Franks' Lemma for Mañé perturbations of Riemannian metrics and applications

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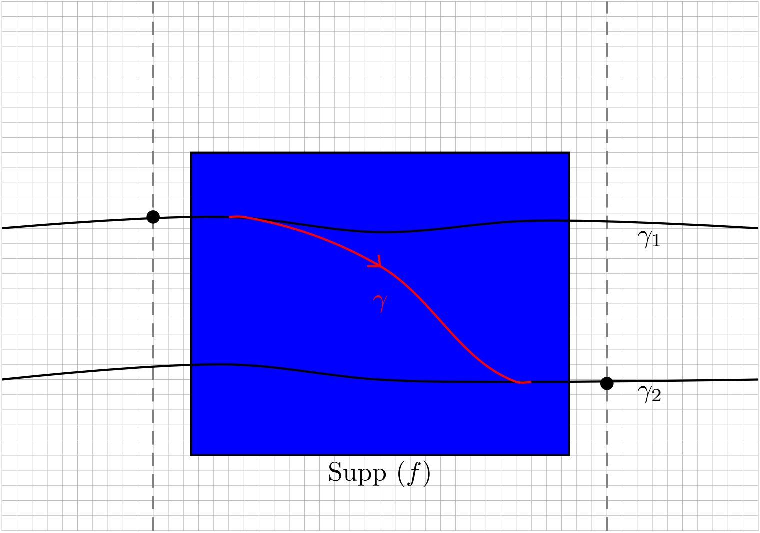
where $f : M \to \mathbb{R}$ is a smooth function.

Remark

If f is close to 0 in C^k topology then the geodesic flow of $\tilde{g} = e^f g$ is close the geodesic flow of g in C^{k-1} topology.

Connecting geodesics



A constructive method I

First define the connecting trajectory by

$$\bar{\gamma}(t) = \alpha(t)\,\gamma_1(t) + (1 - \alpha(t))\,\gamma_2(t).$$

and reparametrize it by arc-length w.r.t. the initial metric g to get a new parametrized curve γ.

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$$H(x,p) = \frac{1}{2}\|p\|_x^2 \quad \text{and} \quad \tilde{H}(x,p) = \frac{e^{-f(x)}}{2}\|p\|_x^2,$$

we would like to construct a real function f satisfying

$$\begin{cases} \dot{\gamma} & = & \frac{\partial \tilde{H}}{\partial p} = e^{-f(\gamma)}\frac{\partial H}{\partial p}(\gamma,p) \\ \dot{p} & = & -\frac{\partial \tilde{H}}{\partial x} = -e^{-f(\gamma)}\frac{\partial H}{\partial x}(\gamma,p) - \tilde{H}(\gamma,p) d_\gamma f. \end{cases}$$

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along γ. This can be done if we force $f = 0$ along γ.

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As a matter of fact, if we impose $f = 0$ along γ then we need

$$\begin{cases} \dot{\gamma}(t) &= \frac{\partial H}{\partial p}(\gamma(t), p(t)) \\ \dot{p}(t) &= -\frac{\partial H}{\partial x}(\gamma(t), p(t)) - \frac{1}{2} d_{\gamma(t)} f. \end{cases}$$

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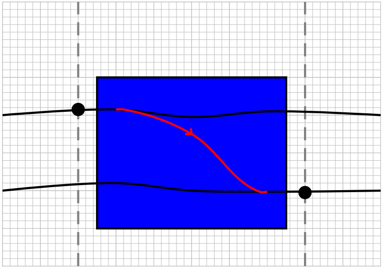

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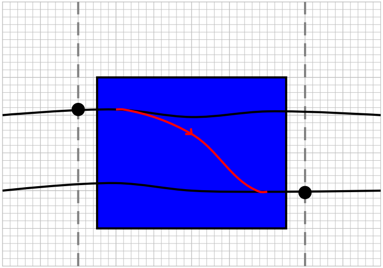

⇝ Closing Lemma for geodesic flows in low topology

The control approach

Define the mapping

$$E : C^\infty([0,\tau],\mathbb{R}^n) \longrightarrow \mathbb{R}^n \times (\mathbb{R}^n)^* \\ u \longmapsto \big(x_u(\tau), p_u(\tau)\big)$$

where $\big(x_u, p_u\big) : [0,\tau] \longrightarrow \mathbb{R}^n \times (\mathbb{R}^n)^*$ is the solution of

$$\begin{cases} \dot{x}(t) = \frac{\partial H}{\partial p}(x(t),p(t)) \\ \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t),p(t)) - u(t), \end{cases}$$

starting at $\big(x_1(0), p_1(0)\big)$.

