a global smooth solution. Indeed, writing the solution u associated with u_0 as

$$u = u_\varepsilon + w_\varepsilon$$

where u_ε is given by Theorem 2.1.2, we have

$$\begin{cases} \partial_t w - \Delta w + w\cdot\nabla w + w\cdot\nabla u_\varepsilon + u_\varepsilon\cdot\nabla w = -\nabla p \\ \operatorname{div} w = 0 \quad \text{et} \quad w_{|t=0} = w_0. \end{cases}$$

Classical $\dot{H}^{\frac{1}{2}}$ estimates allows to prove that, as long as $\|w(t)\|_{\dot{H}^{\frac{1}{2}}} \leq c_0$ (with c_0 small enough), we have

$$\begin{aligned} \|w(t)\|^2_{H^{\frac{1}{2}}} + \int_0^t \|\nabla w(t')\|^2_{\dot{H}^{\frac{1}{2}}} dt' &\leq \|w_0\|^2_{\dot{H}^{\frac{1}{2}}} \exp\Big(C_0 \int_0^t \|\nabla u_\varepsilon(t')\|_{L^\infty} dt'\Big) \\ &\leq \|w_0\|^2_{\dot{H}^{\frac{1}{2}}} \exp\Big(-\frac{C_0}{\varepsilon}\|e^{a|D_3|}v_0\|_{B^{\frac{7}{2}}}\Big) \end{aligned}$$

Thus, if

$$\|w_0\|_{\dot{H}^{\frac{1}{2}}} \leq \frac{c_0}{2}\exp\Big(-\frac{C_0}{\varepsilon}\|e^{a|D_3|}v_0\|_{B^{\frac{7}{2}}}\Big)$$

then $u_{0,\varepsilon} + w_0$ generates a global smooth solution.

2.2 Study of a model problem

In order to motivate the functional setting and to give a flavour of the method used to prove the theorem, let us study for a moment the following simplified model problem for (RNS_ε), in which we shall see in a rather easy way how the same type of method as that of [9] can be used (as a global Cauchy-Kovalevska technic): the idea is to control a nonlinear quantity, which depends on the solution itself. So let us consider the equation

$$\partial_t u + \gamma u + a(D)(u^2) = 0,$$

where u is a scalar, real-valued function, γ is a positive parameter, and $a(D)$ is a Fourier multiplier of order one. We shall sketch the proof of the fact that if the initial data satisfies, for some positive δ and some small enough constant c,

$$\|e^{\delta|D|}u_0\|_{B^{\frac{3}{2}}} \leq c\gamma,$$

then one has a global smooth solution, say in the space $B^{\frac{3}{2}}$ as well as all its derivatives. The idea of the proof is the following: we want to control the same kind of quantity on the solution, but one expects the radius of analyticity of the solution to decay in time. Let us introduce the following notation, which will be used throughout this article. For any locally bounded function Ψ on $\mathbb{R}^+ \times \mathbb{Z}^2 \times \mathbb{R}$ and for any function f, we define

$$f_\Psi(t) \stackrel{\text{def}}{=} \mathcal{F}^{-1}\big(e^{\Psi(t,\cdot)}\widehat{f}(t,\cdot)\big).$$

Let us notice that this notation does not make sense for any f; the following can be made rigourous by a cut-off in Fourier space. This will done in the next section, in the proof of Theorem 2.1.2.

So let us introduce the function $\theta(t)$ which describes the "loss of analyticity" of the solution. We define

$$\dot\theta(t) \stackrel{\text{def}}{=} \|u_\Phi(t)\|_{B^{\frac{3}{2}}} \quad \text{with} \quad \theta(0) = 0 \quad \text{and} \quad \Phi(t,\xi) = (\delta - \lambda\theta(t))|\xi|. \tag{2.2}$$

The parameter λ will be chosen large enough at the end, and we shall prove that $\delta - \lambda\theta(t)$ remains positive for all times. The computations that follow hold as long as that assumption is true (and a bootstrap will prove that in fact it does remain true for all times). Taking the Fourier transform of the equation gives

$$|\widehat{u}(t,\xi)| \leq e^{-\gamma t}|\widehat{u}_0(\xi)| + C\int_0^t e^{-\gamma(t-t')}\,|\xi|\,|\mathcal{F}(u^2)(t',\xi)|\,dt'.$$

