Mass Transportation on the Earth

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Monge's Memoir

MÉMOIRE

SUR LA

THÉORIE DES DÉBLAIS ET DES REMBLAIS.

Par M. MONGE.

LORSQU'ON doit tranſporter des terres d'un lieu dans un autre, on a coutume de donner le nom de *Déblai* au volume des terres que l'on doit tranſporter, & le nom de *Remblai* à l'eſpace qu'elles doivent occuper après le tranſport.

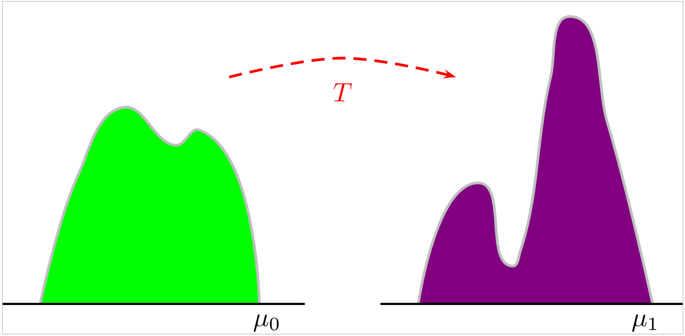
Le prix du tranſport d'une molécule étant, toutes choſes d'ailleurs égales, proportionnel à ſon poids & à l'eſpace qu'on lui fait parcourir, & par conſéquent le prix du tranſport total devant être proportionnel à la ſomme des produits des molécules multipliées chacune par l'eſpace parcouru, il s'enſuit que le déblai & le remblai étant donnés de figure & de poſition, il n'eſt pas indifférent que telle molécule du déblai ſoit tranſportée dans tel ou tel autre endroit du remblai, mais qu'il y a une certaine diſtribution à faire des molécules du premier dans le ſecond, d'après laquelle la ſomme de ces produits ſera la moindre poſſible, & le prix du tranſport total ſera un *minimum*.

In *Histoire de l'Académie Royale des Sciences de Paris, 1781.*

Transport maps

Let μ_0 and μ_1 be **probability measures** on M. We call **transport map** from μ_0 to μ_1 any measurable map $T : M \to M$ such that $T_\sharp \mu_0 = \mu_1$, that is

$$\mu_1(B) = \mu_0\big(T^{-1}(B)\big), \quad \forall B \text{ measurable } \subset M.$$



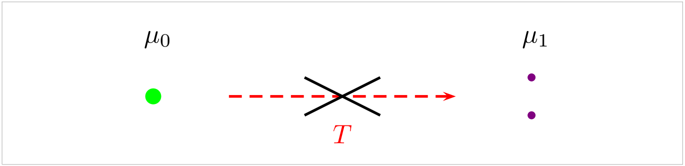
The Monge Problem

Let $M = \mathbb{R}^n$, given two probabilities measures μ_0, μ_1 on M, find a **transport map** $T : M \to M$ from μ_0 to μ_1 which **minimizes** the transportation cost

$$\int_M \| T(x) - x \| \, d\mu_0(x).$$

Examples

- In $M = \mathbb{R}^2$, let $\mu_0 = \delta_x$ and $\mu_1 = \frac{1}{2}\delta_y + \frac{1}{2}\delta_{y'}$ with $y \neq y'$.



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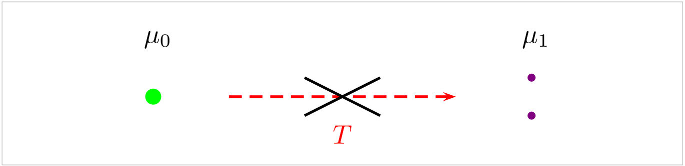
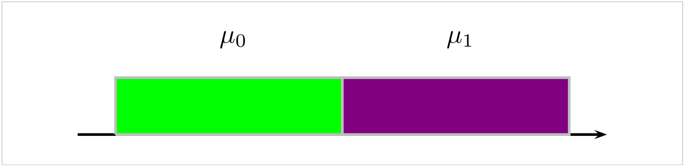


- In $M = \mathbb{R}$, let $\mu_0 = \chi_{[0,1]}$ and $\mu_1 = \chi_{[1,2]}$.



