Control and Dynamics

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Analysis and Geometry in Control Theory
and its Applications

Setting

Let M be a smooth compact manifold of dimension $n \geq 2$ be fixed. Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian of class C^k, with $k \geq 2$, recall that the Hamiltonian vector field reads (in local coordinates)

$$X_H(x,p) = \begin{pmatrix} \frac{\partial H}{\partial p}(x,p) \\ -\frac{\partial H}{\partial x}(x,p) \end{pmatrix}.$$

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Examples:

- $H(x,p) = \|p\|_x^2/2$ (Riemannian)
- $H(x,p) = \|p\|_x^2/2 + V(x)$ (mechanical)
- $H(x,p) = \|p\|_x^2/2 + p \cdot X(x)$ (Mañé)
- Tonelli Hamiltonians

Problems

Two types of problem

1. Change the behavior of an orbit: e.g. close a recurrent orbit or an orbit through a non-wandering point of the Hamiltonian flow into a periodic orbit
   $\rightsquigarrow$ Closing Lemma

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2. Change the behavior of ϕ_t^H along a given orbit
   $\rightsquigarrow$ Franks' Lemma

The Pugh closing lemma

Let X be a C^1 vector field on a compact manifold M and $x \in M$ be a non-wandering point w.r.t to the flow of X.

Proposition

For every $\epsilon > 0$, there is a C^1 vector field Y having x as a periodic point such that $\|Y - X\|_{C^0} < \epsilon$.

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Theorem (Pugh, 1967)

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The Franks Lemma for vector fields

Let $\bar{x}$ be a periodic point for the flow of X of period $T > 0$. Fix a local section Σ transverse to the flow at $\bar{x}$ and consider the **Poincaré first return map**

$$\begin{array}{rccc} P: & \Sigma & \longrightarrow & \Sigma \\ & x & \longmapsto & \phi^X_{\tau(x)}(x). \end{array}$$

It is a local C^1 diffeomorphism fixing $\bar{x}$.

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Lemma (Franks, 1971)

For every $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that for every isomorphism $Q : T_{\bar{x}}\Sigma \to T_{\bar{x}}\Sigma$ satisfying

$$\|Q - d_{\bar{x}}P\| < \delta,$$

there exists a C^1 vector field Y which preserves the orbit of $\bar{x}$ such that

$$\|Y - X\|_{C^1} < \epsilon \quad \text{and} \quad d_{\bar{x}}P = Q.$$

Generic vector fields

Let X be a C^1 vector field on M, we set

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Theorem (Pugh)

Let M be a smooth compact manifolds, the set of C^1 vector fields X on M such that

$$\overline{Per(X)} = \Omega(X),$$

is residual in $\mathcal{X}^1(M)$ (the set of C^1 vector fields on M) .

Closing hamiltonian orbits

Theorem (Pugh-Robinson, 1983)

Let (N, ω) be a symplectic manifold of dimension $2n \geq 2$ and $H : N \to \mathbb{R}$ be a given Hamiltonian of class C^2. Let X be the Hamiltonian vector field associated with H and ϕ^H the Hamiltonian flow. Suppose that $x \in N$ is a non-wandering point of the flow of X and that $\mathcal{U}$ is a neighborhood of X in the C^1 topology. Then there exists $Y \in \mathcal{U}$ such that Y is a Hamiltonian vector field and Y has a closed orbit through x.

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Questions:

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Questions:

- If $(N, \omega) = (T^*M, w_{can})$, can we close a recurrent orbit by adding a small potential $(H \rightsquigarrow H + V)$?
- If $H = (1/2)\|p\|_x^2$, can we close a recurrent orbit by a small perturbation of the Riemannian metric ?

Closing geodesics in low topology

Theorem (Rifford, 2012)

Let M be a smooth compact manifold and g be a Riemannian metric on M of class C^k with $k \geq 3$ and let $(x, v) \in U^g M$ be a non-wandering point for the geodesic flow. Then for every $\epsilon > 0$, there exists a metric $\tilde{g}$ of class C^{k-1} with $\|\tilde{g} - g\|_{C^1} < \epsilon$ such that the geodesic starting from x with initial velocity v is periodic.