Using the fact that

$$\gamma t + (\delta - \lambda\theta(t))\,|\xi| \leq \gamma(t-t') - \lambda|\xi|\int_{t'}^t \dot\theta(t'')\,dt'' \\ + \gamma t' + (\delta - \lambda\theta(t'))\,|\xi-\eta| + (\delta - \lambda\theta(t'))\,|\eta|\,,$$

we infer that

$$e^{\gamma t}|\widehat{u}_\Phi(t,\xi)| \leq e^{\delta|\xi|}|\widehat{u}_0(\xi)| + C\int_0^t e^{-\lambda|\xi|\int_{t'}^t\dot\theta(t'')\,dt''}|\xi|\,e^{\gamma t'}|\mathcal{F}(u_\Phi^2)|(t',\xi)\,dt'.$$

Thus, for any ξ in $2^j\mathcal{C}$, we have

$$e^{\gamma t}|\widehat{u}_\Phi(t,\xi)| \leq e^{\delta|\xi|}|\widehat{u}_0(\xi)| + C\int_0^t e^{-\lambda 2^j \int_{t'}^t \dot\theta(t'')\,dt''} 2^j\, e^{\gamma t'}|\mathcal{F}(u_\Phi^2)|(t',\xi)\,dt'.$$

Taking the $L^2(2^j\mathcal{C}, d\xi)$ norm gives

$$e^{\gamma t}\|u_\Phi(t,\cdot)\|_{L^2_j} \leq \|e^{\delta|D|}u_0\|_{L^2_j} + C\int_0^t e^{-\lambda 2^j \int_{t'}^t \dot\theta(t'')\,dt''} 2^j\, \|e^{\gamma t'}u_\Phi^2(t',\cdot)\|_{L^2_j}\,dt'. \tag{2.3}$$

Now, we need a lemma of paradifferential calculus type. The statement requires the following spaces, introduced in [17].

Definition 2.2.1 *Let s be a real number. We define the space $\widetilde{L}^\infty_T(B^s)$ as the subspace of functions f of $L^\infty_T(B^s)$ such that the following quantity is finite:*

$$\|f\|_{\widetilde{L}^\infty_T(B^s)} \stackrel{\text{def}}{=} \sum_j 2^{js}\|f\|_{L^\infty_T(L^2_j)}.$$

Let us notice that $\widetilde{L}^\infty_T(B^s)$ is obviously included in $L^\infty_T(B^s)$.

We shall also use a very basic version of Bony's decomposition: let us define

$$T_a b \stackrel{\text{def}}{=} \mathcal{F}^{-1}\sum_j \int_{2^j\mathcal{C}\cap\mathcal{B}(\xi,2^j)} \widehat{a}(\xi-\eta)\widehat{b}(\eta)d\eta \quad\text{and}\quad R_a b \stackrel{\text{def}}{=} \mathcal{F}^{-1}\sum_j \int_{2^j\mathcal{C}\cap\mathcal{B}(\xi,2^{j+1})} \widehat{a}(\xi-\eta)\widehat{b}(\eta)d\eta.$$

We obviously have $ab = T_a b + R_b a$.

Lemma 2.2.1 *For any positive s, a constant C exists which satisfies the following properties. For any function Ψ satisfying*

$$\Psi(t,\xi) \leq \Psi(t,\xi-\eta) + \Psi(t,\eta) \tag{2.4}$$

for any function b, a positive sequence $(c_j)_{j\in\mathbb{Z}}$ exists in the sphere of $\ell^1(\mathbb{Z})$ (and depending only on T and b) such that, for any a and any $t \in [0,T]$, we have

$$\|(T_a b)_\Psi(t)\|_{L^2_j} + \|(R_a b)_\Psi(t)\|_{L^2_j} \leq C c_j 2^{-js}\|a_\Psi(t)\|_{B^{\frac{3}{2}}}\|b_\Psi\|_{\widetilde{L}^\infty_T(B^s)}.$$

We prove only the lemma for R, the proof for T being strictly identical. Let us first investigate the case when the function Ψ is identically 0. We first observe that for any ξ in the annulus $2^j\mathcal{C}$, we have

$$\mathcal{F}(R_a b(t))(\xi) = \sum_{j'\geq j-2}\int_{2^{j'}\mathcal{C}\cap B(\xi,2^{j'+1})} \widehat{a}(t,\xi-\eta)\widehat{b}(t,\eta)d\eta.$$