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starting at $\big(x_1(0), p_1(0)\big)$.

$\rightsquigarrow$ If E is open at $u \equiv 0$ then we can connect γ_1 to the geodesics which are sufficiently close to γ_1.

The Franks' Lemma

Let $\varphi : M \to M$ be a C^1 diffeomorphism, consider a finite set of points $S = \{x_1, \ldots, x_m\}$ and set

$$\Pi = \oplus_{i=1}^m T_{x_i} M, \quad \Pi' = \oplus_{i=1}^m T_{\varphi(x_i)} M.$$

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Lemma (Franks, 1971)

There is $\bar{\epsilon} > 0$ such that for every $\epsilon \in (0, \bar{\epsilon})$, there is $\delta = \delta(\epsilon) > 0$ such that for any isomorphism

$$L = (L_1, \ldots, L_m) : \Pi \to \Pi' \text{ s.t. } \|L_i - D_{x_i}\varphi\| < \delta \ \forall i,$$

there exists a C^1 diffeomorphism $\psi : M \to M$ satisfying

(1) $\psi(x_i) = \varphi(x_i) \, \forall i$,

(2) $D_{x_i}\psi = L_i \, \forall i$,

(3) $\|g - f\|_{C^1} < \epsilon$.

Franks' Lemma for geodesic flows I

Given $\theta_0 = (x, v) \in UM$ and $T > 0$, we consider the unit speed geodesic $\gamma_{\theta_0} : [0, T] \to M$ starting at x with initial velocity v and we set $\theta_1 := (\gamma_{\theta_0}(T), \dot{\gamma}_{\theta_0}(T))$. Then denoting by N_0, N_1 the hyperplanes in $T_{\theta_0}UM, T_{\theta_1}UM$ which are orthogonal to the flow at θ_0, θ_1, we consider the (local) **Poincaré mapping** from Σ_0 (tangent to N_0 at θ_0) to Σ_1 (tangent to N_1 at θ_1).

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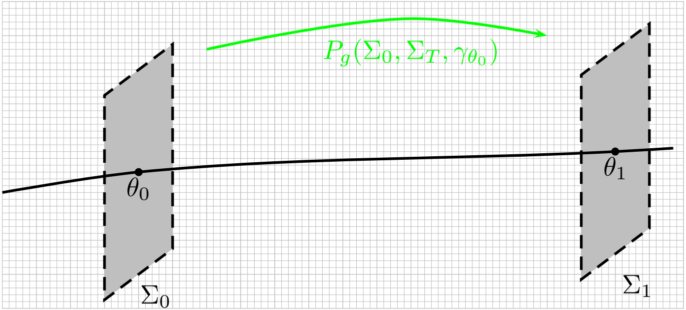

Franks' Lemma for geodesic flows II

Let $\mathrm{Sp}(m)$ be the symplectic group in $M_{2m}(\mathbb{R})$ ($m = n-1$), that is the smooth submanifold of matrices $X \in M_{2m}(\mathbb{R})$ satisfying

$$X^* \mathbb{J} X = \mathbb{J} \quad \text{where } \mathbb{J} := \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}.$$

Choosing a convenient set of coordinates, the differential of the Poincaré mapping $P := P_g(\Sigma_0, \Sigma_T, \gamma_{\theta_0})$ at θ_0 is the symplectic matrix $X(T)$ where $X : [0, T] \to \mathrm{Sp}(m)$ is solution to the Cauchy problem

$$\begin{cases} \dot{X}(t) = A(t)X(t) & \forall t \in [0, T], \\ X(0) = I_{2m}, \end{cases}$$

where $A(t)$ has the form

$$A(t) = \begin{pmatrix} 0 & I_m \\ -K(t) & 0 \end{pmatrix} \qquad \forall t \in [0, T].$$

Franks' Lemma for geodesic flows III

Problem:

Given $\epsilon > 0$,
does the set of differentials of Poincaré maps $P_{\tilde{g}}(\Sigma_0, \Sigma_T, \gamma_{\theta_0})$ at θ_0 associated with smooth conformal factors $f : M \to \mathbb{R}$ such that

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What's the radius of that ball in term of ϵ ?

