Abnormal Singular Foliations and the Sard Conjecture for generic co-rank one distributions

A. Belotto da Silva [*,†] A. Parusiński [‡] L. Rifford [§]

Abstract

Given a smooth totally nonholonomic distribution on a smooth manifold, we construct a singular distribution capturing essential abnormal lifts which is locally generated by vector fields with controlled divergence. Then, as an application, we prove the Sard Conjecture for rank 3 distribution in dimension 4 and generic distributions of corank 1.

1 Introduction

This paper is the second in a series of three focusing on geometrical properties of singular horizontal paths of totally nonholonomic distributions and their links with the Sard Conjecture. Its aim is to extend to the smooth case, possibly generic, some of the results obtained in our first paper [3] dealing with the real-analytic case. Since we do not want to repeat ourselves, we refer the reader to [3] for a general presentation of the Sard Conjecture and a discussion on the importance of the topic.

Throughout all the paper, we consider a smooth connected manifold M of dimension $n \geq 3$ equipped with a totally nonholonomic distribution Δ of rank $m < n$. For convenience, we shall say that both M and Δ are of class $\mathcal{C}$ with $\mathcal{C} = \mathcal{C}^\infty$ if they are C^∞ and of class $\mathcal{C} = \mathcal{C}^\omega$ if they are analytic and we will proceed in the same way for other objects (for example, a $\mathcal{C}$-vector field will refer to a vector field in the category $\mathcal{C}$). Then, as in [3], we denote by ω the canonical symplectic form of T^*M, by $\Delta^\perp \subset T^*M$ the nonzero annihilator of Δ and by $\omega^\perp$ the restriction of ω to $\Delta^\perp$. Our first result is concerned with the description of abnormal distributions constructed in [3] that can be obtained in the smooth case. Before stating the result, we need to introduce a few notions related to singular distributions.

As in [3], we call *distribution* on $\Delta^\perp$ any mapping $\vec{\mathcal{K}}$ which assigns to a point $\mathfrak{a}$ in $\Delta^\perp \subset T^*M$ a vector subspace $\vec{\mathcal{K}}(\mathfrak{a})$ of $T_{\mathfrak{a}}\Delta^\perp$ of dimension $\dim \vec{\mathcal{K}}(\mathfrak{a})$, also called rank,

[*]Université Paris Cité, Institut de Mathématiques de Jussieu-Paris Rive Gauche (UMR CNRS 7586), Bâtiment Sophie Germain, 75205 Paris Cedex 13, France. (andre.belotto@imj-prg.fr)

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[‡]Université Côte d'Azur, CNRS, Labo. J.-A. Dieudonné, UMR CNRS 7351, Parc Valrose, 06108 Nice Cedex 02, France (adam.parusinski@unice.fr)

[§]Université Côte d'Azur, CNRS, Labo. J.-A. Dieudonné, UMR CNRS 7351, Parc Valrose, 06108 Nice Cedex 02, France (ludovic.rifford@math.cnrs.fr)

that may depend upon $\mathfrak{a}$, and a curve $\psi : [0,1] \to \Delta^\perp$ is said to be *horizontal with respect to* $\vec{\mathcal{K}}$ if it is absolutely continuous with derivative in L^2 and satisfies

$$\dot{\psi}(t) \in \vec{\mathcal{K}}(\psi(t)) \subset T_{\psi(t)}\Delta^\perp \quad \text{for a.e. } t \in [0,1].$$

We say that $\vec{\mathcal{K}}$ is *regular* on a set $\mathcal{S} \subset \Delta^\perp$ if its rank is constant over each connected component of $\mathcal{S}$ and otherwise we say that the distribution is *singular*; note that the rank of a regular distribution $\vec{\mathcal{K}}$ on $\mathcal{S}$ may differ from one connected component of $\mathcal{S}$ to another. Then, recalling that a *singular foliation* over a smooth manifold is a partition of that manifold into connected immersed smooth submanifolds called leaves, we say that a singular distribution $\vec{\mathcal{K}}$ on $\Delta^\perp$ is *integrable* if it is associated to a singular foliation, that is, if there exists a singular foliation whose tangent spaces of its leaves are equal to $\vec{\mathcal{K}}$. We refer the reader to [10] for further details on singular foliations. Furthermore, as in [3] we say that a set $\mathcal{S} \subset \Delta^\perp$ is *invariant by dilation* if $\sigma_\lambda(\mathcal{S}) = \mathcal{S}$ for every $\lambda \in \mathbb{R}^*$, where $\sigma_\lambda : T^*M \to T^*M$ is given by $\sigma_\lambda(x,p) = (x, \lambda p)$. Similarly, $\vec{\mathcal{K}}$ is *invariant by dilation* if $d\sigma_\lambda(\vec{\mathcal{K}}(\mathfrak{a})) = \vec{\mathcal{K}}(\sigma_\lambda(\mathfrak{a}))$ for all $\mathfrak{a}$ and λ. Our first result can now be precisely stated:

Theorem 1.1 (Singular distribution capturing essential abnormal lifts)**.** *Let M and Δ be of class $\mathcal{C}$. Then there exist an open and dense set $\mathcal{S}_0 \subset \Delta^\perp$ and an integrable singular distribution $\vec{\mathcal{F}}$ on $\Delta^\perp$, both invariant by dilation, satisfying the following properties:*

(i) ***Specification on $\mathcal{S}_0$.*** *$\vec{\mathcal{F}}$ is regular on $\mathcal{S}_0$ and satisfies $\vec{\mathcal{F}}_{|\mathcal{S}_0} = \ker(\omega^\perp)_{|\mathcal{S}_0}$. In particular, there holds $\dim \vec{\mathcal{F}}_{|\mathcal{S}_0} \equiv m\,(2)$ and $\dim \vec{\mathcal{F}}_{|\mathcal{S}_0} \leq m-2$.*

(ii) ***Specification outside $\mathcal{S}_0$.*** *$\vec{\mathcal{F}}(\mathfrak{a}) = \{0\}$ for all $\mathfrak{a} \in \Sigma := \Delta^\perp \setminus \mathcal{S}_0$.*

(iii) ***Abnormal lifts.*** *Let $\gamma : [0,1] \to M$ be a singular horizontal path and $\psi : [0,1] \to \Delta^\perp$ be an abnormal lift of γ. If $\psi^{-1}(\Sigma) \subset [0,1]$ has Lebesgue measure zero, we call such an abnormal lift* essential*, then ψ is horizontal with respect to $\vec{\mathcal{F}}$. Furthermore, if a horizontal path γ admits a lift $\psi : [0,1] \to \Delta^\perp$ horizontal with respect to $\vec{\mathcal{F}}$, then γ is singular.*

(iv) ***Local generators of $\vec{\mathcal{F}}$.*** *For every point $x \in M$, there is an open neighborhood $\mathcal{V}$ of x and $\mathcal{C}$-vector fields $\{\mathcal{Y}^\alpha,\, \alpha \in \Gamma\}$, where Γ is a finite set, defined on $\tilde{\mathcal{V}} := \Delta^\perp \cap T^*\mathcal{V}$, such that $\vec{\mathcal{F}}_{|\tilde{\mathcal{V}}}$ is generated by $\operatorname{Span}\{\mathcal{Y}^\alpha,\, \alpha \in \Gamma\}$ and each $\mathcal{Y}^\alpha$ is singular over Σ and homogeneous with respect to the p variable (in a local set of symplectic coordinates (x,p)). In addition, if $\vec{\mathcal{F}}$ has constant rank over $\mathcal{S}_0 \cap \tilde{\mathcal{V}}$ then each $\mathcal{Y}^\alpha$ has* controlled divergence*, that is,*

$$\operatorname{div}^{\Delta^\perp}(\mathcal{Y}^\alpha) \in \mathcal{Y}^\alpha \cdot \mathcal{C}(\tilde{\mathcal{V}}).$$

Moreover, if $\vec{\mathcal{F}}$ has rank at most 1 then $|\Gamma| = 1$ and the vector field $\mathcal{Y}^\alpha$ generating $\vec{\mathcal{F}}$ has controlled divergence.

(v) ***Generic case.*** *Assume that Δ is generic (with respect to the Whitney $\mathcal{C}^\infty$ topology). Then Σ is countably smoothly $(2n-m-1)$-rectifiable. Moreover, $\vec{\mathcal{F}}_{|\mathcal{S}_0}$ has rank 0 if m is even, and rank 1 if m is odd.*

According to Theorem 1.1 (v), the singular set $\Sigma := \Delta^\perp \backslash \mathcal{S}_0$ of a generic distribution is countably smoothly $(2n-m-1)$-rectifiable, which means that it can be covered by countably many smooth submanifolds of $\Delta^\perp$ of codimension 1. This result can be improved in the analytic category where one can show that Σ is a proper analytic subset of $\Delta^\perp$, see [3, Th 1.1]. In either way, Σ has Lebesgue measure zero in $\Delta^\perp$. The result of Theorem 1.1 (iv) can also be improved. We can show that if we allow the set Γ to be countable then the $\mathcal{C}$-module of vector fields generated by $\{\mathcal{Y}^\alpha, \alpha \in \Gamma\}$ is involutive, that is, it is stable by Lie-brackets. Moreover, in the case $\mathcal{C} = \mathcal{C}^\omega$, Γ may always be taken to be finite. We refer the reader to Remark 4.3 for further detail.

The property of controlled divergence for generators of the singular distribution given in Theorem 1.1 (iv) was observed and explored in a previous work of the first and third authors in the case $\dim(M) = 3$ where it could be used to prove the strong Sard Conjecture whenever the Martinet surface is smooth (see [5, Theorem 1.1]). Here we use it to establish the Sard Conjecture for corank 1 distributions for which the singular distribution $\vec{\mathcal{F}}$ given by Theorem 1.1 satisfies an extra assumption. We recall that a totally nonholonomic distribution Δ of rank $m < n$ on M is said to satisfy the *Sard Conjecture* if for any $x \in M$, the set of end-points of singular horizontal paths starting from x, denoted by $\mathrm{Abn}_\Delta(x)$, has Lebesgue measure zero in M; we refer to [3] for further detail. We have the following:

Theorem 1.2 (Conditional Sard Conjecture for corank 1 distributions)**.** *Let M and Δ be of class $\mathcal{C}$ and assume that Δ is of corank 1. Assume that the two following properties are satisfied:*

(H1) The distribution $\vec{\mathcal{F}}_{|\mathcal{S}_0}$ has constant rank equal to 0 or 1.

(H2) The singular set Σ of $\vec{\mathcal{F}}$ has Lebesgue measure zero in $\Delta^\perp$.

Then the Sard Conjecture holds true.

Our first application of Theorem 1.2 is concerned with the Sard Conjecture for corank-1 distributions in dimension 4 for which assumptions (H1)-(H2) are automatically satisfied (see Section 2.2):

Corollary 1.3 (Sard Conjecture for rank-3 distribution in dimension 4)**.** *Let M and Δ be $\mathcal{C}^\infty$ with M of dimension 4 and Δ of rank 3. Then the Sard Conjecture holds true.*

By Theorem 1.1 (v), the singular distribution of a generic distribution Δ has constant rank 0 or 1 on $\mathcal{S}_0$ whose complement in $\Delta^\perp$ has Lebesgue measure zero. Therefore, we have the following:

Corollary 1.4 (Sard Conjecture for corank-1 generic distributions)**.** *Let M and Δ be $\mathcal{C}^\infty$ with Δ of corank 1. If Δ is generic (with respect to the Whitney $\mathcal{C}^\infty$ topology) then the Sard Conjecture holds true.*

As a last application, we note that Theorem 1.2 can be combined with [6, Th 2.4] (showing that distributions of corank > 1 satisfy the minimal rank Sard Conjecture) to establish the minimal rank Sard Conjecture for generic distribution (see [3, page 7]):

Corollary 1.5 (Generic minimal rank Sard Conjecture)**.** *Let M and Δ be $\mathcal{C}^\infty$. If Δ is generic (with respect to the Whitney $\mathcal{C}^\infty$ topology) then the minimal rank Sard Conjecture holds true.*

Both Corollaries 1.4 and 1.5 are consequences of Theorem 1.1 (v) which is proven by transversality arguments. The generic property stated in assertion (v) roughly corresponds to the property which is required to obtain Corollaries 1.4 and 1.5 from Theorem 1.2. But in fact much deeper results can be established for generic distributions, as for example the Chitour-Jean-Trélat Theorem [6] stating that all abnormal lifts of generic distributions are essential. This subject will be investigated more deeply in a forthcoming paper [4].

The paper is organized as follows: Several examples illustrating our results are presented in Section 2, Section 3 gathers a few results of importance for the rest of the paper, Sections 4 and 5 are devoted respectively to the proofs of Theorems 1.1 and 1.2 and finally, Appendices A and B provide the proofs of several results stated in the course of the paper.

2 Examples

We gather in this section several examples to illustrate our results. We show in Section 2.1 that Theorem 1.1 provides indeed a singular distribution on M in the case of corank 1 distributions, Sections 2.2, 2.3 are concerned with Sard type results results concerning corank 1 distributions in dimensions 4 and 5, and Section 2.4 features an example of corank 1 distribution in dimension 6 for which the singular set has positive Lebesgue measure.