By definition of $\|\cdot\|_{\widetilde{L}^\infty_T(B^s)}$, we infer that

$$\|R_a b(t)\|_{L^2_j} \leq C\|a(t)\|_{\mathcal{F}(L^1)}\sum_{j'\geq j-2} c_{j'}2^{-j's}\|b\|_{\widetilde{L}^\infty_T(B^s)}.$$

Defining $\widetilde{c}_j = \displaystyle\sum_{j'\geq j-2} 2^{(j-j')s}c_{j'}$ which satisfies $\displaystyle\sum_j \tilde{c}_j \leq C_s$, we obtain

$$\|R_a b(t)\|_{L^2_j} \leq C\tilde{c}_j 2^{-js}\|a(t)\|_{\mathcal{F}(L^1)}\|b\|_{\widetilde{L}^\infty_T(B^s)}. \tag{2.5}$$

As $B^{\frac{3}{2}}$ is included in $\mathcal{F}(L^1)$, the lemma is then proved in the case when the function Ψ is identically 0. In order to treat the general case, let us write that

$$\begin{aligned}|e^{\Psi(t,\xi)}\mathcal{F}(R_a b)(t,\xi)| &= e^{\Psi(t,\xi)}\sum_j \int_{2^j\mathcal{C}\cap B(\xi,2^j)} |\widehat{a}(t,\xi-\eta)|\,|\widehat{b}(t,\eta)|d\eta \\ &\leq \sum_j \int_{2^j\mathcal{C}\cap B(\xi,2^j)} e^{\Psi(t,\xi-\eta)}|\widehat{a}(t,\xi-\eta)|e^{\Psi(t,\eta)}|\widehat{b}(t,\eta)|d\eta.\end{aligned}$$

Estimate (2.5) implies the lemma.

Now let us return to (2.3). We write

$$e^{\gamma t'}u_\Phi^2(t') = T_{u_\Phi(t')}e^{\gamma' t}u_\Phi(t') + R(e^{\gamma' t}u_\Phi(t'),u_\Phi(t'))$$

and Lemma 2.2.1 gives, for all $t'\leq T$ and as long as the function Φ is positive,

$$\|e^{\gamma t'}u_\Phi^2(t',\cdot)\|_{L^2_j} \leq Cc_j(T)2^{-j\frac{3}{2}}\|u_\Phi(t')\|_{B^{\frac{3}{2}}}\|e^{\gamma t'}u_\Phi(t')\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}})}.$$

By definition of the function θ, this gives

$$\|e^{\gamma t'}u_\Phi^2(t',\cdot)\|_{L^2_j} \leq Cc_j(T)2^{-j\frac{3}{2}}\dot\theta(t')\|e^{\gamma t'}u_\Phi(t')\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}})}.$$

Plugging this inequality in (2.3) (after multiplication by $2^{j\frac{3}{2}}$) gives, as long as the function Φ is positive, for all $t\leq T$,

$$2^{j\frac{3}{2}}e^{\gamma t}\|u_\Phi(t,\cdot)\|_{L^2_j} \leq 2^{j\frac{3}{2}}\|e^{\delta|D|}u_0\|_{L^2_j} + Cc_j(T)\|e^{\gamma t}u_\Phi(t)\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}})}\int_0^t e^{-\lambda 2^j\int_{t'}^t \dot\theta(t'')\,dt''}2^j\,\dot\theta(t')\,dt'.$$

As

$$\int_0^t e^{-\lambda 2^j\int_{t'}^t \dot\theta(t)\,dt''}2^j\,\dot\theta(t')\,dt' \leq \frac{1}{\lambda}\,,$$

we get

$$2^{j\frac{3}{2}}e^{\gamma t}\|u_\Phi(t,\cdot)\|_{L^2_j} \leq 2^{j\frac{3}{2}}\|e^{\delta|D|}u_0\|_{L^2_j} + \frac{C}{\lambda}c_j(T)\|e^{\gamma t}u_\Phi(t)\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}})}.$$

Taking the supremum for $t\leq T$, and summing over j, we get, as long as the function Φ is positive,

$$\|e^{\gamma t}u_\Phi(t,\cdot)\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}})} \leq \|e^{\delta|D|}u_0\|_{B^{\frac{3}{2}}} + \frac{C}{\lambda}\|e^{\gamma t}u_\Phi(t)\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}})}.$$