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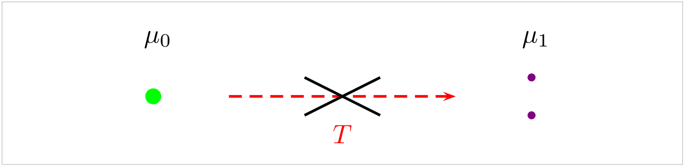

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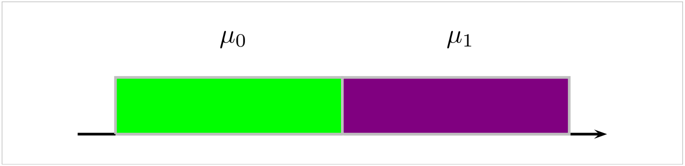

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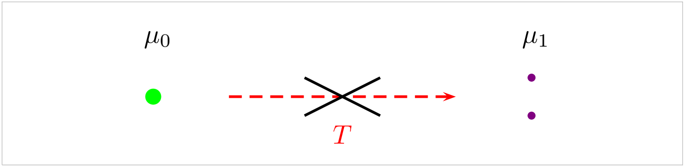

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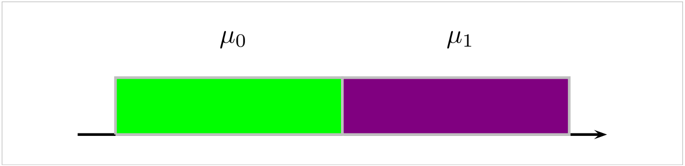

$$T(x) = x + 1 \implies \int_0^1 |T(x) - x|\, dx = 1$$
$$T(x) = 2 - x \implies \int_0^1 |T(x) - x|\, dx = 1$$

Classical Monge's Problem

Theorem (Sudakov '79, Ambrosio '00, Trudinger-Wang '01, Caffarelli-Feldman-McCann '10)

Let μ_0 and μ_1 be two compactly supported probability measures on $\mathbb{R}^n$. Assume that μ_0 is absolutely continuous w.r.t. Lebesgue, then the problem

$$\min \left\{ \int_{\mathbb{R}^n} \|x - T(x)\| \, d\mu_0(x) \,|\, T_\sharp \mu_0 = \mu_1 \right\}$$

has at least one solution.

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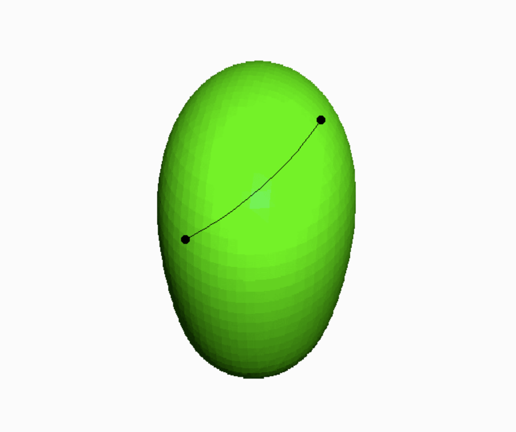
Several minimizers
Very little is known on the regularity of (some) minimizers

An other framework

Let M be a **smooth connected compact surface** in $\mathbb{R}^n$. For any $x, y \in M$, we define the **geodesic distance** between x and y, denoted by $d(x,y)$, as the minimum of the Euclidean lengths of the curves joining x to y that can be drawn on M.