2.1 Corank 1 distributions

In the case of a corank 1 distribution, the nonzero annihilator $\Delta^\perp \subset T^*M$ is a graph (up to dilation) over M, in such a way that all objects given by Theorem 1.1 can indeed be seen in M. The proof of the following result is given in Appendix A (π stands for the canonical projection from T^*M to M):

Theorem 2.1 (Singular distribution for corank 1 distributions)**.** *Let M and Δ be of class $\mathcal{C}$ with Δ of corank 1 (that is, $m = n-1$) and consider $\mathcal{S}_0, \vec{\mathcal{F}}$ and Σ given by Theorem 1.1. Then the open and dense set $\mathcal{R}_0 \subset M$ and the integrable singular distribution $\mathcal{H}$ over M given by*

$$\mathcal{R}_0 := \pi(\mathcal{S}_0) \quad \textit{and} \quad \mathcal{H} := d\pi\big(\vec{\mathcal{F}}\big)$$

satisfy the following properties:

- *(i)* ***Specification on $\mathcal{R}_0$.*** *$\mathcal{H}$ is regular on $\mathcal{R}_0$, $\dim \mathcal{H}_{|\mathcal{R}_0} \equiv m\,(2)$ and $\dim \mathcal{H}_{|\mathcal{R}_0} \leq m-2$.*
- *(ii)* ***Specification outside $\mathcal{R}_0$.*** *$\mathcal{H}(x) = \{0\}$ for all $x \in \sigma := M \setminus \mathcal{R}_0 = \pi(\Sigma)$.*

(iii) ***Singular horizontal paths.*** *Let $\gamma : [0,1] \to M$ be an horizontal path. If γ is singular and $\gamma^{-1}(\sigma)$ has Lebesgue measure zero then γ is horizontal with respect to $\mathcal{H}$. Conversely, if γ is horizontal with respect to $\mathcal{H}$, then it is singular.*

(iv) ***Local generators of $\mathcal{H}$.*** *For every point $x \in M$, there is an open neighborhood $\mathcal{V}$ of x, $d \in \mathbb{N}$, and $\mathcal{C}$-vector fields $\mathcal{Z}^1, \ldots, \mathcal{Z}^d$ defined on $\mathcal{V}$, such that $\mathcal{H}_{|\mathcal{V}}$ is generated by $\mathrm{Span}\{\mathcal{Z}^1, \ldots, \mathcal{Z}^d\}$, and each $\mathcal{Z}^i$ is singular over $\sigma = M \setminus \mathcal{R}_0$ and, if $\mathcal{H}$ has constant rank over $\mathcal{V} \cap \mathcal{R}_0$, then $\mathcal{Z}^i$ has* controlled divergence*, that is,*

$$\mathrm{div}(\mathcal{Z}^k) \in \mathcal{Z}^k \cdot \mathcal{C}(\mathcal{V}).$$

Moreover, if $\mathcal{H}$ has rank at most 1, then $d = 1$.

(v) ***Generic case.*** *Assume that Δ is generic (in respect to the Whitney $\mathcal{C}^\infty$ topology). Then σ is countably smoothly $(n-1)$-rectifiable. Moreover, $\mathcal{H}_{|\mathcal{R}_0}$ has rank 0 if m is even, and rank 1 if m is odd.*

Remark 2.2. *It follows from [3, Th 1.3] that, in the real-analytic category, the singular set σ is a proper analytic subset of M and $\mathcal{H}_{|\mathcal{R}_0}$ has constant rank.*

Remark 2.3. *Since $\mathcal{H} = d\pi(\vec{\mathcal{F}})$ and $\sigma = \pi(\Sigma)$ where Σ is invariant by dilation, the assumptions (H1)-(H2) of Theorem 1.2 are satisfied if and only if $\mathcal{H}_{|\mathcal{R}_0}$ has constant rank equal to 0 or 1 and σ has Lebesgue measure zero in M.*

2.2 Rank 3 distributions in dimension 4

Let M be a connected open set of $\mathbb{R}^4$ and Δ be a rank 3 totally nonholonomic distribution on M generated by three smooth vector fields X^1, X^2, X^3 of the form

$$X^i(x) = \partial_{x_i} + A_i(x)\partial_{x_4} \qquad \forall x \in M, \, \forall i = 1,2,3,$$

where A_1, A_2, A_3 are smooth functions from M to $\mathbb{R}$. Note that up to shrinking M we can always assume that such a property holds true on a neigborhood of a given point in M. By the equation (A.1) used in the proof of Theorem 2.1, the distribution $\mathcal{H}$ given by Theorem 2.1 is generated by the vector field

$$\mathcal{Z} = [X^1, X^2](x_4)\, X^3 + [X^3, X^1](x_4)\, X^2 + [X^2, X^3](x_4)\, X^1$$

where for any $i, j \in \{1,2,3\}$, $[X^i, X^j](x_4)$ stands for the Lie derivative of the function x_4 along $[X^i, X^j]$, that is,

$$[X^i, X^j](x_4) = \partial_{x_i}(A_j) - \partial_{x_j}(A_i) + A_i \partial_{x_4}(A_j) - A_j \partial_{x_4}(A_i).$$

We can easily verify, by using the Jacobi identity, that $\mathcal{Z}$ has controlled divergence. Moreover, we can check that

$$\mathcal{Z} = 0 \quad \Longleftrightarrow \quad [X^1, X^2](x_4) = [X^3, X^1](x_4) = [X^2, X^3](x_4) = 0,$$

which due to the total nonholonomicity of Δ shows that σ has Lebesgue measure zero in M, cf. Lemma 4.5 below. As a consequence, by Theorem 1.2 and Remark 2.3, the Sard Conjecture holds true.

2.3 Rank 4 distributions in dimension 5

We consider here an example in the lowest dimension for which the Sard Conjecture for corank 1 distributions remains open. Let M be a connected open set of $\mathbb{R}^5$ and Δ a rank 4 totally nonholonomic distribution on M generated by four vector fields X^1, X^2, X^3, X^4 of the form

$$X^i(x) = \partial_{x_i} + A_i(x)\partial_{x_5} \qquad \forall x \in M,\, \forall i = 1, 2, 3, 4,$$

where A_1, A_2, A_3, A_4 are analytic functions from M to $\mathbb{R}$. Note that for sake of simplicity we work here with analytic vector fields. In this case (see Theorem 2.1 and Remark 2.2), the open set $\mathcal{R}_0 \subset M$ is the complement of a proper analytic subset of M and the distribution $\mathcal{H}_{|\mathcal{R}_0}$ has constant rank. Moreover, from Proposition 3.1, $\mathcal{H}_{|\mathcal{R}_0}$ corresponds to the projection of $\ker(\mathcal{L}^2) = \ker(\omega^\perp)$ whose dimension coincides with the corank of the 4×4 matrix (see Section 3.1)

$$\tilde{H} = [[X^i, X^j](x_5)]_{i,j}.$$

If $\tilde{H}$ has rank 4 on $\mathcal{R}_0$, then $\mathcal{H}_{|\mathcal{R}_0}$ has rank zero and the Sard Conjecture is easily satisfied. So, we assume that the rank of $\tilde{H}$ is everywhere at most 2 which by Theorem 2.1 (i) means that $\mathcal{H}_{|\mathcal{R}_0}$ has rank 2. Therefore, the pfaffian of the matrix $\tilde{H}$ vanishes everywhere, that is,

$$[X^1, X^2](x_5)[X^3, X^4](x_5) - [X^1, X^3](x_5)[X^2, X^4](x_5) + [X^1, X^4](x_5)[X^2, X^3](x_5) = 0,$$

and by (A.1), the distribution $\mathcal{H}$ is generated by the following vector fields,

$$\begin{aligned}
\mathcal{Z}^1 &= [X^4, X^2](x_5)X^3 + [X^3, X^4](x_5)X^2 + [X^2, X^3](x_5)X^4,\\
\mathcal{Z}^2 &= [X^1, X^4](x_5)X^3 + [X^3, X^1](x_5)X^4 + [X^4, X^3](x_5)X^1,\\
\mathcal{Z}^3 &= [X^1, X^2](x_5)X^4 + [X^4, X^1](x_5)X^2 + [X^2, X^4](x_5)X^1,\\
\mathcal{Z}^4 &= [X^1, X^2](x_5)X^3 + [X^3, X^1](x_5)X^2 + [X^2, X^3](x_5)X^1,
\end{aligned}$$

or equivalently $\mathcal{H}$ is generated by the following 2-derivation,

$$\begin{aligned}
\eta =& [X^1, X^2](x_5)X^3 \wedge X^4 + [X^1, X^4](x_5)X^2 \wedge X^3 + [X^4, X^2](x_5)X^1 \wedge X^3\\
&+ [X^3, X^1](x_5)X^2 \wedge X^4 + [X^2, X^3](x_5)X^1 \wedge X^4 + [X^3, X^4](x_5)X^1 \wedge X^2,
\end{aligned}$$

which can be seen as an analytic section of the bundle of Grassmannian $\mathrm{Gr}(2, M)$. Note that we can easily verify that $\mathcal{Z}^1, \mathcal{Z}^2, \mathcal{Z}^3, \mathcal{Z}^4$ have controlled divergence via the Jacobi identity. In conclusion, the integrable distribution $\mathcal{H}$ given by Theorem 2.1 is generated globally by 4 analytic vector fields with controlled divergence, has rank 2 over $\mathcal{R}_0$ and 0 over $\sigma = M \setminus \mathcal{R}_0$ and the methods of the current paper do not allow us to conclude if the Sard Conjecture is verified or not in this case.

By the results obtained in our previous paper [3] we know that if the involutive distribution $\mathcal{H}_{|\mathcal{R}_0}$ is splittable (see [3, Def. 1.5]), then the Sard Conjecture follows from [3, Thm. 1.6]. But, we do not know if $\mathcal{H}_{|\mathcal{R}_0}$ is always splittable, this question is open. We can nevertheless specialize the example under study in order to make $\mathcal{H}_{|\mathcal{R}_0}$ splittable. For this, we can assume that the functions A_1, A_2, A_3, A_4 satisfy

$$A_1 = 0 \quad \text{and} \quad \frac{\partial A_i}{\partial x_1} = 0 \quad \forall i = 2, 3, 4.$$

Then we can check that $[X^1, X^2] = [X^1, X^3] = [X^1, X^4] = 0$ which gives

$$\begin{aligned}
&\mathcal{Z}^1 = [X^4, X^2](x_5)X^3 + [X^3, X^4](x_5)X^2 + [X^2, X^3](x_5)X^4, \\
&\mathcal{Z}^2 = [X^4, X^3](x_5)X^1, \quad \mathcal{Z}^3 = [X^2, X^4](x_5)X^1, \quad \mathcal{Z}^4 = [X^2, X^3](x_5)X^1.
\end{aligned}$$

Therefore, the distribution $\mathcal{H}_{|\mathcal{R}_0}$ is generated by the two linearly independent vector fields ∂_{x_1} and $\mathcal{Z} = \mathcal{Z}^1$ where $\mathcal{Z}$ does not depend upon x_1, hence the Gaussian (or Ricci) curvature of the 2-dimensionals leaves of $\mathcal{H}_{|\mathcal{R}_0}$ is equal to 0 which shows that $\mathcal{H}_{|\mathcal{R}_0}$ is splittable (see [3, Prop. 7.8 and 7.9]).

2.4 The singular set may have positive measure

We provide here an example of rank 5 totally nonholonomic distribution in $\mathbb{R}^6$ for which the singular set σ given by Theorem 2.1 has positive Lebesgue measure. We consider the distribution Δ in $\mathbb{R}^6$ generated by vector fields X^1, X^2, X^3, X^4, X^5 of the form

$$X^i(x) = \partial_{x_i} + A_i(x)\partial_{x_6} \qquad \forall x \in \mathbb{R}^7, \ \forall i = 1, 2, 3, 4, 5,$$

with $A_1, \ldots, A_5 : \mathbb{R}^6 \to \mathbb{R}$ the smooth functions defined by

$$\left\{ \begin{array}{l}
A_1(x) = A_5(x) = 0 \\
A_2(x) = x_1 \\
A_3(x) = -x_1 \\
A_4(x) = R(x_2 + x_3)
\end{array} \right. \qquad \forall x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6,$$

where $R : \mathbb{R} \to \mathbb{R}$ is a smooth function such that the closed set where R' vanishes has empty interior and positive Lebesgue measure. Then, we check easily that Δ is totally nonholonomic everywhere (we have $[X^1, X^2] = \partial_{x_6}$), and furthermore, we see that the rank of $\mathcal{H}$, corresponding with the dimension of the projection of $\ker(\mathcal{L}^2) = \ker(\omega^\perp)$ coincides with the corank of the 5×5 matrix (see Section 3.1)

$$\tilde{H} = [[X^i, X^j](x_6)]_{i,j} = \begin{bmatrix}
0 & 1 & -1 & 0 & 0 \\
-1 & 0 & 0 & R'(x_2 + x_3) & 0 \\
1 & 0 & 0 & R'(x_2 + x_3) & 0 \\
0 & -R'(x_2 + x_3) & -R'(x_2 + x_3) & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

We conclude that the rank of $\mathcal{H}$ is 1 over the open set $\{R'(x_2 + x_3) \neq 0\}$ and 3 over the closed set $\{R'(x_2 + x_3) = 0\}$. By construction, the set $\{R'(x_2 + x_3) = 0\}$ corresponds to the set σ of Theorem 2.1, it is closed with empty interior and positive Lebesgue measure in $\mathbb{R}^6$.

3 Preliminaries

We gather in this section a few results and notations that will be useful for the proof of Theorem 1.1. Section 3.1 is concerned with the Goh matrix which reprents the $\mathcal{L}^2$ operator introduced in [3] while Section 3.2 introduces several definition and properties on Pfaffians that will be used to define local generators of the singular distribution obtained in Theorem 1.1.

