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REFINEMENT OF THE BENOIST THEOREM ON THE SIZE OF DINI SUBDIFFERENTIALS

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Abstract. Given a lower semicontinuous function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, we prove that the set of points of $\mathbb{R}^n$ where the lower Dini subdifferential has convex dimension k is countably $(n-k)$-rectifiable. In this way, we extend a theorem of Benoist(see [1, Theorem 3.3]), and as a corollary we obtain a classical result concerning the singular set of locally semiconcave functions.

1. **Introduction.** Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be any lower semicontinuous function, the lower Dini subdifferential of f at x in the domain of f (denoted by $\mathrm{dom}(f)$) is defined by

$$\partial^- f(x) = \left\{ \zeta \in \mathbb{R}^n \mid \liminf_{y \to x} \frac{f(y) - f(x) - \langle \zeta, y - x \rangle}{\|y - x\|} \geq 0 \right\}.$$

As it is well-known, for every $x \in \mathrm{dom}(f)$, the set $\partial^- f(x)$ is a possibly empty convex subset of $\mathbb{R}^n$. Now let $k \in \{1, \cdots, n\}$ be fixed; we call k-dimensional Dini singular set of f, denoted by $\mathcal{D}^k(f)$, the set of $x \in \mathrm{dom}(f)$ such that $\partial^- f(x)$ is a nonempty convex set of dimension k. Moreover, we call Dini singular set of f, the set defined by

$$\mathcal{D}(f) := \bigcup_{k \in \{1, \cdots, n\}} \mathcal{D}^k(f).$$

Before stating our result, we recall that, given $r \in \{0, 1, \cdots, n\}$, the set $C \subset \mathbb{R}^n$ is called a r-rectifiable set if there exists a Lipschitz continuous function $\phi : \mathbb{R}^r \to \mathbb{R}^n$ such that $C \subset \phi(\mathbb{R}^r)$. In addition, C is called countably r-rectifiable if it is the union of a countable family of r-rectifiable sets. The aim of the present short note is to extend a result by Benoist, who proved that $\mathcal{D}(f)$ is countably $(n-1)$-rectifiable (see [1, Theorem 3.3]), and to obtain as a corollary a classical result on locally semiconcave functions. We prove the following result.

Theorem 1.1. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Then for every $k \in \{1, \cdots, n\}$, the set $\mathcal{D}^k(f)$ is countably $(n-k)$-rectifiable.*

Let us now recall briefly the notions of semiconcave and locally semiconcave functions; we refer the reader to the book [2] for further details on semiconcavity (see also [4]). Let Ω be an open and convex subset of $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}$ be a continuous

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function, and C be a nonnegative constant. We say that u is C-semiconcave or semiconcave on Ω if

$$\mu u(y) + (1-\mu)u(x) - u(\mu x + (1-\mu)y) \leq \frac{\mu(1-\mu)C}{2}|x-y|^2, \tag{1}$$

for any $\mu \in [0,1]$, and any $x, y \in \mathbb{R}^n$. Consider now an open subset Ω of $\mathbb{R}^n$; the function $u : \Omega \to \mathbb{R}$ is called locally semiconcave on Ω, if for every $x \in \Omega$, there is an open and convex neighborhood of x where u is semiconcave. For every $k \in \{1, \cdots, n\}$, we call k-dimensional singular set of u, denoted by $\Sigma^k(u)$, the set of $x \in \Omega$ such that the Clarke generalized gradient of u at x, denoted by $\partial u(x)$, is a convex set of dimension k (see [2, 3]). In fact, it is easy to deduce from (1), that for any locally semiconcave function $u : \Omega \to \mathbb{R}$ on an open subset Ω of $\mathbb{R}^n$, the sets $\partial u(x)$ and $(-\partial^- u(x))$ coincide at any $x \in \Omega$ (see [2, Theorem 3.3.6 p. 59]). This implies that $\Sigma^k(u) = \mathcal{D}^k(-u)$ for every $k \in \{1, \cdots, n\}$ and yields the following re sult.

