

A few words about the Ma-Trudinger-Wang tensor

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The Ma-Trudinger-Wang tensor

Let M be a **smooth connected (compact) manifold of dimension n** equipped with a **smooth Riemannian metric g** . For any $x, y \in M$, we define the **geodesic distance** between x and y , denoted by $d(x, y)$, as the minimum of the lengths of the curves joining x to y .

Definition (The Ma-Trudinger-Wang tensor)

The **MTW** tensor \mathfrak{S} is defined as

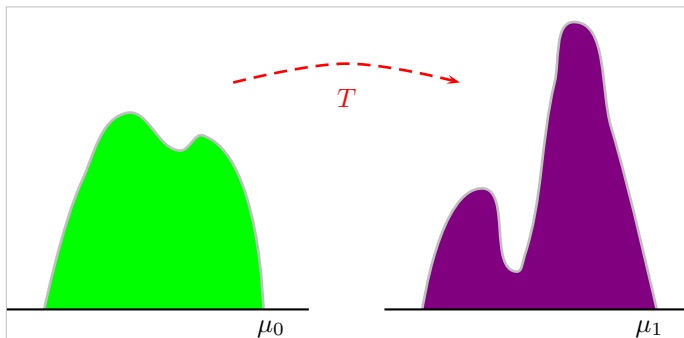
$$\mathfrak{S}_{(x,v)}(\xi, \eta) = -\frac{3}{2} \cdot \frac{d^2}{ds^2} \Big|_{s=0} \frac{d^2}{dt^2} \Big|_{t=0} \frac{d^2}{2} \left(\exp_x(t\xi), \exp_x(v+s\eta) \right)$$

for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_x M$.

Transport maps

Let μ_0 and μ_1 be **probability measures** on M . We call **transport map** from μ_0 to μ_1 any measurable map $T : M \rightarrow M$ such that $T_{\#}\mu_0 = \mu_1$, that is

$$\mu_1(B) = \mu_0(T^{-1}(B)), \quad \forall B \text{ measurable } \subset M.$$



Quadratic Monge's Problem

Given two probabilities measures μ_0, μ_1 sur M , find a transport map $T : M \rightarrow M$ from μ_0 to μ_1 which **minimizes** the quadratic cost ($c = d^2/2$)

$$\int_M c(x, T(x)) d\mu_0(x).$$

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Theorem (McCann '01)

*If μ_0 is absolutely continuous w.r.t. Lebesgue, then there exists a unique optimal transport map T from μ_0 to μ_1 . In fact, there is a **c-convex** function $\varphi : M \rightarrow \mathbb{R}$ satisfying*

$$T(x) = \exp_x(\nabla\varphi(x)) \quad \mu_0 \text{ a.e. } x \in M.$$

(Moreover, for a.e. $x \in M$, $\nabla\varphi(x)$ belongs to the injectivity domain at x .)

The case of \mathbb{R}^n

Quadratic Monge's Problem in \mathbb{R}^n : Given two probability measures μ_0, μ_1 with compact supports in \mathbb{R}^n , we are concerned with transport maps $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ pushing forward μ_0 to μ_1 which **minimize** the transport cost

$$\int_{\mathbb{R}^n} |T(x) - x|^2 d\mu_0(x).$$

Theorem (Brenier '91)

*If μ_0 is absolutely continuous with respect to the Lebesgue measure, there exists a unique optimal transport map with respect to the quadratic cost. In fact, there is a **convex** function $\psi : M \rightarrow \mathbb{R}$ such that*