Thus, choosing $\lambda = 2C$ we infer that

$$\|e^{\gamma t}u_\Phi(t,\cdot)\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}})} \leq 2\|e^{\delta|D|}u_0\|_{B^{\frac{3}{2}}}.$$

As $\|a\|_{L^\infty_T(B^{\frac{3}{2}})} \leq \|a\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}})}$, we get, by definition of θ, as long as the function Φ is positive,

$$\dot\theta(t) \leq 2e^{-\gamma t}\|e^{\delta|D|}u_0\|_{B^{\frac{3}{2}}},$$

which gives $\gamma\theta(t) \leq 2\|e^{\delta|D|}u_0\|_{B^{\frac{3}{2}}}$. If

$$\|e^{\delta|D|}u_0\|_{B^{\frac{3}{2}}} \leq \frac{\delta\gamma}{8C},$$

then we get that the function Φ remains positive for all time and the global regularity is proved.

2.3 The global Cauchy-Kovalevska method

In the light of the computations of the previous section, let us introduce the functional setting we are going to work with to prove the theorem. The proof relies on exponential decay estimates for the Fourier transform of the solution, s let us define the key quantity we wish to control in order to prove the theorem. In order to do so, let us consider the Friedrichs approximation of the original (NS) system

$$\begin{cases} \partial_t u - \Delta u + \mathbb{P}_n(u\cdot\nabla u) = 0 \\ \operatorname{div} u = 0 \\ u_{|t=0} = \mathbb{P}_n\, u_{0,\varepsilon}, \end{cases}$$

where $\mathbb{P}_n$ denotes the orthogonal projection of L^2 on functions the Fourier transform of which is supported in the ball B_n centered at the origin and of radius n. Thanks to the L^2 energy estimate, this approximated system has a global solution the Fourier transform of which is supported in B_n. Of course, this provides an approximation of the rescaled system namely

$$(RNS_{\varepsilon,n}) \begin{cases} \partial_t w^h - \Delta_\varepsilon w^h + \mathbb{P}_{n,\varepsilon}\, M^\perp\big(v\cdot\nabla w^h + w^3\partial_3\overline{v} + \nabla^h q\big) = 0 \\ \partial_t w^3 - \Delta_\varepsilon w^3 + \mathbb{P}_{n,\varepsilon}\, M^\perp\big(v\cdot\nabla w^3 + \varepsilon^2\partial_3 q\big) = 0 \\ \partial_t \overline{v}^h - \varepsilon^2\partial_3^2\overline{v}^h + \mathbb{P}_{n,\varepsilon}\,\partial_3 M(w^3 w^h) = 0 \\ \operatorname{div}(\overline{v}+w) = 0 \\ (\overline{v},w)_{|t=0} = (\overline{v}_0, w_0), \end{cases}$$

where $\mathbb{P}_{n,\varepsilon}$ denotes the orthogonal projection of L^2 on functions the Fourier transform of which is supported in $B_{n,\varepsilon} \stackrel{\text{def}}{=} \{\xi \,/\, |\xi_\varepsilon|^2 \stackrel{\text{def}}{=} |\xi_h|^2 + \varepsilon^2\xi_3^2 \leq n^2\}$. We shall prove analytic type estimates here, meaning exponential decay estimates for the the solution of the above approximated system. In order to make notation not too heavy we shall drop the fact that the solutions we deal with are in fact approximate solutions and not solutions of the original system. A priori bounds on the approximate sequence will be derived, which will clearly yield the same bounds on the solution. In the spirit of [9] (see also (2.2) in the previous section), we define the function θ (we drop also the fact that θ depends on ε in all that follows) by

$$\dot\theta(t) = \|w^3_\Phi(t)\|_{B^{\frac72}} + \varepsilon\|w^h_\Phi(t)\|_{B^{\frac72}} \quad \text{and} \quad \theta(0) = 0 \tag{2.6}$$

where

$$\Phi(t,\xi) = t^{\frac12}|\xi_h| + a|\xi_3| - \lambda\theta(t)|\xi_3| \tag{2.7}$$

for some λ that will be chosen later on (see Section 2.3.2). Notice that the definition of θ takes into account the particular algebraic structure of $(RNS_{\varepsilon,n})$. Since the Fourier transform of w is compactly supported, the above differential equation has a unique global solution on $\mathbb{R}^+$. If we prove that

$$\forall t\in\mathbb{R}^+\,,\ \theta(t) \leq \frac{a}{\lambda}\,, \tag{2.8}$$

this will imply that the sequence of approximated solutions of the rescaled system is a bounded sequence of $L^1(\mathbb{R}^+;\mathrm{Lip})$. So is, for a fixed ε, the family of approximation of the original Navier-Stokes equations. This is (more than) enough to imply that a global smooth solution exists.