3.1 The Goh matrix

We recall here how the Goh matrix is defined locally, we refer the reader to [3] for further details. Given $x \in M$, we consider an open neighborhood $\mathcal{V}$ of x on which Δ is generated by m smooth vector fields $X^1, \ldots, X^m$, we define the Hamiltonians $h^1, \ldots, h^m : T^*\mathcal{V} \to \mathbb{R}$ by

$$h^i(\mathfrak{a}) := h^{X^i}(x,p) = p \cdot X^i(x) \qquad \forall \mathfrak{a} = (x,p) \in T^*\mathcal{V},\, \forall i = 1, \ldots, m$$

and we denote by $\vec{h}^1, \ldots, \vec{h}^m$ the corresponding hamiltonian vector fields. The Goh matrix H at $\mathfrak{a} \in T^*\mathcal{V}$ is the $m \times m$ matrix defined by

$$H_{\mathfrak{a}} := \left[h^{ij}(\mathfrak{a})\right]_{1 \leq i,j \leq m},$$

where, for any $i, j \in \{1, \ldots, m\}$, h^{ij} is the Hamiltonian given by ($\{\,,\}$ stands for the Poisson bracket)

$$h^{ij} := \left\{h^i, h^j\right\}.$$

By construction, the matrix $H_{\mathfrak{a}}$ represents the linear map

$$\mathcal{L}^2_{\mathfrak{a}} : \vec{\Delta}(\mathfrak{a}) := \operatorname{Span}\left\{\vec{h}^1(\mathfrak{a}), \cdots, \vec{h}^m(\mathfrak{a})\right\} \longrightarrow \mathbb{R}^m$$

defined by

$$\left(\mathcal{L}^2_{\mathfrak{a}}(\zeta)\right)_i := \sum_{j=1}^m u_j\, h^{ij}(\mathfrak{a}) \qquad \forall \zeta = \sum_{i=1}^m u_i \vec{h}^i(\mathfrak{a}) \in \vec{\Delta}(\mathfrak{a}),\, \forall i = 1, \ldots, m$$

which satisfies the following result (see [3, Prop. 3.5]):

Proposition 3.1. *For every $\mathfrak{a} \in T^*\mathcal{V} \cap \Delta^\perp$, we have* $\ker(\mathcal{L}^2_{\mathfrak{a}}) = \ker(\omega^\perp_{\mathfrak{a}})$.

As it was recalled in [3, Proposition 3.4], an absolutely continuous curve $\gamma : [0,1] \to M$ which is horizontal with respect to Δ is singular if and only if it admits an anormal lift, that is an absolutely continuous curve of $\psi : [0,1] \to \Delta^\perp$ satisfying $\dot{\psi}(t) \in \ker(\omega^\perp_{\psi(t)})$ for almost every $t \in [0,1]$.

3.2 Pfaffian polynomial of minors

Let $m \in \mathbb{N}$ and let R_m be a sub-ring of the formal power series $\mathbb{R}[\![x_1, \ldots, x_m]\!]$, such as $\mathbb{R}[\![x_1, \ldots, x_n]\!]$ itself or $\mathbb{R}\{x_1, \ldots, x_m\}$, the ring of analytic function germs at the origin of $\mathbb{R}^m$. Denote by $\mathbb{K}_m = \operatorname{Frac}(R_m)$ its field of fractions; we fix m and we denote by $\mathbb{K}$ the field $\mathbb{K}_m$. We consider a $\mathbb{K}$-vector space V of dimension n and we fix an orthonormal basis $(e_1, \ldots, e_n)$ of V. We also fix, once and for all, an ordering on the index set $\{1, \ldots, n\}$ which we assume to be $1 < 2 < \ldots < n$ for simplicity.

Recall that an anti-symmetric bilinear operator over V can be written as

$$A = \sum_{i<j} a_{ij}\, e_i \wedge e_j, \quad a_{ij} \in \mathbb{K}$$

and fix the notation $a_{ji} = -a_{ij}$. Under this convention, A admits a representation as a matrix $M_A = [a_{ij}]_{i,j}$, such that $A(v, w) = v^t \cdot M_A \cdot w$ for all vectors v and w in V.

Definition 3.2. *Suppose that the dimension n of V is even. The Pfaffian polynomial $\varphi(A)$ of an anti-symmetric bilinear operator A over V is defined by*

$$\frac{1}{(n/2)!} \bigwedge^{n/2} A =: \varphi(A)\, e_1 \wedge \ldots \wedge e_n.$$

If V has dimension zero, we fix the convention that $\varphi(A) = 1$. If V has odd dimension, we fix the convention that $\varphi(A) = 0$.

It is clear from the definition that $\varphi(A) \in \mathbb{K}$. We denote by $\mathrm{Det}(A)$ the determinant of the associated matrix M_A of A, it is well known (see, e.g. [16, Section 5.8.1]) that

$$\varphi(A)^2 = \mathrm{Det}(A). \tag{3.1}$$

We are now interested in considering a family of Pfaffian polynomials associated with the minors of A. Given $l \in \{1, \ldots, n\}$, we denote by Λ_l the set of sub-indices $I \subset \{1, \ldots, n\}$ of cardinality l and for every $I \in \Lambda_l$ we define the anti-symmetric bilinear operator

$$A_I := \sum_{i<j \in I} a_{ij}\, e_i \wedge e_j$$

which can be seen as an operator over the subspace $V_I \subset V$ of dimension l. Then, we set

$$\mathrm{Det}(A, I) := \mathrm{Det}(A_I) \quad \text{and} \quad \varphi(A, I) = \varphi(A_I).$$

In order to keep the compatibility of signs between different Pfaffian of minors, we always consider the ordering $\{i_1 < \cdots < i_l\}$ of the elements of I and we fix the convention

$$\bigwedge_{i \in I} e_i = e_{i_1} \wedge \ldots \wedge e_{i_l}.$$

Then, we consider the function ϵ whose input is an index set I and an element $j \in I$, and whose output is a value in $\{-1, 1\}$ defined by

$$e_j \wedge \bigwedge_{i \in I \setminus \{j\}} e_i = \epsilon(I, j) \bigwedge_{i \in I} e_i$$

We are now ready to provide formulas which characterize Pfaffian minors and their derivatives in terms of Pfaffian of smaller orders:

Proposition 3.3. *Let A be an anti-symmetric bilinear operator over the $\mathbb{K}$-vector space V and I be a sub-index of $\{1,\ldots,n\}$ of even cardinality $r = 2s$, then the following properties are satisfied:*

(i) For every $i_0 \in I$, we have:

$$\varphi(A,I) = \frac{1}{s} \sum_{j \in I \setminus \{i_0\}} \epsilon(I,i_0) \cdot \epsilon(I \setminus \{i_0\}, j) \cdot a_{i_0 j} \cdot \varphi(A, I \setminus \{i_0, j\}).$$

(ii) For any $\mathbb{R}$-derivation X over R_m, there holds

$$X\left[\varphi(A,I)\right] = \frac{s}{2} \cdot \sum_{i \neq j \in I} \epsilon(I,i) \cdot \epsilon(I \setminus \{i\}, j) \cdot \varphi(A, I \setminus \{i,j\}) \cdot X(a_{i,j})$$

Proposition 3.3, whose proof is postponed to §B.1, will be used to provide suitable generators of the kernel of A, that is, of the subspace $\ker(A)$ of all vectors $v \in V$ such that $A(v,\cdot) \equiv 0$. In order to make this idea precise, we recall that an even number $r = 2s$ is said to be the rank of A, which we denote by $\operatorname{rank}(A)$, if

$$\bigwedge^{s} A \neq 0 \quad \text{and} \quad \bigwedge^{s+1} A = 0.$$

Note that the kernel of A is a linear subspace of dimension $n - r$. It is, of course, possible to provide generators of $\ker(A)$ via Cramer's rule, but these generators will not satisfy differential properties that will be needed later on. Instead, we now describe generators $\ker(A)$ in terms of the Pfaffian polynomials which are better adapted to our future objectives, c.f. Lemma 4.2 below. The following result holds:

Proposition 3.4. *Let A be an anti-symmetric bilinear operator of rank $r < n$ over V. Then we have*

$$\ker(A) = \operatorname{Span}\Big\{ Z_I \,|\, I \in \Lambda_{r+1} \Big\},$$

where for every sub-index $I \in \Lambda_{r+1}$, the vector $Z_I \in V$ is defined by

$$Z_I := \sum_{i \in I} \epsilon(I,i) \cdot \varphi(A, I \setminus \{i\}) \cdot e_i.$$

The proof of Proposition 3.4 follows easily from Proposition 3.3 (i), it is given in §B.2. As we said before, the formula (ii) of Proposition 3.3 will be used to show that the generators for $\vec{\mathcal{F}}$ in Theorem 1.1 have controlled divergence.

Remark 3.5. *Let $\mathcal{M}$ denote a free R-module, where $R = \mathbb{R}[\![x_1,\ldots,x_m]\!]$ and let A be an anti-symmetric bilinear operator over $\mathcal{M}$. Although all elements Z_I belong to the module $\mathcal{M}$ (because the coefficients of Z_I are polynomials in $a_{ij} \in R$), we do not know if the collection $\{Z_I\}_{I \in \Lambda_{r+1}}$ generates the sub-module $\ker(A) \subset \mathcal{M}$. In general, finding generators of a sub-module is a much more subtle problem than its analogous for vector-spaces, cf. [14, §1 and 55].*

4 Proof of Theorem 1.1

Let M and Δ of class $\mathcal{C}$ be fixed, we divide the proof into three parts.

4.1 Proof of assertions (i)-(iii)

We start by constructing the set $\mathcal{S}_0$ as a union of disjoint open sets. Let d_1 be the minimum of the dimension of $\ker(\omega^\perp)$ over $\Delta^\perp$. By upper semi-continuity of the function $\mathfrak{d} : \mathfrak{a} \in \Delta^\perp \mapsto \dim(\ker(\omega^\perp_{\mathfrak{a}})) \in \mathbb{N}$, the set of points $\mathfrak{a} \in \Delta^\perp$ where $\mathfrak{d}(\mathfrak{a}) = d_1$ is an open subset of $\Delta^\perp$, we denote it by $\mathcal{S}_0^1$. Note that since $\omega^\perp$ is skew-symmetric, we have $d_1 \equiv m\,(2)$ and $d_1 \leq m-2$ (see the proof of [3, Theorem 1.1 (iv)]). Moreover, since $\mathfrak{d}$ is invariant by dilation, the set $\mathcal{S}_0^1$ is invariant by dilation too. If the closed set $\Delta^\perp \setminus \mathcal{S}_0^1$ has empty interior, then we are done and we set $\mathcal{S}_0$. Otherwise, we denote by $\Delta_1^\perp$ the interior of $\Delta^\perp \setminus \mathcal{S}_0^1$, we consider the minimum d_2 of $\mathfrak{d}(\mathfrak{a})$ for $\mathfrak{a} \in \Delta_1^\perp$ and we define $\mathcal{S}_0^2$ the set of points $\mathfrak{a} \in \Delta_1^\perp$ where $\mathfrak{d}(\mathfrak{a}) = d_2$. By construction, $\mathcal{S}_0^2$ is an open subset of $\Delta^\perp$ which is invariant by dilation and does not intersect $\mathcal{S}_0^1$ and in addition we have $d_2 > d_1$, $d_2 \equiv m\,(2)$ and $d_2 \leq m-2$. By continuing this process, we construct in a finite number of steps (because the mapping $i \mapsto d_i$ is increasing) an increasing family of dimensions $d_1, \ldots, d_s$ along with a family of disjoint open subsets $\mathcal{S}_0^1, \ldots, \mathcal{S}_0^s$ of $\Delta^\perp$ such that

$$\mathfrak{d}(\mathfrak{a}) = d_i \qquad \forall \mathfrak{a} \in \mathcal{S}_0^i,\, \forall i = 1, \ldots, s$$

and the set

$$\mathcal{S}_0 := \mathcal{S}_0^1 \cup \cdots \cup \mathcal{S}_0^s$$

is open and dense in $\Delta^\perp$. Now we define the singular distribution $\vec{\mathcal{F}}$ as equal to $\ker(\omega^\perp)$ over $\mathcal{S}_0$ and 0 over its complement. When $\mathcal{C} = \mathcal{C}^\omega$, then $\mathcal{S}_0$ coincides with the set $\mathcal{S}_0^1$ above and [3, Theorem 1.1] implies that $\vec{\mathcal{F}}$ is integrable. In fact, the proof in [3] of this result does not depend on the analyticity of Δ and extends to $\mathcal{C} = \mathcal{C}^\infty$. Therefore, $\vec{\mathcal{F}}$ is a singular integrable distribution on $\Delta^\perp$ satisfying (i) and (ii). Furthermore, assertion (iii) follows easily from the characterization of singular curves as projections of extremal abnormals (see *e.g.* [3, Proposition 3.4]).

4.2 Proof of assertion (iv)

Since the result is local in M, we may assume that Δ is generated by m $\mathcal{C}$-vector fields $X^1, \ldots, X^m$, and that there exists a globally defined symplectic coordinate system $(x,p) = (x_1, \ldots, x_n, p_1, \ldots, p_n)$ over T^*M. For each $l = 1, \ldots, m$, we consider the set of sub-indices

$$\Lambda_l = \Big\{ J \subset \{1, \ldots, m\} \,|\, |J| = l \Big\},$$

where $|J|$ stands for the cardinality of the set J. Then, we denote by $H = [h^{ij}]_{ij}$ the $m \times m$ skew-symmetric matrix associated to the operator $\mathcal{L}^2$ defined in §§3.1, and for every $J \in \Lambda_l$, $l \in \{1, \ldots, m\}$, we set

$$H_I := \left[h^{ij}\right]_{i,j \in J} \quad \text{and} \quad \mathrm{Det}(\mathcal{L}^2, J) := \det(H_J).$$

By construction, all $l \times l$ matrices H_J are skew-symmetric and, by (3.1), we have

$$\mathrm{Det}(\mathcal{L}^2, J) = \varphi(\mathcal{L}^2, J)^2,$$

where $\varphi(\mathcal{L}^2, J)$ is the Pfaffian polynomial associated to H_J and compatible with the ordering of the index set (see Definition 3.2).