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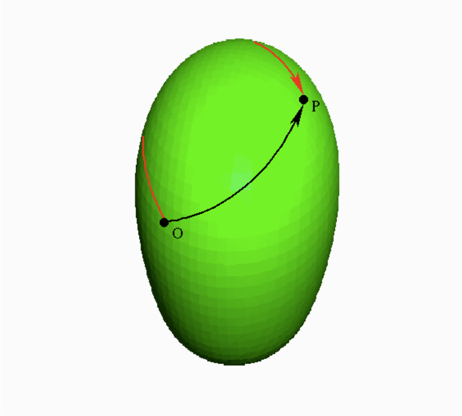

Quadratic Monge's Problem

Given two probabilities measures μ_0, μ_1 on M, find a transport map $T : M \to M$ from μ_0 to μ_1 which **minimizes** the quadratic cost

$$\int_M d^2(x, T(x))d\mu_0(x).$$

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Theorem (McCann '01)

If μ_0 is absolutely continuous w.r.t. Lebesgue, then there exists a **unique** *optimal transport map T from μ_0 to μ_1. In fact, there is a c-***convex** *function $\varphi : M \to \mathbb{R}$ satisfying*

$$T(x) = \exp_x (\nabla \varphi(x)) \qquad \mu_0 \text{ a.e. } x \in M.$$

(Moreover, for a.e. $x \in M$, $\nabla \varphi(x)$ belongs to the injectivity domain at x.)

Back to $\mathbb{R}^n$

Quadratic Monge's Problem in $\mathbb{R}^n$: Given two probability measures μ_0, μ_1 with compact supports in $\mathbb{R}^n$, we are concerned with transport maps $T : \mathbb{R}^n \to \mathbb{R}^n$ pushing forward μ_0 to μ_1 which **minimize** the transport cost

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Theorem (Brenier '91)

If μ_0 is absolutely continuous with respect to the Lebesgue measure, there exists a unique optimal transport map with respect to the quadratic cost. In fact, there is a **convex** *function $\psi : M \to \mathbb{R}$ such that*

$$T(x) = \nabla\psi(x) \qquad \mu_0 \text{ a.e. } x \in \mathbb{R}^n.$$

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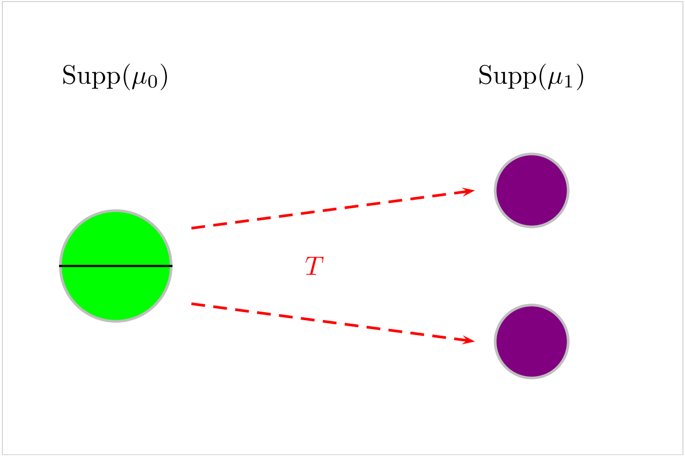
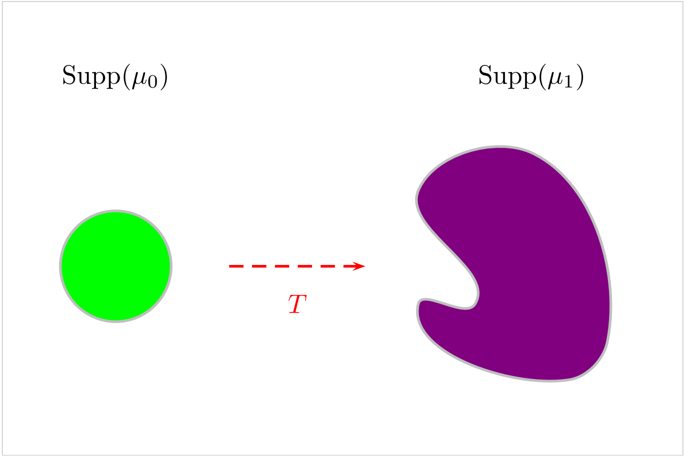
Necessary and sufficient conditions for regularity ?