2.3.1 Main steps of the proof

The proof of Inequality (2.8) will be a consequence of the following two propositions which provide estimates on v^h, w^h and w^3.

Theorem 2.1.2 will be an easy consequence of the following propositions, which will be proved in the coming sections.

The first one uses only the fact that the function Φ is subadditive.

Proposition 2.3.1 *A constant $C_0^{(1)}$ exists such that, for any positive λ, for any initial data v_0, and for any T satisfying $\theta(T) \leq a/\lambda$, we have*

$$\theta(T) \leq \varepsilon \|e^{a|D_3|} w_0^h\|_{B^{\frac{7}{2}}} + \|e^{a|D_3|} w_0^3\|_{B^{\frac{7}{2}}} + C_0^{(1)} \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \theta(T).$$

Moreover, we have the following L^∞-type estimate on the vertical component:

$$\|w_\Phi^3\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq \|e^{a|D_3|} w_0^3\|_{B^{\frac{7}{2}}} + C_0^{(1)} \|v_\Phi\|^2_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.$$

The second one is more subtle to prove, and it shows that the use of the analytic-type norm actually allows to recover the missing vertical derivative on v^h, in a L^∞-type space. It should be compared to the methods described in the model case above.

Proposition 2.3.2 *A constant $C_0^{(2)}$ exists such that, for any positive λ, for any initial data v_0, and for any T satisfying $\theta(T) \leq a/\lambda$, we have*

$$\|v_\Phi^h\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq \|e^{a|D_3|} v_0^h\|_{B^{\frac{7}{2}}} + C_0^{(2)} \Big(\frac{1}{\lambda} + \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\Big) \|v_\Phi^h\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.$$

2.3.2 Proof of the theorem assuming the two propositions

Let us assume these two propositions are true for the time being and conclude the proof of Theorem 2.1.2. It relies on a continuation argument.

For any positive λ and η, let us define

$$\mathcal{T}_\lambda \stackrel{\text{def}}{=} \big\{T \,/\, \max\{\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}, \theta(T)\} \leq 4\eta\big\},$$

As the two functions involved in the definition of $\mathcal{T}_\lambda$ are non decreasing, $\mathcal{T}_\lambda$ is an interval. As θ is an increasing function which vanishes at 0, a positive time T_0 exists such that $\theta(T_0) \leq 4\eta$. Moreover, if $\|e^{a|D|_3|} v_0\|_{B^{\frac{7}{2}}} \leq \eta$ then, since $\partial_t v = \mathbb{P}_n F(v)$ (recall that we are considering Friedrich's approximations), a positive time T_1 (possibly depending on n) exists such that $\|v_\Phi\|_{\widetilde{L}^\infty_{T_1}(B^{\frac{7}{2}})} \leq 4\eta$. Thus $\mathcal{T}_\lambda$ is the form $[0, T^\star)$ for some positive $T^\star$. Our purpose is to prove that $T^\star = \infty$. As we want to apply Propositions 2.3.1 and 2.3.2, we need that $\lambda\theta(T) \leq a$. This leads to the condition

$$4\lambda\eta \leq a. \tag{2.9}$$

From Proposition 2.3.1, defining $C_0 \stackrel{\text{def}}{=} C_0^{(1)} + C_0^{(2)}$, we have, for all $T \in \mathcal{T}_\lambda$,

$$\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq \|e^{a|D_3|} v_0\|_{B^{\frac{7}{2}}} + \frac{C_0}{\lambda} \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} + C_0 \|v_\Phi\|^2_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.$$

Let us choose $\lambda = \dfrac{1}{2C_0}\cdot$ This gives

$$\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq 2\|e^{a|D_3|} v_0\|_{B^{\frac{7}{2}}} + 4C_0\eta \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.$$

Choosing $\eta = \dfrac{1}{12C_0}$, we infer that, for any $T \in \mathcal{T}_\lambda$,

$$\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq 3\|e^{a|D_3|}v_0\|_{B^{\frac{7}{2}}}. \tag{2.10}$$