By keeping the same notations as in the previous section we recall that $\mathcal{S}_0$ is defined as the open dense subset of $\Delta^\perp$ given by

$$\mathcal{S}_0 = \mathcal{S}_0^1 \cup \cdots \cup \mathcal{S}_0^s$$

where $\mathcal{S}_0^1, \ldots, \mathcal{S}_0^s$ is a collection of disjoint open sets in $\Delta^\perp$ associated with a family of integers $d_1 < \cdots < d_s$ and a family of closed sets $\mathcal{C}^1, \ldots, \mathcal{C}^s$ defined recursively by

$$d_1 = \min_{\mathfrak{a} \in \Delta^\perp} \{\mathfrak{d}(\mathfrak{a})\}, \quad \mathcal{S}_0^1 = \mathfrak{d}^{-1}(d_1), \quad \mathcal{C}^1 = \Delta^\perp \setminus \mathcal{S}_0^1$$

and for any integer $i \geq 1$ for which $\mathcal{C}^i$ has nonempty interior,

$$d_{i+1} = \min \{\mathfrak{d}(\mathfrak{a}) \,|\, \mathfrak{a} \in \mathrm{Int}(\mathcal{C}_i)\}, \quad \mathcal{S}_0^{i+1} = \{\mathfrak{a} \in \mathrm{Int}(\mathcal{C}_i) \,|\, \mathfrak{d}(\mathfrak{a}) = d_{i+1}\}, \quad \mathcal{C}^{i+1} = \mathcal{C}^i \setminus \mathcal{S}_0^{i+1}.$$

Note that all sets $\mathcal{S}_0^1, \ldots, \mathcal{S}_0^s$, $\mathcal{C}^1, \ldots, \mathcal{C}^r$ are invariant by dilation. By construction and by Proposition 3.1, for every $i \in \{1, \ldots, s\}$, the linear map $\mathcal{L}^2$ has rank $r_i := m - d_i$ at a point $\mathfrak{a} \in \mathrm{Int}(\mathcal{C}^i)$ (we set $\mathcal{C}^0 := \Delta^\perp$) if, and only if, $\mathfrak{a} \in \mathcal{S}_0^i$. We now define the singular distribution $\vec{\mathcal{F}}$ by using the system of generators given in Proposition 3.4. Given $i \in \{1, \ldots, s\}$, we define for each index set $J \in \Lambda_{r_i+1}$ the smooth vector field

$$\mathcal{Y}_J^i := \sum_{j \in J} \epsilon(J, j) \cdot \varphi(\mathcal{L}^2, J \setminus \{j\}) \cdot \vec{h}^j, \tag{4.1}$$

where the definition of $\epsilon(J, j)$ was introduced in Section 3.2. We note that the vector fields $\mathcal{Y}_J^i$ are all homogeneous with respect to p; indeed all $\vec{h}^i$ are homogeneous vector fields and all $\varphi(\mathcal{L}^2, J)$ are homogeneous functions. The following lemma is a direct consequence of Propositions 3.1 and 3.4.

Lemma 4.1. *For every $i \in \{1, \ldots, s\}$, we have*

$$\ker\bigl(\omega_{\mathfrak{a}}^\perp\bigr) = \mathrm{Span}\left\{\mathcal{Y}_J^i(\mathfrak{a}) \,|\, J \in \Lambda_{r_i+1}\right\} \qquad \forall \mathfrak{a} \in \mathcal{S}_0^i \tag{4.2}$$

and

$$\mathcal{Y}_J^i(\mathfrak{a}) = 0 \qquad \forall \mathfrak{a} \in \mathcal{C}^i, \forall J \in \Lambda_{r_i+1}. \tag{4.3}$$

We need to modify the vector fields $\mathcal{Y}_J^i$ when $s \geq 2$. In this case, we set $\Psi_1 \equiv 1$ and, for each $i \in \{2, \ldots, s\}$, we consider a smooth function $\Psi_i : \Delta^\perp \to [0, \infty)$ homogeneous with respect to the p variable such that

$$\Psi^{-1}(0) = \Delta^\perp \setminus \mathcal{S}_0^i.$$

Note that each function can be taken to be homogeneous because the sets $\mathcal{S}_0^1, \ldots, \mathcal{S}_0^s$ are invariant by dilation. By construction and (4.2), we conclude that

$$\vec{\mathcal{F}}(\mathfrak{a}) = \ker\bigl(\omega_{\mathfrak{a}}^\perp\bigr) = \mathrm{Span}\Big\{\Psi_i \mathcal{Y}_J^i(\mathfrak{a}) \,|\, i \in \{1, \ldots, s\},\, J \in \Lambda_{r_i+1}\Big\} \qquad \forall \mathfrak{a} \in \mathcal{S}_0$$

$$\text{and} \qquad \vec{\mathcal{F}}(\mathfrak{a}) = \{0\} \qquad \forall \mathfrak{a} \in \Sigma := \Delta^\perp \setminus \mathcal{S}_0,$$

which proves the first part of (iv). Furthermore, if $\vec{\mathcal{F}}$ has rank at most 1, then we have $s = 1$, $d_1 = 1$ and $\mathcal{S}_0 = \mathcal{S}_0^1$ (if $s \geq 2$, then the rank of $\vec{\mathcal{F}}$ over $\mathcal{S}_0^2$ would be $d_2 > d_1 = 1$). Hence, we have $r_1 = m - d_1 = m - 1$ which gives $|\Lambda_{r_1+1}| = |\Lambda_m| = 1$ and implies that $\vec{\mathcal{F}}$ is generated by one vector field. It remains to prove the following:

Lemma 4.2. *If $s = 1$, then for any $J \in \Lambda_{r_1+1}$ the vector field $\mathcal{Y}_J := \mathcal{Y}_J^1$ has controlled divergence.*

Proof. Since controlled divergence is invariant by local bi-Lipschitz isomorphism, cf. [2, Lemma 4.2], we can suppose that the metric g is the Euclidean metric on T^*M. In this case we claim that $\mathrm{div}(\mathcal{Y}_J^1) := \mathrm{div}^g(\mathcal{Y}_J^1) = 0$ for all $J \in \Lambda_{r_1+1}$. As a matter of fact, let $J \in \Lambda_{r_1+1}$ be fixed. Since each $\vec{h}^i$ is a Hamiltonian vector field, we know that $\mathrm{div}(\vec{h}^i) = 0$, so (4.1) gives

$$\mathrm{div}\big(\mathcal{Y}_J^1\big) = \sum_{j \in J} \epsilon(J, j) \cdot \Big(\vec{h}^j \cdot \varphi(\mathcal{L}^2, J \setminus \{j\})\Big).$$

Now, from Proposition 3.3 (ii) and usual properties of Poisson algebras, we obtain that

$$\mathrm{div}\big(\mathcal{Y}_J^1\big) = \frac{r_1}{4} \cdot \sum_{j \neq k \neq l \in J} \epsilon_{jkl} \cdot \varphi\left(\mathcal{L}^2, J \setminus \{j, k, l\}\right) \cdot h^{jkl},$$

where we have used the notation $\epsilon_{jkl} := \epsilon(J, j) \cdot \epsilon(J \setminus \{j\}, k) \cdot \epsilon(J \setminus \{j, k\}, l)$. Note that there holds

$$e_j \wedge e_k \wedge e_l = e_l \wedge e_j \wedge e_k = e_k \wedge e_l \wedge e_j \qquad \forall j, k, l \in J$$

from which we conclude that $\epsilon_{jkl} = \epsilon_{ljk} = \epsilon_{klj}$. Therefore, by using Poisson Jacobi identity we infer that

$$\mathrm{div}\big(\mathcal{Y}_J^1\big) = \frac{r_1}{4} \cdot \sum_{j \neq k \neq l \in J} \epsilon_{jkl} \cdot \varphi\left(\mathcal{L}^2, I \setminus \{j, k, l\}\right) \cdot h^{jkl} \equiv 0.$$

Finally, since by Propositions 3.1, 3.4 and the fact that $\vec{h}^j \cdot h^i = h^{ji}$ (see [3, (3.6)]), each h^i (with $i = 1, \dots, m$) is a first integral of $\mathcal{Y}_J^1$, we conclude the proof of (iv) by applying [5, Proposition B.2] m times. $\square$

Remark 4.3. *Suppose that $\mathcal{S}_0 = \mathcal{S}_0^1$, that is, the extra hypothesis of Theorem 1.1(iv) is satisfied, and consider the module $\mathcal{D}$ of vector-fields generated by $\mathcal{Y}_I = \mathcal{Y}_I^1$ with $I \in \Lambda_{r+1}$ and their Lie-brackets. Then $\mathcal{D}$ is involutive and generates the same singular distribution $\vec{\mathcal{F}}$. Moreover, $\mathcal{D} = Span\{\mathcal{Y}_\alpha; \alpha \in \Gamma\}$, where Γ is a countable index-set, and $\mathcal{Y}_\alpha$ is either equal to $\mathcal{Y}_I$ for some $I \in \Lambda_{r+1}$, or can be obtained from a finite number of their Lie-brackets. Note that Γ may always be chosen finite when $\mathcal{C} = \mathcal{C}^\omega$, by Noetherianity. Finally, every $\mathcal{Y}_\alpha$ has controlled divergence. Indeed, it is enough to add the following argument to Lemma 4.2 above: given two vector fields X and Y such that $div(X) = div(Y) = 0$, then $div([Y, X]) = 0$. Indeed, denoting by vol the volume form associated to the Euclidean metric g, we conclude from Cartan's formula that:*

$$div([X, Y])\, vol = d\left(i_{[X,Y]} vol\right) = \mathcal{L}_{[X,Y]} vol = \mathcal{L}_X \mathcal{L}_Y vol - \mathcal{L}_Y \mathcal{L}_X vol = 0$$

since $\mathcal{L}_X vol = div(X)\, vol = 0$ and $\mathcal{L}_Y vol = div(Y)\, vol = 0$.

Remark 4.4. *In general, the set Σ can have positive measure in $\Delta^\perp$, cf. §§2.4. If $rank(\Delta) \leq 3$, nevertheless, then the set Σ is always a rectifiable set of Lebesgue measure zero, and the rank of $\mathcal{L}^2$ is always maximal outside of Σ.*

Indeed, apart from changing the set of generators of Δ, we may suppose that $X^k = \partial_{x_k} + \sum_{i=m+1}^n A_i^k(x)\partial_{x_i}$. Now, from the non-holonomicity, there is a function $h^{ij} = [X^i, X^j] \cdot p$ whose Taylor expansion at (x,p) is non-zero when restricted to $\Delta^\perp$, for all $(x,p) \in \Delta^\perp$. This implies that at every $\mathfrak{a} \in \Delta^\perp$, there exists at least one Pfaffian of a 2×2 minor of H which is formally non-zero at $\mathfrak{a}$. Since H is at most a 3×3 anti-symmetric matrix, its rank is at most 2. We conclude from Lemma 4.5 below.

4.3 Proof of assertion (v)

The proof of Theorem 1.1(v) proceeds by transversality. If we cover M by countably many chart $\varphi_i : \mathcal{D} \to M$ where $\mathcal{D}$ is an open ball in $\mathbb{R}^n$ centered at the origin, then it is sufficient to show that the set of totally nonholonomic distributions on each $\varphi_i(\mathcal{D})$ satisfying the conclusion of Theorem 1.1(v) is generic. Moreover, any smooth distribution on $\mathcal{D}$ can be extended to $\mathbb{R}^n$ and can be generated globally by families of m smooth vector fields (see [13, 15]). So, we can assume from now on that $M = \mathbb{R}^n$ and aim to show that for generic families of linearly independent and bracket-generating vector fields $X^1, \ldots, X^m$ in $\mathbb{R}^n$, the distribution $\Delta = \mathrm{Span}\{X^1, \ldots, X^m\}$ satisfies the desired properties over $\mathbb{R}^n$.

Transversality theory. We recall here the definition of jets of vector fields in $\mathbb{R}^n$ and introduce some notations, we refer the reader to the textbooks [7, 8] for further details on Transversality Theory.

Let d a nonnegative integer be fixed, any real-valued function f smooth in a neighborhood of some $\bar{x} \in \mathbb{R}^n$ admits a Taylor expansion up to order r at $\bar{x}$, that is, it can be written as

$$f(x) \underset{\bar{x},d}{\simeq} f(\bar{x}) + \sum_{k=1}^{d} \sum_{\alpha \in I_k} \frac{1}{\alpha!} \, \partial_\alpha^k f\,(\bar{x})\,(x-\bar{x})^\alpha ,$$

where the symbol $\simeq$ with $\bar{x}, d$ below means that the function in the x variable given by the difference between the left-hand side and the right-hand side has order $> d$ at $\bar{x}$, where for each $k \in \{1, \ldots, d\}$ the set I_k denotes the set of multi-indices $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_1, \ldots, \alpha_k \in \{1, \ldots, n\}$ and $\alpha_1 \leq \ldots \leq \alpha_k$, and where for each multi-index $\alpha = (\alpha_1, \ldots, \alpha_k)$ we set

$$\partial_\alpha^k f(\bar{x}) := \frac{\partial^k f}{\partial x_\alpha}\,(\bar{x}) = \frac{\partial^k f}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_k}}\,(\bar{x}) \quad \text{and} \quad (x-\bar{x})^\alpha := \Pi_{i=1}^k \left(x_{\alpha_i} - \bar{x}_{\alpha_i}\right).$$

Denote by $|I_k|$ the cardinality of I_k for all integer $k \geq 1$. Then, the d-th Taylor expansion at $\bar{x}$ of such function f can be encoded by a tuple

$$\left(\bar{x}, f(\bar{x}), D^1 f\,(\bar{x})\,, \cdots, D^d f\,(\bar{x})\right)$$

in the set

$$J^d\left(\mathbb{R}^n, \mathbb{R}\right) := \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{|I_1|} \times \cdots \times \mathbb{R}^{|I_d|}$$

where $\bar{x}$ is the origin of the expansion and for every $k \in \{1, \ldots, d\}$, $D^k f(\bar{x})$ is the tuple in $\mathbb{R}^{|I_k|}$ given by

$$D^k f\,(\bar{x}) = \left(\partial_\alpha^k f\,(\bar{x})\right)_{\alpha \in I_k}.$$

The set $J^d(\mathbb{R}^n,\mathbb{R})$ is the set of d-jets of smooth function from $\mathbb{R}^n$ to $\mathbb{R}$. To each smooth function $f : \mathbb{R}^n \to \mathbb{R}$ can be associated a smooth function, called d-jet of f, $j^d f : \mathbb{R}^n \to J^d(\mathbb{R}^n,\mathbb{R})$ defined by

$$j^d f(x) := \Big(x, f(x), D^1 f(x), \cdots, D^d f(x)\Big) \qquad \forall x \in \mathbb{R}^n.$$

