The Sard Conjecture on Martinet Surfaces

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Sub-Riemannian structures

Let M be a smooth connected manifold of dimension n.

Definition

A sub-Riemannian structure of rank m in M is given by a pair (Δ, g) where:

- Δ is a **totally nonholonomic distribution** of rank $m \leq n$ on M which is defined locally by

$$\Delta(x) = \mathrm{Span}\left\{X^1(x), \ldots, X^m(x)\right\} \subset T_xM,$$

  where $X^1, \ldots, X^m$ is a family of m linearly independent smooth vector fields satisfying the **Hörmander condition**.
- g_x is a **scalar product** over $\Delta(x)$.

The Hörmander condition

We say that a family of smooth vector fields $X^1, \ldots, X^m$, satisfies the **Hörmander condition** if

$$\mathrm{Lie}\left\{X^1, \ldots, X^m\right\}(x) = T_x M \qquad \forall x,$$

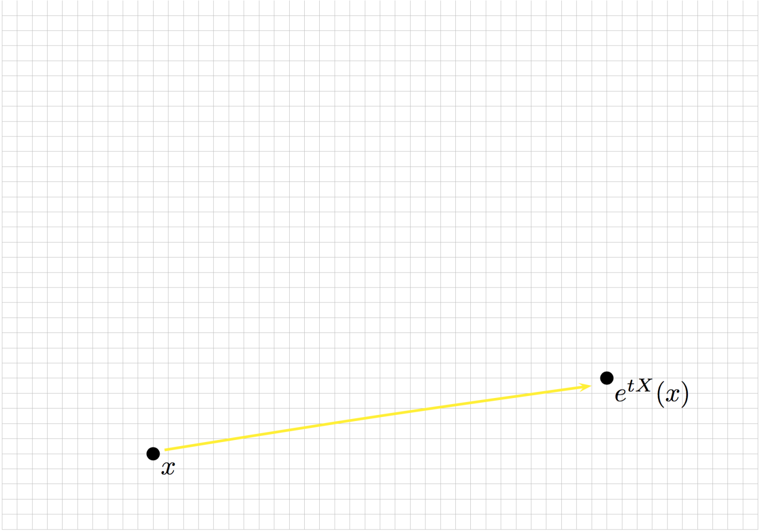
where $\mathrm{Lie}\{X^1, \ldots, X^m\}$ denotes the Lie algebra generated by $X^1, \ldots, X^m$, *i.e.* the smallest subspace of smooth vector fields that contains all the $X^1, \ldots, X^m$ and which is stable under Lie brackets.

Reminder

Given smooth vector fields X, Y in $\mathbb{R}^n$, the Lie bracket $[X, Y]$ at $x \in \mathbb{R}^n$ is defined by

$$[X, Y](x) = DY(x)X(x) - DX(x)Y(x).$$

Lie Bracket: Dynamic Viewpoint



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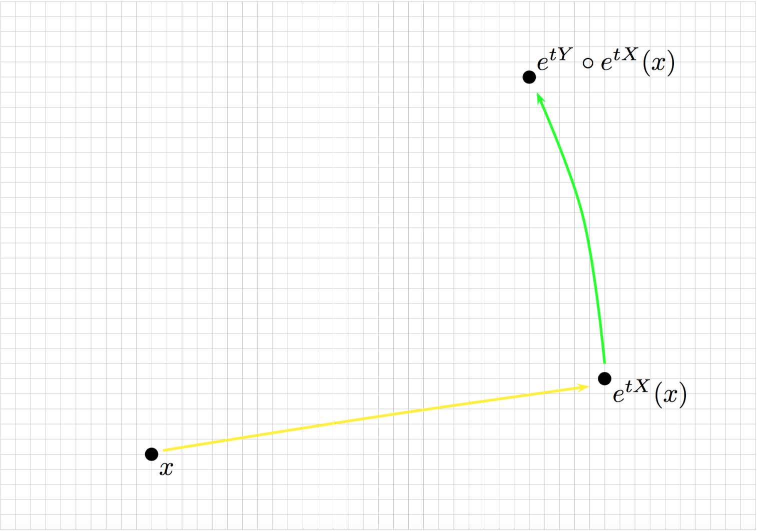

