THE STABILIZATION PROBLEM ON SURFACES

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Abstract. We briefly recall some remarkable result on the stabilization problem of driftless affine control systems on surfaces. Then we remark that an interesting answer to the stabilization problem for such control systems would be to construct what we call smooth repulsive stabilizing feedbacks. Thus we discuss the existence of such feedbacks and present a sufficient condition in the two dimensional case.

Introduction

Let M be a smooth manifold of dimension two. We are concerned with the stabilization problem for control systems of the form

$$\dot{x} = u_1 X(x) + u_2 Y(x), \tag{1}$$

where X, Y are smooth vector fields on the surface M and where the control $u = (u_1, u_2)$ belongs to $\overline{B_2}$ the closed unit ball in $\mathbb{R}^2$. Such a control system is said to be *globally asymptotically controllable* at the point $O \in M$ (abreviated GAC in the sequel) if the two following properties are satisfied:

1. Attractivity: For each $x \in M$ there exists a control $u(\cdot) : [0, \infty) \to \overline{B_2}$ such that the corresponding trajectory $x(\cdot; x, u(\cdot))$ of (1) tends to O as t tends to infinity.
2. Lyapunov stability: For each neighborhood $\mathcal{V}$ of O, there exists some neighborhood $\mathcal{U}$ of O such that if $x \in \mathcal{U}$ then the control $u(\cdot)$ above can be chosen such that $x(t; x, u(\cdot)) \in \mathcal{V}, \forall t \geq 0$.

Given a GAC control system of the form (1), the purpose of the stabilization problem is to study the possible existence of a feedback $k(\cdot) = (k_1(\cdot), k_2(\cdot)) : M \longrightarrow \overline{B_2}$ which makes the closed-loop system

$$\dot{x} = k_1(x) X(x) + k_2(x) Y(x), \tag{2}$$

globally asymptotically stable at the point O (abreviated GAS in the sequel); *i.e.* such that all the trajectories of the closed-loop system converge asymptotically to the point O, and in addition such that the local property of Lyapunov stability is satisfied. As it is widely known, control systems as (1) which are globally asymptotically controllable at one point $O \in M$, do not admit in general a continuous stabilizing feedback. The Brockett's necessary condition makes local obstruction whereas Morse theory brings a global obstruction (we refer the reader to [7] for a detailed exposition of these results). In our paper [6], we proved that if the control system (1) is GAC at $O \in M$, then there exists a feedback $k : M \longrightarrow \overline{B_2}$ which satisfies the following properties:

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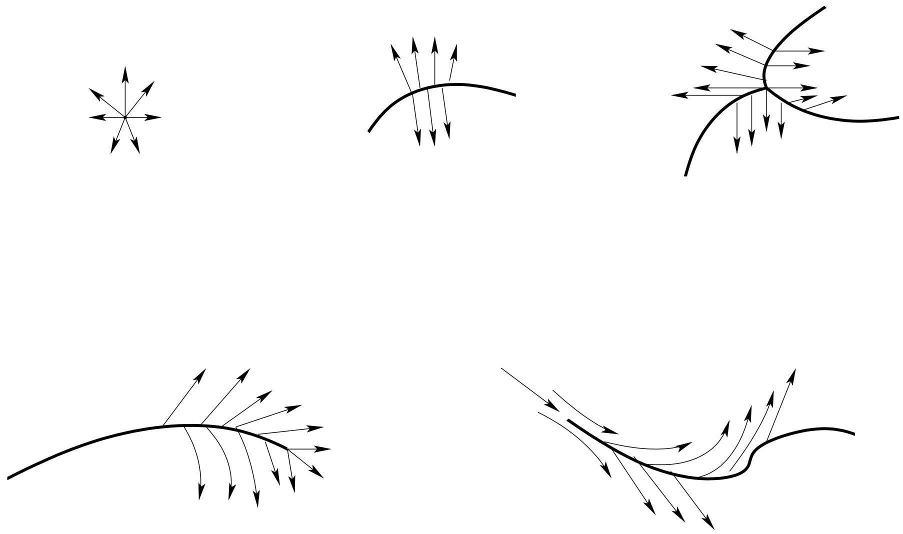

FIGURE 1. Different types of singularities

i) The closed-loop system (2) is GAS (at O) for Carathéodory solutions.
ii) The feedback k is smooth outside a stratified set which is a locally finite union (in $M \setminus \{O\}$) of points and of open submanifolds of dimension one; moreover the type of singularities that appear can be classified as shown in Figure 1.