Perturbation of the differential of P

Set $\gamma := \gamma_{\theta_0}$. We are looking for a smooth function

$$f : M \longrightarrow \mathbb{R}$$

satisfying the following properties

$$f\big(\gamma(t)\big) = 0 \quad \text{and} \quad d_{\gamma(t)} f = 0 \qquad \forall t \in [0, T],$$

with

$$d^2 V\big(\gamma(t)\big) \quad \text{free.}$$

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$$\Longrightarrow \quad d^2 V\big(\gamma(t)\big) \quad \text{is the control.}$$

A controllability problem on Sp(m)

The differential of the Poincaré map $P_{\tilde{g}}(\Sigma_0, \Sigma_T, \gamma_{\theta_0})$ at θ_0 associated with the metric $\tilde{g} = e^f g$ is given by $X_u(T)$ where $X_u : [0, T] \to \mathrm{Sp}(m)$ is solution to the control problem

$$\begin{cases} \dot{X}_u(t) = A(t)X_u(t) + \sum_{i\leq j=1}^m u_{ij}(t)\mathcal{E}(ij)X_u(t), \quad \forall t \in [0, T], \\ X(0) = I_{2m}, \end{cases}$$

where the $2m \times 2m$ matrices $\mathcal{E}(ij)$ are defined by

$$\mathcal{E}(ij) := \begin{pmatrix} 0 & 0 \\ E(ij) & 0 \end{pmatrix},$$

with
$$\begin{cases} \left(E(ii)\right)_{k,l} := \delta_{ik}\delta_{il} \, \forall i = 1, \ldots, m, \\ \left(E(ij)\right)_{k,l} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \, \forall i < j = 1, \ldots, m. \end{cases}$$

Local controllability on Sp(m)

We are considering a bilinear control system on $M_{2m}(\mathbb{R})$ of the form

$$\dot{X}(t) = A(t)X(t) + \sum_{i=1}^{k} u_i(t)B_iX(t) \qquad \forall t \in [0, T].$$

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Moreover, if we assume that $A(t), B_1, \ldots, B_k$ satisfy

$$\mathbb{J}A(t), \mathbb{J}B_1, \ldots, \mathbb{J}B_k \in \mathcal{S}(2m) \qquad \forall t \in [0, T],$$

then any solution $X : [0, T] \to M_{2m}(\mathbb{R})$ starting at $\bar{X} \in \mathrm{Sp}(m)$ satisfies

$$X(t) \in \mathrm{Sp}(m) \qquad \forall t \in [0, T].$$

Local controllability on Sp(m)

Proposition

Define the k sequences of smooth mappings

$$\{B_1^j\}, \ldots, \{B_k^j\} : [0, T] \to T_{I_{2m}} Sp(m)$$

by $$\begin{cases} B_i^0(t) := B_i \\ B_i^j(t) := \dot{B}_i^{j-1}(t) + B_i^{j-1}(t)A(t) - A(t)B_i^{j-1}(t), \end{cases}$$

for every $t \in [0, T]$ and every $i \in \{1, \ldots, k\}$. Assume that there exists some $\bar{t} \in [0, T]$ such that

$$Span\Big\{ B_i^j(\bar{t}) \,|\, i \in \{1, \ldots, k\}, j \in \mathbb{N} \Big\} = T_{I_{2m}} Sp(m).$$

Then for every $\bar{X} \in Sp(m)$, the control system is controllable at first order around $\bar{u} \equiv 0$.

Local controllability on $\mathrm{Sp}(m)$

Sketch of proof.

Let $\bar{X} \in \mathrm{Sp}(m)$ be fixed, we define the mapping $E : L^2([0,T],\mathbb{R}^k) \rightarrow M_{2m}(\mathbb{R})$ by

$$E(u) := X_u(T) \qquad \forall u \in L^2([0,T],\mathbb{R}^k),$$

where X_u is the solution to the control system starting at $\bar{X}$.