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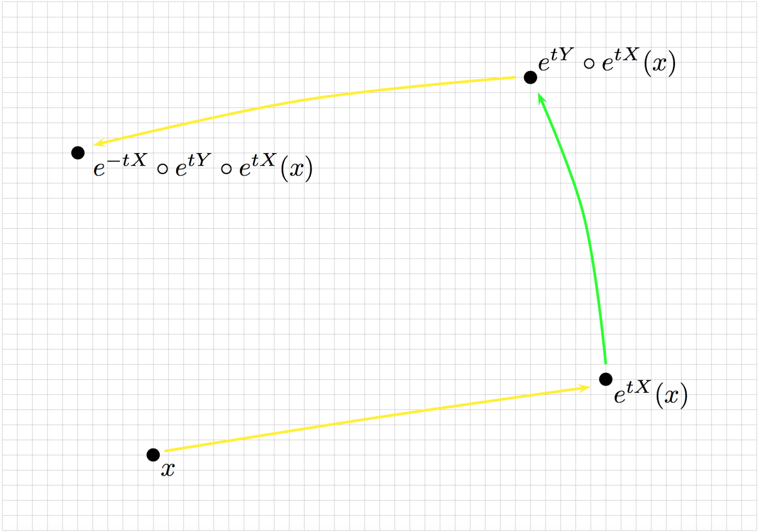

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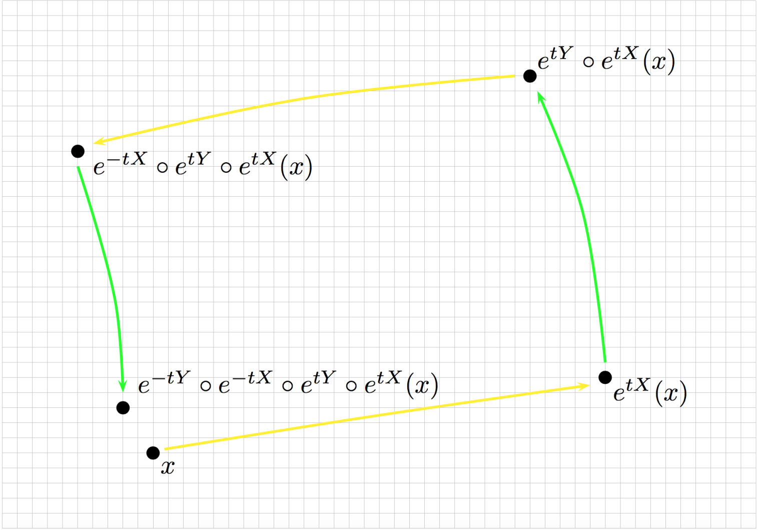

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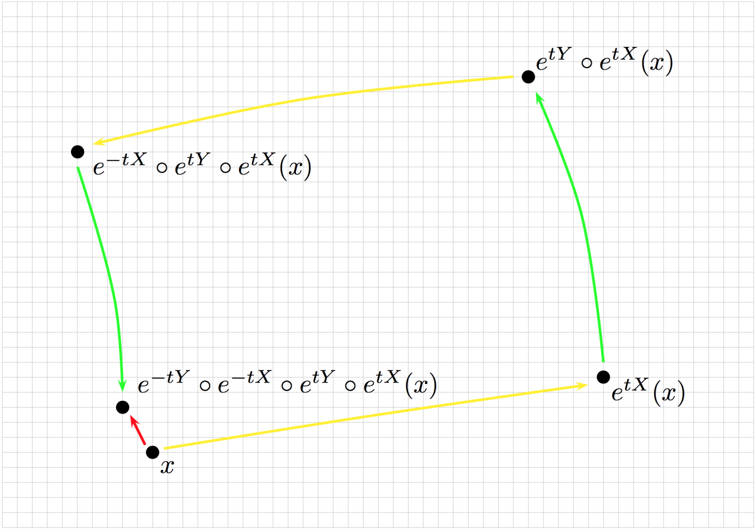
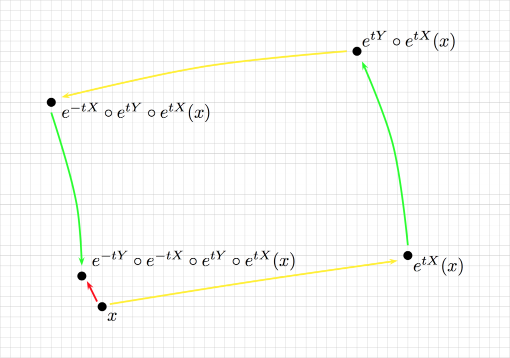