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Remark

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The closing lemma for the geodesic flow in the C^2 topology on the metric is open.

Back to Franks' Lemma

Let M be a smooth compact manifold and $H : T^*M \to \mathbb{R}$ an Hamiltonian of class C^k, with $k \geq 2$.
Let $\bar{\theta} = (\bar{x}, \bar{p})$ be a periodic point for the Hamiltonian flow of positive period $T > 0$.
Fix a local section transversal to the flow at $\bar{\theta}$ and contained in the energy level of $\bar{\theta}$.

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Then consider the **Poincaré first return map**

$$\begin{array}{rcl} P : \Sigma & \longrightarrow & \Sigma \\ \theta & \longmapsto & \phi^H_{\tau(\theta)}(\theta), \end{array}$$

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The Poincaré map is symplectic, i.e. it preserves the restriction of the symplectic form to $T_\theta\Sigma$.

The symplectic group

Let $\mathrm{Sp}(m)$ be the symplectic group in $M_{2m}(\mathbb{R})$ $(m = n - 1)$, that is the smooth submanifold of matrices $X \in M_{2m}(\mathbb{R})$ satisfying

$$X^* \mathbb{J} X = \mathbb{J} \quad \text{where } \mathbb{J} := \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}.$$

Choosing a convenient set of coordinates, the differential of the Poincaré map is the symplectic matrix $X(T)$ where $X : [0, T] \to \mathrm{Sp}(m)$ is solution to the Cauchy problem

$$\begin{cases} \dot{X}(t) = A(t)X(t) & \forall t \in [0, T], \\ X(0) = I_{2m}, \end{cases}$$

where $A(t)$ has the form

$$A(t) = \begin{pmatrix} 0 & I_m \\ -K(t) & 0 \end{pmatrix} \qquad \forall t \in [0, T].$$

Franks's Lemma à la Mañé

Problem:

Given $\epsilon > 0$,
does the set of differentials of Poincaré maps (restricted to $T_\theta\Sigma$) associated with potentials $V : M \to \mathbb{R}$ of class C^k such that

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What's the radius of that ball in term of ϵ ?

Perturbation of the Poincaré map

Let γ be the projection of the periodic orbit passing through $\bar{\theta}$, we are looking for a potential

$$V : M \longrightarrow \mathbb{R}$$

satisfying the following properties

$$V\big(\gamma(t)\big) = 0, \quad dV\big(\gamma(t)\big) = 0,$$

with

$$d^2V\big(\gamma(t)\big) \quad \text{free.}$$

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$$\Longrightarrow \quad d^2V\big(\gamma(t)\big) \quad \text{is the control.}$$

A controllability problem on $\mathrm{Sp}(m)$

The Poincaré map at time T associated with the new Hamiltonian

$$H + V$$

is given by $X_u(T)$ where $X_u : [0, T] \to \mathrm{Sp}(m)$ is solution to the control problem

$$\begin{cases} \dot{X}_u(t) = A(t)X_u(t) + \sum_{i\leq j=1}^{m} u_{ij}(t)\mathcal{E}(ij)X_u(t), & \forall t \in [0, T], \\ X(0) = I_{2m}, \end{cases}$$

where the $2m \times 2m$ matrices $\mathcal{E}(ij)$ are defined by

$$\mathcal{E}(ij) := \begin{pmatrix} 0 & 0 \\ E(ij) & 0 \end{pmatrix},$$

with
$$\begin{cases} (E(ii))_{k,l} := \delta_{ik}\delta_{il} \; \forall i = 1, \ldots, m, \\ (E(ij))_{k,l} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \; \forall i < j = 1, \ldots, m. \end{cases}$$

References:

- "Generic properties of closed orbits of Hamiltonian flows from Mañé's viewpoint"
  L.R., Rafael Ruggiero, IMRN, 2012.
- "Franks' Lemma for C^2-Mañé perturbations of Riemannian metrics and applications to persistence"
  Ayadi Lazrag, L.R., Rafael Ruggiero, Preprint, 2014.

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Thank you for your attention !!