Aubry sets, Hamilton-Jacobi equations, and Mañé Conjecture

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1 Mañé critical value and critical subsolutions

Let (M, g) be a smooth connected compact Riemannian manifold without boundary of dimension $n \geq 2$, and $H : T^*M \to \mathbb{R}$ a C^k *Tonelli Hamiltonian* (with $k \geq 2$), that is, a Hamiltonian of class C^k satisfying the two following properties (here $\| \cdot \|^*$ denotes a norm on T^*M; we note that the properties we introduce are independent of the particular choice of the norm):

(H1) *Superlinear growth:* For every $K \geq 0$ there exists a constant $C^*(K)$ such that

$$H(x,p) \geq K\|p\|_x^* - C^*(K) \qquad \forall\, (x,p) \in T^*M.$$

(H2) *Uniform convexity:* For every $(x, p) \in T^*M$, the second derivative along the fibers $\frac{\partial^2 H}{\partial p^2}(x,p)$ is positive definite.

The *Mañé critical value* of H can be defined as follows:

Definition 1.1. *We call* critical value *of H, and we denote it by $c[H]$, the infimum of the values $c \in \mathbb{R}$ for which there exists a function $u : M \to \mathbb{R}$ of class C^1 satisfying*

$$H(x, du(x)) \leq c \qquad \forall\, x \in M. \tag{1.1}$$

A priori, it is not clear whether the infimum in Definition 1.1 is attained or not, that is, whether there exists a C^1 function satisfying (1.1) with $c = c[H]$. For this reason, we introduce the notion of *critical subsolutions*. (Recall that, by Rademacher's theorem, Lipschitz functions are differentiable almost everywhere.)

Definition 1.2. *A function $u : M \to \mathbb{R}$ is called a* critical subsolution *for H if it is Lipschitz and satisfies*

$$H\big(x, du(x)\big) \leq c[H] \qquad \textit{for a.e. } x \in M. \tag{1.2}$$

Thanks to the coercivity of H in the p variable (see (H1) above), it is not difficult to prove the following:

Proposition 1.3. *The set $\mathcal{SS}[H]$ of critical subsolutions is a nonempty, compact, convex subset of $C^0(M; \mathbb{R})$.*

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The Lagrangian $L : TM \to \mathbb{R}$ associated with H by Legendre-Fenchel duality is defined by

$$L(x,v) := \max_{p \in T_x^*M} \Big\{ \langle p, v \rangle - H(x,p) \Big\}.$$

Thanks to (H1)-(H2), it can be shown (see for instance [5, 15]) that L is a C^k Tonelli Lagrangian, that is, it is of class C^k and satisfies the two following properties ($\|\cdot\|$ denotes a norm on TM):

(L1) *Superlinear growth:* For every $K \geq 0$ there exists a constant $C(K)$ such that

$$L(x,v) \geq K\|v\|_x - C(K) \qquad \forall\, (x,v) \in TM.$$

(L2) *Uniform convexity:* For every $(x,v) \in TM$, $\frac{\partial^2 L}{\partial v^2}(x,v)$ is positive definite.

Note that the Fenchel inequality

$$\langle p, v \rangle \leq L(x,v) + H(x,p) \tag{1.3}$$

holds for any $x \in M$, $v \in T_xM, p \in T_x^*M$, with equality if and only if (in local coordinates)

$$v = \frac{\partial H}{\partial p}(x,p) \Leftrightarrow p = \frac{\partial L}{\partial v}(x,v). \tag{1.4}$$

In addition, L and H are dual, in the sense that H is given in terms of L by

$$H(x,p) = \max_{v \in T_x^*M} \Big\{ \langle p, v \rangle - L(x,v) \Big\}. \tag{1.5}$$

The Legendre-Fenchel duality allows us to characterize the critical subsolutions in a variational way.

Proposition 1.4. *A function $u : M \to \mathbb{R}$ is a critical subsolution if and only if*

$$u\big(\gamma(b)\big) - u\big(\gamma(a)\big) \leq \int_a^b L\big(\gamma(s), \dot\gamma(s)\big)\, ds + c[H]\,(b-a), \tag{1.6}$$

for any Lipschitz curve $\gamma : [a,b] \to M$.

Proof. $\Rightarrow$) Let $u : M \to \mathbb{R}$ be a critical subsolution of class C^1 (the case when u is Lipschitz is easily done by approximation). Then, using (1.3), for any Lipschitz curve $\gamma : [a,b] \to M$ we have

$$\begin{aligned} u\big(\gamma(b)\big) - u\big(\gamma(a)\big) &= \int_a^b \langle du\big(\gamma(s)\big), \dot\gamma(s) \rangle\, ds \\ &\leq \int_a^b L\big(\gamma(s), \dot\gamma(s)\big)\, ds + \int_a^b H\big(\gamma(s), du(\gamma(s))\big)\, ds \\ &\leq \int_a^b L\big(\gamma(s), \dot\gamma(s)\big)\, dt + c[H]\,(b-a). \end{aligned}$$

$\Leftarrow$) First of all, if u satisfies (1.6) then it is easy to show that it is Lipschitz. (Given $x, y \in M$, choose γ to be a constant speed geodesic from x to y parameterized over $[0, d(x,y)]$.)

Now, fix x a differentiability point for u, and let γ be a curve satisfying $(\gamma(0), \dot\gamma(0)) = (x,v)$. Then, by taking the limit as $t \to 0^+$ in the inequality

$$\frac{u\big(\gamma(t)\big) - u\big(\gamma(0)\big)}{t} \leq \frac{1}{t} \int_0^t L\big(\gamma(s), \dot\gamma(s)\big)\, ds + c[H]$$

we get

$$\langle du(x), v \rangle - L(x,v) \leq c[H],$$

and we conclude by (1.5) and the arbitrariness of v. $\square$

2 Lax-Oleinik semigroup and Weak KAM Theorem

The Lax-Oleinik semigroup $\{\mathcal{T}_t\}_{t\geq 0} : C^0(M;\mathbb{R}) \longrightarrow C^0(M;\mathbb{R})$ associated with L is defined as

$$\mathcal{T}_t u(x) := \inf\left\{u\big(\gamma(-t)\big) + \int_{-t}^{0} L\big(\gamma(s),\dot\gamma(s)\big)\,ds\,ds\right\} \qquad \forall\, x\in M,\ t\geq 0, \tag{2.1}$$

where the infimum is taken among all Lipschitz curves $\gamma : [-t,0] \to M$ such that $\gamma(0) = x$. If we define the function $h_t : M\times M \to \mathbb{R}$ by

$$h_t(z,x) := \inf_{\gamma}\left\{\int_0^t L\big(\gamma(s),\dot\gamma(s)\big)\,ds\right\} \qquad \forall\, x,z\in M,\ t>0, \tag{2.2}$$

where the infimum is taken over the set of Lipschitz curves $\gamma : [0,t] \to M$ which satisfy $\gamma(0) = z$ and $\gamma(t) = x$, then $\mathcal{T}_t u$ can also be written as

$$\mathcal{T}_t u(x) := \inf_{z\in M}\Big\{u(z) + h_t(z,x)\Big\} \qquad \forall\, x\in M. \tag{2.3}$$

Under our assumptions it is possible to show that the infimum in the definition of h_t is always attained, and that h_t is Lipschitz on $M\times M$ for any $t > 0$ (see for instance [30, Appendix A]). In particular, one can show that $\mathcal{T}_t$ is well-defined for all $t \geq 0$ and the infimum in (2.1) is always attained.

Using the above definitions it is easy to check that a function $u : M \to \mathbb{R}$ is a critical subsolution if and only if

$$u(x) - u(z) \leq h_t(z,x) + c[H]\,t \qquad \forall\, x,z\in M,\ t>0. \tag{2.4}$$

In fact, $\{\mathcal{T}_t\}_{t\geq 0}$ enjoys the following properties [15, 30]:

Proposition 2.1. *The following properties hold:*

(i) $\mathcal{T}_0 = Id$, *and* $\mathcal{T}_{t+t'} = \mathcal{T}_t \circ \mathcal{T}_{t'}$ *for any* $t,t' \geq 0$.

(ii) For every $t \geq 0$, $\|\mathcal{T}_t u - \mathcal{T}_t v\|_\infty \leq \|u - v\|_\infty$ *for any* $u, v \in C^0(M;\mathbb{R})$.

(iii) For every $u \in C^0(M;\mathbb{R})$, *the map* $t \in [0,\infty) \mapsto \mathcal{T}_t u \in C^0(M;\mathbb{R})$ *is continuous.*

(iv) The set $\mathcal{SS}[H]$ *is invariant with respect to* $\{\mathcal{T}_t\}$.

As shown in [15], if $u \in \mathcal{SS}(H)$ then the functions $u_t := \mathcal{T}_t u + c[H]\,t$ converge uniformly as $t \to +\infty$ to a function u_∞ satisfying $\mathcal{T}_t u_\infty = u_\infty - c[H]\,t$ for all $t \geq 0$.

The following proposition shows that u_∞ is a viscosity solution of the Hamilton-Jacobi equation

$$H(x, du(x)) = c[H], \tag{2.5}$$

see for instance [30, Proposition 3.6] for a proof:

Proposition 2.2. *Let* $u \in C^0(M;\mathbb{R})$. *Then the following properties are equivalent:*

(i) $\mathcal{T}_t u = u - c[H]\,t$ *for all* $t \geq 0$.