An obvious counterexample



The convexity of the targer is necessary



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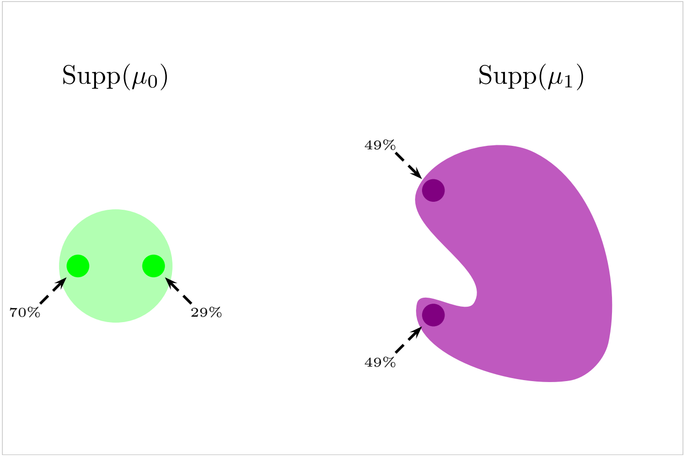

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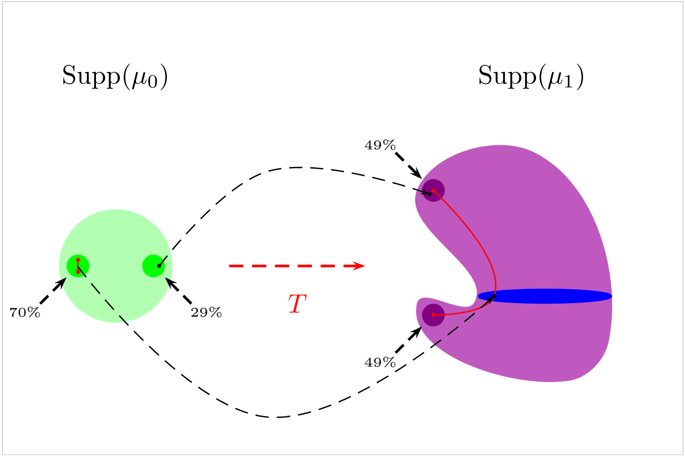

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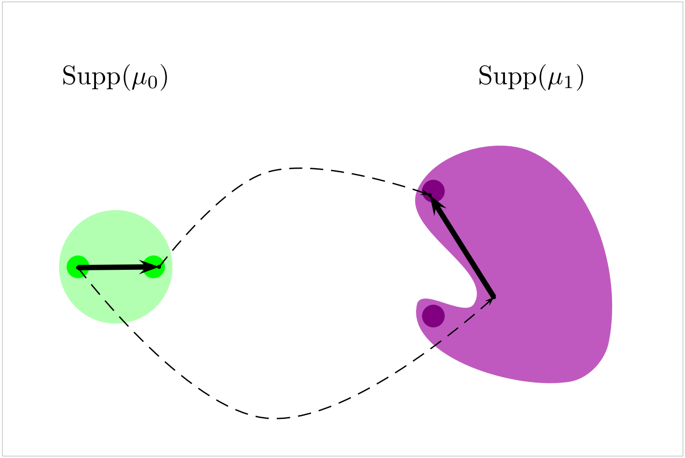

