On the minimizing Sard Conjecture in sub-Riemannian geometry

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Outline

I. Introduction to sub-Riemannian geometry

II. The Minimizing Sard Conjecture

III. A Partial result

Part I

I. Introduction to sub-Riemannian geometry

Sub-Riemannian structures

Let M be a smooth connected manifold of dimension n.

Definition

A **sub-Riemannian structure** of rank $m \leq n$ on M is given by a pair (Δ, g) where:

- Δ is a **totally nonholonomic distribution** of rank $m \leq n$ on M which is defined locally by

$$\Delta(x) = \mathrm{Span}\left\{X^1(x), \cdots, X^m(x)\right\} \subset T_xM,$$

  where $X^1, \ldots, X^m$ is a family of m linearly independent smooth vector fields satisfying the **Hörlmander condition**;

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where $X^1, \ldots, X^m$ is a family of m linearly independent smooth vector fields satisfying the **Hörlmander condition**;
- g is smooth and for every $x \in M$, g_x is a **scalar product** over $\Delta(x)$.

The Hörmander condition

We say that a family of smooth vector fields $X^1, \ldots, X^m$, satisfies the **Hörmander condition** if

$$\mathrm{Lie}\left\{X^1, \ldots, X^m\right\}(x) = T_x M \qquad \forall x,$$

where $\mathrm{Lie}\{X^1, \ldots, X^m\}$ denotes the Lie algebra generated by $X^1, \ldots, X^m$, *i.e.* the smallest subspace of smooth vector fields that contains all the $X^1, \ldots, X^m$ and which is stable under Lie brackets.

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Reminder

Given smooth vector fields X, Y in $\mathbb{R}^n$, the Lie bracket $[X, Y]$ at $x \in \mathbb{R}^n$ is defined by

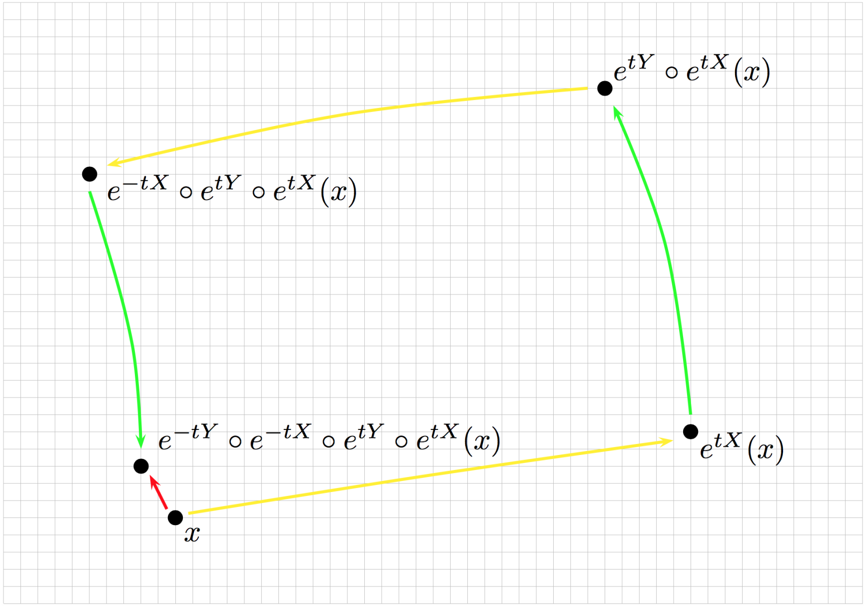
$$[X, Y](x) = DY(x)X(x) - DX(x)Y(x).$$

Lie Bracket: Dynamic Viewpoint

Remark

There holds

$$[X, Y](x) = \lim_{t \downarrow 0} \frac{\left(e^{-tY} \circ e^{-tX} \circ e^{tY} \circ e^{tX}\right)(x) - x}{t^2}.$$



The Chow-Rashevsky Theorem

Definition

We call **horizontal path** any $\gamma \in W^{1,2}([0,1];M)$ such that

$$\dot{\gamma}(t) \in \Delta(\gamma(t)) \qquad \text{a.e. } t \in [0,1].$$

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The following result is the cornerstone of sub-Riemannian geometry. (Recall that M is assumed to be connected.)

Theorem (Chow-Rashevsky, 1938)

Let Δ be a totally nonholonomic distribution on M, then every pair of points can be joined by an horizontal path.