(ii) $u \in \mathcal{SS}(H)$ *and, for every* $x \in M$, *there exists a Lipschitz curve* $\gamma_x : (-\infty, 0] \to M$ *with* $\gamma(0) = x$ *such that*

$$u\big(\gamma_x(b)\big) - u\big(\gamma_x(a)\big) = \int_a^b L\big(\gamma_x(s),\dot\gamma_x(s)\big)\,ds + c[H]\,(b-a) \qquad \forall\, a<b\leq 0. \tag{2.6}$$

(iii) $u \in \mathcal{SS}(H)$ *and for every smooth function* $\phi : M \to \mathbb{R}$ *with* $\phi \le u$ *and all* $x \in M$,

$$\phi(x) = u(x) \quad \Longrightarrow \quad H\big(x, d\phi(x)\big) \ge c[H]. \tag{2.7}$$

If any of this properties holds, then we say that u *is a* weak KAM solution.

Roughly speaking, in the classical KAM theory, weak KAM solutions are smooth and the graphs of their differentials are invariant tori, see [4].

Remark 2.3. It is possible to show that property (i) in Proposition 2.2 uniquely characterize $c[H]$, in the sense that, if some continuous function $v : M \to \mathbb{R}$ satisfies

$$\mathcal{T}_t v = v - c\,t \qquad \forall\, t \ge 0 \tag{2.8}$$

for some constant $c \in \mathbb{R}$, then $c = c[H]$ and v is a weak KAM solution.

Indeed, first of all, using (2.8) it is easy to see that v is Lipschitz and $H(x, dv(x)) \le c$ a.e. (see the proof of Proposition 1.4), so $c \ge c[H]$ by Definition 1.1 and a simple approximation argument (locally use a convolution argument to regularize v, and observe that by convexity in the p variable one gets $H(x, dv_\epsilon(x)) \le c + o(1)$).

Moreover, by the equivalence between (i) and (ii) in Proposition 2.2, it follows that for every $x \in M$ there is a curve $\alpha_x : (-\infty, 0] \to M$, with $\alpha_x(0) = x$, such that

$$v\big(\alpha_x(b)\big) - v\big(\alpha_x(a)\big) = \int_a^b L\big(\alpha_x(s), \dot\alpha_x(s)\big)\, ds + c\,(b-a) \qquad \forall\, a < b \le 0.$$

Applying Proposition 1.4 with $\gamma = \alpha_x$ and u an arbitrary critical subsolution, for any $t > 0$ we get

$$\begin{aligned} -2\|u\|_\infty \le u(x) - u\big(\alpha_x(-t)\big) &\le \int_{-t}^0 L\big(\alpha_x(s), \dot\alpha_x(s)\big)\, ds + c[H]\,t \\ &= v(x) - v\big(\alpha_x(-t)\big) + (c[H] - c)\,t \le 2\|v\|_\infty + (c[H] - c)\,t, \end{aligned}$$

so we conclude by letting $t \to +\infty$.

Example 2.4 (Mechanical and Mañé Lagrangians)**.**
• *Consider a Tonelli Hamiltonian* $H : T^*M \to \mathbb{R}$ *of the form*

$$H(x,p) = \frac{1}{2}\|p\|_x^2 + V(x) \qquad \forall\, (x,p) \in T^*M, \tag{2.9}$$

where $V : M \to \mathbb{R}$ *is a function of class* C^k, $k \ge 2$. *It is easy to check that*

$$c[H] = \max_M V.$$

• *Let* $X : M \to TM$ *be a vector field of class* C^k, $k \ge 2$. *The Mañé Lagrangian* $L_X : TM \to \mathbb{R}$ *associated with* X *is the* C^k *Tonelli Lagrangian defined as*

$$L_X(x,v) = \frac{1}{2}\|v - X(x)\|_x^2 \qquad \forall\, (x,v) \in TM. \tag{2.10}$$

The Hamiltonian H_X *associated to* L_X *by Legendre-Fenchel duality is given by*

$$H_X(x,p) = \frac{1}{2}\|p\|_x^2 + \langle p, X(x)\rangle \qquad \forall\, (x,p) \in T^*M. \tag{2.11}$$

It is not difficult to see that $c[H] = 0$.

We notice that, in both cases, constant functions are weak KAM solution (see [30, Examples 3.9 and 3.10] for more details).

3 Peierls barrier and Aubry set

The *Peierls barrier* $h : M \times M \to \mathbb{R}$ is defined as

$$h(x,y) := \liminf_{t\to+\infty} \Big\{ h_t(x,y) + c[H]t \Big\} \qquad \forall\, x,y \in M.$$

It is easily checked that, for any $x, y, z \in M$ and $t > 0$,

$$h(x,z) \le h(x,y) + h_t(y,z) + c[H]\,t \quad \text{and} \quad h(x,z) \le h_t(x,y) + c[H]\,t + h(y,z). \tag{3.1}$$

In addition, one can show that $h(x,y)$ is finite for any $x, y \in M$, and satisfies the triangle inequality

$$h(x,z) \le h(x,y) + h(y,z) \qquad \forall\, x,y,z \in M. \tag{3.2}$$

Recalling (2.4), we also notice that, for any $u \in \mathcal{SS}(H)$, we have

$$u(y) - u(x) \le h(x,y) \qquad \forall\, x,y \in M, \tag{3.3}$$

which gives in particular

$$h(x,x) \ge 0 \quad \text{and} \quad h(x,y) + h(y,x) \ge 0 \qquad \forall\, x,y \in M. \tag{3.4}$$

The following results relate the Peierls barrier to weak KAM solutions (see for instance [30, Propositions 4.1 and 4.2]):

Proposition 3.1. *Let $u : M \to \mathbb{R}$ be a weak KAM solution, $x \in M$, and $\gamma_x : (-\infty, 0] \to M$ a curve satisfying $\gamma_x(0) = x$ and (2.6). Then any α-limit point z of γ_x, that is any*

$$z \in \bigcap_{t<0} \overline{\gamma_x\big((-\infty,t]\big)},$$

satisfies $h(z,z) = 0$.

Proposition 3.2. *For every $x \in M$, the "pointed" Peierls barrier $h_x : M \to \mathbb{R}$ defined as*

$$h_x(y) := h(x,y)$$

is a weak KAM solution.

Definition 3.3. *We call* projected Aubry set *the nonempty compact subset of M defined by*

$$\mathcal{A}(H) := \Big\{ x \in M \,|\, h(x,x) = 0 \Big\}.$$

The fact that $\mathcal{A}(H)$ is nonempty follows from Proposition 3.1, while the compactness is a consequence of the Lipschitz regularity of h. Proposition 3.1 shows that $\mathcal{A}(H)$ plays the role of a "boundary at infinity" for the Hamilton Jacobi equation (2.5).

In the next proposition we see that the projected Aubry set can be characterized as the set where all critical subsolutions are differentiable, and their gradient is uniquely identified there.

Proposition 3.4. *For any $x \in \mathcal{A}(H)$, the following properties hold:*

(i) There exists a C^2 curve $\gamma_x : \mathbb{R} \to \mathcal{A}(H)$, with $\gamma_x(0) = x$, which solves the Euler-Lagrange equation

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial v}\big(\gamma_x(t), \dot\gamma_x(t)\big) \bigg) = \frac{\partial L}{\partial x}\big(\gamma_x(t), \dot\gamma_x(t)\big) \qquad \forall\, t \in \mathbb{R},$$

and such that, for any $u \in \mathcal{SS}(H)$,

$$u\big(\gamma_x(b)\big) - u\big(\gamma_x(a)\big) = \int_a^b L\big(\gamma_x(s), \dot\gamma_x(s)\big)\, ds + c[H]\,(b-a) \qquad \forall\, a < b \in \mathbb{R}.$$

(ii) Any $u \in \mathcal{SS}(H)$ is differentiable at x, and it holds

$$du(x) = \frac{\partial L}{\partial v}\big(x, \dot\gamma_x(0)\big), \qquad H(x, du(x)) = c[H].$$

(iii) If $\gamma : [a,b] \to M$ is a Lipschitz curve with $a < b$, $0 \in [a,b]$, $\gamma(0) = x$, and

$$u(\gamma(b)) - u(\gamma(a)) = \int_a^b L\big(\gamma(s), \dot\gamma(s)\big)\, ds + c[H]\,(b-a)$$

for some $u \in \mathcal{SS}(H)$, then $\gamma(t) = \gamma_x(t)$ for any $t \in [a,b]$.

On the other hand, for every $x \notin \mathcal{A}(H)$ there is a critical subsolution u which is smooth (say C^∞) in an open neighborhood $\mathcal{V}_x$ of x, and such that $H(x', du(x')) < c[H]$ for any $x' \in \mathcal{V}_x$.

By (i) and (iii) in the proposition above we see that the curve γ_x is unique, while (ii) shows that the gradient of a critical subsolution at a point $x \in \mathcal{A}(H)$ is uniquely identified. In particular, for any $x \in \mathcal{A}(H)$ we can define a covector $P(x) \in T^*_x M$ as

$$P(x) = \frac{\partial L}{\partial v}\big(x, \dot\gamma_x(0)\big). \tag{3.5}$$

This allows us to introduce the Aubry set:

Definition 3.5. *We call* Aubry set *the subset of T^*M defined by*

$$\tilde{\mathcal{A}}(H) := \Big\{(x, P(x)) \in T^*M \mid x \in \mathcal{A}(H)\Big\}.$$

The following result is due to Mather [23, 24]:

Theorem 3.6. *The set $\tilde{\mathcal{A}}(H)$ is a nonempty compact subset of T^*M which is invariant under the Hamiltonian flow. Moreover it is a Lipschitz graph over $\mathcal{A}(H)$.*

Example 3.7 (Mechanical and Mañé Lagrangians)**.**
• *Consider a mechanical Hamiltonian as in* (2.9)*. Then the Aubry set consists of the set of unstable equilibria:*

$$\tilde{\mathcal{A}}(H) = \Big\{(x,0) \mid V(x) = \max_M V\Big\}.$$

• *Let $H_X : T^*M \to \mathbb{R}$ be a Mañé Hamiltonian as in* (2.11)*. Then the projected Aubry set contains the set of recurrent points of the flow of X, and the Aubry set is given by*

$$\tilde{\mathcal{A}}(H) = \Big\{(x,0) \mid x \in \mathcal{A}(H)\Big\}.$$

The Hamiltonian orbits on $\tilde{\mathcal{A}}(H)$ are lifting of orbits of X on T^*M, that is, they are of the form $t \mapsto (x(t), p(t)))$ with $\dot x(t) = X(x(t))$ and $p(t) = 0$.