Now, in order to define the d-jets of smooth vector fields in $\mathbb{R}^n$, we can set

$$J^d\left(\mathbb{R}^n,\mathbb{R}^n\right) := \mathbb{R}^n \times \mathbb{R}^n \times \left(\mathbb{R}^{|I_1|}\right)^n \times \cdots \times \left(\mathbb{R}^{|I_d|}\right)^n$$

and define for every smooth vector field Y in $\mathbb{R}^n$ the d-jet $j^d Y : \mathbb{R}^n \to J^d(\mathbb{R}^n,\mathbb{R}^n)$ by

$$j^d Y(x) := \Big(x, Y(x), D^1 Y(x), \cdots, D^d Y(x)\Big) \qquad \forall x \in \mathbb{R}^n.$$

where each $D^l Y(x)$ has n coordinates $D^l Y_1(x), \ldots D^n Y_1(x)$. Finally, given a family $X = (X^1, \ldots, X^m)$ of smooth vector fields in $\mathbb{R}^n$, we define its d-jet $j^d X : \mathbb{R}^n \to \mathcal{J}^r$ for every $x \in \mathbb{R}^n$ by

$$j^d X(x) := \left(x, X(x), \left(D^1 X^j(x)\right)_{j=1,\ldots,m}, \cdots, \left(D^d X^j(x)\right)_{j=1,\ldots,m}\right),$$

where the set of d-jets of families of m smooth vector fields is defined by

$$\mathcal{J}^d_m := \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \left(\mathbb{R}^{|I_1|}\right)^{n\times m} \times \cdots \times \left(\mathbb{R}^{|I_d|}\right)^{n\times m}.$$

Formal Goh matrix. Set $d \geq n+2$ and fix the coordinate system $x = (x_1, \ldots, x_n)$ over $\mathbb{R}^n$ and a point $\bar{x} \in \mathbb{R}^n$. Without loss of generality, we suppose that $\bar{x} = 0$. Denote by $T : C^\infty(\mathbb{R}^n,\mathbb{R}) \to \mathbb{R}[\![x]\!]$ and $T^m : C^\infty(\mathbb{R}^n,\mathbb{R}^m) \to (\mathbb{R}[\![x]\!])^m$ the Taylor expansion mappings at $\bar{x}$. Recall that T and T^m are surjective mappings by Borel's Theorem (see *e.g.* [12, 1.5.4]). We work formally over $\bar{x}$, essentially motivated by the following observation (we recall that a subset of $\mathbb{R}^n$ is said to be smoothly countably $(n-1)$-rectifiable if it can be covered by countable many smooth submanifolds of $\mathbb{R}^n$ of codimension 1):

Lemma 4.5. *If $f \in C^\infty(\mathbb{R}^n,\mathbb{R})$ is such that $T(f) \not\equiv 0$, then there exists a neighborhood V of $\bar{x}$ such that the set $\{x \in U;\, f(x) = 0\}$ is smoothly countably $(n-1)$-rectifiable.*

Proof. By the Malgrange preparation Theorem (see *e.g.* [9, Th. 7.5.5]), which we can always apply after a linear coordinate change, there exists a neighborhood V of $\bar{x} = 0$, $d \in \mathbb{N}$ and C^∞-functions $U(x)$ and $a_k(x_1, \ldots, x_{n-1})$, $k = 0, \ldots, d-1$, where $a_k(0) = 0$ and $U(x) \neq 0$ for all $x \in V$, such that

$$f_{|V}(x) = U(x)\left(x_n^d + \sum_{k=0}^{d-1} a_k(x_1, \ldots, x_{n-1}) x_n^k\right) \qquad \forall x = (x_1, \cdots, x_n) \in V.$$

Since we are interested in the zero locus of f, we may assume without loss of generality that $U(x) = 1$. The result now follows by induction on d; the case $d = 1$ being clear, assume the result proven for $d-1$. First, by the implicit function Theorem the set $\{f = 0\} \setminus \{\partial_{x_n} f = 0\}$ is rectifiable in V. Second, the zero set of the derivative $\{\partial_{x_n} f = 0\}$ is rectifiable over V by induction. We conclude easily. $\square$

Now, we consider the fiber of the projection $\mathcal{J}^r_m \to \mathbb{R}^n$ over $\bar{x}$, which we denote by $\mathcal{J}^d_m(\bar{x})$. We note that the Taylor expansion mapping T^m (or T) commutes with the d-jet mapping, that is, if we denote by $j^d_{\bar{x}} : (\mathbb{R}[\![x]\!])^m \to \mathcal{J}^d_m(\bar{x})$ the corresponding d-jet, then we have

$$T^m\big(j^d f\big) = j^d_{\bar{x}}\big(T^m(f)\big) \qquad \forall f \in C^\infty(\mathbb{R}^n, \mathbb{R}^m).$$

Let us denote by $\mathcal{D}$ the set of formal vector fields over $\bar{x}$; note that $X \in \mathcal{D}$ means that $X = \sum_{i=1}^n A_i(x)\partial_{x_i}$ where $A_i(x) \in \mathbb{R}[\![x]\!]$. We note that the Taylor expansion T above extends as a surjective function from $\mathrm{Der}_{\mathbb{R}^n}$ to $\mathcal{D}$ which commutes with j^d. In what follows, we consider m-tuples $\widehat{X} = (X^1, \ldots, X^m) \in \mathcal{D}^m$ satisfying an extra property. In fact, we consider an open and dense set $\mathcal{U}^d_m \subset \mathcal{J}^d_m(\bar{x})$ such that, for every $\widehat{X} \in \mathcal{D}^m$ such that $j^d_{\bar{x}}(\widehat{X}) \in \mathcal{U}^d_m(\bar{x})$, we have that $X^1(\bar{x}), \ldots, X^m(\bar{x})$ are linearly independent vectors and from now on we consider m-tuples $\widehat{X}$ in the set $\mathcal{D}_{LI}$ (where LI stands for linearly independent) defined by

$$\mathcal{D}_{LI} := \big(j^d_{\bar{x}}\big)^{-1}\left(\mathcal{U}^d_m(\bar{x})\right).$$

By using the canonical coordinates (x, p) over $T^*\mathbb{R}^n$ (with the canonical projection $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$), given $\widehat{X} = (X^1, \ldots, X^m) \in \mathcal{D}_{LI}$, we consider the functions $h^1, \ldots, h^m$ in $\mathbb{R}[\![x]\!][p]$ defined by

$$h^k := p \cdot X^k, \qquad \forall k = 1, \ldots, m,$$

where we recall that $\mathbb{R}[\![x]\!][p]$ stands for polynomials in p whose coefficients are formal power series in x (in particular, each h^k is 1-homogeneous in p). We define the $m \times m$ matrix $\mathcal{L}^2_{\widehat{X}}$ over $\mathbb{R}[\![x]\!][p]$ by

$$\mathcal{L}^2_{\widehat{X}} := \big[p \cdot [X^i, X^j]\big]_{1 \le i,j \le m} = \big[h^{ij}\big]_{1 \le i,j \le m},$$

where $h^{ij} \in \mathbb{R}[\![x]\!][p]$ are formal power series, and study its rank modulo the ideal

$$\mathcal{I}_{\widehat{X}} = \mathrm{Span}\left(h^1, \ldots, h^m\right) \subset \mathbb{R}[\![x]\!][p].$$

Indeed, recall that we want to study the rank of the Goh matrix when restricted to $\Delta^\perp$. When $\widehat{X}$ is convergent, then $\Delta^\perp$ corresponds to the zero set of $\mathcal{I}_{\widehat{X}}$; but even if $\widehat{X}$ is not convergent, the ideal $\mathcal{I}_{\widehat{X}}$ is well-defined, providing us the precise algebraic counterpart of a "germ of a formal set", which is not defined in this paper. Now, studying the restriction of functions defined in the cotangent bundle to $\Delta^\perp$ corresponds to considering functions of the cotangent bundle quotient-out by $\mathcal{I}_{\widehat{X}}$, that is, over the ring $\mathbb{R}[\![x]\!][p]/\mathcal{I}_{\widehat{X}}$. Therefore, we are interested in the function $\mathcal{R} : \mathcal{D}_{LI} \to \mathbb{N}$ defined by

$$\mathcal{R}(\widehat{X}) = \mathrm{rank}_{\mathbb{R}[\![x]\!][p]/\mathcal{I}_{\widehat{X}}}\big(\mathcal{L}^2_{\widehat{X}}\big) \qquad \forall \hat{X} \in \mathcal{D}_{LI},$$

where we recall that the rank over a principal domain A is defined as the dimension of the associated mapping between $\mathrm{Frac}(A)$-vector-spaces, where $\mathrm{Frac}(A)$ is the field of fractions of A. Note that $\mathcal{R}(\widehat{X})$ is well-defined since, for $\widehat{X} \in \mathcal{D}_{LI}$, the ideal $\mathcal{I}_{\widehat{X}}$ is prime (heuristically, for $\widehat{X} \in \mathcal{D}_{LI}$, the "formal set" $\Delta^\perp$ associated to $\widehat{X}$ is irreducible; it is actually even smooth) and, therefore, the quotient $\mathbb{R}[\![x]\!][p]/\mathcal{I}_{\widehat{X}}$ is a principal domain. We prove that:

Proposition 4.6. *There exists an open dense set $\mathcal{G}(\bar{x}) \subset \mathcal{J}^d_m(\bar{x})$, whose complement is a semi-algebraic set of codimension $n+1$, such that, for every $\widehat{X} \in \mathcal{D}_{LI}$ such that $j^d_{\bar{x}}(\widehat{X}) \in \mathcal{G}(\bar{x})$, we have that $\mathcal{R}(\widehat{X})$ is maximal, that is, if m is even then $\mathcal{R}(\widehat{X}) = m$ and if m is odd then $\mathcal{R}(\widehat{X}) = m-1$.*

Reduction of Theorem 1.1(v) to Proposition 4.6. Let $\bar{x} \in M$ be fixed, and $U \subset M$ be a connected open neighborhood of $\bar{x}$ which admits a globally defined coordinate system $x = (x_1, \dots, x_n)$. Let us consider the set $\mathcal{G} = U \times \mathcal{G}(\bar{x}) \subset \mathcal{J}^d_m$, where $\mathcal{G}(\bar{x})$ is given by Proposition 4.6. Note that $\mathcal{G}$ is a semi-algebraic set of codimension $n+1$ in $\mathcal{J}^d_m$. By Thom's Transversality Theorem (see *e.g.* [8, Theorem 4.9]), the set of vector fields $X \in C^\infty(U, \mathbb{R}^n)^m$ such that $j^d X(U)$ is transverse to $\mathcal{G}$ is a residual set of $C^\infty(\mathbb{R}^n, \mathbb{R}^n)^m$ (in the smooth topology). In particular, since any $j^d X(U)$ is a smooth graph over U in $\mathcal{J}^d_m$, it has dimension n, and since $\mathcal{G}$ has codimension $n+1$, then the set of $X \in C^\infty(U, \mathbb{R}^n)^m$ for which $j^d X(\mathbb{R}^n)$ does not intersect $\mathcal{G}$ is generic in $C^\infty(U, \mathbb{R}^n)^m$. More precisely, there is an open and dense set $\mathcal{O}(U) \subset C^\infty(U, \mathbb{R}^n)^m$ for which $j^d(X) \cap \mathcal{G} = \emptyset$ for all $X \in \mathcal{O}(U)$. Since being totally nonholonomic is an open and dense property in $C^\infty(U, \mathbb{R}^n)^m$, we may as well suppose that $X = (X^1, \dots, X^m)$ generates a totally nonholonomic distribution for every X in $\mathcal{O}(U)$.

Next, we fix $X \in \mathcal{O}(U)$ and suppose that m is even; the odd case follows from a similar argument. Denoting by $\widehat{X}_x = T_x(X)$ the formal expansion of X at $x \in U$, we notice that, since X belongs to $\mathcal{O}(U)$, the rank of the operator $\mathcal{L}^2_{\widehat{X}_x}$ is maximal equal to m for any $x \in U$. Therefore, for every $\mathfrak{a} \in T^*U \cap \Delta^\perp$ the Taylor expansion of $\varphi(\mathcal{L}^2)$ at $\mathfrak{a}$ is a non-identically zero formal power series. By Lemma 4.5, we infer that the zero set of $\varphi(\mathcal{L}^2)$ is smoothly countably $(2n-m-1)$-rectifiable and we conclude by noting that this set coincides with Σ (see the proof of (iv) in Section 4.2).

Proof of Proposition 4.6. Without loss of generality, we may suppose that $\bar{x} = 0$ and $x = (x_1, \dots, x_n)$ is centered at $\bar{x}$. Consider the set $\mathcal{D}_{NF} \subset \mathcal{D}_{LI}$ (where NF stands for "normal form") of formal vector fields $\widehat{X} = (X^1, \dots, X^m)$ of the form

$$X^k = \partial_{x_k} + \sum_{i=m+1}^{n} A^k_i(x)\partial_{x_i}, \quad A^k_i(0) = 0 \qquad i = m+1, \dots, n, \quad k = 1, \dots, m,$$

where $A^k_i \in \mathbb{R}[\![x]\!]$, and denote by $\mathcal{W}^d_m$ the space of d-jets associated to them. We have

$$\mathcal{W}^d_m = \left(\mathbb{R}^{|I_1|}\right)^{(n-m)\times m} \times \cdots \times \left(\mathbb{R}^{|I_d|}\right)^{(n-m)\times m}$$

and we note that for every $\xi \in \mathcal{W}^d_m$ we may consider the unique element $\hat{X}_\xi = (X^1_\xi, \dots, X^m_\xi) \in \mathcal{D}_{NF}$ such that $j^{d+k}(\hat{X}_\xi) = \xi$ for every nonnegative integer k. We start by showing that it is enough to prove Proposition 4.6 over $\mathcal{W}^d_m$:

Lemma 4.7. *There exists a surjective map $\widehat{\psi} : \mathcal{D}_{LI} \to \mathcal{D}_{NF}$ and a surjective semi-algebraic map $\psi : \mathcal{U}^d_m \to \mathcal{W}^d_m$ such that $j^d_{\bar{x}} \circ \widehat{\psi} = \psi \circ j^d_{\bar{x}}$ and:*

(i) For every semi-algebraic set $Z \subset \mathcal{W}^d_m$ of codimension s the set $\psi^{-1}(Z) \subset \mathcal{J}^d_m(\bar{x})$ is a semi-algebraic set of codimension s.