The last type of singularity shown in Figure 1 is said to be of bifurcation type. In that case, there exists one trajectory of the closed-loop system which touches the set of singularities at one time and then go away from this set. Notice that in the absence of bifurcation points (that is, when the only types of singularities allowed are the four firsts given in Figure 1), the Carathéodory solutions of the Cauchy problem

$$\dot{x} = k_1(x)X(x) + k_2(x)Y(x), x(0) = x_0$$

where $x_0 \in M$ do not cross the set of singularities at any positive time, and then are smooth for $t > 0$. In other terms as soon as a trajectory of the closed-loop system (2) leaves the set of singularities, it does not touch it anymore. In particular, this means that the open set defined as the complement (in $M \setminus \{O\}$) of the set of singularities, *i.e.* the set where k is smooth, is invariant with respect to the system (2). So a question arises naturally: Can we cancel the singularities of bifurcation? That is, does there exist some feedback $k : M \to \overline{B_2}$ which satisfies the properties (i) and (ii) above without any singularity of bifurcation (such a feedback will be called a smooth repulsive stabilizing feedback (abreviated SRS feedback))? As shows the next example, the answer is no.

Example 1: Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function which equals 1 on the ball $\overline{B(0,1)}$ and 0 outside the open ball $B(0,2)$. Set on $M = \mathbb{R}^2$ the control system

$$\dot{x} = u_1\phi(x)X(x) + u_2(1 - \phi(x))Y(x),$$

where the vectors fields X and Y are defined by

$$\left\{ \begin{array}{lcl} X(x_1,x_2) & = & (x_2^2 - x_1^2)\frac{\partial}{\partial x_1} - 2x_1x_2\frac{\partial}{\partial x_2}, \\ Y(x_1,x_2) & = & x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}. \end{array} \right.$$

It is straightforward to show that this system is GAC at the origin. Moreover it does not admit a Carathéodory stabilizing feedback without singularity of bifurcation. As a matter of fact by construction of the function ϕ, the control system writes $\dot{x} = u_2 Y(x)$ outside the ball $B(0,2)$. Hence such a feedback should be smooth far from the origin and this would mean that an open set minus a compact set is contractible!

From now, our objective is to show that the bifurcation points can be eliminated under an additional assumption on (1). We prove that the global asymptotic stabilization with SRS feedbacks can be achieved for locally controllable systems without drift on surfaces, that is which satisfy the Hörmander's condition

$$\mathrm{Lie}\{X,Y\}(x) = T_x M, \tag{3}$$

for all x in M.

Theorem 1. *If the control system (1) satisfies the Hörmander's condition (3) then it admits a SRS feedback on M. Moreover the feedback can be taken to be continuous around the origin.*

Example 2: We set on the plane

$$\left\{ \begin{array}{lcl} X(x_1,x_2) & = & (x_1-2)\frac{\partial}{\partial x_1} + x_2\sigma\frac{\partial}{\partial x_2}, \\ Y(x_1,x_2) & = & (\sigma-1)\frac{\partial}{\partial x_1} + (\sigma-1)((x_1-2)x_2+1)\frac{\partial}{\partial x_2}, \end{array} \right.$$

where $\sigma := (x_1-2)^2 + x_2^2$. We let the reader to verify that this control system satisfies (3) on the plane. Let us prove that this control system does not admit a continuous stabilizing feedback. If such a feedback $k = (k_1,k_2)$ exists, then since $X = 0$ on the circle $\sigma = 1$, we have necessarily $k_1 > 0$ on this circle. But since the vector field $k_1X + k_2X$ is continuous, it should admit an equilibrium in the disc $(x_1-2)^2 + x_2^2 \leq 1$, which contradicts the asymptotic stability of the closed-loop system. However Theorem 1 asserts that this control system admits a SRS feedback.

1. Local asymptotic stabilization

This section is principally based on the paper of Kawski [4]. Here we prove that under the assumption of Theorem 1, there exists a continuous feedback which stabilizes the control system (1) locally.