Let us fix $u \in \mathcal{SS}(H)$. By Proposition 3.4 the set of critical subsolutions which coincide with u on $\mathcal{A}(H)$ is convex, compact, and is invariant with respect to the Lax-Oleinik semigroup. Then the same argument used to prove the existence of weak KAM solutions gives:

Proposition 3.8. *For every $u \in \mathcal{SS}(H)$, there exists a weak KAM solution v such that $v = u$ on $\mathcal{A}(H)$.*

This result shows that it is possible to use critical subsolution to prescribe "boundary conditions" on $\mathcal{A}(H)$. In addition, Fathi proved in [15] that there is a comparison theory for weak KAM solutions:

Proposition 3.9. *If u_1, u_2 are two weak KAM solutions such that $u_1 \le u_2$ on $\mathcal{A}(H)$, then $u_1 \le u_2$ on M.*

Hence $\mathcal{A}(H)$ is a "set of uniqueness" for (2.5).

4 The uniqueness issue

Of course, if a given function u is a weak KAM solution for H, then for every constant $a \in \mathbb{R}$ the function $u + a$ is weak KAM solution. Hence we shall say that (2.5) has a unique solution if weak KAM solutions are unique up to an additive constant.

The next theorem proves that the connectedness of $\mathcal{A}(H)$ is strongly related to the uniqueness of weak KAM solutions (see [30, Theorem 6.1] for a proof of (i), and [16] for (ii)):

Theorem 4.1. *Let $H : T^*M \to \mathbb{R}$ be a C^k Tonelli Hamiltonian, $k \geq 2$. Then:*

(i) Assume that (2.5) *has a unique solution. Then $\mathcal{A}(H)$ is connected.*

(ii) Conversely, if $\mathcal{A}(H)$ is connected, $k \geq 2n-2$, and $n \leq 3$, then (2.5) *has a unique solution.*

Although uniqueness does not hold in general, as shown by Mañé [22] it is a generic property: more precisely, given a Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ and a potential $V : M \to \mathbb{R}$ both of class C^k, $k \geq 2$, we define the Hamiltonian $H_V : T^*M \to \mathbb{R}$ by

$$H_V(x,p) := H(x,p) + V(x) \qquad \forall\, (x,p) \in T^*M. \tag{4.1}$$

Denote by $C^k(M)$ the set of C^k potentials on M equipped with the C^k topology. Then, generically on the potential, uniqueness holds:

Theorem 4.2. *Let $H : T^*M \to \mathbb{R}$ be a C^k Tonelli Hamiltonian, $k \geq 2$. Then there is a residual subset (i.e., a countable intersection of open dense sets) $\mathcal{G}$ in $C^k(M)$ such that, for every $V \in \mathcal{G}$, the critical Hamilton-Jacobi equation* (2.5) *associated to H_V has a unique solution.*

Example 4.3 (Mechanical and Mañé Lagrangians)**.**
• *In [25] Mather provides examples of potentials $V : M \to \mathbb{R}$ of class C^k such that the projected Aubry set of the Hamiltonian* (2.9) *is connected but uniqueness fails. However, if $k \geq 2n-2$, then connectedness of $\mathcal{A}(H)$ (with H given by* (2.9)*) does imply uniqueness [16, 32].*
• *In [16, Theorem 6] it is proved that, if $n \leq 3$, then the uniqueness property for Mañé Lagrangians* (2.10) *is related to chain-recurrent properties of the flow of X.*

5 Regularity of critical subsolutions and weak KAM solutions

Fathi and Siconolfi [17] proved that there exist critical subsolutions of class C^1, and Bernard improved this result to $C^{1,1}$ [3]:

Theorem 5.1. *Let $H : T^*M \to \mathbb{R}$ be a C^k Tonelli Hamiltonian, $k \geq 2$. Then there exists a critical subsolution u of class $C^{1,1}$ which is strict outside $\mathcal{A}(H)$, that is $H(x, du(x)) < c[H]$ on $M \setminus \mathcal{A}(H)$.*

In particular, by this result, Proposition 3.4, and Definition 3.5, the Aubry set can also be defined as

$$\tilde{\mathcal{A}}(H) := \bigcap_{u \in \mathcal{S}^1(H)} \Big\{ (x, du(x)) \,|\, x \in M \text{ s.t. } H(x, du(x)) = c[H] \Big\},$$

where $\mathcal{S}^1(H)$ denotes the set of critical subsolutions of class C^1. Indeed, on $\mathcal{A}(H)$ the differential of all critical subsolutions coincide. On the other hand, given $x \notin \mathcal{A}(H)$ and u a critical subsolution such that $H(x, du(x)) < c[H]$, then, for any smooth function φ supported in a small neighborhood of x, the function $u + \epsilon\varphi$ is still a critical subsolution for ϵ sufficiently small. In particular, by choosing φ such that $d\varphi(x) \neq 0$, one can construct two critical subsolutions u_1

and u_2 such that $du_1(x) \neq du_2(x)$.

As shown by Bernard [3], Theorem 5.1 is optimal: there exists H smooth which admits no critical subsolutions of class C^2.

Concerning the regularity of solutions, the following result is proven in [29] (see [5, 29, 31] for the definition of semiconcave functions and their properties):

Theorem 5.2. *Let $u : M \to \mathbb{R}$ be a weak KAM solution. Then u is semiconcave on M, and $C^{1,1}_{loc}$ on an open dense subset $\mathcal{O}$ of M.*

Another result on the regularity of viscosity (sub)solutions is the following theorem of Fathi [14] (see also [29]):

Theorem 5.3. *Let u be a critical viscosity subsolution, and assume that u is a C^1 viscosity solution on some open set $\mathcal{V}$. Then u is $C^{1,1}_{loc}$ inside $\mathcal{V}$.*

Since weak KAM solutions u are Lipschitz, the limiting differential of u at x defined by

$$du^*(x) := \Big\{ \lim du(x_k) \,|\, x_k \to x, u \text{ differentiable at } x_k \Big\},$$

is always a nonempty compact subset of T_x^*M. In [29], the second author proved that there is a one-to-one correspondence between the set of limiting differentials at x and the set of curves $\gamma_x : (-\infty, 0] \to M$ with $\gamma_x(0) = x$ and such that

$$u\big(\gamma_x(b)\big) - u\big(\gamma_x(a)\big) = \int_a^b L\big(\gamma_x(s), \dot\gamma_x(s)\big)\, ds + c[H]\,(b-a) \qquad \forall\, a < b \leq 0.$$

In addition u can be shown to be C^1 at every point $\gamma_x(-t)$ with $t > 0$ (compare with Proposition 3.4(i)-(ii)). Since $\gamma_x(-t)$ tends to the projected Aubry set as $t \to +\infty$ (Proposition 3.1), regularity properties for weak KAM solutions in a neighborhood of $\mathcal{A}(H)$ imply more regularity for u globally (in the spirit of classical results for Dirichlet-type problems [6, 20, 29]). Furthermore, as shown for instance by the following result of Bernard [2], some properties on the behavior of the Hamiltonian flow in a neighborhood of $\tilde{\mathcal{A}}(H)$ can also bring regularity properties:

Theorem 5.4. *Let H be a Tonelli Hamiltonian of class C^k, $k \geq 2$, whose Aubry set is a hyperbolic periodic orbit. Then there is a unique weak KAM solution, and such solution is of class C^k in a neighborhood of $\mathcal{A}(H)$. In particular, there exists a critical subsolution which is of class C^k on M.*

Proof. The key observation of Bernard is that every limiting differential has to be in the unstable manifold of the periodic orbit, and such a manifold is of class C^{k-1}. We refer the reader to [2] for more details. □

6 The Mañé conjecture

We recall the notation H_V for the Hamiltonian $H + V$ (see (4.1)). The Mañé conjecture in C^k topology, $k \geq 2$, can be stated as follows:

Conjecture 6.1 (Mañé Conjecture)**.** *For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class C^k, $k \geq 2$, there is a residual subset (i.e., a countable intersection of open dense sets) $\mathcal{G}$ of $C^k(M)$ such that, for every $V \in \mathcal{G}$, the Aubry set $\tilde{\mathcal{A}}(H_V)$ of the Hamiltonian H_V is either an equilibrium point or a periodic orbit.*

A natural way to attack the Mañé Conjecture in any dimension would be to prove first a density result, then a stability result. Namely, given a Hamiltonian of class C^k satisfying (H1) and (H2), first one could show that the set of potentials $V \in C^k(M)$ such that $\tilde{A}(H_V)$ is either a equilibrium point or a periodic orbit is dense, and then prove that one can make them hyperbolic by adding a second small potential, and that the latter property is open in C^k topology. The stability part is indeed contained in results obtained by Contreras and Iturriaga in [9], so we can consider that the Mañé Conjecture reduces to the density part.

Conjecture 6.2 (Mañé density Conjecture)**.** *For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class C^k, $k \geq 2$, there exists a dense set $\mathcal{D}$ in $C^k(M)$ such that, for every $V \in \mathcal{D}$, the Aubry set of H_V is either an equilibrium point or a periodic orbit.*

In a series of recent papers [18, 19] we made several progress toward a proof of the Mañé Conjecture in C^2 topology. Our approach is based on a combination of techniques coming from finite dimensional control theory and Hamilton-Jacobi theory, together with some of the ideas which were used to prove C^1-closing lemmas for dynamical systems.