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Let Δ be a totally nonholonomic distribution on M, then every pair of points can be joined by an horizontal path.

Since the distribution is equipped with a metric, we can measure the lengths of horizontal paths and consequently we can associate a metric with the sub-Riemannian structure.

Examples of sub-Riemannian structures

Example: Riemannian case

Every Riemannian manifold (M, g) gives rise to a sub-Riemannian structure with $\Delta = TM$.

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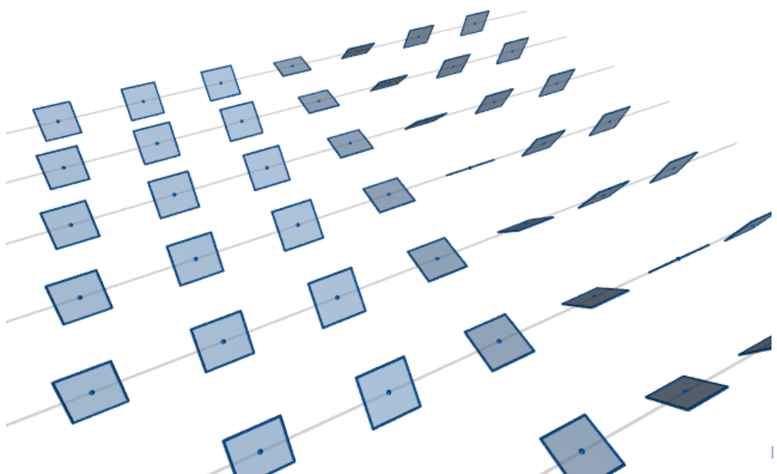
Example: Heisenberg

In $\mathbb{R}^3$, $\Delta = \mathrm{Span}\{X^1, X^2\}$ with

$$X^1 = \partial_x, \quad X^2 = \partial_y + x\partial_z \quad \text{et} \quad g = dx^2 + dy^2.$$



Other examples of sub-Riemannian structures

Example: Martinet

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Since $[X^1, X^2] = 2x\partial_z$ and $[X^1, [X^1, X^2]] = 2\partial_z$, only one bracket is sufficient to generate $\mathbb{R}^3$ if $x \neq 0$, however we needs two brackets if $x = 0$.

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Example: Rank 2 distribution in dimension 4

In $\mathbb{R}^4$, $\Delta = \text{Span}\{X^1, X^2\}$ with

$$X^1 = \partial_x, \quad X^2 = \partial_y + x\partial_z + z\partial_w$$

satisfies $\text{Vect}\{X^1, X^2, [X^1, X^2], [[X^1, X^2], X^2]\} = \mathbb{R}^4$.

The sub-Riemannian distance

The **length** of an horizontal path γ is defined by

$$\text{length}^g(\gamma) := \int_0^T |\dot{\gamma}(t)|^g_{\gamma(t)}\, dt.$$

Definition

Given $x, y \in M$, the **sub-Riemannian distance** between x and y is defined by

$$d_{SR}(x,y) := \inf\left\{\text{length}^g(\gamma) \,|\, \gamma \text{ hor.}, \gamma(0) = x, \gamma(1) = y\right\}.$$

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Proposition

The manifold M equipped with the distance d_{SR} is a metric space whose topology coincides with the one of M (as a manifold).

Sub-Riemannian geodesics

Definition

Given $x, y \in M$, we call **minimizing horizontal path** between x and y any horizontal path $\gamma : [0,1] \to M$ joining x to y satisfying $d_{SR}(x,y) = \text{length}^g(\gamma)$.

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The **energy** of the horizontal path $\gamma : [0,1] \to M$ is given by

$$\text{ener}^g(\gamma) := \int_0^1 \left(|\dot{\gamma}(t)|^g_{\gamma(t)} \right)^2 dt.$$

Definition

We call **minimizing geodesic** between x and y any horizontal path $\gamma : [0,1] \to M$ joining x to y such that

$$d_{SR}(x,y)^2 = \text{ener}^g(\gamma).$$

(M, d_{SR}) gives existence of minimizing geodesics.