Propositions 2.3.1 and 2.3.2 imply that, for all $T \in \mathcal{T}_\lambda$,

$$\theta(T) \leq \varepsilon\|e^{a|D_3|}w_0^h\|_{B^{\frac{7}{2}}} + \|e^{a|D_3|}w_0^3\|_{B^{\frac{7}{2}}} + C_0\eta\theta(T).$$

This implies that

$$\theta(T) \leq 2\varepsilon\|e^{a|D_3|}w_0^h\|_{B^{\frac{7}{2}}} + 2\|e^{a|D_3|}w_0^3\|_{B^{\frac{7}{2}}}.$$

If $2\varepsilon\|e^{a|D_3|}w_0^h\|_{B^{\frac{7}{2}}} + 2\|e^{a|D_3|}w_0^3\|_{B^{\frac{7}{2}}} \leq \eta$ and $\|e^{a|D_3|}v_0^h\|_{B^{\frac{7}{2}}} \leq \eta$, then the above estimate and Inequality (2.10) ensure (2.8). This concludes the proof of Theorem 2.1.2.

For the proof of the two propositions, we refer to the paper by I. Gallgher, M. Paicu and the author "Global regularity for some classes of large solutions to the Navier-Stokes equations", *Annals of Mathematics*, **173**, 2011, pages 986–1012.

Bibliography

[1] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematishcen Wissenschaften, **343**, Springer-Verlag Berlin Heidelberg, 2011.

[2] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, *Annales de l'École Normale Supérieure*, **14**, 1981, pages 209–246.

[3] L. Caffarelli, R. Kohn et L. Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations, *Communications on Pure and Applied Mathematics*, **35**, 1982, pages 771–831.

[4] M. Cannone, Y. Meyer et F. Planchon, Solutions autosimilaires des équations de Navier-Stokes, Séminaire "Équations aux Dérivées Partielles de l'École polytechnique, Exposé VIII, 1993–1994.

[5] M. Cannone, *Ondelettes, paraproduits et Navier-Stokes*, Diderot Éditeur, Arts et Sciences, 1995.

[6] J.-Y. Chemin, Remarques sur l'existence pour le système de Navier-Stokes incompressible, *SIAM Journal of Mathematical Analysis*, **23**, 1992, pages 20–28.

[7] J.-Y. Chemin, *Fluides parfaits incompressibles*, Astérisque, **230**, 1995.

[8] J.-Y. Chemin, Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel. *Journal d'Analyse Mathématique*, **77**, 1999, pages 27–50.

[9] J.-Y. Chemin, Le système de Navier-Stokes incompressible soixante dix ans après Jean Leray, Séminaire et Congrès, **9**, 2004, page 99–123.

[10] J.-Y. Chemin, Localization in Fourier space and Navier-Stokes system, *Phase Space Analysis of Partial Differential Equations*, Proceedings 2004, CRM series, pages 53–136.

[11] J.-Y. Chemin, About weak-strong uniqueness for uniqueness for the 3D incompressible Navier-Stokes, *to appear in Communications in Pure and Applied Mathematics.*

[12] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Fluids with anisotropic viscosity, *Modélisation Mathématique et Analyse Numérique*,**34**, 2000, pages 315–335.

[13] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Basics of Mathematical Geophysics, *Mathematical Geophysics; an introduction to rotating fluids and Navier-Stokes equations*, Oxford Lecture series in Mathematics and its applications, **32**, Oxford University Press, 2006.

[14] J.-Y. Chemin and I. Gallagher, On the global wellposedness of the 3-D Navier-Stokes equations with large initial data, *Annales de l'École Normale Supérieure*, **39**, 2006, pages 679–698.

[15] J.-Y. Chemin and I. Gallagher, Wellposedness and stability results for the Navier-Stokes equations in $\mathbb{R}^3$,*Annales de l'Institut Henri Poincaré, Analyse Non Linéaire*, **26**, 2009, pages 599–624.

[16] J.-Y. Chemin and I. Gallagher, Large, global solutions to the Navier-Stokes equations, slowly varying in one direction, *Transactions of the Americal Mathematical Society*, **362**, 2010, pages 2859–2873.

[17] J.-Y. Chemin et N. Lerner, Flot de champs de vecteurs non-lipschitziens et équations de Navier-Stokes, *Journal of Differential Equations*, **121**, 1995, pages 314–328.