(ii) For all $\widehat{X} \in \mathcal{D}_{LI}$, we have that $\mathcal{R}(\widehat{X}) = \mathcal{R}(\widehat{\psi}(\widehat{X}))$.

We postpone the proof to appendix B.3. The Lemma follows from standard ideas and computations: we make a linear change of coordinates and a systematic study of the changes of generators of $\mathrm{Span}(X^1, \dots, X^m)$ which are necessary to obtain the normal forms in $\mathcal{D}_{NL}$. We are now ready to prove Proposition 4.6:

Proof of Proposition 4.6. By Lemma 4.7, it is enough to prove that there exists an open dense set $\mathcal{O} \subset \mathcal{W}_m^d$ whose complement is a semi-algebraic set of codimension $n+1$, such that for every $\widehat{X} \in \mathcal{D}_{NF}$ such that $j^d(\widehat{X}) \in \mathcal{O}$, we have that $\mathcal{R}(\widehat{X})$ is maximal. We start by remarking that for every $\widehat{X} \in \mathcal{D}_{NF}$, we have

$$h^k = p_k + \sum_{i=m+1}^{n} A_i^k(x) p_i \qquad \forall k = 1, \dots, m,$$

so that the matrix $\mathcal{L}^2_{\widehat{X}}$ does not depend upon the variables $p_1, \dots, p_m$. It follows that

$$\mathcal{R}(\widehat{X}) = \mathrm{rank}_{\mathbb{R}[\![x]\!][p]/\mathcal{I}_{\widehat{X}}}\big(\mathcal{L}^2_{\widehat{X}}\big) = \mathrm{rank}_{\mathbb{R}[\![x]\!][p]}\big(\mathcal{L}^2_{\widehat{X}}\big).$$

Next, we slightly abuse notation, and we also denote by j^d the extension of the truncated mapping $\mathbb{R}[\![x]\!] \to \mathbb{R}[x]$ to $\mathbb{R}[\![x]\!][p] \to \mathbb{R}[x][p]$ (where j^d acts as the identity over p). Note that the rank of $\mathcal{L}^2_{\widehat{X}}$ can only decrease when we truncate its Taylor expansion, and that $j^{d-1}(\mathcal{L}^2_{\widehat{X}})$ only depends on $\xi = j^d(\widehat{X})$, that is,

$$\mathrm{rank}\big(\bar{\mathcal{L}}^2_{\xi}\big) \leq \mathrm{rank}\big(\mathcal{L}^2_{\widehat{X}}\big) \quad \text{with} \quad \bar{\mathcal{L}}^2_{\xi} := j^{d-1}\big(\mathcal{L}^2_{\widehat{X}_{\xi}}\big) = j^{d-1}\big(\mathcal{L}^2_{\widehat{X}}\big).$$

We now prove the existence of $\mathcal{O}$ in an inductive way. To that end, we consider the point $\mathfrak{a} = (\bar{x}, p_0) = (0, p_0)$, where $p_0 = (0, \dots, 0, 1)$. Moreover, given a sub-index $I \subset \{1, \dots, m\}$, we recall that $\varphi(\bar{\mathcal{L}}^2_{\xi}, I)$ denotes the associated Pfaffian; its evaluation at p_0, which yields a series in $\mathbb{R}[\![x]\!]$, will be denoted by $\varphi(\bar{\mathcal{L}}^2_{\xi}, I)_{|p_0}$ as all its Lie derivatives along vector fields. We are ready to state the inductive claim:

Claim 4.8. *For every even $0 < r \leq m$ and every index $I \in \Lambda_r$, there exists a semi-analytic set $B_I \subset \mathcal{W}_m^d$ of codimension $n+1$, such that $\varphi(\bar{\mathcal{L}}^2_{\xi}, I)_{|p_0} \not\equiv 0$ for $\xi \in B_I$.*

Note that the Proposition easily follows from the Claim. We therefore turn to its proof, which follows by induction on r; for $r = 0$ there is nothing to prove. Suppose the Claim proved up until $r-2$, and fix an index set $I \in \Lambda_r$. Up to re-ordering, we may suppose that $I = \{1, \dots, r\}$. Consider the mapping

$$Tr : \mathcal{W}_m^d \to \mathbb{R}^d$$

defined as

$$Tr(\xi) \cdot e_k = (X^1_{\xi})^{k-1}(\varphi(\bar{\mathcal{L}}_{\xi}, I))_{|p_0} \qquad \forall k = 1, \dots, d,$$

where $e_1, \dots, e_d$ denotes the vectors of the canonical basis in $\mathbb{R}^d$. Then, recalling that for each $\xi \in \mathcal{W}_m^d$, $X_{\xi} = (X^1_{\xi}, \dots, X^m_{\xi})$ denotes the tuple of polynomial vector fields such that $j^{d+k}(\hat{X}_{\xi}) = \xi$ for every nonnegative integer k, we may write

$$X^2_{\xi} = \partial_{x_2} + \sum_{i=m+1}^{n} A^2_i(x,\xi)\partial_{x_n}, \qquad \text{and} \qquad A^2_n(x,\xi) = \sum_{k=1}^{d} \gamma_k x_1^k + \tilde{A}(x,\xi)$$

where $\tilde{A}(x,\xi)$ is such that $A(x_1,0,\dots,0,\xi) \equiv 0$ and $\tilde{A}(x,\xi)$ is independent of the coefficients $\gamma_1,\dots,\gamma_d$. In what follows, we compute the derivatives of Tr with respect to the variables $\gamma_1,\dots,\gamma_d$ at $\gamma_1 = \dots = \gamma_d = 0$. We start by some simple observations, we have

$$(p \cdot [X_\xi^1, X_\xi^2])_{|p_0} = \gamma_1 + R_1(\xi)$$

$$\text{and} \quad (X_\xi^1)^{k-1}(p \cdot [X_\xi^1, X_\xi^2])_{|p_0} = \gamma_k + R_k(\xi) \qquad \forall k = 1,\dots,d,$$

where $R_k : \mathcal{W}_m^d \to \mathbb{R}$ is a function independent of the variables $\gamma_k,\dots,\gamma_d$. Furthermore, we have that, for every $j \in I \setminus \{2\}$

$$\varphi(\bar{\mathcal{L}}_\xi^2, I \setminus \{2,j\})_{|p_0} \text{ is independent of } \gamma_1,\dots,\gamma_d$$

and for every $j > 1$ and every $k = 1,\dots,d$

$$(X_\xi^1)^{k-1}(p \cdot [X_\xi^2, X_\xi^j])_{|p_0} \text{ is independent of } \gamma_k,\dots,\gamma_d.$$

Now, by Proposition 3.3 (i):

$$\begin{aligned}
\varphi(\bar{\mathcal{L}}_\xi^2, I)_{|p_0} &= \frac{1}{(r/2)} \sum_{j \in I \setminus \{2\}} -\epsilon(I \setminus \{2\}, j) \cdot ([X_\xi^2, X_\xi^j])_{|p_0} \cdot \varphi(\bar{\mathcal{L}}_\xi^2, I \setminus \{2,j\})_{|p_0} \\
&= \frac{1}{(r/2)} \left[\gamma_1 \cdot \varphi(\bar{\mathcal{L}}_\xi^2, I \setminus \{1,2\})_{|p_0} + S_1(\xi)\right]
\end{aligned}$$

where $S_1 : \mathcal{W}_m^d \to \mathbb{R}$ is independent of $\gamma_1,\dots,\gamma_d$. Next, by deriving $\varphi(\bar{\mathcal{L}}_\xi^2, I)$ with respect to X_ξ^1, we get:

$$(X_\xi^1[\varphi(\bar{\mathcal{L}}_\xi, I)])_{|p_0} = \frac{1}{(r/2)} \left[\gamma_2 \cdot \varphi(\bar{\mathcal{L}}_\xi, I \setminus \{1,2\})_{|p_0} + S_2(\xi)\right],$$

where $S_2 : \mathcal{W}_m^d \to \mathbb{R}$ is independent of $\gamma_2,\dots,\gamma_d$. Repeating this process we get for every $k = 1,\dots,d$,

$$((X_\xi^1)^{k-1}[\varphi(\bar{\mathcal{L}}_\xi, I)])_{|p_0} = \frac{1}{(r/2)} \left[\gamma_k \cdot \varphi(\bar{\mathcal{L}}_\xi^2, I \setminus \{1,2\})_{|p_0} + S_k(\xi)\right],$$

where $S_k : \mathcal{W}_m^d \to \mathbb{R}$ is independent of $\gamma_k,\dots,\gamma_d$. Therefore, the Jacobian of Tr in respect to the variables $\gamma_1,\dots,\gamma_d$ at the origin has a determinant equal to

$$\varphi(\bar{\mathcal{L}}_\xi^2, I \setminus \{1,2\})_{|p_0}^d$$

It follows from the induction hypothesis, that outside a semialgebraic set of codimension $n+1$, the mapping Tr is a submersion. We conclude easily. $\square$

5 Proof of Theorem 1.2

Let M and Δ be of class $\mathcal{C}$ and Δ a totally nonholonomic distribution of corank 1 and let $\vec{\mathcal{F}}$ be the integrable distribution given by Theorem 1.1 which is assumed to satisfy properties (H1)-(H2) of Theorem 1.2. From Theorem 2.1 and Remark 2.3, we infer that the distribution $\mathcal{H}_{|\mathcal{R}_0} := d\pi(\vec{\mathcal{F}}_{|\mathcal{S}_0})$ has constant rank 0 or 1 and that the singular set $\sigma := \pi(\Sigma)$ has Lebesgue measure zero in M. If $\mathcal{H}_{|\mathcal{R}_0}$ has rank 0 then, by Theorem 2.1 (iii), all (non-trivial) singular horizontal paths must be contained in σ and as a consequence for any $x \in M$, the set $\mathrm{Abn}_\Delta(x)$ is contained in σ which has Lebesgue measure 0, so the Sard Conjecture is satisfied. It remains to show that the Sard Conjecture holds true whenever $\mathcal{H}_{|\mathcal{R}_0}$ has rank 1. Our proof follows closely the proof given in [5]. We fix a smooth Riemannian metric g on M and denote by d^g its geodesic distance and by $\mathcal{H}^1$ the corresponding 1-dimensional Hausdorff measure in M, then we start with the following Lemma which can be proved in the exact same way as [5, Lemma 2.2] (we refer the reader to the discussion before [5, Lemma 2.2] for the definition of $\partial\omega_z$):

Lemma 5.1. *Assume that $\mathcal{H}_{|\mathcal{R}_0}$ has rank 1 and that there is $x \in M$ such that $Abn_\Delta(x)$ has positive Lebesgue measure. Then there is $\bar{x} \in \sigma$ such that for every neighborhood $\mathcal{V}$ of $\bar{x}$ in M, there are two closed sets S_0, S_∞ in M satisfying the following properties:*

(i) $S_0 \subset \mathcal{V}$ and S_0 has positive Lebesgue measure,

(ii) $S_\infty \subset \sigma \cap \mathcal{V}$,

(iii) for every $z \in S_0$, there is a half-orbit ω_z of the line foliation $\mathcal{H}_{|\mathcal{R}_0}$ which is contained in $\mathcal{V}$ such that $\mathcal{H}^1(\omega_z) \leq 1$ and $\partial\omega_z \in S_\infty$.

To conclude the proof of Theorem 1.2, we assume that $\mathcal{H}_{|\mathcal{R}_0}$ has rank 1, we suppose that there is $x \in M$ such that $\mathrm{Abn}_\Delta(x)$ has positive Lebesgue measure and we apply the above Lemma. By Theorem 2.1(iv), there are a relatively compact open neighborhood $\mathcal{V}$ of $\bar{x}$ in M and a set of coordinates x in $\mathcal{V}$ such that $\bar{x} = 0$ and the distribution $\mathcal{H}_{|\mathcal{V}}$ is generated by a vector field $\mathcal{Z}$ with controlled divergence. The latter property implies that, apart from shrinking $\mathcal{V}$, there exists $K > 0$ such that (c.f. [5, Lemma 2.3])

$$|\mathrm{div}_x(\mathcal{Z})| \leq K\,|\mathcal{Z}(x)| \qquad \forall x \in \mathcal{V}. \tag{5.1}$$

Now, by Lemma 5.1, there are two closed sets $S_0, S_\infty \subset \mathcal{V}$ satisfying properties (i)-(iii). Denote by φ_t the flow of $\mathcal{Z}$. For every $z \in S_0$, there is $\epsilon \in \{-1, 1\}$ such that $\omega_z = \{\varphi_{\epsilon t}(z)\,|\,t \geq 0\}$. Then, there is $\epsilon \in \{-1, 1\}$ and $S_0^\epsilon \subset S_0$ of positive Lebesgue measure such that for every $z \in S_0^\epsilon$ there holds

$$\omega_z = \Big\{\varphi_{\epsilon t}(z)\,|\,t \geq 0\Big\} \subset \mathcal{V}, \quad \mathcal{H}^1(\omega_z) \leq 1, \quad \text{and} \quad \lim_{t\to+\infty} d\left(\varphi_{\epsilon t}(z), S_\infty\right) = 0, \tag{5.2}$$

where $d(\cdot, S_\infty)$ stands for the distance function to S_∞ with respect to g. Set for every $t \geq 0$,

$$S_t := \varphi_{\epsilon t}\left(S_0^\epsilon\right)$$

and denote by vol the volume associated with the Riemannian metric g on M. Since S_∞ has volume zero (because $S_\infty \subset \sigma$ with σ of Lebesgue measure zero), by the dominated convergence Theorem, the last property in (5.2) yields

$$\lim_{t\to+\infty} \mathrm{vol}\left(S_t\right) = 0. \tag{5.3}$$

Moreover, there is $C > 0$ such that for every $z \in S_0^\epsilon$ and every $t \geq 0$, we have ($|\cdot|$ denotes the norm with respect to g)

$$\int_0^t |\mathcal{Z}(\varphi_{\epsilon s}(z))| \, ds \leq C\mathcal{H}^1(\omega_z) \leq C.$$

Therefore by Proposition [5, Prop. B2] and (5.1), we have for every $t \geq 0$

$$\begin{aligned}
\mathrm{vol}(S_t) = \mathrm{vol}\left(\varphi_{\epsilon t}(S_0^\epsilon)\right) &= \int_{S_0^\epsilon} \exp\left(\int_0^t \mathrm{div}_{\varphi_{\epsilon s}(z)}(\epsilon\,\mathcal{Z})\,ds\right) d\mathrm{vol}(z) \\
&\geq \int_{S_0^\epsilon} \exp\left(-K\int_0^t |\mathcal{Z}(\varphi_{\epsilon s}(z))| \, ds\right) d\mathrm{vol}(z) \\
&\geq e^{-KC}\,\mathrm{vol}(S_0),
\end{aligned}$$

which contradicts (5.3). The proof of Theorem 1.2 is complete.

