Mass Transportation on the Earth
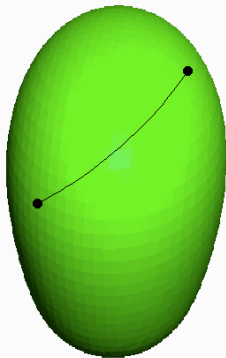
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The framework
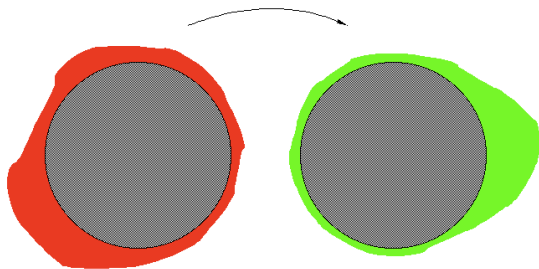
Let M be a **smooth connected compact surface** in $\mathbb{R}^n$. For any $x, y \in M$, we define the geodesic distance between x and y, denoted by $d(x, y)$, as the minimum of the lengths of the curves (drawn on M) joining x to y.

Transport maps

Let μ_0 and μ_1 be **probability measures** on M. We call **transport map** from μ_0 to μ_1 any measurable map $T : M \rightarrow M$ such that $T_\sharp\mu_0 = \mu_1$, that is

$$\mu_1(B) = \mu_0\big(T^{-1}(B)\big), \quad \forall B \text{ measurable } \subset M.$$

Quadratic Monge's Problem

Given two probabilities measures μ_0, μ_1 sur M, find a transport map $T : M \to M$ from μ_0 to μ_1 which minimizes the quadratic cost ($c = d^2/2$)

$$\int_M c(x, T(x)) d\mu_0(x).$$

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Existence ?

Uniqueness ?

Regularity ?

The Brenier Theorem

Quadratic Monge's Problem in $\mathbb{R}^n$: Given two probability measures μ_0, μ_1 with compact supports in $\mathbb{R}^n$, we are concerned with transport maps t $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ pushing forward μ_0 to μ_1 which minimize the transport cost

$$\int_{\mathbb{R}^n} |T(x) - x|^2 \, d\mu_0(x).$$

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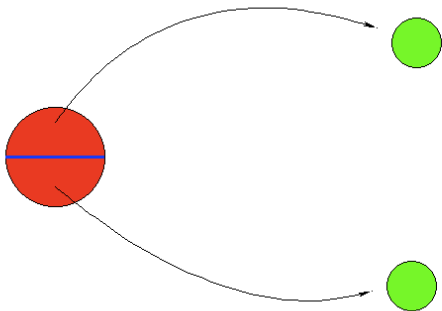
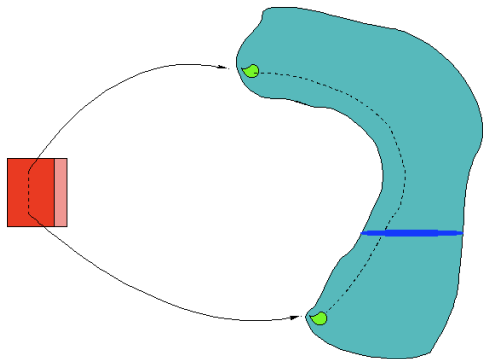$$\int_{\mathbb{R}^n} |T(x) - x|^2 \, d\mu_0(x).$$

Theorem (Brenier '91)

*If μ_0 is absolutely continuous with respect to the Lebesgue measure, there exists a unique optimal transport map with respect to the quadratic cost. In fact, there is a **convex** function $\psi : M \to \mathbb{R}$ such that*

