Optimal Transportation on Sub-Riemannian Manifolds

Ludovic Rifford

Université de Nice - Sophia Antipolis

(Joint work with A. Figalli)

Monge's Optimal Transportation Problem

Let M be a separable metric space equipped with its Borel σ-algebra, $c : M \times M \longrightarrow \mathbb{R}$ be a cost function and μ, ν be two compactly supported probability measures in M. Find a measurable map $T : M \rightarrow M$ satisfying

$$T_\sharp \mu = \nu,$$

and in such a way that T minimizes the transportation cost given by

$$\int_M c(x, T(x)) d\mu(x).$$

When the transport condition $T_\sharp \mu = \nu$ is satisfied, we say that T is a *transport map*.

Two questions

- **Existence of an optimal transport map ?**

- **Uniqueness ?**

Example 1: The Euclidean case

Assume that $M = \mathbb{R}^n$ and that the cost c is given by

$$c(x, y) = |x - y|^2.$$

Theorem (Brenier's Theorem,1991)

If μ is absolutely continuous with respect to the Lebesgue measure, there is a unique optimal transport map T. It is characterized by the existence of a convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that

$$T(x) = \nabla\psi(x) \qquad \text{for } \mu \text{ a.e. } x \in \mathbb{R}^n.$$

Example 2: The Riemannian case

Assume that (M, g) is a smooth complete Riemannian manifold and denote by $d_g(\cdot, \cdot)$ the Riemannian distance on $M \times M$. Assume that the cost c is given by

$$c(x, y) = d_g(x, y)^2.$$

Theorem (McCann's Theorem, 2001)

If μ is absolutely continuous with respect to the Lebesgue measure on M, there is a unique optimal transport map T. It is characterized by the existence of a semiconvex function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that

$$T(x) = \exp_x (\nabla \psi(x)) \qquad \text{for } \mu \text{ a.e. } x \in \mathbb{R}^n.$$

References

Papers:

- "Optimal transportation under nonholonomic constraints" by A. Agrachev and P. Lee. Trans. Amer. Math. Soc, to appear. Available on *http://people.sissa.it/~agrachev/*.
- "Optimal Transportation on Sub-Riemannian Manifolds" by A. Figalli and L. Rifford. Preprint, 2008. Available on *http://math.unice.fr/~rifford/*.

Books:

- "Topics in Mass Transportation" by C. Villani. *Graduate Studies in Mathematics Surveys*, Vol. 58. American Mathematical Society, 2003.
- "Optimal transport, old and new" by C. Villani. To appear in the *Grundlehren des mathematischen Wissenschaften* Springer series.

A Weak Formulation: Kantorovitch's Problem

Let M be a separable metric space equipped with its Borel σ-algebra, $c : M \times M \to \mathbb{R}$ be a cost function and μ, ν be two compactly supported probability measures in M. Find a probability measure γ on $M \times M$ having marginals μ and ν, i.e.

$$(\pi_1)_\sharp \gamma = \mu \qquad \text{and} \qquad (\pi_2)_\sharp \gamma = \nu,$$

(where $\pi_1 : M \times M \to M$ and $\pi_2 : M \times M \to M$ are the canonical projections), which minimizes the transportation cost given by

$$\int_{M \times M} c(x, y) d\gamma(x, y).$$

When the transport condition $(\pi_1)_\sharp \gamma = \mu, (\pi_2)_\sharp \gamma = \nu$ is satisfied, we say that γ is a *transport plan*, and if γ minimizes also the cost we call it an *optimal transport plan*.

Kantorovitch's Duality

Theorem

There are two continuous function $\phi_1, \phi_2 : M \rightarrow \mathbb{R}$ *satisfying*

$$\phi_1(x) = \inf_{y \in M} \{c(x,y) - \phi_2(y)\} \qquad \forall x \in M,$$

$$\phi_2(y) = \inf_{x \in M} \{c(x,y) - \phi_1(x)\} \qquad \forall y \in M.$$

such that the following holds: a transport plan γ *is optimal if and only if one has*

$$\phi_1(x) - \phi_2(y) = c(x,y) \qquad \text{for } \gamma \text{ a.e. } (x,y) \in M \times M.$$

As a consequence, to obtain that an optimal transport plan corresponds to a Monge's optimal transport map, we have to show that γ is concentrated on a graph.