In the next section we will give a more detailed description of the results in [18, 19]. Here we mention just a weak form of some of the results obtained there:

Theorem 6.3. *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class C^k with $k \geq 4$, and assume that there exists a critical subsolution of class C^{k+1}. Then, for any $\epsilon > 0$ there exists a potential $V : M \to \mathbb{R}$ of class C^{k-1}, with $\|V\|_{C^2} < \epsilon$, such that $c[H_V] = c[H]$ and the Aubry set of H_V is either an equilibrium point or a periodic orbit.*

This theorem, combined with Theorem 5.4 and the stability results by Contreras and Iturriaga [9], shows that, for smooth Hamiltonians (i.e., of class C^∞, or at least of class C^k with $k \geq 4$), the Mañé Conjecture in C^2 topology is equivalent to the following conjecture on the generic smoothness of critical subsolutions (here, for simplicity, we state the conjecture with C^∞):

Conjecture 6.4. *For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class C^∞ there is a set $\mathcal{D} \subset C^\infty(M)$ which is dense in $C^2(M)$ (with respect to the C^2 topology) such that, for every $V \in \mathcal{D}$, the Hamiltonian H_V admits a critical subsolution of class C^∞.*

7 Some results on the Mañé conjecture

The starting point of [18, 19] is the following remark:

Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class C^k with $k \geq 2$, and fix $\epsilon \in (0, 1)$. Without loss of generality, up to adding a constant to H (which does not change the dynamics), we can assume that $c[H] = 0$. Let L denote the Lagrangian associated to H. Then, in order to prove the conjecture in C^k topology, we claim that it is sufficient to find a potential $V : M \to \mathbb{R}$ of class C^k with $\|V\|_{C^k} < \epsilon$, together with a C^1 function $v : M \to \mathbb{R}$, and a curve $\gamma : [0, T] \to M$ with $\gamma(0) = \gamma(T)$, such that the following properties are satisfied:

(P1) $H_V\big(x, dv(x)\big) \leq 0 \quad \forall\, x \in M.$

(P2) $\int_0^T L_V(\gamma(t), \dot{\gamma}(t))\, dt = 0.$

Indeed, if we are able to do so, then (P1) implies that $c[H_V] \leq 0$ (by Definition 1.1 of $c[H_V]$), while (P2) implies that $h_T(\gamma(t), \gamma(t)) \leq c[H_V]T$ for any $t \in [0, T]$ (see (2.2)). Since by (3.4) $h(x, x) \geq 0$, the only possibility is $h_T(\gamma(t), \gamma(t)) = c[H_V] = 0$. In particular this implies that $h(\gamma(t), \gamma(t)) = 0$, so by Definition 3.3 the closed curve $\Gamma := \gamma([0, T]) \subset M$ is contained in $\mathcal{A}(H_V)$.

Now, if $W : M \to \mathbb{R}$ is any smooth function such that $W = 0$ on Γ, $W > 0$ outside Γ, and $\|W\|_{C^k} < \epsilon - \|V\|_{C^k}$, then v is a critical subsolution of $H_{V-W} = H + V - W$ which is strict outside Γ. By proposition 3.4 this implies that $\mathcal{A}(H_{V-W}) = \Gamma$, which concludes the proof.

Let us remark that, by the discussion above, if $\mathcal{A}(H)$ already contains a fixed point or a periodic orbit then the proof is trivial, since it suffices to add to H a smooth potential which vanishes either on the point or on the orbit, and it is strictly negative outside.

From now on, we assume that the Aubry set $\mathcal{A}(H)$ does not contain an equilibrium point or a periodic orbit, and we choose a recurrent point $\bar{x} \in \mathcal{A}(H)$, i.e., there exists a sequence of times $t_k \to +\infty$ such that

$$\lim_{k\to\infty} \pi^*\left(\phi_{t_k}^H(\bar{x}, du(\bar{x}))\right) = \bar{x},$$

for some critical subsolution $u : M \to \mathbb{R}$. (Here and in the sequel, $\phi_t^H : T^*M \to T^*M$ denotes the Hamiltonian flow. Note that, since $\bar{x} \in \mathcal{A}(H)$, because of Proposition 3.4 the definition of recurrent point does not depends on the particular subsolution u.)

We denote by $\mathcal{O}^+(\bar{x})$ the positive orbit of $\bar{x}$ in the projected Aubry set, that is,

$$\mathcal{O}^+(\bar{x}) := \Big\{\pi^*\left(\phi_t^H(\bar{x}, du(\bar{x}))\right) \mid t \geq 0\Big\},$$

where $u : M \to \mathbb{R}$ is again an arbitrary critical viscosity subsolution. (As before, the above definition does not depend on u.)

The rough idea is now the following: Since $\bar{x}$ is recurrent, the curve $t \mapsto \pi^*\left(\phi_t^H\left(\bar{x}, du(\bar{x})\right)\right)$ passes near $\bar{x}$ infinitely many times. Hence one would like to:
(a) Choose a time $T \gg 1$ such that $\bar{x}_T := \pi^*\left(\phi_T^H\left(\bar{x}, du(\bar{x})\right)\right)$ is sufficiently close to $\bar{x}$, and "close" the trajectory by adding a potential.
(b) When closing the trajectory, make sure to control the action so that (P2) holds.
(c) Finally, construct a critical subsolution (see (P1)) to ensure that such a curve belongs to the projected Aubry set of the new Hamiltonian.

There are many points to address here:

(a) If we add a potential V small in C^k topology, it means that the Hamiltonian vector field associated to H_V is close in C^{k-1} topology to the Hamiltonian vector field of H.

Hence this can been seen as a more involved version of the classical *closing lemma*: fixed $j \geq 0$, one asks whether, given a vector field X with a recurrent point $\tilde{x}$, one can find a vector field Y close to X in C^j topology which has a periodic orbit. The "cheap strategy" of closing the trajectory in one step allows to prove the validity of the closing lemma when $j = 0$. Indeed, this is due to the fact that, when connecting a trajectory to another which is close to it, one needs to perform a modification of X which is compactly supported near the trajectory, otherwise that would destroy all the dynamics.

To give an idea, if $X = e_1$ and we want to move a trajectory up by ϵ in the direction e_2 over an interval of time of length 1, then we would like to use $e_1 + \epsilon e_2$. However, as just explained, in order not to destroy the dynamics we need to add a cut-off function both in x_1 and x_2. While in the direction of the flow x_1 we have some space to introduce our cut-off, even if the points we want to connect were chosen as the "closest ones" (see Figure 2 below) we need to put a cut-off function of the form $\varphi(x_1, x_2/\epsilon)$ in order not to touch the other trajectories nearby (see Figure 1). So our vector field Y looks like

$$Y = X + \epsilon\varphi(x_1, x_2/\epsilon)e_2,$$

which satisfies $\|Y - X\|_{C^0} \lesssim \epsilon$ but $\|Y - X\|_{C^1} \simeq 1$.

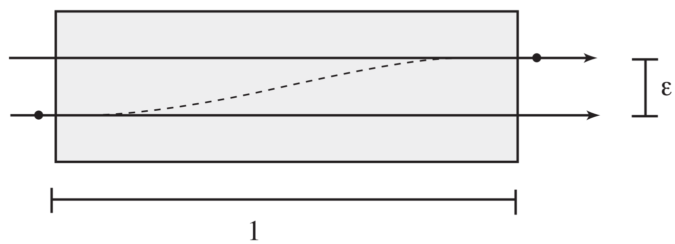

Figure 1: The dashed line represents the trajectory of Y, and the rectangle is the support of $\varphi(x_1, x_2/\epsilon)$.

Hence, this solves the problem only for $j = 0$. For $j = 1$ new deep ideas have been introduced to solve the problem [26, 27, 28, 21], while for $j \geq 2$ the problem is still open, though many results suggest that the result may be false when j is sufficiently large.

In our case the analog of the closing lemma is the following: If we hope to close the orbit in only one step, then V can be small only in C^1 topology.

If we want to be able to close the trajectory with a potential V small in C^2 topology, we may try to adapt the strategy used to solve the closing lemma in C^1 topology: roughly speaking, fixed an error size $\epsilon > 0$ and a small radius r which "ideally" represents the distance between $\bar{x}$ and $\bar{x}_T := \pi^*\left(\phi_T^H\left(\bar{x}, du(\bar{x})\right)\right)$, the idea is to close the trajectory in $1/\epsilon$ steps where at each step we "move" $\bar{x}_T$ in the direction of $\bar{x}$ by a size ϵr. However, when doing this, we need to make sure that the modification done at every "approaching step" does not influence any of the modifications performed before, and in addition that it does not "destroy" the property of $\bar{x}$ of being recurrent.

As we will see, under some suitable assumptions on u this can be done, and actually we can choose our connecting points so that we have a neighborhood of size r where we can make the modification. In order words, if we reconsider the case of vector fields (we use the same notation as above), to move one point up by ϵr we would use the vector field

$$Y = X + \epsilon r \varphi(x_1, x_2/r) e_2.$$

In this way $\|X - Y\|_{C^1} \lesssim \epsilon$.

Of course, being the C^2 closing lemma an open problem, the case $k \geq 3$ is at the moment out of reach (at least with this approach).

In order to perform the above strategy, we need to be able to go from one point to another by adding a small potential. To this aim we employ techniques and results from control theory which allow us to connect points by Hamiltonian trajectories.

(b) Point (a) above deals with the "closing part" of our statement, that is, finding a closed orbit for H_V. Now we need to ensure that (P2) is satisfied, and for this we use again delicate techniques from control theory.

(c) The construction of a critical subsolution is different in the case $k = 1$ and $k = 2$.