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where X_u is the solution to the control system starting at $\bar{X}$. If E is not a submersion at $\bar{u} \equiv 0$, then there is a nonzero matrix Y such that $X_0(T)\mathbb{J}Y \in \mathcal{S}(2m)$ and

$$\mathrm{Tr}\left(Y^* D_0 E(v)\right) = 0 \qquad \forall v \in L^2([0, T], \mathbb{R}^k).$$

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$$\mathrm{Tr}\left(Y^* D_0 E(v)\right) = 0 \qquad \forall v \in L^2([0,T],\mathbb{R}^k).$$

The latter can be written as (with $\dot{S} = AS, S(0) = I_{2m}$)

$$\sum_{i=1}^{k} \int_0^T v_i(t)\mathrm{Tr}\left(Y^* S(T) S(t)^{-1} B_i X_0(t)\right)\, dt = 0 \quad \forall v.$$

Back to Franks' lemmas

In our case, we have

$$\begin{cases} \dot{X}_u(t) = A(t)X_u(t) + \sum_{i\leq j=1}^{m} u_{ij}(t)\mathcal{E}(ij)X_u(t), & \forall t \in [0,T], \\ X(0) = I_{2m}, \end{cases}$$

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Corollary (Contreras-Paternain, Contreras, Visscher, Lazrag)

Assume that there is $\bar{t} \in [0,T]$ such that the $m \times m$ symmetric matrix K has simple eigenvalues, then the Franks' Lemma for Mané perturbations holds at first order.

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What happens if the algebraic condition on K is not satisfied ?

Local controllability on Sp(m)

Proposition

Assume that $B_i B_j = 0$ for all i, j and define the k sequences of smooth mappings $\{B_1^j\}, \ldots, \{B_k^j\} : [0, T] \to T_{I_{2m}} Sp(m)$ as before. If the following properties are satisfied with $\bar{t} = 0$:

$$\left[B_i^j(\bar{t}), B_i\right] \in Span\left\{B_r^s(\bar{t}) \,|\, r = 1, .., k,\ s \geq 0\right\} \quad \forall i,\ \forall j = 1, 2,$$

and

$$Span\left\{B_i^j(\bar{t}), [B_i^1(\bar{t}), B_l^1(\bar{t})] \,|\, i, l = 1, .., k \text{ and } j = 0, 1, 2\right\} = T_{I_{2m}} Sp(m).$$

Then, for every $\bar{X} \in Sp(m)$, the control system is controllable at second order around $\bar{u} \equiv 0$.

Local controllability on Sp(m)

If $Q : \mathcal{U} \to \mathbb{R}$ is a quadratic form, its negative index is defined by

$$\mathrm{ind}_-(Q) := \max\Big\{\dim(L) \mid Q_{|L\setminus\{0\}} < 0\Big\}.$$

Theorem

Let $F : \mathcal{U} \to \mathbb{R}^N$ be a mapping of class C^2 on an open set $\mathcal{U} \subset X$ and $\bar{u} \in \mathcal{U}$ be a critical point of F of corank r. If

$$ind_-\left(\lambda^*\left(D^2_{\bar{u}}F\right)_{|Ker(D_{\bar{u}}F)}\right) \geq r \qquad \forall \lambda \in \left(Im\left(D_{\bar{u}}F\right)\right)^\perp \setminus \{0\},$$

then the mapping F is locally open at second order at $\bar{u}$.

Applications of Franks' Lemma

Theorem

Let (M, g) be a smooth compact Riemannian manifold of dimension ≥ 2 such that the periodic orbits of the geodesic flow are C^2-persistently hyperbolic from Mañé's viewpoint. Then the closure of the set of periodic orbits of the geodesic flow is a hyperbolic set.

Applications of Franks' Lemma

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Corollary

Let (M, g) be a smooth compact Riemannian manifold, suppose that either M is a surface or $\dim M \geq 3$ and (M, g) has no conjugate points. Assume that the geodesic flow is C^2 persistently expansive from Mañé's viewpoint, then the geodesic flow is Anosov.

The End

Thank you for your attention !!