Corollary 1. *Let Ω be an open subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a locally semiconcave function. Then for every $k \in \{1, \cdots, n\}$, the set $\Sigma^k(u)$ is countably $(n-k)$-rectifiable.*

Our proofs combine techniques developed by Benoist in [1] and Cannarsa, Sinestrari in [2].

Notations: Throughout this paper, we denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively, the Euclidean scalar product and norm in $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and any $r > 0$, we set $B(x,r) := \{y \in \mathbb{R}^n \mid |y-x| < r\}$ and $\bar{B}(x,r) := \{y \in \mathbb{R}^n \mid |y-x| \leq r\}$. Finally, we use the abbreviations $B_r := B(0,r)$, $\bar{B}_r := \bar{B}(0,r)$, $B := B_1$, and $\bar{B} := \bar{B}_1$.

2. **Preliminary results.** Let $k \in \{1, \cdots, n-1\}$, we call k-planes the k-dimensional subspaces of $\mathbb{R}^n$. Given a k-plane Π, we denote by $\Pi^\perp$ its orthogonal complement in $\mathbb{R}^n$. Given $x \in \mathbb{R}^n$, we denote by $p_\Pi(x)$ and $p_{\Pi^\perp}(x)$ the orthogonal projections of x onto Π and $\Pi^\perp$ respectively. If Π, Π' are two given k-planes, we set

$$d(\Pi, \Pi') := \|p_\Pi - p_{\Pi'}\|,$$

where $\|\cdot\|$ is the operator norm of a linear operator in $\mathbb{R}^n$. We notice that the set of k-planes, denoted by $\mathcal{P}^k$, equipped with the distance d, is a compact metric space. Hence it admits a dense countable family $\{\Pi_i^k\}_{i\geq 1}$. In the sequel, we denote by $B_d^k(\Pi, \epsilon)$ the set of $\Pi' \in \mathcal{P}^k$ such that $d(\Pi, \Pi') \leq \epsilon$.

Given a compact set $K \subset \mathbb{R}^n$, we recall that the support function σ_K of K is defined by

$$\forall h \in \mathbb{R}^n, \quad \sigma_K(h) := \max\{\langle w, h\rangle \mid w \in K\}.$$

We notice that if $\mathrm{conv}(K)$ denotes the convex hull of K, then we have

$$\sigma_{\mathrm{conv}(K)} = \sigma_K.$$

Moreover if K, K' are two compact sets such that $K \subset K'$, then $\sigma_K \leq \sigma_{K'}$.

Given a k-plane Π, we define the function $\bar{\sigma}_\Pi : \mathbb{R}^n \to \mathbb{R}$ by

$$\forall h \in \mathbb{R}^n, \quad \bar{\sigma}_\Pi(h) := \max\left\{\langle w, h\rangle \mid w \in \Pi \cap \bar{B}\right\}.$$

The following result is useful for the proof of our theorem.

Lemma 2.1. *Let Π, Π' be two k-planes and $h \in \mathbb{R}^n$, then we have*

$$|\bar{\sigma}_\Pi(h) - \bar{\sigma}_{\Pi'}(h)| \leq d(\Pi, \Pi')|h|. \tag{2}$$

Proof. There is $w \in \Pi \cap \bar{B}$ such that $\bar{\sigma}_{\Pi}(h) = \langle w, h \rangle$. Set

$$d := |p_{\Pi'}(w)|\,.$$

Notice, that since $w \in \bar{B}$, we have necessarily $d \leq 1$, which means that $p_{\Pi'}(w)$ belongs to $\Pi' \cap \bar{B}$. Hence we have

$$\begin{array}{rcl} \bar{\sigma}_{\Pi'}(h) & \geq & \langle p_{\Pi'}(w), h \rangle \\ & = & \langle p_{\Pi'}(w) - p_{\Pi}(w), h \rangle + \langle w, h \rangle \\ & \geq & -\|p_{\Pi'}(w) - p_{\Pi}(w)\|\,|h| + \bar{\sigma}_{\Pi}(h) \\ & \geq & -\|p_{\Pi'} - p_{\Pi}\|\,|w||h| + \bar{\sigma}_{\Pi}(h) \\ & \geq & -d(\Pi, \Pi')|h| + \bar{\sigma}_{\Pi}(h). \end{array}$$