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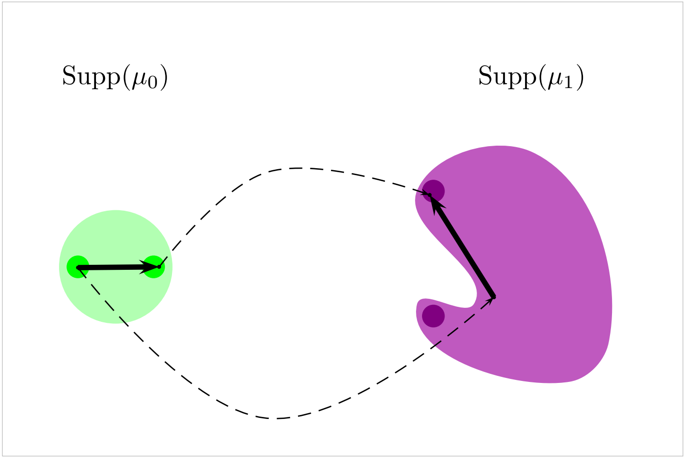

T gradient of a convex function

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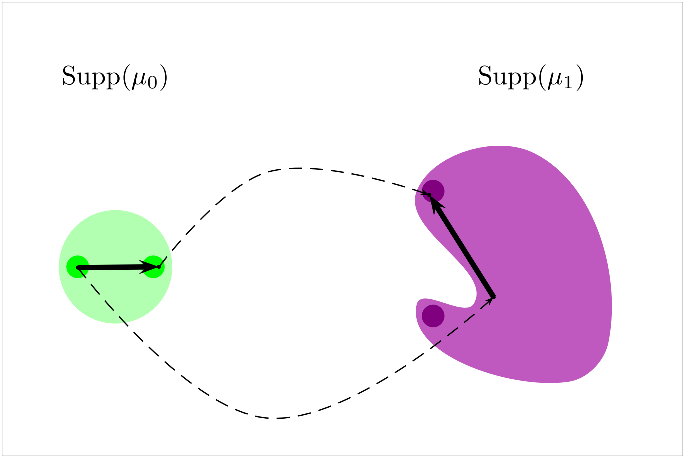

T gradient of a convex function $\Longrightarrow \langle y - x, T(y) - T(x) \rangle \geq 0$!!!

Caffarelli's Regularity Theory

If μ_0 and μ_1 are associated with densities f_0, f_1 w.r.t. Lebesgue, then

$$T_\sharp \mu_0 = \mu_1 \Longleftrightarrow \int_{\mathbb{R}^n} \zeta\big(T(x)\big) f_0(x) dx = \int_{R^n} \zeta(y) f_1(y) dy \quad \forall \zeta.$$

$\rightsquigarrow$ ψ weak solution of the **Monge-Ampère equation** :

$$\det\left(\nabla^2\psi(x)\right) = \frac{f_0(x)}{f_1(\nabla\psi(x))}.$$

Theorem (Caffarelli '90s)

Let Ω_0, Ω_1 be connected and bounded open sets in $\mathbb{R}^n$ and f_0, f_1 be probability densities resp. on Ω_0 and Ω_1 such that $f_0, f_1, 1/f_0, 1/f_1$ are **essentially bounded**. *Assume that μ_0 and μ_1 have respectively densities f_0 and f_1 w.r.t. Lebesgue and that Ω_1 is* **convex**. *Then the quadratic optimal transport map from μ_0 to μ_1 is continuous.*

Back to surfaces

Given two probabilities measures μ_0, μ_1 on M, find a transport map $T : M \to M$ from μ_0 to μ_1 which minimizes the quadratic cost

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Definition

We say that the surface $M \subset \mathbb{R}^n$ satisfies the **Transport Continuity Property (TCP)** if for any pair of probability measures μ_0, μ_1 associated locally with **continuous positive densities** ρ_0, ρ_1 on M, the optimal transport map from μ_0 to μ_1 is **continuous**.

Exponential mapping and injectivity domains

Let $x \in M$ be fixed.

- For every $v \in T_xM$, we define the **exponential** of v by

$$\exp_x(v) = \gamma_{x,v}(1),$$

  where $\gamma_{x,v} : [0,1] \to M$ is the unique geodesic starting at x with speed v.