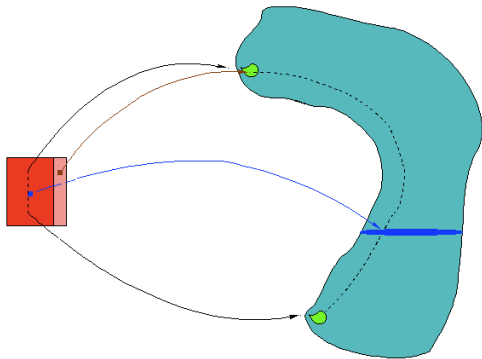$$T(x) = \nabla \psi(x) \qquad \mu_0 \text{ a.e. } x \in \mathbb{R}^n.$$
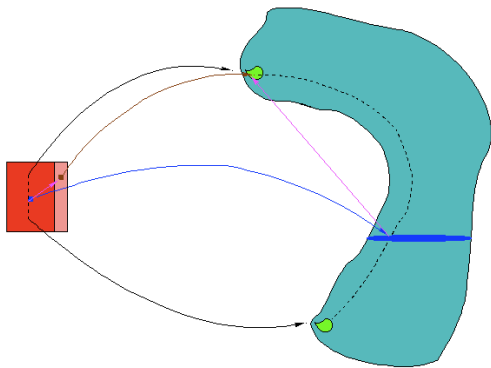
T gradient of a convex function
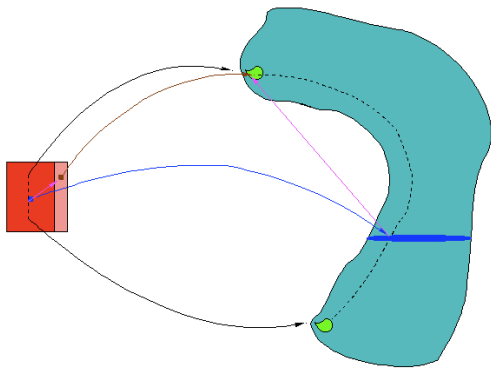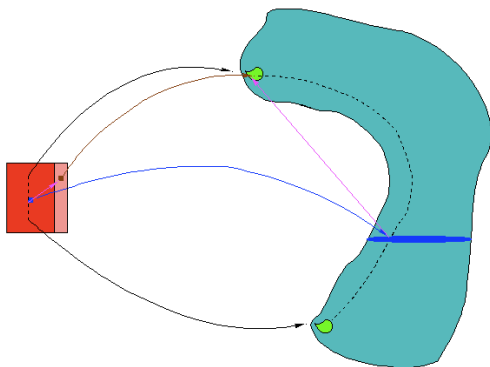
T gradient of a convex function $\implies \langle y-x, T(y)-T(x)\rangle \geq 0$!!!

Caffarelli's Regularity Theory

If μ_0 and μ_1 are sont associated with densities f_0, f_1 w.r.t. Lebesgue, then

$$T_\sharp \mu_0 = \mu_1 \iff \int_{\mathbb{R}^n} \zeta\big(T(x)\big) f_0(x) dx = \int_{R^n} \zeta(y) f_1(y) dy \quad \forall \zeta.$$

$\rightsquigarrow \psi$ weak solution of the l'**Monge-Ampère equation** :

$$\det\big(\nabla^2 \psi(x)\big) = \frac{f_0(x)}{f_1(\nabla\psi(x))}.$$

Theorem (Caffarelli '90s)

Let Ω_0, Ω_1 be connected and bounded open sets in $\mathbb{R}^n$ and f_0, f_1 be probability densities resp. on Ω_0 and Ω_1 such that $f_0, f_1, 1/f_0, 1/f_1$ are essentially bounded. Assume that μ_0 and μ_1 have respectively densities f_0 and f_1 w.r.t. Lebesgue and that Ω_1 is convex. Then the quadratic optimal transport map from μ_0 to μ_1 is continuous.

The McCann Theorem

Given two probability measures μ_0, μ_1 on M, we are concerned with transport maps $T : M \rightarrow M$ ($T_\sharp \mu_0 = \mu_1$) which minimize the quadratic transport cost ($c = d^2/2$)

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$$\int_M c(x, T(x)) d\mu_0(x).$$

Theorem (McCann '01)

*If μ_0 is absolutely continuous w.r.t. Lebesgue, there exists a unique optimal transport map from μ_0 to μ_1. In fact, there is a c-**convex** function $\varphi : M \to \mathbb{R}$ satisfying*

$$T(x) = \exp_x \left(\nabla \varphi(x) \right) \qquad \mu_0 \text{ a.e. } x \in \mathbb{R}^n.$$

Moreover, for a.e. $x \in M$, $\nabla \varphi(x)$ belongs to the injectivity at x.

Exponential mapping and injectivity domains

Let $x \in M$ be fixed.

- For every $v \in T_x M$, we define the **exponential** of v by

$$\exp_x(v) = \gamma_{x,v}(1),$$

where $\gamma_{x,v} : [0,1] \to M$ is the unique geodesic starting at x with speed v.

Exponential mapping and injectivity domains

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  where $\gamma_{x,v} : [0,1] \to M$ is the unique geodesic starting at x with speed v.

- We call **injectivity domain** at x, the subset of $T_x M$ defined by

$$\mathcal{I}(x) := \left\{ v \in T_x M \;\middle|\; \begin{array}{c} \exists t > 1 \text{ s.t. } \gamma_{tv} \text{ is the unique} \\ \text{minim. geod. between } x \text{ and } \exp_x(tv) \end{array} \right\}$$

  It is a star-shaped (w.r.t. $0 \in T_x M$) bounded open set with Lipschitz boundary.

