On the Mather Quotient

Ludovic Rifford

Université de Nice - Sophia Antipolis

(Joint work with A. Fathi and A. Figalli)

Framework

Let M denote a compact, connected, smooth manifold.

Let $L : TM \to \mathbb{R}$ be a C^k $(k \geq 2)$ Lagrangian satisfying the following properties:

- **Strict convexity.** $\forall (x,v) \in TM$, the second derivative along the fibers $\frac{\partial^2 L(x,v)}{\partial v^2}$ is positive definite.
- **Superlinear growth.** For every $K \geq 0$ there exists a finite constant $C(K)$ such that

$$\forall (x,v) \in TM, \quad L(x,v) \geq K\|v\|_x + C(K).$$

Critical Value of L

Definition

The critical value of L is defined as

$$c[L] = -\inf\left\{\frac{1}{T}\int_0^T L\left(\gamma(t),\dot{\gamma}(t)\right)dt\right\},$$

where the infimum is taken over the set of smooth curves $\gamma:\mathbb{R}\to M$ such that $\gamma(0)=\gamma(T)$ for some $T>0$.

Legendre-Fenchel Duality

The Hamiltonian $H : T^*M \longrightarrow \mathbb{R}$ associated to L is defined by

$$\forall (x,p) \in T_x^*M, \quad H(x,p) = \max_{v \in T_xM} \{p(v) - L(x,v)\}.$$

It is C^k and satisfies the following properties:

- **Strict convexity.** $\forall (x,p) \in T^*M$, the second derivative along the fibers $\frac{\partial^2 H(x,p)}{\partial p^2}$ is positive definite.
- **Superlinear growth.** For every $K \geq 0$ there exists a finite constant $C(K)$ such that

$$\forall (x,p) \in TM, \quad H(x,p) \geq K\|p\|_x + C(K).$$

Fathi's Weak KAM Theorem

The Lax-Oleinik semi-group $(T_t^-)_{t\geq 0}$ acting on $C^0(M,\mathbb{R})$ is defined as

$$T_t^- u(x) = \inf\left\{u(\gamma(0)) + \int_0^t L\left(\gamma(s),\dot\gamma(s)\right) ds\right\},$$

where the infimum is taken over the set of smooth curves $\gamma : [0,t] \longrightarrow M$ such that $\gamma(t) = x$.

Theorem (Fathi's Weak KAM Theorem)

There exists a Lipschitz function $u : M \longrightarrow \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that

$$T_t^- u + ct = u, \quad \forall t \geq 0.$$

Fathi's Weak KAM Theorem

Proposition

Let $u : M \to \mathbb{R}$ be a continuous function satisfying

$$T_t^- u + ct = u, \quad \forall t \geq 0.$$

Then u satisfies the two following properties:

- *For any Lipschitz curve $\gamma : [a, b] \to M$, one has*

$$u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds + c(b-a).$$

- *For every $x \in M$, there is a C^1 curve $\gamma_x : (-\infty, 0] \to M$ such that*

$$u(\gamma_x(0)) = u(\gamma_x(-t)) + \int_{-t}^0 L(\gamma_x(s), \dot{\gamma}_x(s))\, ds + ct, \forall t \geq 0.$$

Fathi's Weak KAM Theorem

Proposition

There is a unique $c = c[H] \in \mathbb{R}$ for which there is a Lipschitz function $u : M \longrightarrow \mathbb{R}$ such that $T_t^- u + ct = u$, for all $t \geq 0$.

Proof. Let (u_1, c_1) and (u_2, c_2) two pairs of solutions with $c_1 > c_2$. Let $x \in M$. There is $\gamma_x^1 : (-\infty, 0] \rightarrow M$ such that

$$u_1(\gamma_x^1(0)) = u_1(\gamma_x^1(-t)) + \int_{-t}^0 L\left(\gamma_x^1(s), \dot{\gamma}_x^1(s)\right) ds + c_1 t, \quad \forall t \geq 0.$$

But we have for any $t \geq 0$,

$$u_2(\gamma_x^1(0)) - u_2(\gamma_x^1(-t)) \leq \int_{-t}^0 L\left(\gamma_x^1(s), \dot{\gamma}_x^1(s)\right) ds + c_2 t$$

Fathi's Weak KAM Theorem

Hence we have for any $t \geq 0$,

$$\left[u_2(\gamma_x^1(0)) - u_2(\gamma_x^1(-t))\right] - \left[u_1(\gamma_x^1(0)) - u_1(\gamma_x^1(-t))\right] \leq (c_2 - c_1)t.$$

Taking $t \to \infty$, we obtain a contradiction ($c_2 - c_1 < 0$!). $\square$

Definition

The constant $c[H]$ is called the critical value of H. A Lipschitz function $u : M \to \mathbb{R}$ satisfying, $T_t^- u + c[H]t = u, \forall t \geq 0$, is called a weak KAM solution or a critical viscosity solution.