$$T(x) = \nabla\psi(x) \quad \mu_0 \text{ a.e. } x \in \mathbb{R}^n.$$

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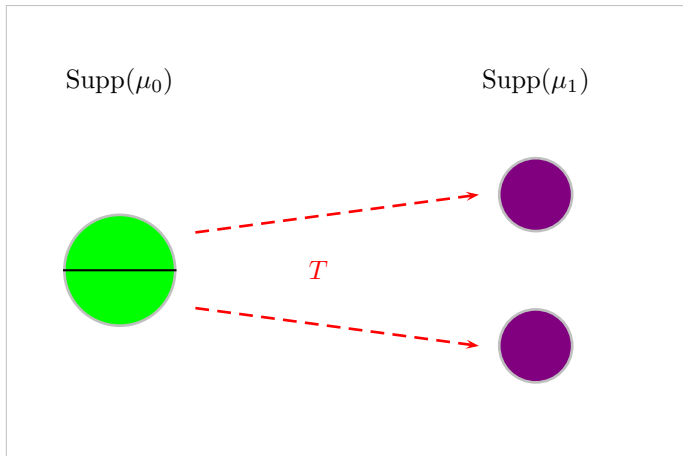
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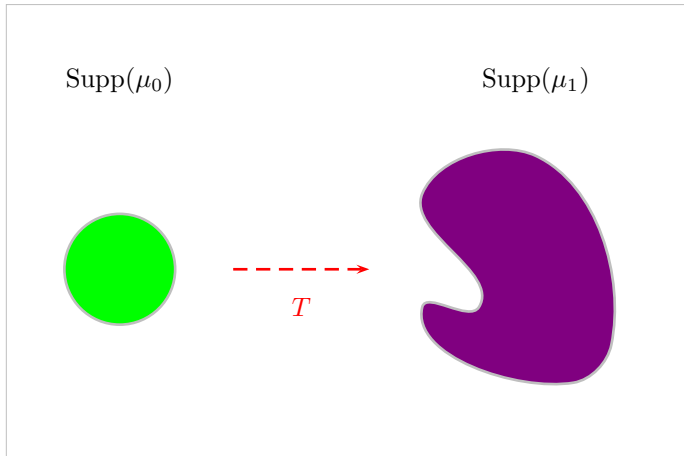
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Necessary and sufficient conditions for regularity ?

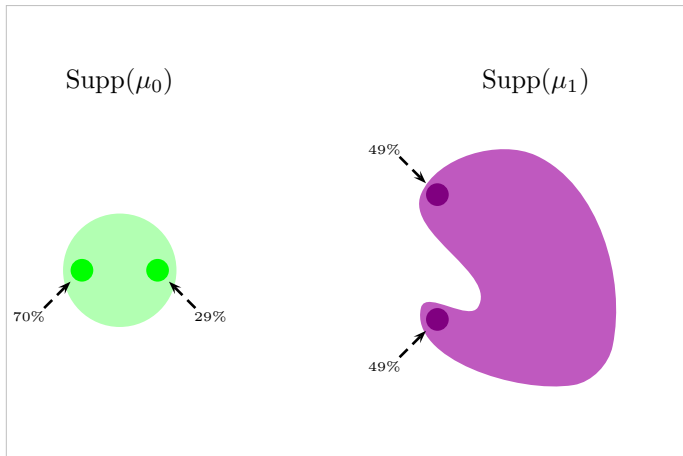
An obvious counterexample



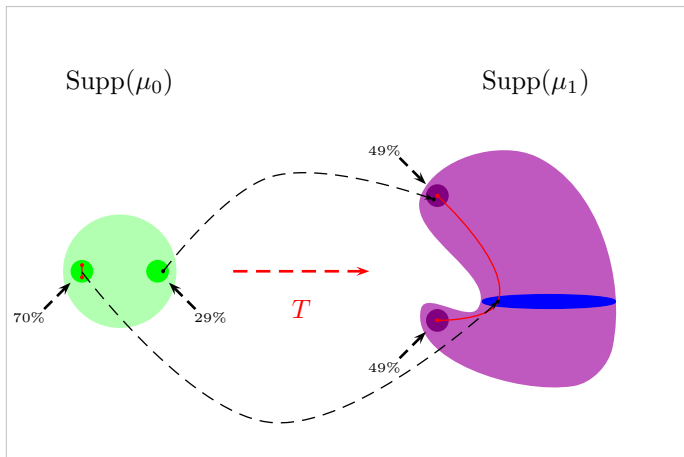
The convexity of the target is necessary