Lie Bracket: Dynamic Viewpoint

Exercise

There holds

$$[X,Y](x) = \lim_{t\downarrow 0} \frac{\left(e^{-tY}\circ e^{-tX}\circ e^{tY}\circ e^{tX}\right)(x) - x}{t^2}.$$



The Chow-Rashevsky Theorem

Definition

We call **horizontal path** any $\gamma \in W^{1,2}([0,1]; M)$ such that

$$\dot{\gamma}(t) \in \Delta(\gamma(t)) \qquad \text{a.e. } t \in [0,1].$$

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The following result is the cornerstone of the sub-Riemannian geometry. (Recall that M is assumed to be connected.)

Theorem (Chow-Rashevsky, 1938)

Let Δ be a totally nonholonomic distribution on M, then every pair of points can be joined by an horizontal path.

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Let Δ be a totally nonholonomic distribution on M, then every pair of points can be joined by an horizontal path.

Since the distribution is equipped with a metric, we can measure the lengths of horizontal paths and consequently we can associate a metric with the sub-Riemannian structure.

Examples of sub-Riemannian structures

Example (Riemannian case)

Every Riemannian manifold (M, g) *gives rise to a sub-Riemannian structure with* $\Delta = TM$.

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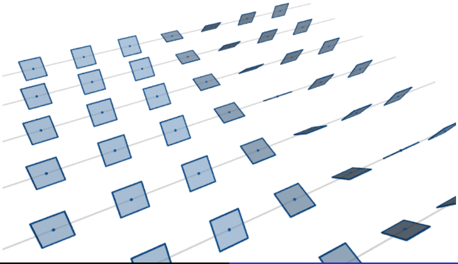
Example (Heisenberg)

In $\mathbb{R}^3$, $\Delta = Span\{X^1, X^2\}$ with

$$X^1 = \partial_x, \quad X^2 = \partial_y + x\partial_z \quad et \quad g = dx^2 + dy^2.$$



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Example (Martinet)

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Since $[X^1, X^2] = 2x\partial_z$ *and* $[X^1, [X^1, X^2]] = 2\partial_z$, *only one bracket is sufficient to generate* $\mathbb{R}^3$ *if* $x \neq 0$, *however we needs two brackets if* $x = 0$.

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Example (Rank 2 distribution in dimension 4)

In $\mathbb{R}^4$, $\Delta = Span\{X^1, X^2\}$ *with*

$$X^1 = \partial_x, \quad X^2 = \partial_y + x\partial_z + z\partial_w$$

satisfies $Vect\{X^1, X^2, [X^1, X^2], [[X^1, X^2], X^2]\} = \mathbb{R}^4$.

The sub-Riemannian distance

The **length** of an horizontal path γ is defined by

$$\mathrm{length}^g(\gamma) := \int_0^T |\dot{\gamma}(t)|^g_{\gamma(t)}\, dt.$$

Definition

Given $x, y \in M$, the **sub-Riemannian distance** between x and y is defined by

$$d_{SR}(x,y) := \inf\Big\{\mathrm{length}^g(\gamma) \,|\, \gamma \text{ hor.}, \gamma(0) = x, \gamma(1) = y\Big\}.$$

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Proposition

The manifold M equipped with the distance d_{SR} is a metric space whose topology coincides the one of M (as a manifold).

Sub-Riemannian geodesics

Definition

Given $x, y \in M$, we call **minimizing horizontal path** between x and y any horizontal path $\gamma : [0, 1] \to M$ joining x to y satisfying $d_{SR}(x, y) = \text{length}^g(\gamma)$.