In fact, the function $u : M \to \mathbb{R}$ is a weak KAM solution if and only if it is continuous and satisfies the following property: for any smooth function $\phi : M \to \mathbb{R}$ with $\phi \leq u$, we have for any $x \in M$,

$$\phi(x) = u(x) \Longrightarrow H(x, d_x\phi) = c[H].$$

In particular, if u is differentiable at x, then $H(x, d_x u) = c[H]$.

The Peierls barrier

Define for any $t \geq 0$, $h_t : M \times M \longrightarrow \mathbb{R}$ by

$$h_t(x,y) = \inf \left\{ \int_0^t L\left(\gamma_x(t), \dot{\gamma}_x(t)\right) dt \right\},$$

where the infimum is taken over all smooth paths $\gamma : [0,t] \longrightarrow M$ such that $\gamma(0) = x$ and $\gamma(t) = y$.

Definition

The Peierls barrier $h : M \times M \longrightarrow \mathbb{R}$ is defined by

$$h(x,y) = \liminf_{t \rightarrow \infty} \left\{ h_t(x,y) + c[H]t \right\}.$$

Proposition

For every $x \in M$, the function $h_x = h(x,\cdot) : y \in M \mapsto h(x,y)$ is a weak KAM solution.

The projected Aubry set

Definition

The projected Aubry set $\mathcal{A} \subset M$ is defined by

$$\mathcal{A} = \{x \in M \mid h(x,x) = 0\}.$$

Proposition

The projected Aubry set is a nonempty compact subset of M.

Proof. Let u be a weak KAM solution and $x \in M$. There is a C^1 curve $\gamma_x : (-\infty, 0] \to M$ such that

$$u(\gamma_x(0)) = u(\gamma_x(-t)) + \int_{-t}^{0} L(\gamma_x(s), \dot{\gamma}_x(s))\, ds + ct, \forall t \geq 0.$$

Assume that $\{t_n\}_n \uparrow \infty$ is a sequence of times such that $\gamma(-t_n)$ tends to some $y \in M$. Thanks to the equality above, it is easy to show that $y \in \mathcal{A}$. $\square$

$c[L]$ vs. $c[H]$

Recall that

$$c[L] = -\inf \left\{ \frac{1}{T} \int_0^T L\left(\gamma(t), \dot{\gamma}(t)\right) dt \right\},$$

where the infimum is taken over the set of smooth curves $\gamma : \mathbb{R} \rightarrow M$ such that $\gamma(0) = \gamma(T)$ for some $T > 0$.

Proposition

$$c[H] = c[L].$$

Proposition

The critical value $c[H]$ can be seen as the infimum of the constants $c \in \mathbb{R}$ such that there is a smooth function $v : M \rightarrow \mathbb{R}$ satisfying $H(x, d_x v) \leq c, \forall x \in M$.

The Aubry set

Proposition

Any weak KAM solution is differentiable on $\mathcal{A}$. Moreover, its differential at $x \in \mathcal{A}$ does not depend on u.

Definition

The Aubry set $\tilde{\mathcal{A}} \subset TM$ is defined as the set of $(x, v) \in TM$ such that $x \in \mathcal{A}$ and v is the unique element in T_xM such that $d_xu = \partial L/\partial v(x, v)$ for any weak KAM solution u.

Proposition

The Aubry set is a nonempty compact subset of TM which is invariant with respect to the Euler-Lagrange flow ϕ_t^L associated with L.

Fathi's extension theorem

Definition

We call critical subsolution of H any Lipschitz function $v : M \longrightarrow \mathbb{R}$ satisfying

$$H(x, d_x v) \leq c(H), \quad \text{for a.e. } x \in M.$$

Theorem

Let v be a critical subsolution of H. Then, there exists a weak KAM solution $u : M \longrightarrow \mathbb{R}$ such that $u_{|\mathcal{A}} = v_{|\mathcal{A}}$. Moreover, it is unique.

As a consequence, we deduce that any critical subsolution of H is differentiable on the projected Aubry set. We also deduce that two weak KAM solutions which coincide on the Aubry set coincide everywhere.

How many weak KAM solutions ?

Question : Let $x \in M$ be fixed. How many weak KAM solutions u satisfying $u(x) = 0$ do exist ?