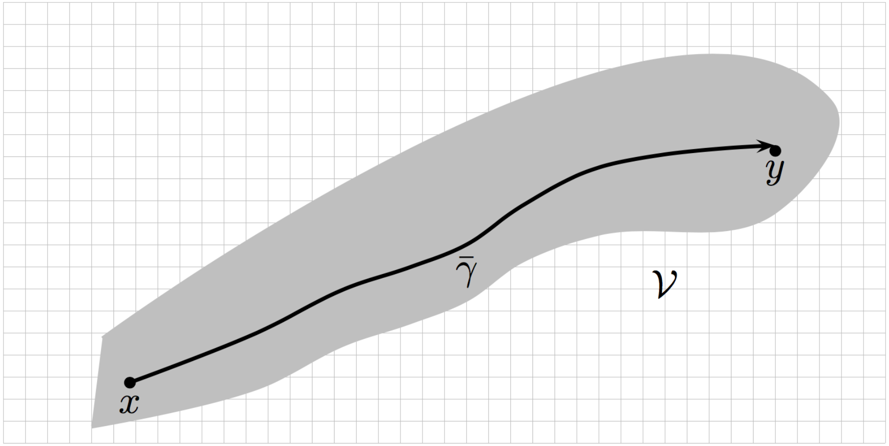
Part II

II. The Minimizing Sard Conjecture

Study of minimizing geodesics I

Let $x, y \in M$ and $\bar{\gamma}$ be a **minimizing geodesic** between x and y be fixed. The SR structure admits an orthonormal parametrization along $\bar{\gamma}$, which means that there exists a neighborhood $\mathcal{V}$ of $\bar{\gamma}([0,1])$ and an orthonomal family of m vector fields $X^1, \ldots, X^m$ such that

$$\Delta(z) = \mathrm{Span}\left\{X^1(z), \ldots, X^m(z)\right\} \qquad \forall z \in \mathcal{V}.$$



Study of minimizing geodesics II

There exists a control $\bar{u} \in L^2\big([0,1];\mathbb{R}^m\big)$ such that

$$\dot{\bar{\gamma}}(t) = \sum_{i=1}^{m} \bar{u}_i(t)\, X^i\big(\bar{\gamma}(t)\big) \qquad \text{a.e. } t \in [0,1].$$

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Moreover, any control $u \in \mathcal{U} \subset L^2([0,1];\mathbb{R}^m)$ (u sufficiently close to $\bar{u}$) gives rise to a trajectory γ_u solution of

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Furthermore, for every horizontal path $\gamma : [0,1] \rightarrow \mathcal{V}$ there exists a unique control $u \in L^2\big([0,1];\mathbb{R}^m\big)$ for which the above equation is satisfied.

Study of minimizing geodesics III

Consider the **End-Point mapping**

$$E^{x,1} \,:\, L^2\big([0,1];\mathbb{R}^m\big) \longrightarrow M$$

defined by

$$E^{x,1}(u) := \gamma_u(1),$$

and set $C(u) = \|u\|_{L^2}^2$, then $\bar{u}$ is a solution to the following **optimization problem with constraints**:

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(Since the family $X^1, \ldots, X^m$ is orthonormal, we have

$$\text{ener}^g(\gamma_u) = C(u) \qquad \forall u \in \mathcal{U}.)$$

Study of minimizing geodesics IV

Proposition (Lagrange Multipliers)

There exist $p \in T_y^*M \simeq (\mathbb{R}^n)^*$ and $\lambda_0 \in \{0,1\}$ with $(\lambda_0, p) \neq (0,0)$ such that

$$p \cdot d_{\bar{u}} E^{x,1} = \lambda_0 d_{\bar{u}} C.$$

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As a matter of fact, the function given by

$$\Phi(u) := \big(C(u), E^{x,1}(u)\big)$$

cannot be a submersion at $\bar{u}$. Otherwise $D_{\bar{u}}\Phi$ would be surjective and so open at $\bar{u}$, which means that the image of Φ would contain some points of the form $(C(\bar{u}) - \delta, y)$ with $\delta > 0$ small.

$\rightsquigarrow$ Two cases may appear: $\lambda_0 = 1$ or $\lambda_0 = 0$.

Study of minimizing geodesics V

First case : $\lambda_0 = 1$

This is the good case, the Riemannian-like case. The minimizing geodesic can be shown to be solution of a "geodesic equation". In fact, it is the projection of a **normal extremal**. It is smooth, there is a "geodesic flow"...

Second case : $\lambda_0 = 0$

In this case, we have

$$p \cdot D_{\bar{u}} E^{x,1} = 0 \text{ with } p \neq 0,$$

which means that $\bar{u}$ is **singular** as a critical point of the mapping $E^{x,1}$.