When $k = 1$, it is very delicate to construct the subsolution near the trajectory: indeed, the fact that the potential constructed in steps (a) and (b) is small only in C^1 (and not in C^2) topology may create conjugate points along the closed trajectory, and this creates problem when trying to construct solutions of (2.5) using the method of characteristics (since, as we will see later, we need to ensure that the solution is at least $C^{1,1}$ near the new trajectory). On the other hand, once this problem is taken care of, then it is easy to extend this subsolution from a neighborhood of the trajectory to the whole M by an interpolation procedure.

When $k = 2$, the situation is completely reversed: while it is easy to construct the subsolution near the trajectory, extending it to the whole M is much more delicate and needs some additional assumptions.

Let us start to describe more in detail our results.

7.1 The case $k = 1$

As explained above, the rough idea of choosing a time $T \gg 1$ such that $\pi^*\left(\phi_T^H(\bar{x}, du(\bar{x}))\right)$ is sufficiently close to $\bar{x}$, and then "closing" the trajectory in one step, does work if one wants to use a potential which is small in C^1 topology.

However, we need to close the trajectory making sure that (P2) holds. Hence, we do the following: First we wait enough time so that there are many points as close as we want to $\bar{x}$, and among them we choose two points $z_1^0 := \pi^*\left(\phi_{t_1}^H(\bar{x}, du(\bar{x}))\right)$ and $z_2^0 := \pi^*\left(\phi_{t_2}^H(\bar{x}, du(\bar{x}))\right)$, with $t_1 > t_2$, which are the closest ones, see Figure 2.

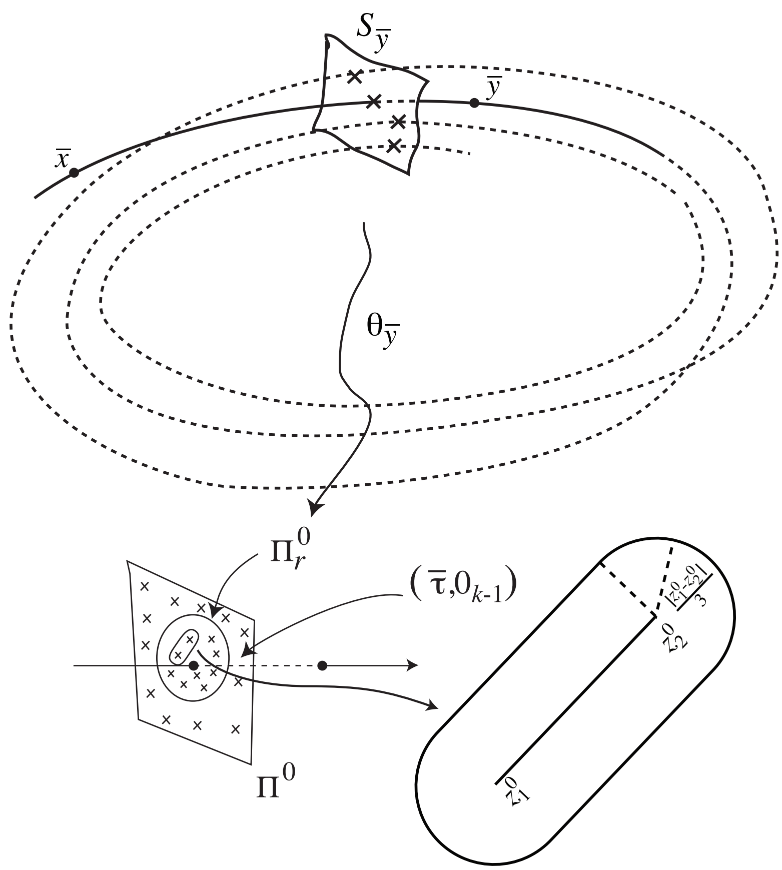

Figure 2: We fix a time $T \ll 1$ and among all points on the orbit which are close to $\bar{x}$ choose the two closest points.

Then, we add a first potential to connect the orbit passing through z_1^0 to the one passing through z_2^0, and, while doing this, we make sure that the support of the potential does not intersect any other point on the trajectory $t \mapsto \pi^*\left(\phi_t^H(\bar{x}, du(\bar{x}))\right)$ for $0 \le t \le t_1$: in this way, since $t_1 > t_2$, this ensures that the orbit is now closed.

The connection of the trajectory can be done in two ways: either by using techniques of control theory to show that we can go from a point to a close one with a small potential [18, Proposition 3.1], or by doing a construction "by hand" of the potential where we explicitly define our connecting trajectory (by taking a convex combination of the original trajectories and a suitable time rescaling) and the potential [19, Proposition 2.1]. With respect to the "control theory approach" used in [18, Proposition 3.1], this second construction has the advantage of forcing the connecting trajectory to be "almost tangent" to the Aubry set, and this is crucial for the proof of Theorem 7.5 below. However, as a counterpart, the second approach requires more regularity assumptions on the Hamiltonian. We also mention that, in either cases, we still need to use control theory techniques to add a second potential in order to ensure that (P2) is satisfied [19, Lemma 5.4], see the potential $\bar{V}_0$ in Figure 3.

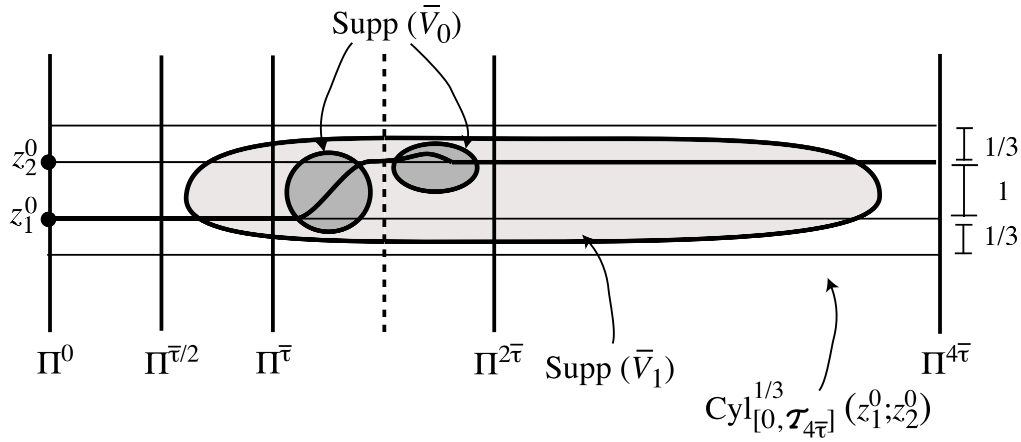

Figure 3: The potential $\bar{V}_0$ is constructed in two steps: first, on the left, $\bar{V}_0$ is used to connect the trajectory passing through z_1^0 to the one passing through z_2^0; then, on the right, it is used to control the action of the connecting curve, making sure that the trajectory remains closed. After that, by adding to $\bar{V}_0$ a nonpositive potential $\bar{V}_1$ which vanishes together with his gradient along the connecting trajectory $Z_{\bar{V}_0}\left(\cdot\,;z_1^0\right)$, we can also ensure that the characteristics associated to $H_{\bar{V}_0+\bar{V}_1}$ do not cross near $Z_{\bar{V}_0}\left(\cdot\,;z_1^0\right)$. In this way, using the theory of characteristics, we can construct a viscosity solution of class $C^{1,1}$ in a uniform neighborhood of $Z_{\bar{V}_0}\left(\cdot\,;z_1^0\right)$.

As mentioned before, since now we do not control the C^2 norm of the potential $\bar{V}_0$, the Hamilton-Jacobi equation associated to $H+\bar{V}_0$ may have conjugate points along the connecting trajectory.

This has the following issue: in order to construct a critical solution in a neighborhood of the trajectory we would like to take a $C^{1,1}$ critical subsolution for H (whose existence is provided by Theorem 5.1) as boundary datum on the hyperplane Π^0 in a neighborhood of z_1^0 (see Figure 3), and then use the theory of characteristics to construct a solution for some positive time.

To make this strategy work, we add to $H+\bar{V}_0$ a smooth nonpositive potential $\bar{V}_1$ which satisfies $\bar{V}_1 = \nabla\bar{V}_1 = 0$ along the connecting curve, and which is very "concave" in the transversal directions. This has the feature of making the "curvature" of the Hamiltonian system sufficiently negative near the connecting curve, so that the characteristics associated to $H+\bar{V}_0+\bar{V}_1$ will not cross there, see Figure 3 (we refer to [19, Section C.3] for more details on this delicate construction).

Finally, a simple interpolation argument allows to make the subsolution global, see Figure 4. Hence, we obtain the following result [19, Theorem 1.2]:

Theorem 7.1. *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class C^k with $k \geq 4$. Then,*

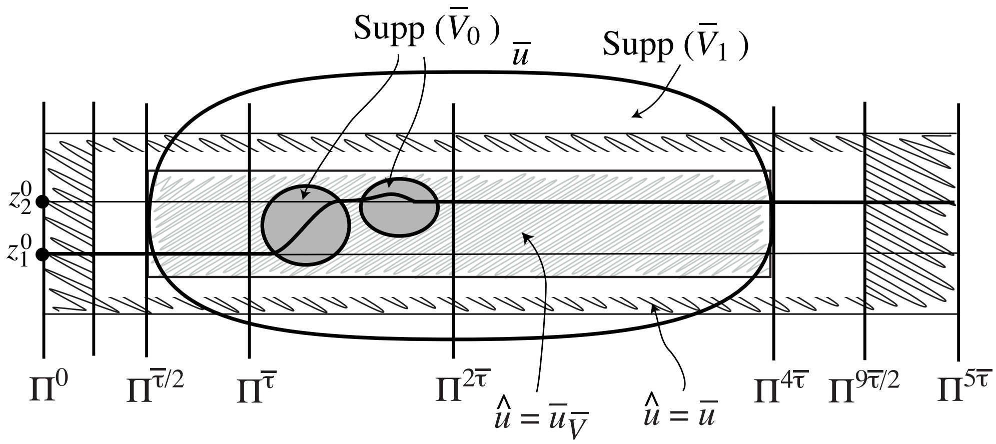

Figure 4: The function $\hat{u}$ is obtained by interpolating (using a cut-off function) between $\bar{u}$ (the $C^{1,1}$ critical viscosity subsolution for H used as boundary datum on Π^0) and $\bar{u}_{\bar{V}}$ (the viscosity solution for $H_{\bar{V}_0+\bar{V}_1}$) inside the cylinder $\mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar{\tau}}]}\left(\hat{z}_1^0;\hat{z}_2^0\right)$. Since $V_1 \leq 0$, the function $\bar{u}$ is a viscosity subsolution to $H_{\bar{V}_0+\bar{V}_1}\left(z,\nabla\bar{u}(z)\right) \leq 0$ outside $\mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar{\tau}}]}\left(\hat{z}_1^0;\hat{z}_2^0\right)$. So, we can find a nonpositive potential $\bar{W}$, small in C^1 topology and supported inside $\mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar{\tau}}]}\left(\hat{z}_1^0;\hat{z}_2^0\right)$, such that $H_{\bar{V}_0+\bar{V}_1+\bar{W}}\left(z,\nabla\hat{u}(z)\right) \leq 0$ on the whole manifold M.