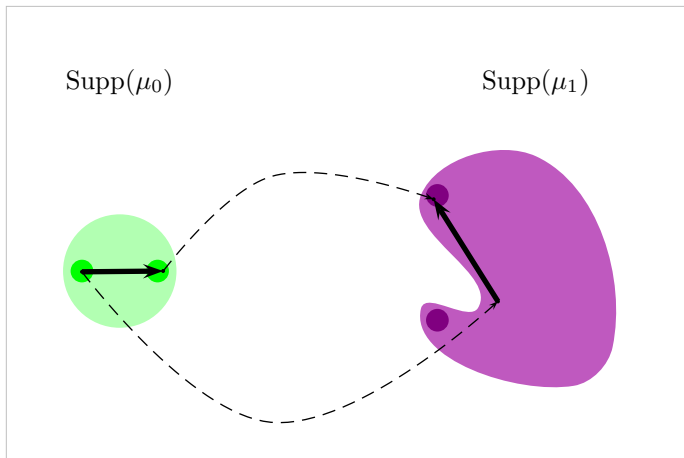
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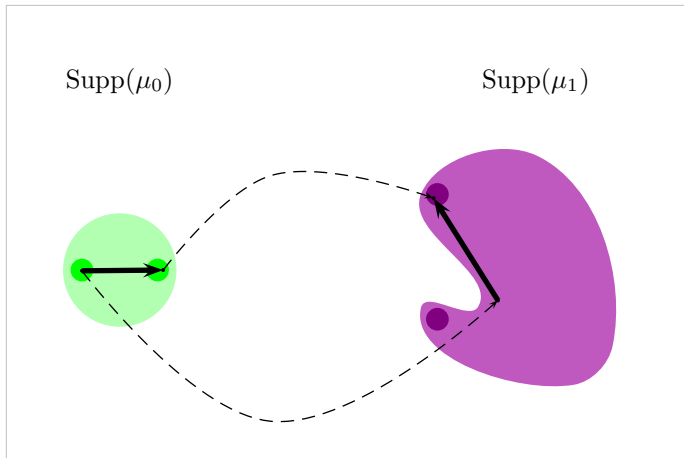
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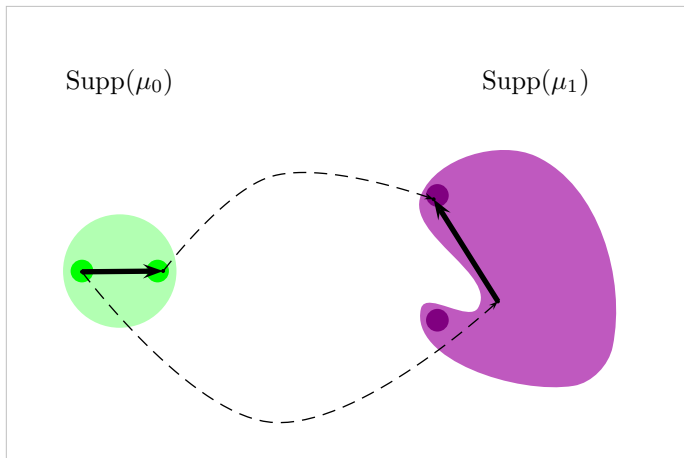


The convexity of the target is necessary



T gradient of a convex function

The convexity of the target is necessary



T gradient of a convex function $\implies \langle y-x, T(y)-T(x) \rangle \geq 0!!!$

Caffarelli's Regularity Theory

If μ_0 and μ_1 are associated with densities f_0, f_1 w.r.t. Lebesgue, then

$$T_{\#}\mu_0 = \mu_1 \iff \int_{\mathbb{R}^n} \zeta(T(x)) f_0(x) dx = \int_{\mathbb{R}^n} \zeta(y) f_1(y) dy \quad \forall \zeta.$$

$\rightsquigarrow \psi$ weak solution of the **Monge-Ampère equation** :

$$\det(\nabla^2 \psi(x)) = \frac{f_0(x)}{f_1(\nabla \psi(x))}.$$

Theorem (Caffarelli '90s)

Let Ω_0, Ω_1 be connected and bounded open sets in \mathbb{R}^n and f_0, f_1 be probability densities resp. on Ω_0 and Ω_1 such that $f_0, f_1, 1/f_0, 1/f_1$ are **essentially bounded**. Assume that μ_0 and μ_1 have respectively densities f_0 and f_1 w.r.t. Lebesgue and that Ω_1 is **convex**. Then the quadratic optimal transport map from μ_0 to μ_1 is continuous.

