Optimal Transport on Surfaces

Ludovic Rifford

Université de Nice - Sophia Antipolis

Optimal transport on Riemannian manifolds

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Denote by d_g the geodesic distance on M and define the quadratic cost $c : M \times M \longrightarrow [0, \infty)$ by

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Given two Borelian probability measures μ_0, μ_1 on M, find a mesurable map $T : M \rightarrow M$ satisfying

$$T_\sharp \mu_0 = \mu_1 \quad \text{(i.e. } \mu_1(B) = \mu_0\left(T^{-1}(B)\right), \forall B \text{ borelian } \subset M),$$

and minimizing

$$\int_M c(x, T(x)) d\mu_0(x).$$

The Brenier-McCann Theorem

Theorem (McCann '01)

Let μ_0, μ_1 be two probability measures on M. If μ_0 is absolutely continuous w.r.t. the Lebesgue measure, then there is a unique optimal transport map $T : M \to M$ satisfying $T_\sharp \mu_0 = \mu_1$ and minimizing

$$\int_M c(x, T(x)) d\mu_0(x).$$

It is characterized by the existence of a semiconvex function $\psi : M \to \mathbb{R}$ such that

$$T(x) = \exp_x \left(\nabla \psi(x)\right) \qquad \text{for } \mu_0 \text{ a.e. } x \in \mathbb{R}^n.$$

The Transport Continuity Property

We say that (M, g) satisfies the **Transport Continuity Property (TCP)** if the following property is satisfied:
For any pair of probability measures μ_0, μ_1 associated locally with **continuous positive densities** ρ_0, ρ_1, that is

$$\mu_0 = \rho_0 \mathcal{L}^n, \quad \mu_1 = \rho_1 \mathcal{L}^n,$$

the optimal transport map between μ_0 and μ_1 is **continuous**.

Characterization of surfaces satisfying **TCP**

Theorem (Figalli-LR-Villani '10)

The surface (M, g) satisfies **(TCP)** *if and only if the two following properties hold:*

- *all the injectivity domains are convex,*
- *the cost c is regular.*

Injectivity domains

Let $x \in M$ be fixed. We call exponential mapping from x, the mapping defined as

$$\begin{array}{rcl} \exp_x : T_xM & \longrightarrow & M \\ v & \longmapsto & \exp_x(v) := \gamma_v(1), \end{array}$$

where $\gamma_v : [0,1] \to M$ is the unique geodesic starting at x with speed $\dot{\gamma}_v(0) = v$.

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$$\mathcal{I}(x) := \left\{ v \in T_xM \;\middle|\; \begin{array}{c} \exists t > 1 \text{ s.t. } \gamma_{tv} \text{ is the unique minimizing} \\ \text{geodesic between } x \text{ and } \exp_x(tv) \end{array} \right\}.$$

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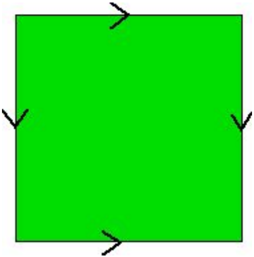
It is a star-shaped (w.r.t. $0 \in T_xM$) domain with Lipschitz boundary.

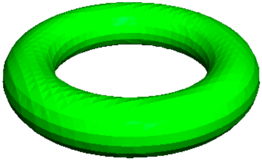
Injectivity domains: Examples...

Flat tori : all injectivity domains are convex.



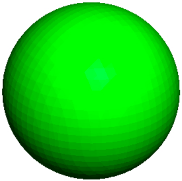
Injectivity domains: Examples...

Tori of revolution: injectivity domains are not necessarily convex.



Injectivity domains: Examples...

Round spheres: all injectivity domains are convex.

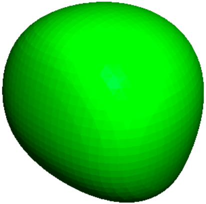


Injectivity domains: Examples...

C^4 small perturbations of round spheres:

Theorem (Figalli-LR '09)

Any small deformation of the round sphere $(\mathbb{S}^2, g^0)$ in C^4 topology has all its injectivity domains convex.



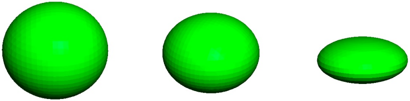
Injectivity domains: examples...

Oblate ellipsoids of revolution:

$$E_\mu : \quad x^2 + y^2 + \left(\frac{z}{\mu}\right)^2 = 1 \qquad \mu \in (0,1].$$

Theorem (Bonnard-Caillau-LR '10)

The injectivity domain on an oblate ellipsoid of revolution is convex for any point if and only if the ratio between the minor and the major axis is greater or equal to $1/\sqrt{3}$.