for any $\epsilon > 0$ there exists a potential $V : M \to \mathbb{R}$ of class C^{k-2}, with $\|V\|_{C^1} < \epsilon$, such that $c[H_V] = c[H]$ and the Aubry set of H_V is either an equilibrium point or a periodic orbit.

7.2 The case $k = 2$

In this situation, as explained before, we cannot simply choose a time $T \gg 1$ such that $\pi^*\left(\phi_T^H\left(\bar{x}, du(\bar{x})\right)\right)$ is sufficiently close to $\bar{x}$ and then close the trajectory in one step. Instead, we exploit some techniques introduced by Mai to prove the closing lemma in C^1 topology.

7.2.1 The Mai Lemma

The Mai Lemma was introduced in [21] to give a new and simpler proof of the closing lemma in C^1 topology, first proved by Pugh [26, 27], and Pugh and Robinson [28].

Let $\{E_i\}_{i\in\mathbb{N}}$ be a countable family of ellipsoids in $\mathbb{R}^k$, that is, a countable family of compact sets in $\mathbb{R}^k$ associated with invertible linear maps $P_i : \mathbb{R}^k \to \mathbb{R}^k$ such that

$$E_i = \Big\{ v \in \mathbb{R}^k \mid |P_i(v)| \leq \|P_i\| \Big\}.$$

For every $y \in \mathbb{R}^k$, $r > 0$ and $i \in \mathbb{N}$, we call E_i-ellipsoid centered at y with radius r the set defined by

$$E_i(y,r) := \Big\{ y + rv \mid v \in E_i \Big\} = \Big\{ y' \mid |P_i(y'-y)| \leq r\|P_i\| \Big\}.$$

We note that such an ellipsoid contains the open ball $B(y,r)$. The Mai Lemma can be stated as follows (see also Figure 5):

Lemma 7.2. *Let $\hat{N} \geq 2$ be an integer. There exist a real number $\hat{\rho} \geq 3$ and an integer $\eta > 0$, which depend on the family $\{E_i\}$ and on $\hat{N}$ only, such that the following holds: For every $\bar{r} > 0$ and every finite set $Y = \{w_0, \ldots, w_J\} \subset \mathbb{R}^k$ such that $Y \cap B_{\bar{r}}$ contains at least two points, there exist η points $\hat{w}_1, \ldots, \hat{w}_\eta$ in $\mathbb{R}^k$ and η positive real numbers $\hat{r}_1, \ldots, \hat{r}_\eta$ satisfying:*

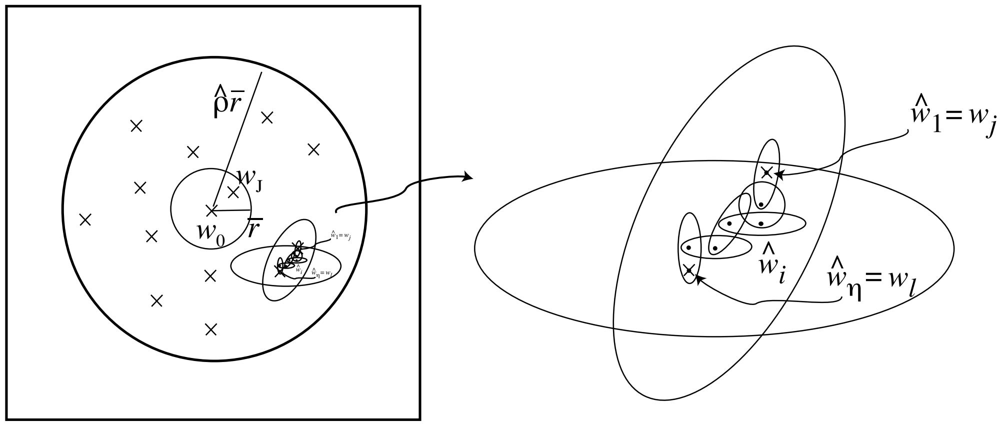

Figure 5: An illustration of The Mai Lemma: there exist two points w_j, w_l, which can be connected using a sequence of $\eta - 1$ small ellipsoids $E_i\big(\hat{w}_i, \hat{r}_i/\hat{N}\big)$, so that none of the points w_k $(k \neq j, l)$ belongs to $E_i\big(\hat{w}_i, \hat{r}_i\big)$ for $i = 1, \dots, \eta - 1$ (in the figure above, we just drew two of the ellipsoids $E_i\big(\hat{w}_i, \hat{r}_i\big)$).

(i) there exist $j, l \in \{1, \dots, J\}$, with $j > l$, such that $\hat{w}_1 = w_j$ and $\hat{w}_\eta = w_l$;

(ii) $\forall\, i \in \{1, \dots, \eta - 1\}$, $E_i\big(\hat{w}_i, \hat{r}_i\big) \subset B_{\hat{\rho}\bar{r}}$;

(iii) $\forall\, i \in \{1, \dots, \eta - 1\}$, $E_i\big(\hat{w}_i, \hat{r}_i\big) \cap \big(Y \setminus \{w_j, w_l\}\big) = \emptyset$;

(iv) $\forall\, i \in \{1, \dots, \eta - 1\}$, $\hat{w}_{i+1} \in E_i\big(\hat{w}_i, \hat{r}_i/\hat{N}\big)$.

We refer the reader to [21] or the monograph [1] for a proof of the above result.

7.2.2 How to close a trajectory in C^1 topology

Here we denote by u a viscosity solution to (2.5). In order to perform the argument below, we will need to make some assumptions on u, but at the moment we try to be informal.

Given $\epsilon > 0$ small, we fix a small neighborhood $\mathcal{U}_{\bar{x}} \subset M$ of $\bar{x}$, and a smooth diffeomorphism $\theta_{\bar{x}} : \mathcal{U}_{\bar{x}} \to B^n(0,1)$, such that

$$\theta_{\bar{x}}(\bar{x}) = 0_n \quad \text{and} \quad d\theta_{\bar{x}}(\bar{x})\big(\dot{\bar{\gamma}}(0)\big) = e_1.$$

Then, we choose a point $\bar{y} = \bar{\gamma}(\bar{t}) \in \mathcal{A}(H)$, with $\bar{t} > 0$, such that, after a smooth diffeomorphism $\theta_{\bar{y}} : \mathcal{U}_{\bar{y}} \to B^n(0,2)$, $\theta_{\bar{y}}(\bar{y}) = (\bar{\tau}, 0_{n-1})$ (the point $\bar{y}$ is chosen in such a way that some controllability assumptions on the Hamiltonian system in a neighborhood of $\bar{y}$ holds[1], see [18, Section 5.2] for more details). We denote by $\bar{u} : B^n(0,2) \to \mathbb{R}$ the function given by $\bar{u}(z) = u\big(\theta_{\bar{y}}^{-1}(z)\big)$ for $z \in B^n(0,2)$, and by $\bar{H} : B^n(0,2) \times \mathbb{R}^n \to \mathbb{R}$ the Hamiltonian associated with the H through $\theta_{\bar{y}}$. Finally, we denote by Π^0 the hyperplane passing through the origin which is orthogonal to the vector e_1 in $\mathbb{R}^n$, $\Pi^0_r := \Pi^0 \cap B^n(0,r)$ for every $r > 0$, and $\Pi^{\bar{\tau}} := \Pi^0 + \bar{\tau} e_1$, where $\bar{\tau} \in (0,1)$ is small but fixed, see Figure 6.

[1] To be precise, this kind of construction should be used also in the case $k = 1$. Indeed, in order to find a potential which closes the trajectory and such that (P2) holds, it is important to choose the point $\bar{y}$ in a suitable way, so that the action can be controlled near the connecting trajectory. However, in order to make the presentation simpler and to focus more on the second part of the construction (i.e., how to build a critical subsolution to get (P1)), in the previous section we have decided to neglect this point. Here instead, since one of the main issues is exactly the connecting part, we prefer to be more rigorous.