Since Hörmander's condition is satisfied at the origin we may without loss of generality assume that $X(0) \neq 0$. Now by a change of local coordinates (*i.e.* so that locally $X = \frac{\partial}{\partial x_1}$), we may transform the control system (1) to the form

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & u_1 + u_2 Y_1(x_1,x_2), \\ \dot{x}_2 & = & u_2 Y_2(x_1,x_2), \end{array} \right.$$

where the vector field Y writes (in the new coordinates) $Y = Y_1\frac{\partial}{\partial x_1} + Y_2\frac{\partial}{\partial x_2}$. Obviously, modifying the feedback if necessary (that is setting $k_1' = k_1 - k_2Y_1, k_2' = k_2$), it is sufficient to stabilize locally the control system

$$\begin{cases} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2 Y_2(x_1, x_2). \end{cases}$$

On the other hand, in a suitable small neighborhood of the origin, both sets $\mathrm{Lie}\{X, (0, Y_2)^T\}(x_1, x_2)$ and $\mathrm{Lie}\{X, Y\}(x_1, x_2)$ coincide. This means that the new control system above satisfies the Hörmander's condition at the origin; in particular, there exists some integer $r \geq 0$ such that $\frac{\partial^r Y_2}{\partial x_1^r} \neq 0$. Therefore setting $k_2(x_1, x_2) := x_1$ or x_1^2, we deduce that the single-input control system (with control u_1)

$$\begin{cases} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= k_2(x_1, x_2) Y_2(x_1, x_2). \end{cases}$$

is small time locally controllable at the origin. Approximating the control system by an analytic one (see [2, 3]), and applying the Kawski's result (see [4]), we get the

Proposition 1. *If the control system (1) satisfies the Hörmander's condition at the origin, then it admits locally a continuous stabilizing feedback.*

Remark 1. *Actually, it can be proven that if the control system (1) is analytic and GAC, then it can be stabilized locally by a SRS feedback. This result does not hold globally; see for instance Example 1.*

2. Cancellation of bifurcation points

Here we sketch the proof of Theorem 1. For sake of simplicity, we assume that everything is analytic. From Proposition 1 and the main result of our paper [6], there exists a feedback $k : M \to \overline{B_2}$ which is continuous in some neighborhood of the origin and wich satisfies properties (i) and (ii) that we described in the introduction. Let us show how the singularities of bifurcation can be eliminated.

Let us denote by $\mathcal{S} \subset M$ the set of singularities of the feedback k, which is closed and stratified in $M \setminus \{O\}$. By (ii), we know that this set can be stratified into a locally finite union of points and of open submanifolds of dimension one. Let S be an open submanifold of dimension one in M associated with some $x_0 \in M$ such that the set $S \cup \{x_0\}$ forms a singularity of bifurcation as shown in figure 1 and such that the point x_0 is the point of bifurcation. This means that there exists some Carathéodory trajectory $x(\cdot)$ of the closed-loop system (2) and some constant $\delta > 0$ such that $x(0) = x_0$ and $x(t) \notin S \cup \{x_0\}$ for any $t \in (-\delta, \delta)$. Let $T < 0$ be the infimum of the set of $t < 0$ such that $x(s) \notin \mathcal{S}$ for all $s \in (t, 0)$; notice that without loss of generality (multiplying the feedback k by some positive function if necessary), we can assume that the trajectory $x(\cdot)$ is defined on $(-\infty, +\infty)$. Two cases appear.

First case: $T \neq -\infty$.
Obviously, $x(T) \in \mathcal{S}$. Here, three cases appear. In each case, we show in