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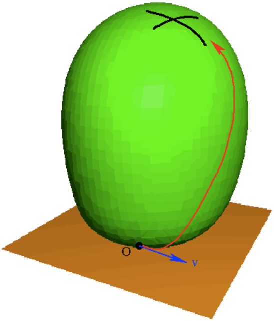
- We call **injectivity domain** at x, the subset of T_xM defined by

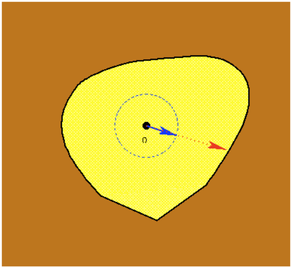
$$\mathcal{I}(x) := \left\{ v \in T_xM \,\middle|\, \begin{array}{c} \exists t > 1 \text{ s.t. } \gamma_{tv} \text{ is the unique} \\ \text{minim. geod. between } x \text{ and } \exp_x(tv) \end{array} \right\}$$

  It is a star-shaped (w.r.t. $0 \in T_xM$) bounded open set with Lipschitz boundary.

The distance to the cut locus

Using the exponential mapping, we can associate to each unit tangent vector its **distance to the cut locus**.





In that way, the **injectivity domain** $\mathcal{I}(x)$ is the open set which is enclosed by the **tangent cut locus** $\mathrm{TCL}(x)$.

A necessary and sufficient condition for **TCP**

Theorem (Figalli-R-Villani '10)

Let M be a smooth connected compact surface in $\mathbb{R}^n$. Then

M satisfies **TCP**

$$\Updownarrow$$

All the injectivity domains are **convex**,
and for any $x, x' \in M$ the function

$$F_{x,x'} \, : \, v \in \mathcal{I}(x) \longmapsto d^2\big(x, \exp_x(v)\big) - d^2\big(x', \exp_x(v)\big)$$

is **quasiconvex** *(all its sublevel sets are convex).*

Almost characterization of quasiconvex functions

Lemma (Sufficient condition)

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class C^2. Assume that for every $v \in U$ and every $w \in \mathbb{R}^n \setminus \{0\}$ the following property holds :

$$\langle \nabla_v F, w \rangle = 0 \Longrightarrow \langle \nabla_v^2 F\, w, w \rangle > 0.$$

Then F is quasiconvex.

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Lemma (Necessary condition)

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class C^2. Assume that F is quasiconvex. Then for every $v \in U$ and every $w \in \mathbb{R}^n \setminus \{0\}$ the following property holds :

$$\langle \nabla_v F, w \rangle = 0 \implies \langle \nabla_v^2 F\, w, w \rangle \geq 0.$$

Proof of the easy lemma

Proof.

Let $v_0, v_1 \in U$ be fixed. Set $v_t := (1-t)v_0 + tv_1$, for every $t \in [0,1]$. Define $h : [0,1] \rightarrow \mathbb{R}$ by

$$h(t) := F(v_t) \qquad \forall t \in [0,1].$$

If $h \not\leq \max\{h(0), h(1)\}$, there is $\tau \in (0,1)$ such that

$$h(\tau) = \max_{t \in [0,1]} h(t) \; > \; \max\{h(0), h(1)\}.$$

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There holds

$$\dot{h}(\tau) = \langle \nabla_{v_\tau} F, \dot{v}_\tau \rangle \quad \text{et} \quad \ddot{h}(\tau) = \langle \nabla^2_{v_\tau} F \, \dot{v}_\tau, \dot{v}_\tau \rangle.$$

Since τ is a local maximum, one has $\dot{h}(\tau) = 0$.
Contradiction !! □

The Ma-Trudinger-Wang tensor

The **MTW** tensor $\mathfrak{S}$ is defined as

$$\mathfrak{S}_{(x,v)}(\xi,\eta) = -\frac{3}{4} \left.\frac{d^2}{ds^2}\right|_{s=0} \left.\frac{d^2}{dt^2}\right|_{t=0} d^2\left(\exp_x(t\xi), \exp_x(v+s\eta)\right),$$

for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$.