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$\rightsquigarrow$ As shown by R. Montgomery, the case $\lambda_0 = 0$ cannot be ruled out.

Singular horizontal paths and Examples

Definition

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Let $\Delta(x) = T_xM$, any path in $W^{1,2}$ is horizontal. There are no singular curves.

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Example: Riemannian case

Let $\Delta(x) = T_x M$, any path in $W^{1,2}$ is horizontal. There are no singular curves.

Example: Heisenberg, fat distributions

In $\mathbb{R}^3$, Δ given by $X^1 = \partial_x, X^2 = \partial_y + x\partial_z$ does not admin nontrivial singular horizontal paths.

Martinet-like distributions

In $\mathbb{R}^3$, let $\Delta = \text{Vect}\{X^1, X^2\}$ with X^1, X^2 of the form

$$X^1 = \partial_{x_1} \quad \text{and} \quad X^2 = \left(1 + x_1 \phi(x)\right) \partial_{x_2} + x_1^2 \partial_{x_3},$$

where ϕ is a smooth function and let g be a metric over Δ.

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Theorem (Montgomery, 1991)

There exists $\bar{\epsilon} > 0$ such that for every $\epsilon \in (0, \bar{\epsilon})$, the singular horizontal path

$$\gamma(t) = (0, t, 0) \qquad \forall t \in [0, \epsilon],$$

is minimizing (w.r.t. g) among all horizontal paths joining 0 to $(0, \epsilon, 0)$.

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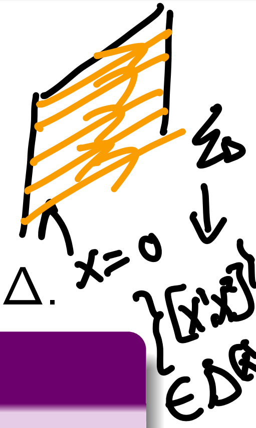

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is minimizing (w.r.t. g) among all horizontal paths joining 0 to $(0, \epsilon, 0)$. Moreover, if $\{X^1, X^2\}$ is orthonormal w.r.t. g and $\phi(0) \neq 0$, then γ is not the projection of a normal extremal $(\lambda_0 = 1)$.

Summary

Given a complete sub-Riemannian structure (Δ, g) on M and a minimizing geodesic γ from x to y, two cases may happen:

- The geodesic γ is the projection of a normal extremal so it is smooth..
- The geodesic γ is a singular curve and could be non-smooth..

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Questions:

When? How many? How?

The Sard Conjecture

Given $x \in M$, we denote by $\mathrm{Sing}_{\Delta}^{x}$ the set of points $y \in M$ for which there is a **singular horizontal path** joining x to y, it is a closed subset of M containing x.

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Given $x \in M$, we denote by Sing_Δ^x the set of points $y \in M$ for which there is a **singular horizontal path** joining x to y, it is a closed subset of M containing x.

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The set Sing_Δ^x has Lebesgue measure zero in M.

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The result is known in very few cases:

- Rank 2 in dimension 3 (much stronger result by Belotto, Figalli, Parusinski, R).
- Some cases of Carnot groups.

The Minimizing Sard Conjecture

Given $x \in M$, we denote by $\mathrm{Abn}^{min}(x)$ (or $\mathrm{Sing}_{\Delta,g}^{x,min}$) the set of points $y \in M$ for which there is a **singular minimizing geodesic** joining x to y, it is a closed subset of M containing x.

Minimizing Sard Conjecture

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Only few cases are known. In general, we have

Theorem (Agrachev, 2009)

Let M be a smooth connected manifold of dimension n equipped with a complete sub-Riemannian structure (Δ, g). Then for every $x \in M$, the closed set $\mathrm{Abn}^{min}(x)$ has empty interior.