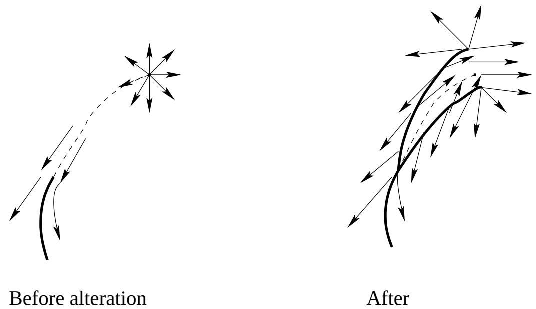

FIGURE 2

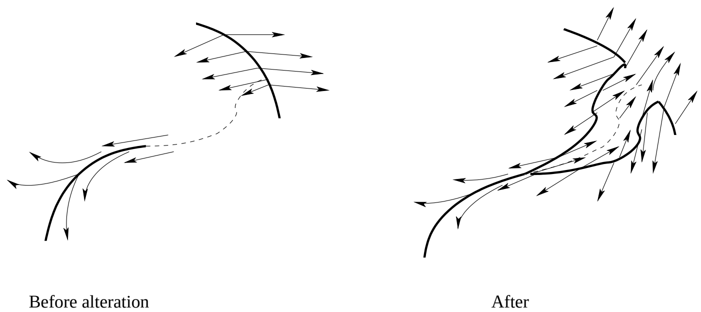

FIGURE 3

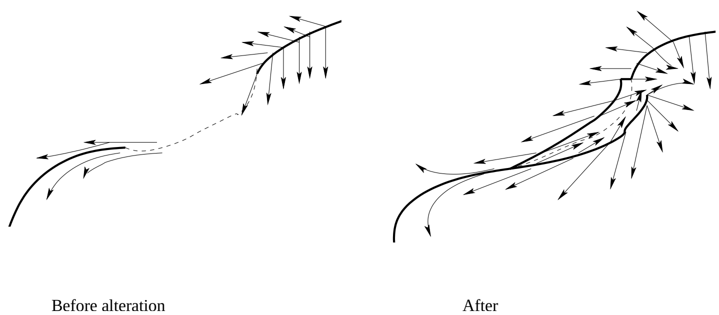

FIGURE 4

Figures 2-4, how by pasting new strata to the set $\mathcal{S}$ amd modifying the feedback in some regions, we can transform the feedback k in such a way that the point x_0 is not a bifurcation point for the new closed-loop system.

Second case: $T = -\infty$
Note that this situation may appear only if the manifold M is noncompact. Define for each $x \in M$, the set of velocities given by (1) as,

$$F(x) := \{u_1 X(x) + u_2 Y(x) : (u_1, u_2) \in \overline{B_2}\}.$$

For any $x \in M$ this set is a convex subset of $T_x M$ of dimension at least one. If the set $F(x(t))$ has dimension two for some $t \leq 0$ (for instance, it is the case for $t = 0$), then it is easy to modify locally the feedback in a neighborhood of the trajectory $x(\cdot)$ in such way that the new feedback is discontinuous on the trajectory as in the second type of singularity shown in

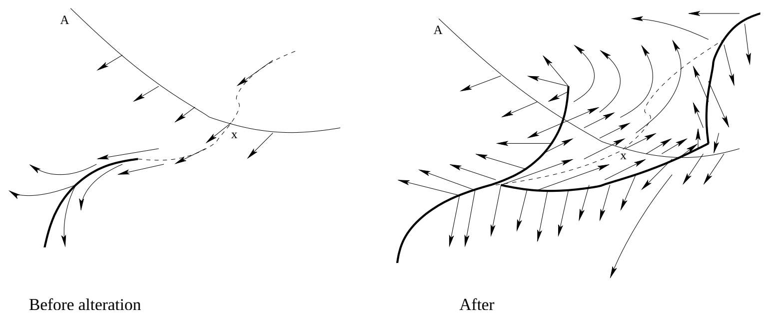

FIGURE 5

Figure 1. However, this cannot be done if $F(x(t))$ has dimension one. But since we assumed everything to be analytic and since the control system (1) satisfies the Hörmander's condition, the set of $x \in M$ where $F(x)$ has dimension one is analytic on the compact subsets of M. In addition, the set of $t \in [-\infty, 0]$ where the convex set $F(x(t))$ has dimension one is discrete. Hence If $x := x(t)$ is such a point, this means that the trajectory crosses at x the set of points in M where $F(x)$ has dimension one. But this set can be locally stratified by submanifolds of dimension zero and one. Assume for example that this set consists in a single submanifold A of dimension one in a neighborhood of x. In that case, as shown in Figure 5, we can modify the initial feedback k into a new feedback associated to a new set of singularities.

3. Concluding remarks

We do not know if Theorem 1 holds in dimension greater than two. However we are able to prove the following :

Theorem 2. *Assume that $n = 3$. If the control system (1) satisfies the Hörmander's condition at the origin,*

$$\mathit{Lie}\ \{f_1, \cdots, f_m\}(0) = \mathbb{R}^3,$$

then there exists a local SRS feedback.

Furthermore we can also prove the existence of SRS feedback in a very special case of distributions; the result can be stated as follows (We refer the reader to [1] and [9] for its proof and for the definition of fat distribution).

Theorem 3. *If control system (1) defines a fat distribution then it admits a SRS feedback.*

This theorem applies for instance in the case of the nonholonomic integrator.

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