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Proposition (Villani '09, Figalli-R-Villani '10)

Let $M \subset \mathbb{R}^n$ be a compact surface with convex injectivity domains. Then the following properties are equivalent:

- *All the functions $F_{x,x'}$ are quasiconvex.*
- *The* **MTW** *tensor $\mathfrak{S}$ is $\succeq 0$, that is for any $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$,*

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi,\eta) \geq 0.$$

Back to the characterization of **TCP**

Theorem (Figalli-R-Villani '10)

Let M be a smooth connected compact surface in $\mathbb{R}^n$. It satisfies **TCP** *if and only if the two following properties holds:*

- *all the injectivity domains are convex,*
- $\mathfrak{S} \succeq 0$.

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Thanks to an observation by Loeper, for every $x \in M$ and for any pair of unit orthogonal tangent vectors $\xi, \eta \in T_x M$, there holds

$$\mathfrak{S}_{(x,0)}(\xi, \eta) = \sigma_x,$$

where σ_x denotes the **gaussian curvature** of M at x.

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Consequently,

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$$\textbf{TCP} \Longrightarrow \sigma \geq 0.$$

In particular, if $M \subset \mathbb{R}^3$ satisfies **TCP** , then it is **convex**.

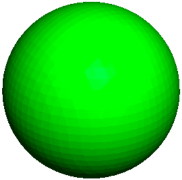
Examples: The Sphere

Loeper checked that the **MTW** tensor of the round unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ satisfies for every $x \in \mathbb{S}^2$, $v \in \mathcal{I}(x)$ and $\xi, \eta \in T_x\mathbb{S}^2$,

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi, \eta) \geq |\xi|^2 |\eta|^2.$$

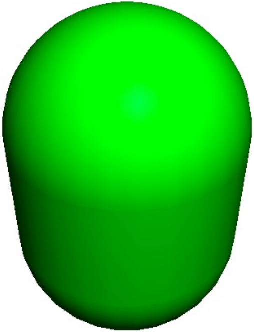
Theorem (Loeper '06)

The sphere $\mathbb{S}^2$ satisfies **TCP**.



Examples: The Pill

The surface made of two hemispheres and a cylindrical tube does not satisfy **TCP**.



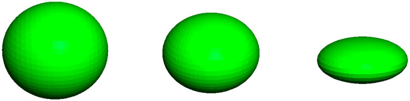
Examples: Ellipsoids of revolution

Ellipsoids of revolution (oblate case):

$$E_\mu : \quad x^2 + y^2 + \left(\frac{z}{\mu}\right)^2 = 1 \quad \mu \in (0, 1].$$

Theorem (Bonnard-Caillau-R '10)

The injectivity domains of an oblate ellipsoid of revolution are all convex if and only if and only if the ratio between the minor and the major axis is greater or equal to $1/\sqrt{3}\,(\simeq 0.58)$.



Examples: Ellipsoids of revolution

Ellipsoids of revolution (oblate case):

$$E_\mu : \quad x^2 + y^2 + \left(\frac{z}{\mu}\right)^2 = 1 \quad \mu \in (0, 1].$$

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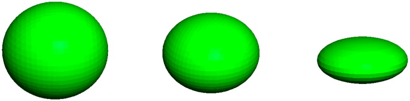

Consequently, sufficiently flat ellipsoids do not satisfy **TCP**.

Spheres

Theorem (Loeper '06)

The **MTW** *tensor on the round (unit) sphere* $\mathbb{S}^2$ *satisfies* $\mathfrak{S} \succeq 1$, *that is for any* $x \in \mathbb{S}^2, v \in \mathcal{I}(x)$ *and* $\xi, \eta \in T_x\mathbb{S}^2$,

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi, \eta) \geq |\xi|^2 |\eta|^2.$$

In particular, the round sphere $\mathbb{S}^2$ *satisfies* **TCP**.