Back to Riemannian manifolds

Given two probabilities measures μ_0, μ_1 sur M , find a transport map $T : M \rightarrow M$ from μ_0 to μ_1 which minimizes the quadratic cost ($c = d^2/2$)

$$\int_M c(x, T(x)) d\mu_0(x).$$

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Definition

We say that the Riemannian manifold (M, g) satisfies the **Transport Continuity Property (TCP)** if for any pair of probability measures μ_0, μ_1 associated locally with **continuous positive densities** ρ_0, ρ_1 , that is

$$\mu_0 = \rho_0 \text{vol}_g, \quad \mu_1 = \rho_1 \text{vol}_g,$$

the optimal transport map from μ_0 to μ_1 is **continuous**.

Exponential mapping and injectivity domains

Let $x \in M$ be fixed.

- For every $v \in T_x M$, we define the **exponential** of v by

$$\exp_x(v) = \gamma_{x,v}(1),$$

where $\gamma_{x,v} : [0, 1] \rightarrow M$ is the unique geodesic starting at x with speed v .

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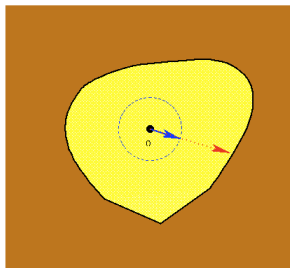
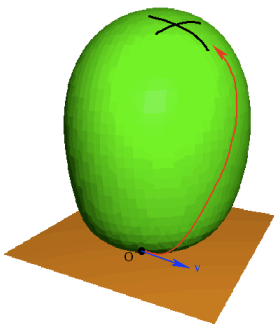
- We call **injectivity domain** at x , the subset of $T_x M$ defined by

$$\mathcal{I}(x) := \left\{ v \in T_x M \mid \exists t > 1 \text{ s.t. } \gamma_{tv} \text{ is the unique minim. geod. between } x \text{ and } \exp_x(tv) \right\}$$

It is a star-shaped (w.r.t. $0 \in T_x M$) bounded open set with Lipschitz boundary.

The distance to the cut locus

Using the exponential mapping, we can associate to each unit tangent vector its **distance to the cut locus**.



In that way, we define the so-called **injectivity domain** $\mathcal{I}(x)$ whose boundary is the **tangent cut locus** $\text{TCL}(x)$.

A necessary condition for **TCP**

Theorem (Figalli-R-Villani '10)

Let (M, g) be a smooth compact Riemannian manifold satisfying **TCP**. Then the following properties hold:

- all the injectivity domains are convex,
- for any $x, x' \in M$, the function

$$F_{x,x'} : v \in \mathcal{I}(x) \mapsto c(x, \exp_x(v)) - c(x', \exp_x(v))$$

is **quasiconvex** (its sublevel sets are always convex).

Almost characterization of quasiconvex functions

Lemma (Sufficient condition)

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \rightarrow \mathbb{R}$ be a function of class C^2 . Assume that for every $v \in U$ and every $w \in \mathbb{R}^n \setminus \{0\}$ the following property holds :

$$\langle \nabla_v F, w \rangle = 0 \implies \langle \nabla_v^2 F w, w \rangle > 0.$$

Then F is quasiconvex.

Almost characterization of quasiconvex functions

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Lemma (Necessary condition)

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \rightarrow \mathbb{R}$ be a function of class C^2 . Assume that F is quasiconvex. Then for every $v \in U$ and every $w \in \mathbb{R}^n \setminus \{0\}$ the following property holds :

$$\langle \nabla_v F, w \rangle = 0 \implies \langle \nabla_v^2 F w, w \rangle \geq 0.$$

Proof of the easy lemma

Proof.