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The **energy** of the horizontal path $\gamma : [0,1] \to M$ is given by

$$\text{ener}^g(\gamma) := \int_0^1 \left(|\dot{\gamma}(t)|^g_{\gamma(t)} \right)^2 dt.$$

Definition

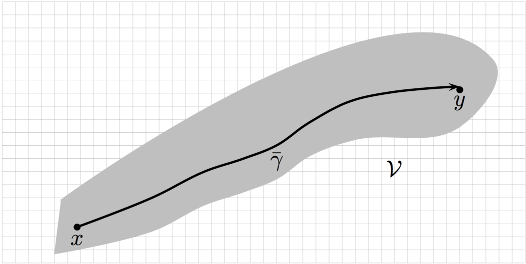
We call **minimizing geodesic** between x and y any horizontal path $\gamma : [0,1] \to M$ joining x to y such that

$$d_{SR}(x,y)^2 = \text{ener}^g(\gamma).$$

Study of minimizing geodesics

Let $x, y \in M$ and $\bar{\gamma}$ be a **minimizing geodesic** between x and y be fixed. The SR structure admits an orthonormal parametrization along $\bar{\gamma}$, which means that there exists a neighborhood $\mathcal{V}$ of $\bar{\gamma}([0,1])$ and an orthonomal family of m vector fields $X^1, \ldots, X^m$ such that

$$\Delta(z) = \mathrm{Span}\left\{X^1(z), \ldots, X^m(z)\right\} \qquad \forall z \in \mathcal{V}.$$



Study of minimizing geodesics

There exists a control $\bar{u} \in L^2\big([0,1];\mathbb{R}^m\big)$ such that

$$\dot{\bar{\gamma}}(t) = \sum_{i=1}^{m} \bar{u}_i(t)\, X^i\big(\bar{\gamma}(t)\big) \qquad \text{a.e. } t \in [0,1].$$

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Moreover, any control $u \in \mathcal{U} \subset L^2([0,1];\mathbb{R}^m)$ (u sufficiently close to $\bar{u}$) gives rise to a trajectory γ_u solution of

$$\dot{\gamma}_u = \sum_{i=1}^{m} u^i\, X^i\big(\gamma_u\big) \quad \text{sur } [0,T], \quad \gamma_u(0) = x.$$

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Furthermore, for every horizontal path $\gamma : [0,1] \to \mathcal{V}$ there exists a unique control $u \in L^2\big([0,1];\mathbb{R}^m\big)$ for which the above equation is satisfied.

Study of minimizing geodesics

Consider the **End-Point mapping**

$$E^{x,1} \ : \ L^2\big([0,1];\mathbb{R}^m\big) \longrightarrow M$$

defined by

$$E^{x,1}(u) := \gamma_u(1),$$

and set $C(u) = \|u\|_{L^2}^2$, then $\bar{u}$ is a solution to the following **optimization problem with constraints**:

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$\bar{u}$ minimize $C(u)$ among all $u \in \mathcal{U}$ s.t. $E^{x,1}(u) = y$.

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$$\bar{u} \text{ minimize } C(u) \text{ among all } u \in \mathcal{U} \text{ s.t. } E^{x,1}(u) = y.$$

(Since the family $X^1, \ldots, X^m$ is orthonormal, we have

$$\text{ener}^g(\gamma_u) = C(u) \qquad \forall u \in \mathcal{U}.)$$

Study of minimizing geodesics

Proposition (Lagrange Multipliers)

There exist $p \in T_y^*M \simeq (\mathbb{R}^n)^*$ *and* $\lambda_0 \in \{0,1\}$ *with* $(\lambda_0, p) \neq (0,0)$ *such that*

$$p \cdot d_{\bar{u}} E^{x,1} = \lambda_0 d_{\bar{u}} C.$$

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As a matter of fact, the function given by

$$\Phi(u) := \big(C(u), E^{x,1}(u)\big)$$

cannot be a submersion at $\bar{u}$. Otherwise $D_{\bar{u}}\Phi$ would be surjective and so open at $\bar{u}$, which means that the image of Φ would contain some points of the form $(C(\bar{u}) - \delta, y)$ with $\delta > 0$ small.

$\rightsquigarrow$ Two cases may appear: $\lambda_0 = 1$ or $\lambda_0 = 0$.

Study of minimizing geodesics

First case : $\lambda_0 = 1$

This is the good case, the Riemannian-like case. The minimizing geodesic can be shown to be solution of a geodesic equation. It is smooth, there is a "geodesic flow"...

Second case : $\lambda_0 = 0$

In this case, we have

$$p \cdot D_{\bar{u}} E^{x,1} = 0 \text{ with } p \neq 0,$$

which means that $\bar{u}$ is **singular** as a critical point of the mapping $E^{x,1}$.