Proof of Brenier-McCann's Theorem

- The function $x \mapsto d_g(x,y)^2$ is locally Lipschitz on M.
- The function ϕ_1 is locally Lipschitz on M. As a consequence, by Rademacher's Theorem, it is differentiable μ-a.e.
- Let $\bar{x} \in \mathrm{supp}(\mu)$ be such that ϕ_1 is differentiable at $\bar{x}$. Let $\bar{y}$ be such that

$$\phi_1(\bar{x}) = d_g(\bar{x},\bar{y})^2 - \phi_2(\bar{y}).$$

Then we have,

$$d_g(x,\bar{y})^2 \geq \phi_1(x) + \phi_2(\bar{y}) \qquad \forall x \in M.$$

Which implies that $\bar{y} = \exp_{\bar{x}}\left(-\frac{1}{2}\nabla\phi_1(\bar{x})\right)$. We set

$$\psi := -\frac{1}{2}\phi_1.$$

Strategy of proof

TWO ISSUES

- Show that ϕ_1 is differentiable μ-a.e. (for instance, by showing that ϕ_1 is locally Lipschitz on M).
- Deduce that, if ϕ_1 is differentiable at $\bar{x} \in \mathrm{supp}(\mu)$, then there is a unique $\bar{y} \in M$ such that

$$\phi_1(\bar{x}) = c(\bar{x}, \bar{y}) - \phi_2(\bar{y}).$$

The sub-Riemannian Optimal Transport Problem

Let (M, Δ, g) be a complete sub-Riemannian structure of dimension n and rank $m < n$. Let $d_{SR}(\cdot, \cdot)$ be the sub-Riemannian distance on $M \times M$. Let μ, ν be two compactly supported probability measures on M. Find a measurable map $T : M \to M$ satisfying

$$T_\sharp \mu = \nu,$$

and in such a way that T minimizes the transportation cost given by

$$\int_M d_{SR}(x, T(x))^2 d\mu(x).$$

A Theorem of Existence and Uniqueness

Let us denote by D the diagonal in $M \times M$.

Theorem (A. Figalli, L. Rifford, 2008)

Assume that there exists an open set $\Omega \subset M \times M$ such that $supp(\mu \times \nu) \subset \Omega$, and d_{SR}^2 is locally Lipschitz on $\Omega \setminus D$. Let ϕ be the function provided by Kantorovitch's duality. Then, there is an open set $\mathcal{M}^\phi$ such that and ϕ is locally Lipschitz in a neighborhood of $\mathcal{M}^\phi \cap supp(\mu)$. There exists a unique optimal transport map which is defined μ-a.e. by

$$T(x) := \begin{cases} \exp_x(-\frac{1}{2}\, d\phi(x)) & \text{if } x \in \mathcal{M}^\phi \cap supp(\mu), \\ x & \text{if } x \in \left(M \setminus \mathcal{M}^\phi\right) \cap supp(\mu). \end{cases}$$

Examples

- Example 1: Two generating distributions

Proposition (A. Agrachev, P. Lee, 2008)

If Δ is two-generating on M, then the squared sub-Riemannian distance function is locally Lipschitz on $M \times M$.

- Example 2: Generic sub-Riemannian structures

Proposition (Y. Chitour, F. Jean, E. Trélat, 2006)

Let (M, g) be a complete Riemannian manifold of dim ≥ 4. Then, for any generic distribution of rank ≥ 3, the squared sub-Riemannian distance function is locally semiconcave (hence locally Lipschitz) on $M \times M \setminus D$.

Examples again

- Example 3: Medium-fat distributions
  The distribution Δ is called *medium-fat* if, for every $x \in M$ and every vector field X on M such that $X(x) \in \Delta(x) \setminus \{0\}$, there holds

$$T_x M = \Delta(x) + [\Delta, \Delta](x) + [X, [\Delta, \Delta]](x).$$

Proposition

Assume that Δ is medium-fat. Then the squared sub-Riemannian distance function is locally Lipschitz on $M \times M \setminus D$.

Thank you for your attention !