Regular costs

The cost $c = d^2/2 : M \times M \longrightarrow \mathbb{R}$ is called **regular**, if for every $x \in M$ and every $v_0, v_1 \in \mathcal{I}(x)$, there holds

$$v_t := (1-t)v_0 + tv_1 \in \mathcal{I}(x) \qquad \forall t \in [0,1],$$

and

$$c(x, y_t) - c(x', y_t) \leq \max\Big(c(x, y_0) - c(x', y_0),\, c(x, y_1) - c(x', y_1) \Big),$$

for any $x' \in M$, where $y_t := \exp_x v_t$.

An obvious remark

Remark

Assume that all the injectivity domains of (M,g) are convex. Then the cost c is regular if and only if for every $x, x' \in M$, the mapping

$$F_{x,x'} \ : \ v \in \mathcal{I}(x) \longmapsto c\big(x, \exp_x(v)\big) - c\big(x', \exp_x(v)\big)$$

is quasiconvex (its sublevels sets are always convex).

An easy lemma

Lemma

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class C^2. Assume that for every $v \in U$ and every $w \in \mathbb{R}^n \setminus \{0\}$, the following property holds

$$\langle \nabla_v F, w \rangle = 0 \Longrightarrow \langle \nabla_v^2 F \, w, w \rangle > 0.$$

Then F is quasiconvex.

Proof of the easy lemma

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Let $v_0, v_1 \in U$ be fixed. Set $v_t := (1-t)v_0 + tv_1$, for every $t \in [0,1]$, and define $h : [0,1] \longrightarrow \mathbb{R}$ by

$$h(t) := F(v_t) \qquad \forall t \in [0,1].$$

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$$h(t) := F(v_t) \qquad \forall t \in [0,1].$$

If $h \not\leq \max\{h(0), h(1)\}$, there is $\tau \in (0,1)$ such that

$$h(\tau) = \max_{t \in [0,1]} h(t) \; > \; \max\{h(0), h(1)\}.$$

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There holds

$$\dot{h}(\tau) = \langle \nabla_{v_\tau} F, \dot{v}_\tau \rangle \quad \text{and} \quad \ddot{h}(\tau) = \langle \nabla^2_{v_\tau} F\, \dot{v}_\tau, \dot{v}_\tau \rangle.$$

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Since τ is a local maximum, we get a contradiction. $\square$

Exercice 1

The following lemma is false !!

FALSE Lemma

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class C^2. Assume that for every $v \in U$ and every $w \in \mathbb{R}^n$, the following property holds

$$\langle \nabla_v F, w \rangle = 0 \implies \langle \nabla_v^2 F\, w, w \rangle \geq 0.$$

Then F is quasiconvex.

Exercice 2

However, the following result holds true.

Lemma

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class C^2. Assume that there is a constant $C > 0$ such that

$$\langle \nabla_v^2 F\, w, w \rangle \geq -C \left| \langle \nabla_v F, w \rangle \right| |w| \qquad \forall v \in U, \forall w \in \mathbb{R}^n.$$

Then F is quasiconvex.

The Ma-Trudinger-Wang tensor

The **MTW** tensor $\mathfrak{S}$ is defined as

$$\mathfrak{S}_{(x,v)}(\xi,\eta) = -\frac{3}{2}\,\frac{d^2}{ds^2}\Big|_{s=0}\,\frac{d^2}{dt^2}\Big|_{t=0} c\left(\exp_x(t\xi), \exp_x(v+s\eta)\right),$$

for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$.

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for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi,\eta \in T_xM$.

Proposition (Villani '09, Figalli-LR-Villani '10)

Assume that all the injectivity domains are convex. Then, the two following properties are equivalent:

- *the cost c is regular,*
- *the* **MTW** *tensor $\mathfrak{S}$ is $\succeq 0$, that is, for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi,\eta \in T_xM$,*

$$\langle \xi,\eta\rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi,\eta) \geq 0.$$

Another characterization of **(TCP)** surfaces

Theorem (Figalli-LR-Villani '10)

Let (M,g) be a compact surface. Then, it satisfies **(TCP)** *if and only if the following properties hold:*

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Loeper noticed that for every $x \in M$ and for any pair of unit orthogonal tangent vectors $\xi, \eta \in T_xM$, there holds

$$\mathfrak{S}_{(x,0)}(\xi,\eta) = \kappa_x,$$

where κ_x denotes the gaussian curvature of M at x.
Consequently, any (M,g) satisfying **TCP** must have nonnegative gaussian curvatures.