Let $v_0, v_1 \in U$ be fixed. Set $v_t := (1 - t)v_0 + tv_1$, for every $t \in [0, 1]$. Define $h : [0, 1] \rightarrow \mathbb{R}$ by

$$h(t) := F(v_t) \quad \forall t \in [0, 1].$$

If $h \not\leq \max\{h(0), h(1)\}$, there is $\tau \in (0, 1)$ such that

$$h(\tau) = \max_{t \in [0, 1]} h(t) > \max\{h(0), h(1)\}.$$

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There holds

$$\dot{h}(\tau) = \langle \nabla_{v_\tau} F, \dot{v}_\tau \rangle \quad \text{et} \quad \ddot{h}(\tau) = \langle \nabla_{v_\tau}^2 F \dot{v}_\tau, \dot{v}_\tau \rangle.$$

Since τ is a local maximum, one has $\dot{h}(\tau) = 0$.

Contradiction !!



The Ma-Trudinger-Wang tensor

The **MTW** tensor \mathfrak{G} is defined as

$$\mathfrak{G}_{(x,v)}(\xi, \eta) = -\frac{3}{2} \frac{d^2}{ds^2} \Big|_{s=0} \frac{d^2}{dt^2} \Big|_{t=0} \frac{d^2}{2} \left(\exp_x(t\xi), \exp_x(v+s\eta) \right),$$

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for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_x M$.

Proposition (Villani '09, Figalli-R-Villani '10)

Let (M, g) be such that all injectivity domains are convex.

Then the following properties are equivalent:

- *All the functions $F_{x,x'}$ are quasiconvex.*
- *The **MTW** tensor \mathfrak{G} is $\succeq 0$, that is for any $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_x M$,*

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{G}_{(x,v)}(\xi, \eta) \geq 0.$$

When geometry enters the problem

Theorem (Loeper '06)

For every $x \in M$ and for any pair of unit orthogonal tangent vectors $\xi, \eta \in T_x M$, there holds

$$\mathfrak{G}_{(x,0)}(\xi, \eta) = \kappa_x(\xi, \eta),$$

where the latter denotes the **sectional curvature** of M at x along the plane spanned by $\{\xi, \eta\}$.

Corollary (Loeper '06)

$$\mathbf{TCP} \implies \mathfrak{G} \succeq 0 \implies \kappa \geq 0.$$

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$$\mathbf{TCP} \implies \mathfrak{G} \succeq 0 \implies \kappa \geq 0.$$

Caution!!! $\kappa \geq 0 \not\Rightarrow \mathfrak{G} \succeq 0$.

Sufficient conditions for **TCP**

Theorem (Figalli-R-Villani '10)

Let (M, g) be a compact smooth Riemannian **surface**. It satisfies **TCP** if and only if the two following properties holds:

- all the injectivity domains are convex,
- $\mathcal{G} \succeq 0$.

Sufficient conditions for **TCP**

Theorem (Figalli-R-Villani '10)

Let (M, g) be a compact smooth Riemannian **surface**. It satisfies **TCP** if and only if the two following properties holds:

- all the injectivity domains are convex,
- $\mathfrak{S} \succeq 0$.

Theorem (Figalli-R-Villani '10)

Assume that (M, g) satisfies the two following properties:

- all its injectivity domains are **strictly convex**,
- the **MTW** tensor \mathfrak{S} is $\succ 0$, that is for any $x \in M, v \in \mathcal{I}(x)$, and $\xi, \eta \in T_x M$,

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi, \eta) > 0.$$

Then, it satisfies **TCP**.

Examples and counterexamples

Examples:

- Flat tori (Cordero-Erausquin '99)
- Round spheres (Loeper '06)
- Product of spheres (Figalli-Kim-McCann '13)
- Quotients of the above objects
- Perturbations of round spheres (Figalli-R-Villani '12)

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Counterexamples:



- How is the set of metrics satisfying $\mathcal{G} \succ 0$?
- Does $\mathcal{G} \succeq 0$ imply convexity of injectivity domains ?
- Does $\mathcal{G} \succeq 0$ imply more topological obstructions than $\kappa \geq 0$?