[18] J.-Y. Chemin and P. Zhang, On the global wellposedness of the 3-D incompressible anisotropic Navier-Stokes equations, *Communications in Mathematical Physics*, **272**, 2007, pages 529–566.

[19] F. Colombini and N. Lerner, Hyperbolic operators with non-Lipschitz coefficients, *Duke Mathematical Journal*, **77**, 1995, pages 657–698.

[20] P. Constantin et C. Foias, *Navier-Stokes equations*, Chigago University Press, 1988.

[21] B. Desjardins, A few remarks on ordinary differential equations, *Communications in Partial Differential Equations*, **21**, 1996, pages 1667–1703.

[22] B. Desjardins, Linear transport equations with values in Sobolev spaces and application to the Navier-Stokes equations, *Differential and Integral Equations*, **10**, 1997, pages 577–586.

[23] R. Di Perna and P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, *Inventiones Mathematicae*, **98**, 1989, pages 511–549.

[24] L. Escauriaza, G. Seregin and V. Šverak, On the $L^{3,\infty}$ solutions to the Navier-Stokes equations and backward uniqueness, *Russ. Math. Surv.*, **58**, 2003, pages 211–250.

[25] T.M. Fleet, *Differential analysis*, Cambridge University Press, 1980.

[26] H. Fujita and T. Kato, On the Navier-Stokes initial value problem I, *Archiv for Rational Mechanic Analysis*, **16**, 1964, pages 269–315.

[27] G. Furioli, P.-G. Lemarié-Rieusset and E. Terraneo, Unicité des solutions mild des équations de Navier-Stokes dans $L^3(\mathbb{R}^3)$ et d'autres espaces limites, *Revista Matematica Iberoamericana*, **16**, 2000, pages 605–667.

[28] I. Gallagher, D. Iftimie, and F. Planchon, Asymptotics and stability for global solutions to the Navier-Stokes equations, *Annales de l' Institut Fourier*, **53**, 2003, pages 1387–1424.

[29] I. Gallagher and M. Paicu, Remarks on the blow-up of solutions to a toy model for the Navier-Stokes equations, *Proceedings of the American Mathematical Society*, **137**, 2009, pages 2075–2083.

[30] I. Gallagher and F. Planchon, On global infinite energy solutions to the Navier-Stokes equations in two dimensions, *Archive for Rational Mechanics and Analysis*, **161**, 2002, pages 307-337.

[31] P. Germain, Weak-strong uniqueness for the isentropic compressible Navier-Stokes system, *Journal of Mathematical Fluid Mechanics*, in press

[32] Z. Grujic et I. Kukavica, Space analycity for the Navier-Stokes and related equations with initial data in L^p, *Journal of Functional Analysis*, **152**, 1998, pages 247–466.

[33] Y. Giga, Solutions for semi linear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system, *Journal of Differential Equations*, **61**, 1986, pages 186–212.

[34] Y. Giga et T. Miyakawa, Solutions in L^r of the Navier-Stokes initial value problem, *Archiv for Rational Mechanics and Analysis*, **89**, 1985, pages 267–281.

[35] Y. Giga et T. Miyakawa, Navier-Stokes flows in $\mathbf{R}^3$ and Morrey spaces, *Communications in Partial differential Equations*, **14**, 1989, pages 577–618.

[36] D. Iftimie, A uniqueness result for the Navier-Stokes equations with vanishing vertical viscosity, *SIAM Journal of Mathematical Analysis*, **33**, 2002, page 1483–1493.

[37] T. Kato, Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$, *Journal of functionnal Analysis*, **9**, 1972, pages 296–305.

[38] T.Kato, Strong L^p-solutions of the Navier-Stokes equation in $\mathbf{R}^m$ with applications to weak solutions, *Mathematische Zeitschrift*, **187**, 1984, pages 471–480.

[39] T. Kato et G. Ponce, Well-posedness of the Euler and Navier-Stokes equations in the Lebesgue spaces $L^p_s(\mathbb{R}^2)$, *Revista Matemática Iberoamericana*, **2**, 1986, pages 73–88.

[40] H. Koch and D. Tataru: Well-posedness for the Navier-Stokes equations, *Advances in Mathematics*, **157**, 2001, pages 22–35.

[41] H. Kozono et M. Yamazaki, Semilinear heat equations and the Navier-Stokes equations with distributions in new function spaces as initial data, *Communications in Partial Differential Equations*, **19**, 1994, pages 959–1014.