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Is this result stable ?

Issues

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  On $\mathbb{S}^2$, the **MTW** tensor is given by

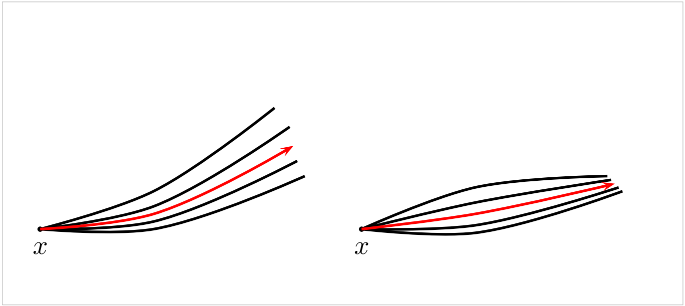
$$\begin{aligned}
&\mathfrak{S}_{(x,v)}\left(\xi, \xi^{\perp}\right) \\
&\quad = 3\left[\frac{1}{r^2} - \frac{\cos(r)}{r\sin(r)}\right]\xi_1^4 + 3\left[\frac{1}{\sin^2(r)} - \frac{r\cos(r)}{\sin^3(r)}\right]\xi_2^4 \\
&\qquad + \frac{3}{2}\left[-\frac{6}{r^2} + \frac{\cos(r)}{r\sin(r)} + \frac{5}{\sin^2(r)}\right]\xi_1^2\xi_2^2,
\end{aligned}$$

  with
  $x \in \mathbb{S}^2, v \in \mathcal{I}(x), r := |v|, \xi = (\xi_1, \xi_2), \xi^{\perp} = (-\xi_2, \xi_1)$.

Focalization

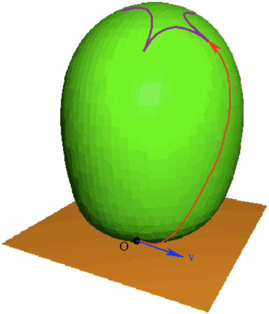
Definition

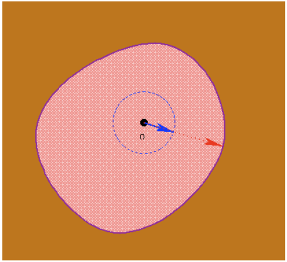
Let $x \in M$ and v be a unit tangent vector in T_xM. The vector v is **not conjugate** at time $t \geq 0$ if for any $t' \in [0, t+\delta]$ ($\delta > 0$ small) the geodesic from x to $\gamma(t')$ is locally minimizing.



The distance to the conjugate locus

Again, using the exponential mapping, we can associate to each unit tangent vector its **distance to the conjugate locus**.



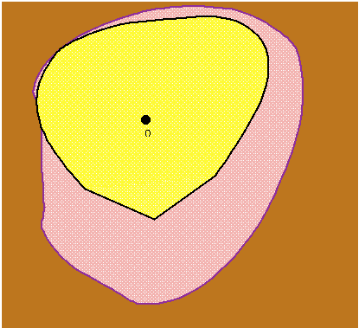


In that way, we define the so-called **nonfocal domain** $\mathcal{NF}(x)$ whose boundary is the **tangent conjugate locus** $\mathrm{TFL}(x)$.

Fundamental inclusion

The following inclusion holds

$$\text{injectivity domain} \subset \text{nonfocal domain}.$$



Mass Transportation on the Earth

Theorem (Figalli-R '09)

Any small deformation of $\mathbb{S}^2$ *in* C^5 *topology satisfies* $\overline{\mathfrak{S}} \succeq 1/2$*, has convex injectivity domains and so satisfies* **TCP**.

Thank you for your attention !!