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$\rightsquigarrow$ As shown by R. Montgomery, the case $\lambda_0 = 0$ cannot be ruled out.

Singular horizontal paths and Examples

Definition

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Let $\Delta(x) = T_x M$, any path in $W^{1,2}$ is horizontal. There are no singular curves.

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Example 2: Heisenberg, fat distributions
In $\mathbb{R}^3$, Δ given by $X^1 = \partial_x, X^2 = \partial_y + x\partial_z$ does not admin nontrivial singular horizontal paths.

Examples

Example 3: Martinet-like distributions

In $\mathbb{R}^3$, let $\Delta = \mathrm{Vect}\{X^1, X^2\}$ with X^1, X^2 of the form

$$X^1 = \partial_{x_1} \quad \text{and} \quad X^2 = \big(1 + x_1\phi(x)\big)\,\partial_{x_2} + x_1^2\partial_{x_3},$$

where ϕ is a smooth function and let g be a metric over Δ.

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Theorem (Montgomery)

There exists $\bar{\epsilon} > 0$ such that for every $\epsilon \in (0, \bar{\epsilon})$, the singular horizontal path

$$\gamma(t) = (0, t, 0) \qquad \forall t \in [0, \epsilon],$$

is minimizing (w.r.t. g) among all horizontal paths joining 0 to $(0, \epsilon, 0)$.

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is minimizing (w.r.t. g) among all horizontal paths joining 0 to $(0, \epsilon, 0)$. Moreover, if $\{X^1, X^2\}$ is orthonormal w.r.t. g and $\phi(0) \neq 0$, then γ is not the projection of a normal extremal ($\lambda_0 = 1$).

The Sard Conjectures

Let (Δ, g) be a SR structure on M and $x \in M$ be fixed.

$$\mathcal{S}^x_{\Delta,min^g} = \left\{\gamma(1) | \gamma : [0,1] \to M, \gamma(0) = x, \gamma \text{ hor., sing., min.} \right\}.$$

Conjecture (SR or minimizing Sard Conjecture)

The set $\mathcal{S}^x_{\Delta,min^g}$ has Lebesgue measure zero.

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The Brown-Morse-Sard Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function of class C^k.

Definition

- We call **critical point** of f any $x \in \mathbb{R}^n$ such that $d_x f : \mathbb{R}^n \to \mathbb{R}^m$ is not surjective and we denote by C_f the set of critical points of f.
- We call **critical value** any element of $f(C_f)$. The elements of $\mathbb{R}^m \setminus f(C_f)$ are called **regular values**.

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H.C. Marston Morse

(1892-1977)

Arthur B. Brown

(1905-1999)

Anthony P. Morse

(1911-1984)

Arthur Sard

(1909-1980)

The Brown-Morse-Sard Theorem

Theorem (Arthur B. Brown, 1935)

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be of class C^k. If $k = \infty$ (or large enough) then $f(C_f)$ has empty interior.

Theorem (Anthony P. Morse, 1939)

Assume that $m = 1$ and $k \geq m$, then $f(C_f)$ has Lebesgue measure zero.

Theorem (Arthur Sard, 1942)

If $k \geq \max\{1, n-m+1\}$, $\mathcal{L}^m(f(C_f)) = 0$.

Remark

Thanks to a construction by Hassler Whitney (1935), the assumption in Sard's theorem is sharp.

Infinite dimension (Bates-Moreira, 2001)

The Sard Theorem is false in infinite dimension. Let $f : \ell^2 \to \mathbb{R}$ be defined by

$$f\left(\sum_{n=1}^{\infty} x_n\, e_n\right) = \sum_{n=1}^{\infty} \left(3 \cdot 2^{-n/3} x_n^2 - 2x_n^3\right).$$

The function f is polynomial ($f^{(4)} \equiv 0$) with critical set

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and critical values

$$f(C(f)) = \left\{ \sum_{n=1}^{\infty} \delta_n\, 2^{-n} \,|\, \delta_n \in \{0,1\} \right\} = [0,1].$$

Back to the Sard Conjecture

Let (Δ, g) be a SR structure on M and $x \in M$ be fixed. Set

$$\Delta^\perp := \left\{(x,p) \in T^*M \,|\, p \perp \Delta(x)\right\} \subset T^*M$$

and (we assume here that Δ is generated by m vector fields $X^1, \ldots, X^m$) define

$$\vec{\Delta}(x,p) := \mathrm{Span}\left\{\vec{h}^1(x,p), \ldots, \vec{h}^m(x,p)\right\} \quad \forall (x,p) \in T^*M,$$

where $h^i(x,p) = p \cdot X^i(x)$ and $\vec{h}^i$ is the associated Hamiltonian vector field in T^*M.