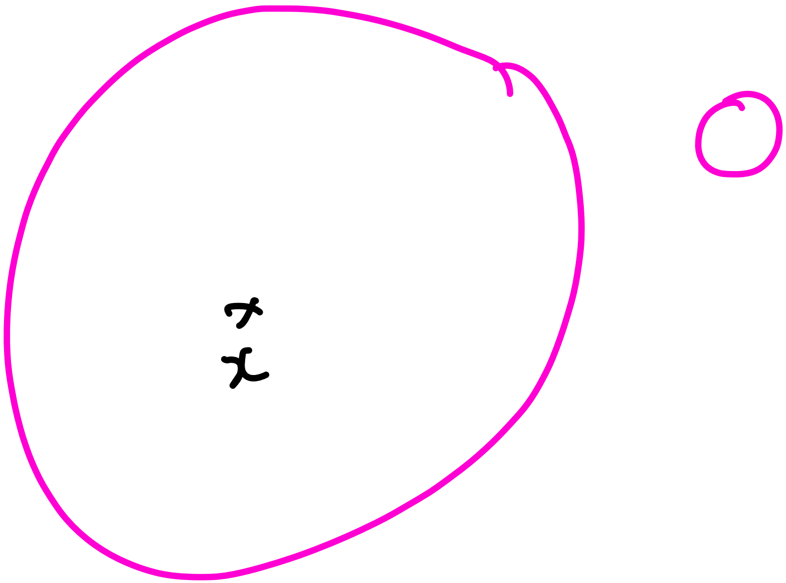
Part III

III. A Partial result

A Partial result

Theorem (R, 2023)

Let M be equipped with a complete SR structure (Δ, g) and $x \in M$ be fixed. If for almost every $y \in M$ all minimizing horizontal paths from x to y have Goh-rank at most 1, then the closed set $\mathrm{Abn}^{min}(x)$ has Lebesgue measure zero in M.



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Corollary (R, 2023)

If M is equipped with a complete SR structure (Δ, g) having minimizing co-rank 1 almost everywhere, then the Minimizing Sard Conjecture holds true.

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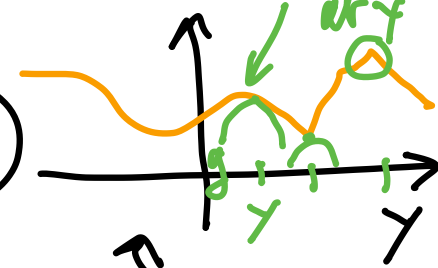
Corollary (R, 2023)

If M is equipped with a complete SR structure (Δ, g) of rank $m \geq 2$ where Δ is generic, then the Minimizing Sard Conjecture holds true.

Sketch of proof I

Let (Δ, g) be a SR structure and $x \in M$ be fixed, we define $f_x : M \to \mathbb{R}$ by

$$f_x(y) := \frac{1}{2} d_{SR}(x,y)^2$$



The following properties are equivalent:

- The set $\mathrm{Abn}^{min}(x)$ has Lebesgue measure zero in M.
- For a.e. $y \in M$, $\partial^- f_x(y) \neq \emptyset$.
- The set $\mathrm{Lip}^-(f_x)$ has full Lebesgue measure in M.
- The function f_x is smooth on an open subset of M of full Lebesgue measure.

Sketch of proof II

Roughly speaking, our assumptions allow to show that f_x is Lipschitz along an hyperplane at each $y \neq x$. The main ingredient of the proof is the:

Proposition

Let $\varphi : (a, b) \to \mathbb{R}$ be a continuous function. Then, for a.e. $x \in (a, b)$, at least one of the following properties is satisfied:

(i) φ is differentiable at x.

(ii) There is a sequence $\{x_k\}$ converging to x such that $0 \in \partial^- \varphi(x_k)$ for all k.

In fact, we need a more precise version of (ii), which is given by the Denjoy-Young Saks Theorem.

The Denjoy-Young-Saks Theorem

Theorem

Let $f : (a,b) \to \mathbb{R}$ be a function. Then for a.e. $x \in (a,b)$, one of the following assertion holds:

(1) f is differentiable at x,

(2) $D^+f(x) = D^-f(x) = +\infty$, $D_+f(x) = D_-f(x) = -\infty$,

(3) $D^+f(x) = +\infty$, $D_-f(x) = -\infty$, $D_+f(x) = D^-f(x) \in \mathbb{R}$,

(4) $D^-f(x) = +\infty$, $D_+f(x) = -\infty$, $D_-f(x) = D^+f(x) \in \mathbb{R}$.

Here, D^+f, D_+f, D^-f, D_-f stand for the Dini derivatives of f defined by ($T_{c,d} := (f(d) - f(c))/(d-c)$)

$$D^+f(x) = \limsup_{h \to 0^+} T_{x,x+h}, \quad D_+f(x) = \liminf_{h \to 0^+} T_{x,x+h},$$

$$D_-f(x) = \liminf_{h \to 0^+} T_{x-h,x}, \quad D^-f(x) = \limsup_{h \to 0^+} T_{x-h,x},$$

Thank you for your attention !!