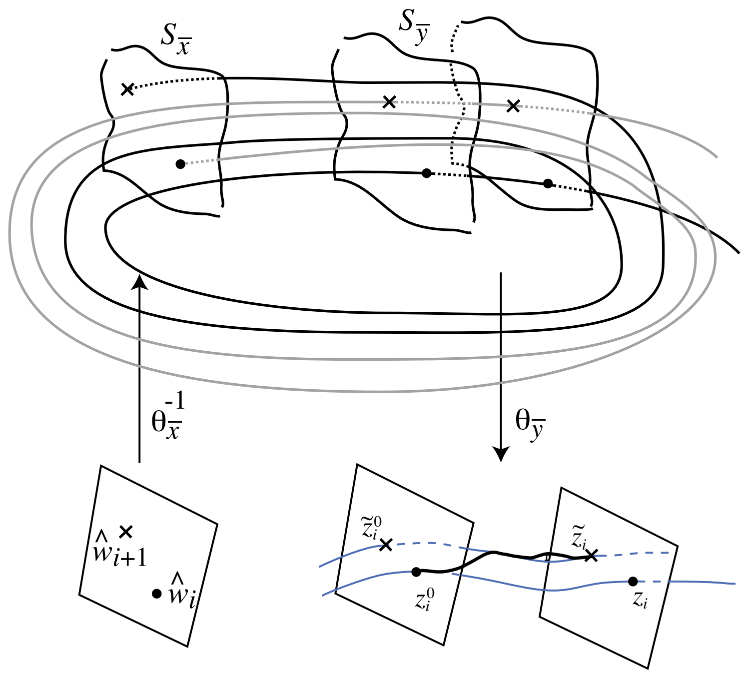

Figure 6: The point z_i^0 (resp. $\tilde{z}_i^0$) is obtained by considering the i-th intersection of the curve $t \mapsto \Phi(t, \hat{w}_i)$ (resp. $t \mapsto \Phi(t, \hat{w}_{i+1})$) with the hypersurface $S_{\bar{y}}$. Then, we use [18, Proposition 5.2] to connect z_i^0 to $\tilde{z}_i$.

Define the function $\Psi : [0, +\infty) \times M \to M$ by

$$\Psi(t, z) := \pi^* \left(\phi_t^H(z, du(z))\right).$$

We now fix $\bar{r} > 0$ small enough, and we use the recurrence assumption on $\bar{x}$ to find a time $T_{\bar{r}} > 0$ such that $\Psi(T_{\bar{r}}, \bar{x}) \in \theta_{\bar{x}}^{-1}\left(\Pi_{\bar{r}}^0\right)$, and we look at the set of points

$$W := \left\{ w_0 := \theta_{\bar{x}}(\bar{x}), w_1 := \theta_{\bar{x}}\left(\bar{\gamma}(t_1')\right), \ldots, w_J := \theta_{\bar{x}}\left(\bar{\gamma}(T_{\bar{r}})\right) \right\} \subset \Pi^0 \cap \mathcal{A} \subset \Pi^0 \simeq \mathbb{R}^{n-1} \tag{7.1}$$

(see [18, Equation (5.18)]) obtained by intersecting the curve

$$[0, T_{\bar{r}}] \ni t \mapsto \bar{\gamma}(t) := \Psi(t, \bar{x})$$

with $\theta_{\bar{x}}^{-1}\left(\Pi_{\bar{\delta}/2}^0\right)$, where $\bar{r} \ll \bar{\delta} \ll 1$ (more precisely, $\bar{\delta} \in (0, 1/4)$ is provided by [18, Proposition 5.2]). We also consider the maps $\Phi_i : \Pi_{\bar{\delta}_i}^0 \to \Pi_{\bar{\delta}/2}^0$ corresponding to the i-th intersection of the curve $t \mapsto \pi^*\left(\phi_t^H(\theta_{\bar{x}}^{-1}(w), du(\theta_{\bar{x}}^{-1}(w)))\right)$ with $\theta_{\bar{y}}^{-1}(\Pi_{\bar{\delta}/2}^0)$ (see [18, Equation (5.14)] and thereafter).

If we assume that u is C^2 at the point $\bar{x}$ (we will properly explain later what this means), then all the maps Φ_i are C^1. Hence, we define the ellipsoids E_i associated to $P_i = D\Phi_i(0_{n-1})$, and we apply Lemma 7.2 to $Y = W$ with $N \simeq 1/\epsilon$. In this way we get a sequence of points $\hat{w}_1, \ldots, \hat{w}_\eta$ in $\Pi_{\hat{\rho}\bar{r}}^0$ connecting w_j to w_l, where $\hat{\rho} \geq 3$ is fixed and depends on ϵ but not on $\bar{r}$. Then, we use the flow map to send the points $\theta_{\bar{x}}^{-1}(\hat{w}_i)$ onto the "hyperplane" $S_{\bar{y}} := \theta_{\bar{y}}^{-1}\left(\Pi_{\bar{\delta}/2}^0\right)$ in the following way (see [18, Subsection 5.3, Figure 5]):

$$z_i^0 := \theta_{\bar{y}}\left(\Phi_i(\hat{w}_i)\right), \qquad z_i := \mathcal{P}(z_i^0), \qquad \tilde{z}_i^0 := \theta_{\bar{y}}\left(\Phi_i(\hat{w}_{i+1})\right), \qquad \tilde{z}_i := \mathcal{P}(\tilde{z}_i^0), \tag{7.2}$$

where $\mathcal{P}$ is the Poincaré mapping from $\Pi^0_{1/2}$ to $\Pi^{\bar{\tau}}_1$ (see [18, Lemma 5.1(ii)]).

We have now reduced ourselves to the following problem: connect two trajectories who are r/N apart, knowing that the support of the potential has to be contained inside a cylinder of width r, see Figure 7.

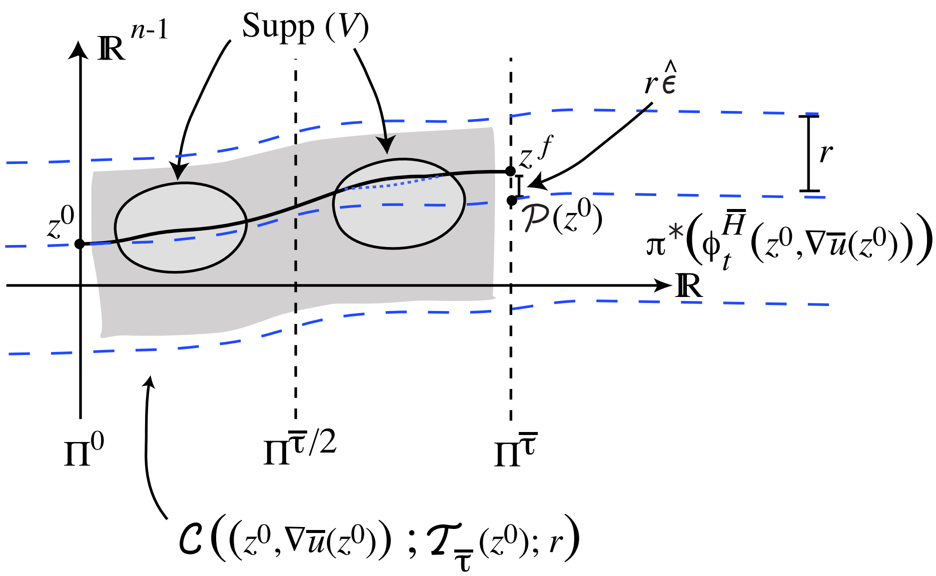

Figure 7: By using first [18, Proposition 3.1] on $[0, \bar{\tau}/2]$ we can add a first potential to connect the trajectories, and then, by [19, Proposition 4.1] on $[\bar{\tau}/2, \bar{\tau}]$, we can add a second potential to fit the action without changing the starting and final point of the trajectory.

By using control theory techniques, we then find C^2-small potentials V_i, supported inside some suitable disjoints cylinders $\mathcal{C}_i$, which allow to connect z_i^0 to $\tilde{z}_i$ with a control on the action. Then the closed curve $\tilde{\gamma} : [0, t_f] \to M$ is obtained by concatenating $\gamma_1 : [0, \tilde{t}_\eta] \to M$ with $\gamma_2 : [\tilde{t}_\eta, t_f] \to M$, where

$$\gamma_2(t) := \pi^* \left(\phi^H_{t-\tilde{t}_\eta} \left(\theta_{\bar{y}}^{-1}(z_\eta^0), du\big(\theta_{\bar{y}}^{-1}(z_\eta^0)\big)\right)\right) \qquad \text{connects } \theta_{\bar{y}}^{-1}(z_\eta^0) \text{ to } x,$$

while γ_1 is obtained as a concatenation of $2\eta - 1$ pieces: for every $i = 1, \ldots, \eta - 1$, we use the flow $(t, z) \mapsto \pi^* \left(\phi_t^{H+V}(z, du(z))\right)$ to connect $\theta_{\bar{y}}^{-1}(z_i^0)$ to $\theta_{\bar{y}}^{-1}(\tilde{z}_i)$ on a time interval $[\tilde{t}_i, \tilde{t}_i + T_i^f]$, while on $[0, \tilde{t}_1]$ and on $[t_i + T_i^f, t_{i+1}]$ $(i = 1, \ldots, \eta - 1)$ we just use the original flow $(t, z) \mapsto \pi^* \left(\phi_t^H(z, du(z))\right)$ to send, respectively, $\theta_{\bar{x}}^{-1}(\hat{w}_1)$ onto $\theta_{\bar{y}}^{-1}(z_1^0)$ and $\theta_{\bar{y}}^{-1}(\tilde{z}_i)$ onto $\theta_{\bar{y}}^{-1}(z_{i+1}^0)$. (See [18, Subsection 5.3] for more detail.) Moreover, since u is a viscosity solution we have that the relation (2.6) holds along all curves $t \mapsto \pi^* \left(\phi_t^H(\hat{w}_i, du(\hat{w}_i))\right)$, and this allows us to control the action and ensure that property (P2) holds.

Then, using the characteristic theory for solutions to the Hamilton-Jacobi equation we construct $C^{1,1}$ viscosity solutions $\bar{u}_i$ for the new Hamiltonian inside $\mathcal{C}_i$.

Finally, to conclude we need to "glue" these function with $\bar{u}$ outside of $\mathcal{C}_i$. With respect to the case $k = 1$ this is much more delicate, since one needs to control the closeness of $\bar{u}_i$ to $\bar{u}$ up to the second order [18, Subsection 5.5]. Still, this can be done [18, Lemma 5.5], and then the "gluing" can be performed as shown in Figure 8.