A Proof of Theorem 2.1

Assertions (i), (ii), (iii) and (v) are easy consequences of the definitions of $\mathcal{H}$ and $\mathcal{R}_0$, and Theorem 1.1. Let us now prove (iv). Since it is enough to verify the result locally, we may assume that Δ is generated on M by m $\mathcal{C}$-vector fields $X^1, \ldots, X^m$ of the form

$$X^i = \partial_{x_i} + A_i(x)\partial_{x_n} \qquad \forall i = 1, \ldots, m = n-1$$

in such a way that (we assume that we have a local set of symplectic coordinates (x,p))

$$\Delta^\perp = \Big\{(x,p) \in T^*\mathcal{V} \,|\, p \neq 0 \text{ and } p_i + A_i(x)p_n = 0 \ \forall i = 1, \ldots, n-1\Big\}$$

and

$$\left[X^i, X^j\right] = \left(\partial_{x_i}(A_j) - \partial_{x_j}(A_i) + A_i\partial_{x_4}(A_j) - A_j\partial_{x_4}(A_i)\right)\partial_{x_n} \qquad \forall i,j = 1, \ldots, m.$$

Thus the Goh matrix (see Section 3.1) and the Pfaffians (see Section 3.2) have the form

$$H_{\mathfrak{a}} = p_n\tilde{H}(x) \quad \text{and} \quad \varphi\big(\mathcal{L}^2_{\mathfrak{a}}, I\big) = \varphi_I(x)\,p_n^{|I|} \qquad \forall \mathfrak{a} = (x,p) \in \Delta^\perp, \forall I \subset \{1, \ldots, m\}.$$

Set $\sigma = \pi(\Sigma)$ and $\mathcal{H} = d\pi(\vec{\mathcal{F}})$. Since Σ and $\vec{\mathcal{F}}$ are invariant by dilation, σ is a closed $\mathcal{C}$-set and $\mathcal{H}$ has constant rank over each connected component of $M \setminus \sigma$. Now, first consider the extra hypothesis of the Theorem, that is, that $\mathcal{H}$ has constant rank over $M \setminus \sigma$, which is equivalent to asking that $\vec{\mathcal{F}}$ has constant rank over $\mathcal{S}_0$. In this case:

$$\sigma = M \setminus \mathcal{R}_0 = \Big\{x \in M \,|\, \varphi_I(x) = 0 \ \forall I \in \Lambda_r\Big\},$$

where r stands for the rank of the Goh matrix outside of Σ. Next, by Lemma 4.1, the local generators of the distribution $\mathcal{H}$ over $\Delta^\perp \cap \mathcal{S}_0$ are of the form:

$$\mathcal{Y}_I := \sum_{i \in I} \epsilon(I,i) \cdot \varphi(\mathcal{L}^2, I \setminus \{i\}) \cdot \vec{h}^i = p_n^r \left(\sum_{i \in I} \epsilon(I,i) \cdot \varphi_{I\setminus\{i\}}(x)\,\vec{h}^i\right),$$

where $\vec{h}^i$ is a degree zero vector-field in respect to the cotangent variable p. We conclude that $\mathcal{Y}_I = p_n^r \left(\mathcal{Z}_I + \tilde{\mathcal{Z}}_I \right)$, where

$$\mathcal{Z}_I = \sum_{i \in I} \epsilon(I,i) \cdot \varphi_{I \setminus \{i\}}(x) \, X^i, \tag{A.1}$$

can be seen as a section of TM such that $d\pi(\mathcal{Y}_I) = \mathcal{Z}_I$, and $\tilde{\mathcal{Z}}_I$ belongs to the sub-module generated by ∂_{p_i}, with $i = 1, \ldots, n$; in particular $d\pi(\tilde{\mathcal{Z}}_I) = 0$. Now, given two 0-homogeneous vector-fields X^1 and X^2, denote by $X^1 = \mathcal{X}^1 + \widetilde{\mathcal{X}}^1$ the analogous decomposition, and note that

$$[X^1, X^2] = [\mathcal{X}^1, \mathcal{X}^2] + \text{ rest depending on the } \partial_{p_i} \text{ vectors.}$$

Combining this observation with Lemma 4.1, we conclude that at every point $x \in \mathcal{R}_0$, the sub-module of vector-fields generated by $\{\mathcal{Z}_I, \, I \in \Lambda_{r+1}\}$ is closed by the Lie-bracket operation. By Frobenius Theorem, and the fact that the singular locus of $\mathcal{Z}_I$ contains σ, we conclude that the singular distribution $\mathcal{F}$ generated by $\text{Span}\{\mathcal{Z}_I, \, I \in \Lambda_{r+1}\}$ is integrable. Furthermore, it is clear by the construction that $\mathcal{F}$ is regular over $\mathcal{R}_0$.

Finally, recall that $\mathcal{Y}_I$ is of controlled divergence by Lemma 4.2. Since $\vec{h}^i(x_j) = \delta_{ij}$ for $i, j = 1, \ldots, n-1$, we have that $\text{div}(\mathcal{Y}_I)$ belongs to the ideal $(\mathcal{Y}_I(x_1), \ldots, \mathcal{Y}_I(x_{n-1}))$ and $\mathcal{Y}_I(p_n)$ belongs to the ideal $p_n(\mathcal{Y}_I(x_1), \ldots, \mathcal{Y}_I(x_{n-1}))$. By using the fact that $\mathcal{Y}_I$ is homogeneous, we also have that $\text{div}(\mathcal{Y}_I)$ belongs to the ideal $p_n^r(\mathcal{Z}_I(x_1), \ldots, \mathcal{Z}_I(x_{n-1}))$ and $\mathcal{Y}_I(p_n)$ belongs to the ideal $p_n^{r+1}(\mathcal{Z}_I(x_1), \ldots, \mathcal{Z}_I(x_{n-1}))$. Now:

$$\text{div}(\mathcal{Z}_I) = \text{div}\left(\frac{1}{p_n^r} \mathcal{Y}_I \right) - \frac{\mathcal{Y}_I(p_n)}{p_n^{r+1}} \in (\mathcal{Z}_I(x_1), \ldots, \mathcal{Z}_I(x_{n-1})),$$

which proves that $\mathcal{Z}_I$ has controlled divergence. Finally, the general case (that is, when the hypothesis that $\mathcal{H}$ has constant rank along $M \setminus \sigma$ is not satisfied) follows by combining the above argument with the formalism introduced in the proof of Theorem 1.1 in order to treat each connected component of $TM \setminus \Sigma$ separately. The necessary adaptations are straightforward, and we omit the details in here.

B Proofs of auxiliary results

B.1 Proof of Proposition 3.3

Let us prove (i). By hypothesis, the cardinality of $|I|$ is $r = 2s$ for some s. Fix $i_0 \in I$, and consider the decomposition

$$A_I = v_{i_0} + B_{i_0}, \qquad \text{where } v_{i_0} = \sum_{j \in I} a_{i_0 j} \, e_{i_0} \wedge e_j,$$

(and recall the notation $a_{i_0 j} := -a_{j i_0}$ whenever $j < i_0$). It is straightforward that $v_{i_0} \wedge v_{i_0} \equiv 0$ and $\bigwedge^s B_i \equiv 0$. Therefore

$$\begin{aligned}
\frac{1}{(r/2)!} \bigwedge^{r/2} A_I &= \frac{1}{(r/2)!} v_{i_0} \wedge \bigwedge^{s-1} B_{i_0} \\
&= \frac{1}{(r/2)!} \sum_{j \in I \setminus \{i_0\}} a_{i_0 j}\, e_{i_0} \wedge e_j \wedge \bigwedge^{s-1} B_{i_0} \\
&= \frac{1}{(r/2)!} \sum_{j \in I \setminus \{i_0\}} a_{i_0 j}\, e_{i_0} \wedge e_j \wedge \bigwedge^{s-1} A_{I \setminus \{i_0, j\}} \\
&= \frac{1}{(r/2)} \sum_{j \in I \setminus \{i_0\}} a_{i_0 j} \cdot \varphi(A, I \setminus \{i_0, j\})\, e_{i_0} \wedge e_j \wedge \bigwedge_{k \in I \setminus \{i_0, j\}} e_k
\end{aligned}$$

and the formula easily follows from the definition of the function ϵ, which concludes the proof of (i).

Next, it will be convenient to establish some extra notation for determinants of non-symmetric minors of A. Given I and $J \in \Lambda_l$, we consider

$$A_{I,J} = [a_{ij}]_{i \in I, j \in J}, \quad \mathrm{Det}(A, I, J) := \det(A_{I,J})$$

and note that $A_I = A_{I,I}$. The following result about a special case of $\mathrm{Det}(A, I, J)$ is crucial for the proof of (ii):

Lemma B.1. *Let A be an anti-symmetric bilinear operator over a $\mathbb{K}$-vector space V and T be a sub-index of $\{1, \ldots, n\}$ of odd cardinality. Then, for any fixed i, $j \in T$, we have*

$$\mathrm{Det}(A, T \setminus \{i\}, T \setminus \{j\}) = \varphi(A, T \setminus \{i\}) \cdot \varphi(A, T \setminus \{j\})$$

Proof of Lemma B.1. Fix a sub-index T of cardinality $2s - 1 \leq n$ with $s > 0$. If $i = j$, the result is straightforward, so we assume that $i \neq j$. Moreover, without loss of generality we may suppose that $T = \{1, \ldots, 2s - 1\}$. Let $y = (y_1, \ldots, y_{2s-1}) \in \mathbb{R}^{2s-1}$ be fixed, we consider (tr denotes the transpose)

$$B(y) = \begin{bmatrix} A_T & y^{\mathsf{tr}} \\ -y & 0 \end{bmatrix}$$

and note that $B(y)$ is a skew-symmetric matrix. On the one hand, from the usual properties of the determinant, we have

$$\begin{aligned}
\mathrm{Det}(B(y)) &= \sum_{i,j=1}^{2s-1} (-1)^{i+j} y_i \cdot y_j \cdot \mathrm{Det}(A, T \setminus \{i\}, T \setminus \{j\}) \\
&= \sum_{i=1}^{2s-1} y_i^2 \cdot \varphi(A, T \setminus \{i\})^2 + 2 \cdot \sum_{i<j} (-1)^{i+j} \cdot y_i \cdot y_j \cdot \mathrm{Det}(A, T \setminus \{i\}, T \setminus \{j\}).
\end{aligned}$$

On the other hand, since $B(y)$ is skew-symmetric and $\mathrm{Det}(B(y))$ is quadratic homogenous with respect to y, we conclude that there exists $f_1, \ldots, f_{2s-1} \in \mathbb{K}$ such that

$$\mathrm{Det}(B(y)) = \left(\sum_{i=1}^{2s-1} y_i f_i \right)^2 .$$

Since the equality must hold for every $y \in \mathbb{R}^{2s-1}$, we conclude that

$$f_i = (-1)^i \cdot \varphi(A, T \setminus \{i\})$$

and the result easily follows. $\square$

We now turn to the proof of (ii). By the usual properties of the derivative of the determinant, we know that

$$X\left[\varphi(A, I)\right] = \frac{1}{2 \cdot \varphi(A, I)} \cdot \mathrm{tr}\left(\mathrm{Adj}(A_I) \cdot X[A_I]\right),$$

where $\mathrm{Adj}(\cdot)$ denotes the adjoint matrix and $X[A_I]$ denotes the matrix $\left[X(a_{jk})\right]_{j,k \in I}$. In particular, since the adjoint matrix is the transpose of the cofactor matrix, we have

$$\left[\mathrm{Adj}(A_I)\right]_{i,j} = \epsilon(I, i) \cdot \epsilon(I, j) \cdot \mathrm{Det}(A, I \setminus \{j\}, I \setminus \{i\}).$$

Therefore, using the fact that $a_{ii} = 0$, we have

$$X\left[\varphi(A, I)\right] = \frac{1}{2 \cdot \varphi(A, I)} \cdot \sum_{i \neq j \in I} \epsilon(I, i) \cdot \epsilon(I, j) \cdot \mathrm{Det}(A, I \setminus \{j\}, I \setminus \{i\}) \cdot X(a_{ji}). \quad \text{(B.1)}$$

Now, by (i), we have for all $i \in I$

$$\varphi(A, I) = \frac{1}{(r/2)} \epsilon(I, i) \cdot \sum_{k \in I} \epsilon(I \setminus \{i\}, k) \cdot a_{ik} \cdot \varphi(A, I \setminus \{i, k\}),$$

which, by using the definition of the determinant and Lemma B.1 with $T = I \setminus \{i\}$, for every $i \neq j \in I$, yields

$$\begin{aligned}
D(A, I \setminus \{j\}, I \setminus \{i\}) &= \epsilon(I \setminus \{j\}, i) \cdot \sum_{k \in I} \epsilon(I \setminus \{i\}, k) \cdot a_{ik} \cdot D(A, I \setminus \{i, j\}, I \setminus \{i, k\}) \\
&= \epsilon(I \setminus \{j\}, i) \cdot \varphi(A, I \setminus \{i, j\}) \cdot \sum_{k \in I} \epsilon(I \setminus \{i\}, k) \cdot a_{ik} \cdot \varphi(A, I \setminus \{i, k\}) \\
&= \epsilon(I \setminus \{j\}, i) \cdot \varphi(A, I \setminus \{i, j\}) \cdot \epsilon(I, i) \cdot (r/2) \cdot \varphi(A, I).
\end{aligned}$$

Combining this last equality with (B.1) yields:

$$X\left[\varphi(A, I)\right] = \frac{r}{4} \cdot \sum_{i \neq j \in I} \epsilon(I, j) \cdot \epsilon(I \setminus \{j\}, i) \cdot \varphi(A, I \setminus \{i, j\}) \cdot X(a_{ji})$$

and we conclude easily by interchanging i and j.