We deduce that $\bar{\sigma}_{\Pi'}(h) - \bar{\sigma}_{\Pi}(h) \geq -d(\Pi, \Pi')\|h\|$. By symmetry, we obtain the inequality (2). □

3. **Proof of the theorem.** Let $k \in \{1, \cdots, n\}$ be fixed. Let us choose a sequence $(v_j)_{j\geq 1}$ which is dense in $\mathbb{R}^n$ and let us define, for $\omega = (r, i, j, l) \in I := (\mathbb{N}^*)^4$, the set D_ω constituted of elements x belonging to the closed ball $\bar{B}_r$ such that $f(x) \leq r$, and such that there exist $\Pi \in B^k_d\left(\Pi_i, \frac{1}{4r}\right)$, $\rho \geq \frac{9}{r}$ and $\zeta \in \bar{B}\left(v_j, \frac{1}{2r}\right)$ satisfying:

$$\forall y \in B\left(x, \frac{1}{l}\right), \quad f(y) \geq f(x) + \langle \zeta, y - x \rangle + \rho\bar{\sigma}_{\Pi}(y - x) - \frac{1}{2r}|y - x|. \tag{3}$$

Lemma 3.1. *We have the following inclusion:*

$$\mathcal{D}^k(f) \subset \bigcup_{\omega \in I} D_\omega.$$

Proof. Denote by $e^k_1, \cdots, e^k_k$ the standard basis in $\mathbb{R}^k$ and choose a constant $\nu^k > 0$ such that

$$\bar{B}^k_{\nu^k} \subset \operatorname{conv}\left(\pm e^k_1, \cdots, \pm e^k_k\right), \tag{4}$$

where $\bar{B}^k_{\nu^k}$ denotes the closed ball centered at the origin with radius ν^k in $\mathbb{R}^k$. Let $x \in \mathcal{D}^k(f)$; there are $\zeta \in \mathbb{R}^n$ and $\mu > 0$ such that the convex set $\partial^- f(x)$ contains the k-ball $\mathcal{B}$ defined as,

$$\mathcal{B} := \bar{B}(\zeta, \mu) \cap H,$$

where H denotes the affine subspace of dimension k which is spanned by $\partial^- f(x)$ in $\mathbb{R}^n$. Choose $r \geq 1$ such that $|x| \leq r, f(x) \leq r$, and $\mu \geq \frac{9}{\nu^k r}$. By (4), there are k vectors $e_1, \cdots, e_k \in \mathbb{R}^n$ of norm 1 such that

$$\bar{B}_{\nu^k\mu} \cap \Pi \subset \mu E \subset \bar{B}_\mu, \tag{5}$$

where Π and E are defined by

$$\Pi := \text{SPAN}\{e_1, \cdots, e_k\} \quad \text{and} \quad E := \operatorname{conv}(\pm e_1, \cdots, \pm e_k).$$

For every $m \in \{1, \cdots, k\}$ and every $\epsilon = \pm 1$, the vector $\zeta + \mu\epsilon e_m$ belongs to $\mathcal{B}$, then there exists a neighborhood $\mathcal{V}_{m,\epsilon}$ of x such that

$$\forall y \in \mathcal{V}_{m,\epsilon}, \quad f(y) \geq f(x) + \langle \zeta + \mu\epsilon e_m, y - x \rangle - \frac{1}{2r}|y - x|.$$