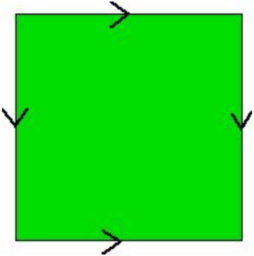
The flat torus

The **MTW** tensor of the flat torus $\left(\mathbb{T}^n, g^0\right)$ satisfies

$$\mathfrak{S}_{(x,v)} \equiv 0 \qquad \forall x \in \mathbb{T}^n, \forall v \in \mathcal{I}(x)$$

Theorem (Cordero-Erausquin '99)

The flat torus $\left(\mathbb{T}^n, g^0\right)$ *satisfies* **TCP**.



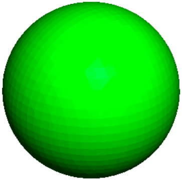
Round spheres

Loeper checked that the **MTW** tensor of the round sphere $\left(\mathbb{S}^n, g^0\right)$ satisfies for any $x \in \mathbb{S}^n$, $v \in \mathcal{I}(x)$ and $\xi, \eta \in T_x\mathbb{S}^n$,

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi, \eta) \geq \|\xi\|_x^2 \|\eta\|_x^2.$$

Theorem (Loeper '06)

The round sphere $\left(\mathbb{S}^n, g^0\right)$ satisfies **TCP**.



Riemannian quotients

Let G be a discrete group of isometries of (M, g) acting freely and properly. Then there exists on the quotient manifold $N = M/G$ a unique Riemannian metric h such that the canonical projection $p : M \longrightarrow N$ is a Riemannian covering map.

Theorem (Delanoe-Ge '08)

If (M, g) satisfies **TCP**, *then $\left(N = M/G, h\right)$ satisfies* **TCP**.

Examples: $\left(\mathbb{RP}^n, g^0\right)$, the flat Klein bottle.

On $(\mathbb{S}^2, g^0)$, the **MTW** tensor is given by

$$\begin{aligned}\mathfrak{S}_{(x,v)}(\xi,\xi^\perp) &= 3\left[\frac{1}{r^2}-\frac{\cos(r)}{r\sin(r)}\right]\xi_1^4 + 3\left[\frac{1}{\sin^2(r)}-\frac{r\cos(r)}{\sin^3(r)}\right]\xi_2^4 \\ &\quad + \frac{3}{2}\left[-\frac{6}{r^2}+\frac{\cos(r)}{r\sin(r)}+\frac{5}{\sin^2(r)}\right]\xi_1^2\xi_2^2,\end{aligned}$$

with $x \in \mathbb{S}^2, v \in \mathcal{I}(x), r := \|v\|_x, \xi = (\xi_1,\xi_2), \xi^\perp = (-\xi_2,\xi_1)$.

Small deformations of $(\mathbb{S}^2, g^0)$

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&\qquad\qquad + \frac{3}{2}\left[-\frac{6}{r^2} + \frac{\cos(r)}{r\sin(r)} + \frac{5}{\sin^2(r)}\right]\xi_1^2\xi_2^2,
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Theorem (Figalli-LR '09)

Any small deformation of the round sphere $(\mathbb{S}^2, g^0)$ in C^4 topology satisfies **TCP**.

Oblate ellipsoids

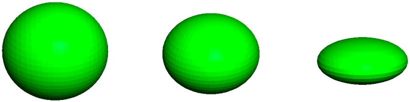
Any oblate ellipsoids of revolution

$$E_\mu : \quad x^2 + y^2 + \left(\frac{z}{\mu}\right)^2 = 1$$

with

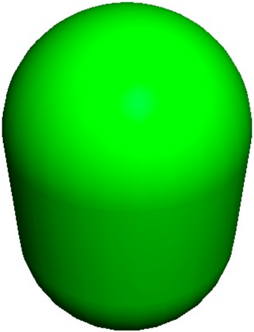
$$\mu < \frac{1}{\sqrt{3}}$$

does not satisfy **TCP**.



Jump of curvature

The surface made with two half-balls joined by a cylinder has not a regular cost.



Then, it does not satisfy **TCP**.

Thank you for your attention !

Greater dimension: Mind the gap !

Theorem (Necessary conditions)

Assume that (M^n, g) satisfies **(TCP)** *then the following properties hold:*

- *all the injectivity domains are convex,*
- $\mathfrak{S} \succeq 0$.

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Theorem (Sufficient conditions)

Assume that (M^n, g) satisfies the following properties:

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- $\mathfrak{S} \succ 0$.

Then (M, g) satisfies **TCP**.

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Then (M, g) *satisfies* **TCP**.

There is a gap !