B.2 Proof of Proposition 3.4

Fix $I \in \Lambda_{r+1}$ with $r = 2s$ together with an index $l \in \{1, \dots, n\}$ and consider the anti-symmetric bi-linear operator $A_{I,l}$ over $W = \mathbb{K}^{n+1} = V \times \mathbb{K}$ defined by

$$A_{I,l} = \sum_{i<j\in I} a_{ij}\, e_i \wedge e_j + \sum_{i\in I} a_{il}\, e_i \wedge e_{n+1}$$

whose associated matrix is given by

$$M_{I,l} = \begin{bmatrix} M_I & v^t \\ -v & 0 \end{bmatrix} \qquad \text{with} \quad v = \left(a_{i_1 l}, \dots, a_{i_{r+1} l}\right).$$

We notice that $\varphi(A_{I,l}) = 0$. As a matter of fact, either $l \in I$ and the result is straightforward (the cardinality of I is odd), or $l \notin I$ and, apart from re-ordering, $M_{I,l}$ is a sub-matrix of size $r+2$ of M_A which has rank r, implying that $\det(M_{I,l}) = \varphi(A_{I,l})^2 = 0$. Then, by applying Proposition 3.3 (i) to the operator $A_{I,l}$ with $J = I \cup \{n+1\}$ and $j_0 = n+1$, we obtain

$$\begin{aligned} 0 = \varphi\left(A_{I,l}, J\right) &= \frac{1}{s+1} \sum_{j \in J\setminus\{j_0\}} \epsilon(J, j_0) \cdot \epsilon(J \setminus \{j_0\}, j) \cdot (-a_{jl}) \cdot \varphi\left(A_{I,l}, J \setminus \{j_0, j\}\right) \\ &= \frac{\epsilon(J, j_0)}{s+1} \sum_{i\in I} \epsilon(I, i) \cdot a_{li} \cdot \varphi\left(A, I \setminus \{i\}\right) \\ &= \frac{\epsilon(J, j_0)}{s+1} A_{I,l} \left(e_l, \sum_{i\in I} \epsilon(I, i) \varphi\left(A, I \setminus \{i\}\right) \cdot e_i \right) = \frac{\epsilon(J, j_0)}{s+1} A_{I,l}\left(e_l, \mathcal{Z}_I\right). \end{aligned}$$

Since the above equality is verified for all $l \in \{1, \dots, n\}$, we infer that $\{Z_I\}_{I\in\Lambda_{r+1}} \subset \ker(A)$.

Then, we notice that the dimension of $\ker(A)$ must be $n - \operatorname{rank}(A) = n - r$. In particular, there exists $J \in \Lambda_r$ such that $\varphi(A, J) \neq 0$ and, without loss of generality, we may assume $J = \{1, \dots, r\}$. Consider $I_l = J \cup \{l\}$ for every $l = r+1, \dots, n$ and note that the vectors $\{\mathcal{Z}_{I_l}\}_{l=r+1,\dots,n}$ are all linear independent. This implies that the dimension of $\{Z_I\}_{I\in\Lambda_{r+1}} \subset \ker(A)$ is at least $n - r$, concluding the result.

B.3 Proof of Lemma 4.7

The morphism ψ (and its extension $\widehat{\psi}$) will be obtained as a composition of surjective semi-algebraic morphisms satisfying property (i) and (ii):

$$\varphi : \mathcal{U}^d_m \to \mathcal{V}^d_m(1), \quad \Phi_j : \mathcal{V}^d_m(j) \to \mathcal{Z}^d_m(j), \quad \Psi_j : \mathcal{Z}^d_m(j) \to \mathcal{V}^d_m(j+1),$$

for $j = 1, \dots, m$, where $\mathcal{V}^d_m(m+1) = \mathcal{W}^d_m$ and $\psi = \Psi_m \circ \Phi_m \circ \cdots \circ \Psi_1 \circ \Phi_1 \circ \varphi$. In what follows, we introduce each one of these morphisms in the level of formal power series (we will denote them by $\widehat{\varphi}$, $\widehat{\Phi}_j$ and $\widehat{\Psi}_j$), and we will then show the properties of their restriction to jets. Let us start by defining the source and targets of each morphism:

The set $\mathcal{V}^d_m(j)$: It is the d-jets at $\bar{x}$ of vector-fields $\{X^1, \dots, X^m\}$ of the form:

$$X^k = \partial_{x_k} + \sum_{i=1}^{m} A^k_i(x) \partial_{x_i}, \quad A^k_i(0) = 0,\ i = 1, \dots, n,\ k = 1, \dots, m,$$

such that $A_i^k(x) \equiv 0$ for $i = 1, \dots, j-1$ and $k = 1, \dots, m$. Note that

$$\mathcal{V}_m^d(j) = \left(\mathbb{R}^{|I_1|}\right)^{(n-j+1)\times m} \times \cdots \times \left(\mathbb{R}^{|I_r|}\right)^{(n-j+1)\times m}.$$

The set $\mathcal{Z}_m^d(j)$*:* It is the d-jets at $\bar{x}$ of vector-fields $\{X^1, \dots, X^m\}$ of the form:

$$X^k = \partial_{x_k} + \sum_{i=1}^{n} A_i^k(x)\partial_{x_i}, \quad A_i^k(0) = 0,\ i = 1, \dots, n,\ k = 1, \dots, m,$$

such that $A_i^k \equiv 0$ for $i = 1, \dots, j-1$ and $A_j^j \equiv 0$. Note that

$$\mathcal{Z}_m^d(j) = \left(\mathbb{R}^{|I_1|}\right)^{(n-j+1)\times m-1} \times \cdots \times \left(\mathbb{R}^{|I_r|}\right)^{(n-j+1)\times m-1}.$$

Let us now explicitly define the morphisms φ, Ψ_j and Φ_j:

Morphism $\widehat{\varphi}$*:* First, fix $\widehat{X} = (X^1, \dots, X^m) \in \mathcal{D}_{LI}$. By hypothesis, there exists a linear change of coordinates $\rho : \mathbb{R}^n \to \mathbb{R}^n$, which only depends on the values of $(X^1(\bar{x}), \dots, X^m(\bar{x}))$ such that $(\rho^* X^1, \dots, \rho^* X^m) = (Y^1, \dots, Y^m)$ are such that $Y^j(\bar{x}) = \partial_{x_j}$ for $j = 1, \dots, m$.

We claim that ρ may be chosen in such a way that it is semi-algebraic in respect to $(X^1(\bar{x}), \dots, X^m(\bar{x}))$. In fact, if $m = n$, then the claim is trivial since ρ is chosen canonically; in the general case, ρ may be chosen in different ways, depending on how one completes the list of vectors $(X^1(\bar{x}), \dots, X^m(\bar{x}))$ in order to form a local basis of $T_{\bar{x}}\mathbb{R}^n$. We may always find a semi-algebraic stratification the space of parameters $\mathcal{U}_m^0$ and a locally defined coordinate system so that, in each strata, the choice of ρ becomes canonical (for example, via a choice of ordering of coordinates in $\mathbb{R}^n$, that is, we chose to complete it with $(e_1, \dots, e_{n-m})$ first; if not possible, by $(e_1, \dots, e_{n-m-1}, e_{n-m+1})$, etc). We may now define:

$$\widehat{\varphi}(X^1, \dots, X^m) = (\rho^* X^1, \dots, \rho^* X^m).$$

and φ is semi-algebraic by construction. Property (ii) is immediate; we now argue in a fiber-wise way that property (i) is satisfied. We fiber $\mathcal{U}_m^d$ via the parameters $\sigma = (X^1(0), \dots, X^m(0))$; denote by F_σ one of these fibers. Note that ρ is constant along this fiber. It is, furthermore, invertible, implying that $\varphi_{|F_\sigma}$ is a linear bijection, implying (i).

Morphism $\widehat{\Phi}_j$*:* is defined by:

$$\widehat{\Phi}_1(X^1, \dots, X^m) = \left(X^1, \dots, X^{j-1}, U_j(x)X^j, X^{j+1}, \dots, X^m\right)$$

where $U_j(x) = 1/(1 + A_j^j(x))$. Note that Φ_j is clearly surjective and semi-algebraic (in fact, it is polynomial). Property (ii) easily follows from the fact that:

$$[U_j X^j, X^k] \cdot p \mod \mathcal{I}_{\widehat{X}} = U_j h^{j,k} \mod \mathcal{I}_{\widehat{X}},$$

for all $k = 1, \dots, m$. We now argue in a fiber-wise way that property (i) is satisfied. Fiber $\mathcal{V}_m^d(j)$ via the parameters $\lambda = (A_j^j)$; denote by F_λ one of these fibers. Note that the unit $U_j(x)$ is constant along each one of the fibers F_λ, implying that $(\Phi_j)_{|F_\lambda}$ is a linear mapping. Furthermore, dividing by $U_j(x)$ would provide an inverse for $(\Phi_j)_{|F_\lambda}$, implying that $(\Phi_j)_{|F_\lambda}$ is a linear bijection, implying (i).

Morphism $\widehat{\Psi}_j$*:* is defined by:

$$\widehat{\Psi}_j(X^1, \ldots, X^m) = \big(X^1 - A^1_j X^j, \ldots, X^{j-1} - A^{j-1}_j X^j, X^j,\\ X^{j+1} - A^{j+1}_j X^j, \ldots, X^m - A^m_j X^j\big).$$

Note that Ψ_j is clearly surjective and semi-algebraic (in fact, it is polynomial). In order to prove property (ii) note that:

$$[X^k - A^k_j X^j, X^l - A^l_j X^j] \cdot p \mod \mathcal{I}_{\widehat{X}} = h^{kl} - A^l_j h^{kj} - A^k_j h^{jl} \mod \mathcal{I}_{\widehat{X}},$$

for all $k, l = 1, \ldots, m$. In particular, $\mathcal{L}^2_{\widehat{\Psi}_j(\widehat{X})}$ can be obtained from $\mathcal{L}^2_{\widehat{X}}$ by the following operation: we subtract to the k-line of $\mathcal{L}^2_{\widehat{X}}$ its j-line times A^k_j, and we do the symmetric operation for columns. This operation does not change the rank of the matrix, implying property (ii). We now argue in a fiber-wise way that property (i) is satisfied. Fiber $\mathcal{Z}^d_m(j)$ via the parameters $\lambda = (A^2_1(x), \ldots, A^m_1(x))$; denote by F_λ one of these fibers. Note that $(\Psi_j)_{|F_\lambda}$ is a linear mapping which admits an inverse, implying that $(\Psi_j)_{|F_\lambda}$ is a linear bijection, implying (i).

References

[1] A. Agrachev, A. Barilari and U. Boscain. *A comprehensive introduction to sub-Riemannian geometry. From the Hamiltonian viewpoint. With an appendix by Igor Zelenko.* Cambridge Studies in Advanced Mathematics, 181. Cambridge University Press, Cambridge, 2020.

[2] A. Belotto da Silva, A. Figalli, A. Parusiński and L. Rifford. Strong Sard Conjecture and regularity of singular minimizing geodesics for analytic sub-Riemannian structures in dimension 3. Inventiones Mathematicae, 229, pages 395–448 (2022).

[3] A. Belotto da Silva, A. Parusiński and L. Rifford. Abnormal subanalytic distributions and minimal rank Sard Conjecture, hal-03741318, [matj.DG], 2022.

[4] A. Belotto da Silva, A. Parusiński and L. Rifford. Generic abnormal distributions. In preparation.

[5] A. Belotto da Silva and L. Rifford. The sub-Riemannian Sard conjecture on Martinet surfaces. *Duke Math. J.* 167 (2018), no. 8, 1433–1471.

[6] Y. Chitour, F. Jean and E. Trélat. Genericity results for singular curves. *J. Diff. Geom.*, 73 (2006), no. 1, 45–73.

[7] Y. Eliashberg and N. Mishachev. *Introduction to the h-principle.* Graduate Studies in Mathematics, 48. American Mathematical Society, Providence, RI, 2002.

[8] M. Golubitsky and V. Guillemin. *Stable mappings and their singularities.* Graduate Texts in Mathematics, Vol. 14. Springer-Verlag, New York-Heidelberg, 1973.

[9] L. Hörmander., The analysis of linear partial differential operators I, Grundl. Math. Wissenschaft., vol. 256, Springer, second edition (1990).

[10] S. Lavau, A short guide through integration theorems of generalized distributions. Differ. Geom. Appl. (2018) 61:42-58.

[11] R. Montgomery. *A tour of sub-Riemannian geometries, their geodesics and applications.* Mathematical Surveys and Monographs, Vol. 91. American Mathematical Society, Providence, RI, 2002.

[12] R. Narasimhan. *Analysis on real and complex manifolds.* Advanced Studies in Pure Mathematics, Vol. 1, Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[13] L. Rifford. *Sub-Riemannian Geometry and Optimal Transport.* Springer Briefs in Mathematics, Springer, New York, 2014.

[14] A. Seidenberg, *Constructions in Algebra*, Trans. Amer. Math. Soc. 197 (1974), 273-313.

[15] H.J. Sussmann. Smooth distributions are globally finitely spanned. In Analysis and design of nonlinear control systems, Springer, Berlin, 3–8, 2008.

[16] S. Winitzki. *Linear Algebra via Exterior Products.* Published by lulu.com, ISBN 978-1-4092-9496-2, (2010).