Two remarks:

- Let u_1, u_2 be two distinct weak KAM solutions. Then, by convexity of H in the p variable, for every $\lambda \in (0, 1)$, the function $v = \lambda u_1 + (1 - \lambda)u_2$ is a critical subsolution. Thus, by the theorem above, there is a infinite number of distinct weak KAM solutions.
- As soon as $\mathcal{A}$ is not connected, then we can construct two distinct weak KAM solutions.

Examples

Example 1. Let $V : M \to [0, \infty)$ be a function of class C^k ($k \geq 2$) such that $\{V = 0\} \neq \emptyset$ and $L_V : TM \to \mathbb{R}$ be the Lagrangian defined by

$$L_V(x, v) = \frac{1}{2}\|v\|_x^2 + V(x).$$

The associated Hamiltonian is given by

$$H_V(x, p) = \frac{1}{2}\|p\|_x^2 - V(x).$$

One has obviously

$$c[H] = 0, \ \mathcal{A} = \{x \in M \mid V(x) = 0\}, \text{ and } \tilde{\mathcal{A}} = \mathcal{A} \times \{0\}.$$

Moreover, the function $v \equiv 0$ is a critical subsolution for H_V.

Examples

Assume now that $\dim M = 4$.
Let $f : M \longrightarrow \mathbb{R}$ be a C^3 counterexample to the Sard Theorem, that is, a function of class C^3 whose the set $\mathcal{C}$ of critical points is connected and such that its image by f $f(\mathcal{C})$ is a nontrivial interval.
Define $V : M \longrightarrow \mathbb{R}$ by

$$V(x) = \frac{1}{2}\|d_x f\|_x^2.$$

The Lagrangian and the Hamiltonian are C^2.
By construction, the function f is a weak KAM solution and $v \equiv 0$ is a critical subsolution of H_V.
Hence there is an infinite number of distinct weak KAM solutions for H_V !

Examples

Example 2. If X is a C^k vector field on M, with $k \geq 2$, the Mañé Lagrangian $L_X : TM \to \mathbb{R}$ associated to X is defined by

$$L_X(x,v) = \frac{1}{2}\|v - X(x)\|_x^2, \quad \forall (x,v) \in TM.$$

Its associated Hamiltonian $H_X : T^*M \to \mathbb{R}$ is given by

$$H_X(x,p) = \frac{1}{2}\|p\|_x^2 + p(X(x)).$$

The function $u \equiv 0$ is a viscosity solution of $H_X(x, d_x u) = 0$. Therefore, $c[H] = 0$.

Examples

Theorem

Let X be a $C^k, k \geq 2$ vector field on the compact connected smooth manifold M. Assume that one of the following conditions hold:

(1) *The dimension of M is 1 or 2.*

(2) *The dimension of M is 3, and the vector field X never vanishes.*

(3) *The dimension of M is 3, and X is of class $C^{3,1}$.*

Then the projected Aubry set $\mathcal{A}$ of the Mañé Lagrangian $L_X : TM \to \mathbb{R}$ associated to X is the set of chain-recurrent points of the flow of X on M. Moreover, the constants are the only weak KAM solutions for L_X if and only if every point of M is chain-recurrent under the flow of X.

The Mather quotient

The Peierls barrier satisfies the triangle inequality

$$\forall x, y, z \in M, \quad h(x,z) \leq h(x,y) + h(y,z).$$

Hence the function $d_M : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ defined as

$$\forall x, y \in \mathcal{A}, \quad d_M(x,y) := h(x,y) + h(y,x),$$

is a semi-distance on the projected Aubry set. The quotient Aubry set $(\mathcal{A}_M, d_M)$ is the metric space obtained by identifying two points in $\mathcal{A}$ if their semi-distance d_M vanishes.

Mather's Problem. If L is C^∞, is the set $\mathcal{A}_M$ totally disconnected, *i.e.* is each connected component of $\mathcal{A}_M$ reduced to a single point?

Sard-type results and Mather quotient

Proposition

Assume that $\dim M = 1, 2$ *and* H *of class* C^2 *or* $\dim M = 3$ *and* H *of class* $C^{k,1}$ *with* $k \geq 3$*. If* $u_1, u_2 : M \to \mathbb{R}$ *are two critical subsolutions, then the set* $(u_1 - u_2)(\mathcal{A})$ *has Lebesgue measure zero.*

Theorem

If $\dim M = 1, 2$ *and* H *of class* C^2 *or* $\dim M = 3$ *and* H *of class* $C^{k,1}$ *with* $k \geq 3$*, then* $(\mathcal{A}_M, d_M)$ *is totally disconnected.*