We say that the surface $M \subset \mathbb{R}^n$ satisfies the **Transport Continuity Property (TCP)** if the following property is satisfied:
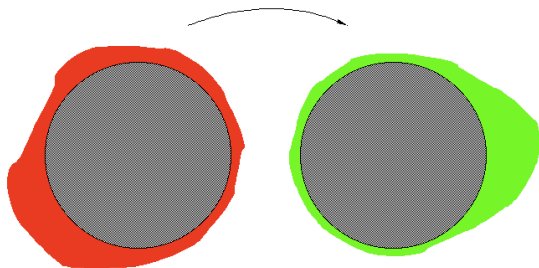
For any pair of probability measures μ_0, μ_1 associated locally with **continuous positive densities** ρ_0, ρ_1, that is

$$\mu_0 = \rho_0 \text{vol}_g, \quad \mu_1 = \rho_1 \text{vol}_g,$$

the optimal transport map from μ_0 to μ_1 is **continuous**.

Characterization of **TCP** on surfaces

Theorem (Figalli-R-Villani '10)

Let M be a smooth compact surface in $\mathbb{R}^n$. It satisfies **TCP** *if and only if the following properties hold:*

- *all the injectivity domains are convex,*
- *the cost $c = \frac{1}{2}d^2$ is regular, that is for any $x, x' \in M$, the function*

$$F_{x,x'} : v \in \mathcal{I}(x) \longmapsto c\big(x, \exp_x(v)\big) - c\big(x', \exp_x(v)\big)$$

is **quasiconvex** *(its sublevel sets are always convex).*

An easy lemma

Lemma

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class C^2. Assume that for every $v \in U$ and every $w \in \mathbb{R}^n \setminus \{0\}$ the following property holds :

$$\langle \nabla_v F, w \rangle = 0 \implies \langle \nabla_v^2 F \, w, w \rangle > 0.$$

Then F is quasiconvex.

Proof of the easy lemma

Proof.

Let $v_0, v_1 \in U$ be fixed. Set $v_t := (1 - t)v_0 + tv_1$, for every $t \in [0, 1]$. Define $h : [0, 1] \to \mathbb{R}$ by

$$h(t) := F(v_t) \qquad \forall t \in [0, 1].$$

If $h \nleq \max\{h(0), h(1)\}$, there is $\tau \in (0, 1)$ such that

$$h(\tau) = \max_{t \in [0,1]} h(t) > \max\{h(0), h(1)\}.$$

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$$h(t) := F(v_t) \qquad \forall t \in [0,1].$$

If $h \not\leq \max\{h(0), h(1)\}$, there is $\tau \in (0,1)$ such that

$$h(\tau) = \max_{t \in [0,1]} h(t) > \max\{h(0), h(1)\}.$$

There holds

$$\dot{h}(\tau) = \langle \nabla_{v_\tau} F, \dot{v}_\tau \rangle \quad \text{et} \quad \ddot{h}(\tau) = \langle \nabla^2_{v_\tau} F \, \dot{v}_\tau, \dot{v}_\tau \rangle.$$

Since τ is a local maximum, one has $\dot{h}(\tau) = 0$.
Contradiction !! □

Exercises

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class C^2.

False Lemma

Assume that for every $v \in U$ and every $w \in \mathbb{R}^n$, the following property holds:

$$\langle \nabla_v F, w \rangle = 0 \implies \langle \nabla_v^2 F \, w, w \rangle \geq 0.$$

Then F is quasiconvex.

True Lemma

Assume that there is a constant $C > 0$ such that

$$\langle \nabla_v^2 F \, w, w \rangle \geq -C \, |\langle \nabla_v F, w \rangle| \, |w| \qquad \forall v \in U, \forall w \in \mathbb{R}^n.$$

Then F is quasiconvex.

Back to our problem

Assume that all the injectivity domains are convex and fix $x, x' \in M$. Recall that

$$F_{x,x'}(v) = F(v) = c(x, \exp_x(v)) - c(x', \exp_x(v)).$$

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- F is not smooth.
- For generic segments, $t \mapsto F(v_t)$ is smooth outside a finite set of "convex" times.
- If it is smooth at v, then $\nabla_v^2 F$ has the form

$$\nabla_v^2 F(h, h) = - \int_0^1 (1 - t) \frac{\partial^4 c}{\partial^2 x \partial^2 y}(*)(*) \, dt$$

The Ma-Trudinger-Wang tensor

The **MTW** tensor $\mathfrak{S}$ is defined as

$$\mathfrak{S}_{(x,v)}(\xi,\eta) = -\frac{3}{2} \left.\frac{d^2}{ds^2}\right|_{s=0} \left.\frac{d^2}{dt^2}\right|_{t=0} c\left(\exp_x(t\xi), \exp_x(v+s\eta)\right),$$

for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_x M$.