Hence we deduce that for every $y \in \bigcap_{m\in\{1,\cdots,k\},\epsilon=\pm 1} \mathcal{V}_m$, we have

$$f(y) \geq f(x) + \langle \zeta, y - x \rangle$$
$$+ \max\left\{\mu\langle \epsilon e_m, y - x \rangle \mid m = 1, \cdots, k, \epsilon = \pm 1\right\} - \frac{1}{2r}|y - x|.$$

But by (5), we have for every $h \in \mathbb{R}^n$,

$$\max\{\mu\langle \epsilon e_m, h\rangle \mid m = 1, \cdots, k, \epsilon = \pm 1\} = \sigma_{\mu E}(h) \geq \sigma_{\left(\bar{B}_{\nu^k\mu} \cap \Pi\right)}(h) = \nu^k \mu \bar{\sigma}_{\Pi}(h).$$

We conclude easily by density of the families $\{\Pi_i^k\}_{i\geq 1}, \{v_j\}_{j\geq 1}$. □

Set for every $i \geq 1$, the cone

$$K_i := \left\{ h \in \mathbb{R}^n \mid \bar{\sigma}_{\Pi_i}(h) \leq \frac{1}{2}\|h\| \right\}.$$

We have the following lemma.

Lemma 3.2. *For every $\omega = (r, i, j, l) \in I$ and every $x \in D_\omega$, we have*

$$D_\omega \cap \bar{B}\left(x, \frac{1}{l}\right) \subset \{x\} + K_i.$$

Proof. Let $y \in D_\omega \cap \bar{B}\left(x, \frac{1}{l}\right)$ be fixed. There are $\Pi_y \in B_d^k\left(\Pi_i, \frac{1}{4r}\right), \rho_y \geq \frac{9}{r}$ and $\zeta_y \in \bar{B}\left(v_j, \frac{1}{2r}\right)$ such that

$$\forall z \in \bar{B}\left(y, \frac{1}{l}\right), \quad f(z) \geq f(y) + \langle \zeta_y, z - y\rangle + \rho_y \bar{\sigma}_{\Pi_y}(z - y) - \frac{1}{2r}|z - y|.$$

In particular, for $z = x$, this implies

$$\begin{aligned} f(x) &\geq f(y) + \langle \zeta_y, x - y\rangle + \rho_y \bar{\sigma}_{\Pi_y}(x - y) - \frac{1}{2r}|y - x| \\ &\geq f(y) + \langle \zeta_y, x - y\rangle - \frac{1}{2r}|y - x|. \end{aligned} \tag{6}$$

But since $x \in D_\omega$, there are $\Pi_x \in B_d^k\left(\Pi_i, \frac{1}{4r}\right), \rho_x \geq \frac{9}{r}$ and $\zeta_x \in \bar{B}\left(v_j, \frac{1}{2r}\right)$ such that

$$f(y) \geq f(x) + \langle \zeta_x, y - x\rangle + \rho_x \bar{\sigma}_{\Pi_x}(y - x) - \frac{1}{2r}|y - x|. \tag{7}$$

Summing the inequalities (6) and (7), we obtain

$$0 \geq \langle \zeta_x - \zeta_y, y - x\rangle + \rho_x \bar{\sigma}_{\Pi_x}(y - x) - \frac{1}{r}|y - x|.$$

But $|\zeta_x - \zeta_y| \leq \frac{1}{r}$, hence

$$\rho_x \bar{\sigma}_{\Pi_x}(y - x) \leq \frac{2}{r}|y - x|.$$

Which gives by (2)

$$\begin{aligned} \bar{\sigma}_{\Pi_i}(y - x) &= \left(\bar{\sigma}_{\Pi_i}(y - x) - \bar{\sigma}_{\Pi_x}(y - x)\right) + \bar{\sigma}_{\Pi_x}(y - x) \\ &\leq d(\Pi_i, \Pi_x)|y - x| + \frac{2}{\rho_x r}|y - x| \\ &\leq \frac{1}{4r}|y - x| + \frac{1}{4}|y - x| \\ &\leq \frac{1}{2}|y - x|. \end{aligned}$$