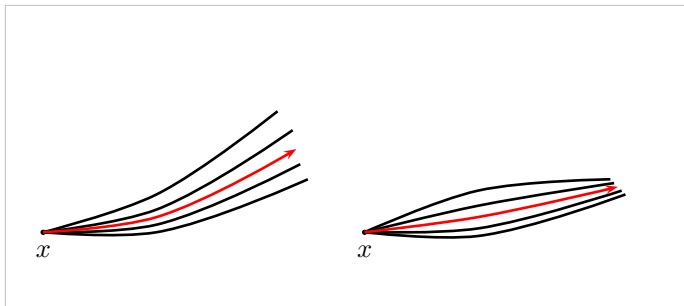
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Focalization is the major obstacle

Focalization

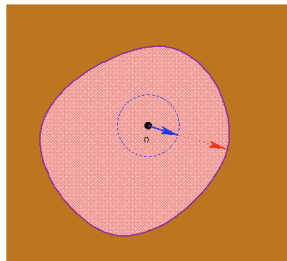
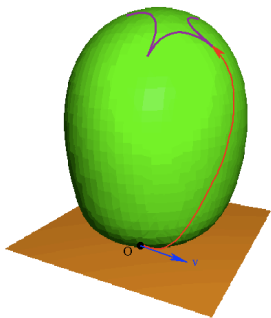
Definition

Let $x \in M$ and v be a unit tangent vector in $T_x M$. The vector v is **not conjugate** at time $t \geq 0$ if for any $t' \in [0, t + \delta]$ ($\delta > 0$ small) the geodesic from x to $\gamma(t')$ is locally minimizing.



The distance to the conjugate locus

Again, using the exponential mapping, we can associate to each unit tangent vector its **distance to the conjugate locus**.

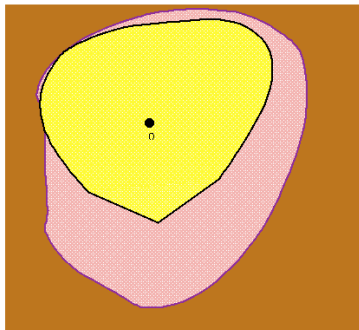


In that way, we define the so-called **nonfocal domain** $\mathcal{NF}(x)$ whose boundary is the **tangent conjugate locus** $\text{TFL}(x)$.

Fundamental inclusion

The following inclusion holds

injectivity domain \subset nonfocal domain.



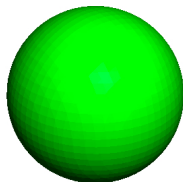
Spheres

Loeper checked that the **MTW** tensor of the round sphere \mathbb{S}^2 satisfies for any $x \in \mathbb{S}^2$, $v \in \mathcal{I}(x)$ and $\xi, \eta \in T_x \mathbb{S}^2$,

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{G}_{(x,v)}(\xi, \eta) \geq |\xi|^2 |\eta|^2.$$

Theorem (Loeper '06)

*The round sphere \mathbb{S}^2 satisfies **TCP**.*



Small deformations of \mathbb{S}^2

On \mathbb{S}^2 , the **MTW** tensor is given by

$$\begin{aligned} \mathfrak{G}_{(x,v)}(\xi, \xi^\perp) &= 3 \left[\frac{1}{r^2} - \frac{\cos(r)}{r \sin(r)} \right] \xi_1^4 + 3 \left[\frac{1}{\sin^2(r)} - \frac{r \cos(r)}{\sin^3(r)} \right] \xi_2^4 \\ &\quad + \frac{3}{2} \left[-\frac{6}{r^2} + \frac{\cos(r)}{r \sin(r)} + \frac{5}{\sin^2(r)} \right] \xi_1^2 \xi_2^2, \end{aligned}$$

with $x \in \mathbb{S}^2$, $v \in \mathcal{I}(x)$, $r := |v|$, $\xi = (\xi_1, \xi_2)$, $\xi^\perp = (-\xi_2, \xi_1)$.

Theorem (Figalli-R '09)

*Any small deformation of \mathbb{S}^2 in C^4 topology satisfies **TCP**.*

The Villani Conjecture

Conjecture (Villani's Conjecture)

Any Riemannian manifold satisfying $\mathcal{G} \succeq 0$ has convex injectivity domains.

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Any Riemannian manifold satisfying $\mathfrak{S} \succeq 0$ has convex injectivity domains.

Theorem (Figalli-Gallouët-R '14)

*Let (M, g) be a smooth compact Riemannian manifold, assume that it is **nonfocal**. Then all its injectivity domains are convex.*

Thank you for your attention !!