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Proposition

An horizontal path $\gamma : [0,1] \to M$ is singular if and only if it is the projection of a path $\psi : [0,1] \to \Delta^\perp \setminus \{0\}$ which is horizontal w.r.t. $\vec{\Delta}$.

The case of Martinet surfaces

Let M be a smooth manifold of dimension 3 and Δ be a totally nonholonomic distribution of rank 2 on M. We define the **Martinet surface** by

$$\Sigma_\Delta = \{x \in M \mid \Delta(x) + [\Delta, \Delta](x) \neq T_x M\}$$

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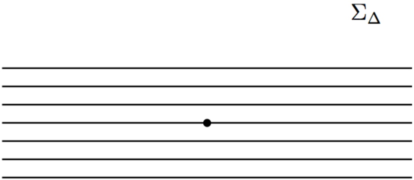
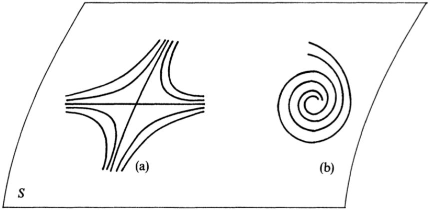
$\rightsquigarrow$ Let us fix x on Σ_Δ and see how its orbit look like.

Transverse case



Generic tangent case

(Zelenko-Zhitomirskii, 1995)



The Sard Conjecture on Martinet surfaces

Let M be of dimension 3 and Δ of rank 2.

$$\mathcal{S}_\Delta^x = \{\gamma(1) | \gamma : [0,1] \to M, \gamma(0) = x, \gamma \text{ hor., sing.}\}.$$

Conjecture (Sard Conjecture)

The set $\mathcal{S}_\Delta^x$ has vanishing $\mathcal{H}^2$-measure.

Theorem (Belotto-R, 2016)

The above conjecture holds true under one of the following assumptions:

- *The Martinet surface is smooth;*
- *All datas are analytic and*

$$\Delta(x) \cap T_x Sing(\Sigma_\Delta) = T_x Sing(\Sigma_\Delta) \quad \forall x \in Sing(\Sigma_\Delta).$$

Ingredients of the proof

- Control of the divergence of vector fields which generates the trace of Δ over Σ_Σ of the form

$$\left|\mathrm{div}\mathcal{Z}\right| \leq C \left|\mathcal{Z}\right|.$$

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- Control of the divergence of vector fields which generates the trace of Δ over Σ_Σ of the form

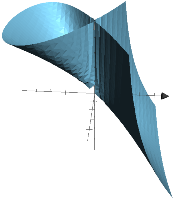
$$\left|\mathrm{div}\mathcal{Z}\right| \leq C \left|\mathcal{Z}\right|.$$

- Resolution of singularities.

An example

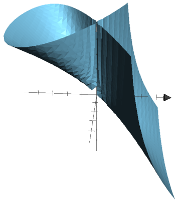
In $\mathbb{R}^3$,

$$X = \partial_y \quad \text{and} \quad Y = \partial_x + \left[\frac{y^3}{3} - x^2 y(x+z) \right] \partial_z.$$

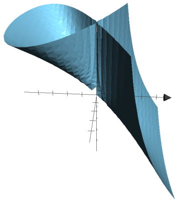


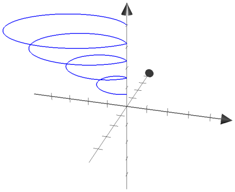
Martinet Surface: $\Sigma_\Delta = \left\{ y^2 - x^2(x+z) = 0 \right\}$.

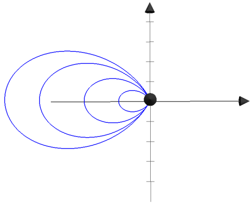
An example



An example







Thank you for your attention !!

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