□

Lemma 3.3. *Let $\omega = (r, i, j, l) \in I$ and $\bar{x} \in D_\omega$ be fixed; set*

$$A := p_{\Pi_i^\perp}\left(D_\omega \cap \bar{B}\left(\bar{x}, \frac{1}{2l}\right)\right).$$

For every $y \in A$, there exists a unique $z = z_y \in \Pi_i$ such that

$$y + z \in D_\omega \cap \bar{B}\left(\bar{x}, \frac{1}{2l}\right).$$

Moreover, the mapping $\psi_\omega : y \in A \mapsto z_y$ is Lipschitz continuous.

Proof. First of all, for every $y \in A$, there is, by definition of A, some $x \in D_\omega \cap \bar{B}\left(\bar{x}, \frac{1}{2l}\right)$ such that $y = p_{\Pi_i^\perp}(x)$. Since $x - y \in \Pi_i$, this proves the existence of z_y. To prove the uniqueness, we argue by contradiction. Let $y \in A$, assume that there are $z \neq z' \in \Pi_i$ such that $y + z$ and $y + z'$ belong to $D_\omega \cap \bar{B}\left(\bar{x}, \frac{1}{2l}\right)$. Since $y + z \in D_\omega$, by the previous lemma, we know that

$$D_\omega \cap \bar{B}\left(y + z, \frac{1}{l}\right) \subset \{y + z\} + K_i.$$

But since both $y + z$ and $y + z'$ belong to $\bar{B}\left(\bar{x}, \frac{1}{2l}\right)$, $y + z'$ belongs clearly to $D_\omega \cap \bar{B}\left(y + z, \frac{1}{l}\right)$. Hence $y+z' \in \{y+z\}+K_i$. Which means that $(y+z')-(y+z) = z'-z$ belongs to K_i. But since $z' - z \in \Pi_i$, we have that $\bar{\sigma}_{\Pi_i}(z' - z) = |z' - z| > \frac{1}{2}|z' - z|$. We find a contradiction. Let us now prove that the map ψ_ω is Lipschitz continuous. Let $y, y' \in A$ be fixed. By the proof above we know that $\psi_\omega(y) = x - y$ (resp. $\psi_\omega(y') = x' - y'$) where x is such that $y = p_{\Pi_i^\perp}(x)$ (resp. $y = p_{\Pi_i^\perp}(x)$). Set $z := \psi_\omega(y), z' := \psi_\omega(y')$ and $h := x'-x$. Since $x = y+y$ and $x' = y'+z'$ where $y, y' \in \Pi_i^\perp$ and $z, z' \in \Pi_i$, we have $|h|^2 = |z' - z|^2 + |y' - y|^2$. But $\bar{\sigma}_{\Pi_i}(h) = |z' - z| \leq \frac{1}{2}|h|$. Hence we obtain that

$$|z' - z| \leq |x' - x| = |h| \leq \frac{2}{\sqrt{3}}|y' - y|.$$

The proof of the lemma is completed. □

From the lemma above, for every $\omega = (r, i, j, l) \in I$ and every $\bar{x} \in D_\omega$, the map $\phi : A \to \mathbb{R}^n$ defined as,

$$\forall y \in A, \quad \phi(y) = y + \psi_\omega(y),$$

is Lipschitz continuous and satisfies

$$D_\omega \cap \bar{B}\left(\bar{x}, \frac{1}{2l}\right) \subset \phi(A).$$

Since $A \subset \Pi_i^\perp$, such a map can be extended into a Lipschitz continuous map from $\Pi_i^\perp$ into $\mathbb{R}^n$. Since $\Pi_i^\perp$ has dimension $(n-k)$, we deduce that the set $D_\omega \cap \bar{B}\left(\bar{x}, \frac{1}{2l}\right)$ is $(n - k)$-rectifiable. The fact that any set D_ω can be covered by a finite number of balls of radius $\frac{1}{2l}$ completes the proof of the theorem.

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