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for every $x \in M, v \in \mathcal{I}(x)$, and $\xi, \eta \in T_x M$.

Proposition (Villani '09, Figalli-R-Villani '10)

Let M be a surface all of whose injectivity domains are convex. Then the following properties are equivalent:

- *The cost $c = d^2/2$ is regular.*
- *The **MTW** tensor is $\succeq 0$, that is for any $x \in M, v \in \mathcal{I}(x)$, and $\xi, \eta \in T_x M$,*

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi,\eta) \geq 0.$$

Caracterization of **TCP** on surfaces

Theorem (Figalli-R-Villani '10)

*Let M be a surface in $\mathbb{R}^n$. It satisfies **TCP** if and only if the two following properties holds:*

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Loeper noticed that for every $x \in M$ and for any pair of unit orthogonal tangent vectors $\xi, \eta \in T_x M$, there holds

$$\mathfrak{S}_{(x,0)}(\xi, \eta) = \sigma_x,$$

where σ_x denote the gaussian curvature of M at x. As a consequence,

$$\textbf{TCP} \implies \sigma \geq 0.$$

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Therefore, if $M \subset \mathbb{R}^3$ satisfies **TCP**, then it is **convex**.

Examples and Counterexamples

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- Flat tori (Cordero-Erausquin, '99).
- Round spheres (Loeper, '06).
- Quotients of the above objects.

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Counterexamples:

Spheres

Theorem (Loeper '06)

The **MTW** tensor on the round (unit) sphere $\mathbb{S}^2$ satisfies $\mathfrak{S} \succeq 1$, that is for any $x \in \mathbb{S}^2, v \in \mathcal{I}(x)$ and $\xi, \eta \in T_x\mathbb{S}^2$,

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi, \eta) \geq |\xi|^2|\eta|^2.$$

In particular, the round sphere $\mathbb{S}^2$ satisfies **TCP**.

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Is this result stable ?

Two issues

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  On $\mathbb{S}^2$, the **MTW** is given by

  $$
  \mathfrak{S}_{(x,v)}(\xi, \xi^\perp)
  = 3 \left[\frac{1}{r^2} - \frac{\cos(r)}{r \sin(r)} \right] \xi_1^4 + 3 \left[\frac{1}{\sin^2(r)} - \frac{r \cos(r)}{\sin^3(r)} \right] \xi_2^4
  $$
  $$
  + \frac{3}{2} \left[-\frac{6}{r^2} + \frac{\cos(r)}{r \sin(r)} + \frac{5}{\sin^2(r)} \right] \xi_1^2 \xi_2^2,
  $$

  with
  $x \in \mathbb{S}^2, v \in \mathcal{I}(x), r := |v|, \xi = (\xi_1, \xi_2), \xi^\perp = (-\xi_2, \xi_1)$.

Sketch of proof

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- We extend the Ma-Trudinger-Wang tensor beyond boundaries of injectivity domains.

- The uniform convexity of the nonfocal domains $\mathcal{NF}(x)$ is stable.

- The positivity of the extended tensor $\overline{\mathfrak{S}}$ is stable.

- $\overline{\mathfrak{S}} \succ 0 +$ (uniform) convexity of the $\mathcal{NF}(x)$'s
  $\implies \mathfrak{S} \succ 0 +$ (uniform) convexity of the $\mathcal{I}(x)$'s.

Mass Transportation on the Earth

Theorem (Figalli-R '09)

Any small deformation of $\mathbb{S}^2$ in C^5 topology satisfies $\overline{\mathfrak{S}} \succeq 1/2$, has convex injectivity domains and satisfies **TCP**.

Thank you for your attention !!

Greater dimensions

Let (M, g) be a smooth connected compact Riemannian manifold of dimension $n \geq 2$.

Theorem (Figalli-R-Villani '10)

*Assume that (M, g) satisfies (**TCP**). Then*

- *all its injectivity domains are convex,*
- *the **MTW** tensor is $\succeq 0$.*

Theorem (Figalli-R-Villani '10)

Assume that (M, g) satisfies the two following properties:

- *all its injectivity domains are strictly convex,*
- *the **MTW** tensor is $\succ 0$,*

*Then, it satisfies **TCP**.*