[42] O. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, T Mathematics and its Applications, Vol. 2, Gordon and Breach, Science Publishers, New York-London-Paris 1969

[43] S. Leibovich, A. Mahalov, E. Titi, Invariant helical subspaces for the Navier-Stokes equations, *Archiv for Rational Mechanics and Analysis*, **112**, 1990, pages 193–222.

[44] P.-G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem*, Research Notes in Mathematics, **431**, 2002.

[45] J. Leray, Étude de diverses équations intégrales non linéaires et quelques problèmes que pose l'hydrodynamique, *Journal de Mathématiques Pures et Appliquées*, **9**(12),1933, pages 1–82.

[46] J. Leray, Essai sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Matematica*, **63**, 1933, pages 193–248.

[47] A. Mahalov and B. Nicolaenko, Global solvability of three-dimensional Navier-Stokes equations with uniformly high initial vorticity, (Russian. Russian summary), *Uspekhi Mat. Nauk*, **58**, 2003, pages 79–110.

[48] Y. Meyer, *Wavelets, paraproducts and Navier-Stokes*, Current Developments in Mathematics, International Press, Cambridge, Massachussets, 1996.

[49] S. Montgomery-Smith, Finite-time blow up for a Navier-Stokes like equation, *Proc. Amer. Math. Soc.*, **129**, 2001, pages 3025–3029.

[50] J. Neças, M. Ružička and V. Šverak, On Leray's self-similar solutions of the Navier-Stokes equations *Acta Mathematica*, **176**, 1996, pages 283–294.

[51] C. Oseen, Sur les formules de Green généralisées qui se présentent dans l'hydrodynamique et sur quelques unes de leurs applications, (première partie) *Acta Mathematica*, **34**, 1911, pages 205–284.

[52] C. Oseen, Sur les formules de Green généralisées qui se présentent dans l'hydrodynamique et sur quelques unes de leurs applications, (seconde partie) *Acta Mathematica*, **35**, 1912, pages 97–192.

[53] M. Paicu, Équation anisotrope de Navier-Stokes dans des espaces critiques, *Revista Rev. Matemtica Iberomericana*, **21**, 2005, pages 179–235.

[54] G. Ponce, R. Racke, T.C. Sideris, and E.S. Titi, Global stability of large solutions to the 3D Navier-Stokes equations, *Communications in Mathematical Physics*, **159**, 1994, pages 329–341.

[55] G. Raugel and G.R. Sell, Navier-Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions, *Journal of the American Mathematical Society*, **6**, 1993, pages 503–568.

[56] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, de Gruyter Series in Nonlinear Analysis and Applications, 3. Walter de Gruyter, Berlin, 1996.

[57] M. Sammartino and R.E. Caflisch, Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations, *Communications in Mathematical Physics*, **192**, 1998, pages 433–461.

[58] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.

[59] E.M. Stein, *Harnomic Analysis*, Princeton University Press, 1993.

[60] M. Taylor, *Tools for PDE. Pseudodifferential operators, paradifferential operators, and layer potentials*, Mathematical Surveys and Monographs, 81, American Mathematical Society, Providence, 2000.

[61] R. Temam, *Navier-Stokes equations, Theory and Numerical analysis*, North Holland, Amsterdam, **1984**.

[62] H. Triebel, *Interpolation theory, function spaces, differential operators*, North Holland (1978).

[63] H. Triebel, *Theory of function spaces*, Birkhäuser, Basel, 1983.

[64] M. Ukhovskii and V. Iudovich, Axially symmetric flows of ideal and viscous fluids filling the whole space, *Prikl. Mat. Meh.*, **32**, pages 59–69 (Russian).

[65] F. Vigneron, Spatial decay of the velocity field of an incompressible viscous fluid in $\mathbb{R}^d$, *Nonlinear Analysis*, **63**, 2005, page 525549.

[66] W. von Wahl, *The equations of Navier-Stokes and abstract parabolic equations*, Aspect der Mathematik, Vieweg & Sohn, Wiesbaden, 1985.

[67] M. Wiegner, Decay results for the weak solutions of the Navier-Stokes equations on $\mathbf{R}^n$, *Journal of the London Mathematical Society*, **35**, 1987, pages 303–313.

[68] F.Weissler, The Navier-Stokes initial value problem in L^p, *Archiv for Rational Mechanics and Analysis*, **74**, 1980, pages 219–230.