7.2.3 Statement of the result

In order to properly state the result described above, let us first formalize the concept of a $C^{1,1}$ function being C^2 at one point. Let $v : \mathcal{V} \to \mathbb{R}$ be a function of class $C^{1,1}$ in an open

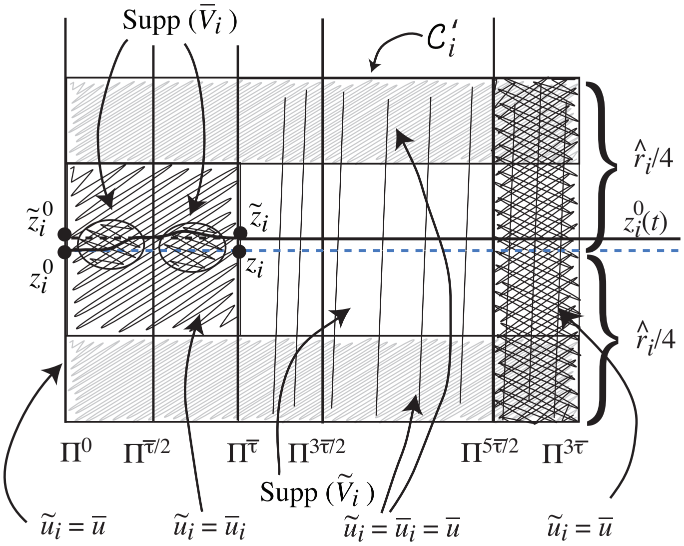

Figure 8: The function $\tilde{u}_i$ is obtained by interpolating (using a cut-off function) between $\bar{u}$ (the viscosity solution for $\bar{H}$) and $\bar{u}_i$ (the viscosity solution for $\bar{H}_{\bar{V}_i}$) inside the "cylinder" $\mathcal{C}_i' := \mathcal{C}\left(\left(z_i^0, \nabla\bar{u}(z_i^0)\right); \mathcal{T}_{3\bar{\tau}}(z_i^0); \hat{r}_i/4\right)$. Then, by adding a new potential $\tilde{V}_i$, small in C^2 topology and supported inside $\mathcal{C}_i' \cap \{z = (z_1, \hat{z}) \mid z_1 \in [\bar{\tau}, 3\bar{\tau}]\}$, we can ensure that $\bar{H}_{\bar{V}_i + \tilde{V}_i}(z, \nabla\tilde{u}_i(z)) \leq 0$ on the whole ball $B^n(0,2)$. Since the cylinders $\mathcal{C}_i'$ are disjoint, we can repeat this construction for $i = 1, \ldots, \eta - 1$ to find $\tilde{u} : B^n(0,2) \to \mathbb{R}$ and $\tilde{V} : B^n(0,2) \to \mathbb{R}$ so that (P1) and (P2) hold.

set $\mathcal{V} \subset M$. Thanks to Rademacher's Theorem, its differential dv is differentiable almost everywhere in M. Let $\mathrm{Dom}(\mathrm{Hess}^g v) \subset \mathcal{V}$ be the set of points where dv is differentiable. Then, for every $x \in \mathrm{Dom}(\mathrm{Hess}^g v)$, the function v is two times differentiable at x, and its Hessian with respect to the metric g is the symmetric bilinear form on T_xM defined as

$$\mathrm{Hess}^g v(x)[\xi, \eta] := \left\langle \left(\nabla^g_\xi dv\right)(x), \eta \right\rangle \qquad \forall\, \xi, \eta \in T_xM,$$

where ∇^g denotes the covariant derivative with respect to g. We call *generalized Hessian* of v at $x \in \mathcal{V}$ the set of symmetric bilinear form on T_xM defined by

$$\mathcal{H}\mathrm{ess}^g v(x) := \mathrm{conv}\left(\left\{\lim_{k\to\infty} \mathrm{Hess}^g v(x_k) \,|\, x_k \to x,\ x_k \in \mathrm{Dom}(\mathrm{Hess}^g v)\right\}\right),$$

where conv denotes the convex hull, and the limit is taken in the fiber bundle of symmetric bilinear forms on the fibers of TM. By construction, $\mathcal{H}\mathrm{ess}^g v(x)$ is a nonempty compact convex set of symmetric bilinear forms on T_xM for any $x \in M$. Then, the informal sentence "v is C^2 at a point x" means that $\mathcal{H}\mathrm{ess}^g v(x)$ is a singleton. (This definition is motivated by the fact that a $C^{1,1}$ function is C^2 on an open set $\mathcal{V}$ if and only if its generalized Hessian is a singleton at every point of $\mathcal{V}$.)

Recall that, by Theorem 5.3, C^1 viscosity solutions are $C^{1,1}$. So it make sense to talk about their generalized Hessian. The strategy described in the previous section allows to prove the following result [18, Theorem 2.1]:

Theorem 7.3. *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class C^k with $k \geq 2$. Assume that there are a recurrent point $\bar{x} \in \mathcal{A}(H)$, a critical viscosity subsolution $u : M \to \mathbb{R}$, and an open neighborhood $\mathcal{V}$ of $\mathcal{O}^+(\bar{x})$ such that the following properties are satisfied:*

(i) u is of class C^1 in $\mathcal{V}$;

(ii) $H(x, du(x)) = c[H]$ for every $x \in \mathcal{V}$;

(iii) $\mathcal{H}ess^g u(\bar{x})$ is a singleton.

Then, for any $\epsilon > 0$ there exists a potential $V : M \to \mathbb{R}$ of class C^k, with $\|V\|_{C^2} < \epsilon$, such that $c[H_V] = c[H]$ and the Aubry set of H_V is either an equilibrium point or a periodic orbit.

Recalling that constant functions are viscosity solutions for Mañé Lagrangians, see Example 2.4, as a corollary we get the following closing-type result:

Corollary 7.4. *Let X be a vector field on M of class C^k with $k \geq 2$. Then for every $\epsilon > 0$ there is a potential $V : M \to \mathbb{R}$ of class C^k, with $\|V\|_{C^2} < \epsilon$, such that the Aubry set of $H_X + V$ is either an equilibrium point or a periodic orbit.*

7.2.4 Equivalence between Mañé conjecture and Conjecture 6.4

Motivated by proving the equivalence between the Mañé conjecture and the generic smoothness of smooth critical subsolutions (the fact that the former implies the latter follows from Theorem 5.4, so we only need to show the converse implication), we want to address the case when we only have a sufficiently smooth subsolution.

The strategy used before to prove Theorem 7.3 does not easily generalize to the case of subsolutions. Indeed, if u is just a subsolution then the relation (2.6) holds only along curves in the Aubry set, and so in particular it may fail along the curves $t \mapsto \pi^*\big(\phi_t^H(\hat{w}_i, du(\hat{w}_i))\big)$ (since the points $\hat{w}_i$ may not be in the Aubry set). However, as shown before, the fact that (2.6) holds along such curves was crucial in the proof of Theorem 7.3 to control the action and ensure the validity of (P2).

Hence, we use instead the following strategy: If u is a critical subsolution which is smooth in a neighborhood $\mathcal{V}$ of $\mathcal{O}^+(\bar{x})$, we define the potential $V_0 : \mathcal{V} \to \mathbb{R}$ by

$$V_0(x) := -H\big(x, du(x)\big) \qquad \forall\, x \in \mathcal{V},$$

so that u becomes a solution of

$$H_{V_0}\big(x, du(x)\big) = 0 \qquad \forall\, x \in \mathcal{V}. \tag{7.3}$$

In this way we can apply the strategy explained before to find a small potential V_ϵ which allows to close the orbit $\mathcal{O}^+\big(\bar{x}\big)$. However, the problem is that V_0 is not small, so to conclude the proof we need to replace V_0 by another potential $V_1 : M \to \mathbb{R}$, which has small C^2-norm and such that "the Aubry sets of $H + V_0 + V_\epsilon$ and $H + V_1 + V_\epsilon$ coincide" (see [18, Subsection 6.2]).

This construction is much easier in dimension 2. Indeed, the fact that $\bar{x}$ is recurrent implies that, for every $t \in [0, \bar{t}_\eta]$, there are points of $\mathcal{A}(H)$ which are arbitrarily close to $\bar{\gamma}(t)$ and "transversal" to $\bar{\gamma}$. In two dimension this implies that $d^2V_0 = 0$ on $\bar{\Gamma}_1$, and the construction of V_1 becomes pretty easy [18, Section 6]. On the hand, in higher dimension we can only deduce that d^2V_0 is small in the "directions tangent to $\mathcal{A}(H)$". This fact creates much more difficulties, since we will need to know that the connecting trajectories can be chosen to belong to "the tangent space to $\mathcal{A}(H)$". To do this, we need a version of Lemma 7.2 with constraints (so that the connecting points $\hat{w}_i$ lie almost in a fixed subspace) to be able to refine our connecting trajectories. We refer to [19, Sections 3 and 4] for more details on this delicate construction.

In this way, we finally obtain the following result [19, Theorem 1.1]:

Theorem 7.5. *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class C^k with $k \geq 4$. Assume that there are a recurrent point $\bar{x} \in \mathcal{A}(H)$, a critical viscosity subsolution $u : M \to \mathbb{R}$, and an open neighborhood $\mathcal{V}$ of $\mathcal{O}^+(\bar{x})$ such that u is at least C^{k+1} on $\mathcal{V}$. Then, for any $\epsilon > 0$ there exists a potential $V : M \to \mathbb{R}$ of class C^{k-1}, with $\|V\|_{C^2} < \epsilon$, such that $c[H_V] = c[H]$ and the Aubry set of H_V is either an equilibrium point or a periodic orbit.*

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