Corollary

Assume that $\dim M = 1, 2$ *and* H *of class* C^2 *or* $\dim M = 3$ *and* H *of class* $C^{k,1}$ *with* $k \geq 3$*, and that* $\mathcal{A}$ *is connected. Then, there is only one weak KAM solution (up to a constant).*

Proof of the theorem

Proof. Let $x, y \in \mathcal{A}$ such that $d_M(x,y) > 0$ be fixed. Set $w = h_x - h_y$. We have

$$\begin{aligned} d_M(x,y) &= h(x,y) + h(y,x) \\ &= (h_x(y) - h_x(x)) + (h_y(x) - h_y(y)) \\ &= w(y) - w(x) > 0. \end{aligned}$$

By the proposition above, since h_x and h_y are both critical viscosity solutions, the set $w(\mathcal{A})$ has Lebesgue measure zero. This implies that there exists $t_0 \notin w(\mathcal{A})$ such that

$$w(y) > t_0 > w(x).$$

Since w is continuous with respect to d_M, that tells us that x and y are not in the same connected component. □

Preparatory results

Lemma (Bates Lemma)

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function of class $C^{1,1}$. Then the set of critical values of g has Lebesgue measure zero.

Theorem (Bernard's Theorem)

Let u be a weak KAM solution, then there is a critical subsolution v of class $C^{1,1}$ such that $v_{|\mathcal{A}} = u_{|\mathcal{A}}$.

Lemma

Let $E \subset \mathbb{R}^n$ be a measurable set, $f : E \to \mathbb{R}$ continuous. If $n \geq 2$ and f satisfies

$$|f(x) - f(y)| \leq C|x - y|^n \quad \forall x, y \in E,$$

then the set $f(E)$ has Lebesgue zero.

Norton's generalization of Morse Vanishing Lemma

Lemma (The Generalized Morse Vanishing Lemma)

Suppose M is an n-dimensional (separable) manifold endowed with a distance d coming from a Riemannian metric. Let $k \in \mathbb{N}$ and $\alpha \in [0,1]$. Then for any subset $A \subset M$, we can find a countable family $B_i, i \in \mathbb{N}$ of C^1-embedded compact disks in M of dimension $\leq n$ and a countable decomposition of $A = \cup_i A_i$, with $A_i \subset B_i$, for every i, such that every $f \in C^{k,\alpha}(M, \mathbb{R})$ vanishing on A satisfies, for each i,

$$\forall y \in A_i,\, x \in B_i, \quad |f(x) - f(y)| \leq M_i d(x,y)^{k+\alpha}$$

for a certain constant M_i (depending on f).

Projected stationary Aubry set

The projected stationary Aubry set $\mathcal{A}^0 \subset \mathcal{A}$ is defined by

$$\mathcal{A}^0 = \left\{ x \in \mathcal{A} \mid (x,0) \in \tilde{A} \right\}.$$

Theorem

Suppose that L is at least C^2, and that the restriction $x \mapsto L(x,0)$ of L to the zero section of TM is of class $C^{k,1}$. Then $(\mathcal{A}^0_M, \delta_M)$ has vanishing Hausdorff measure in dimension $2\dim M/(k+3)$. In particular, if $k \geq 2\dim M - 3$ then $\mathcal{H}^1(\mathcal{A}^0_M, \delta_M) = 0$, and if $x \mapsto L(x,0)$ is C^∞ then $(\mathcal{A}^0_M, \delta_M)$ has zero Hausdorff dimension.

Projected periodic Aubry set

Let $\mathcal{A}^p$ the set of $x \in \mathcal{A}$ which are projection of a point $(x, v) \in \tilde{\mathcal{A}}$ whose orbit under the the Euler-Lagrange flow ϕ_t^L is periodic with strictly positive period. We call this set the projected periodic Aubry set.

Theorem

If $\dim M \geq 2$ *and* H *of class* $\mathrm{C}^{k,1}$ *with* $k \geq 2$*, then* $(\mathcal{A}_M^p, \delta_M)$ *has vanishing Hausdorff measure in dimension* $8 \dim M/(k+8)$*. In particular, if* $k \geq 8 \dim M - 8$ *then* $\mathcal{H}^1(\mathcal{A}_M^p, \delta_M) = 0$*, and if* H *is* C^∞ *then* $(\mathcal{A}_M^p, \delta_M)$ *has zero Hausdorff dimension.*

Compact surfaces

If M is a compact surface, then, using the finiteness of exceptional minimal sets of flows, we have:

Theorem

If M is a compact surface of class C^∞ and H is of class C^∞, then $(\mathcal{A}_M, \delta_M)$ has zero Hausdorff dimension.

Thank you for your attention !