Feedback Stabilization and Lyapunov Functions

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Abstract

Given a locally defined, nondifferentiable but Lipschitz Lyapunov function, we construct a (discontinuous) feedback law which stabilizes the underlying system to any given tolerance. A further result shows that suitable Lyapunov functions of this type exist under mild assumptions. We also establish a robustness property of the feedback relative to measurement error commensurate with the sampling rate of the control implementation scheme.

Introduction

Consider a standard control system of the form

$$\dot{x}(t) = f(x(t), u(t)) \quad a.e., u(t) \in \mathcal{U}, \tag{1}$$

and let V be a Lyapunov function for the system : We have $V(x) \geq 0, V(x) = 0$ iff $x = 0, V(x) \to \infty$ as $x \to \infty$, and (for some function W) *the Infinitesimal Decrease Condition*

$$\min_{u \in \mathcal{U}} \quad \langle \nabla V(x), f(x, u) \rangle \leq -W(x) < 0, x \neq 0. \tag{2}$$

It is well-known (but true) that the existence of V implies (open-loop) *Asymptotic Controllablity* to the origin :
for every $\alpha \in \mathbb{R}^n$, there is a control $u(t)$ such that the solution $x(\cdot)$ of (1) with initial condition $x(0) = \alpha$ satisfies $x(t) \to 0$ as $t \to \infty$ (and in addition, convergence to zero takes place in a certain uniform and stable manner that we will not dwell upon here). A related and important goal in many situations is to produce a state feedback $k(\cdot) : \mathbb{R}^n \to \mathcal{U}$ which stabilizes the system; i.e., such that the system $\dot{x} = f(x, k(x))$ is globally asymptotically stable. This article explores the question of how to define such a feedback law through the use of a given Lyapunov function V.

The *ideal case*, a well-known heuristic useful for motivational purposes, is the one in which we can find a continuous function $k(x)$ that selects a value of $u \in \mathcal{U}$ attaining (or almost) the minimum in (2) :

$$\langle \nabla V(x), f(x, k(x)) \rangle \leq -W(x) \quad \forall x \neq 0.$$

Then any solution of $\dot{x} = f(x, k(x))$ is such that

$$\frac{d}{dt} V(x(t)) = \langle \nabla V(x(t)), \dot{x}(t) \rangle \leq -W(x) < 0,$$

a monotonicity conclusion that, together with the growth property of V, assures that $x(t) \to 0$ as $t \to \infty$.
There are two fundamental difficulties with this ideal picture, and both concern regularity issues. The first is that a differentiable Lyapunov function may not exist, and the second is that even when a smooth V exists, the continuous selection $k(\cdot)$ does not generally exist. If we have recourse to a discontinuous feedback $k(\cdot)$,then the issue arises of how to interpret the discontinuous differential equation $\dot{x} = f(x, k(x))$.
The primary goal of this article is to give a general answer to the problem of defining a (discontinuous) stabilizing feedback based upon a given (nondifferentiable) Lyapunov function, one for which infinitesimal decrease is known to hold only on a restricted set. The construction is described in section 1, while section 2 establishes that under mild conditions, a Lyapunov function of the type required in the previous section always exists. In the final section, we

address the issue of robustness of the feedback with respect to measurement error and small perturbations of the dynamics, a particularly important issue when discontinuity is present. Some works and general references related to the results of this article include [2, 3, 8, 11, 12, 13, 15, 20, 22, 23, 25, 29, 30]. We proceed now to situate our result with respect to the literature.

The possible nonexistence of continuous stabilizing feedback was brought to light in the seminal work of Sontag and Sussmann [28] and of Brockett [4]. The latter who developed a necessary condition for continuous stabilizability and adduced the following example, the "non-holonomic integrator" :

$$
\begin{cases}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_1 u_2 - x_2 u_1
\end{cases}
\qquad (u_1, u_2) \in \bar{B} =: \mathcal{U}.
$$

This system is globally asymptotically controllable yet fails to admit a continuous stabilizing feedback (by Brockett's condition). In considering the use of discontinuous feedback laws $k(\cdot)$, one could have recourse to the Filippov solution concept [14] : x is a solution of $\dot{x} = f(x, k(x)) =: g(x)$ provided that we have

$$
\dot{x} \in \bigcap_{\substack{\delta > 0 \\ \mathrm{meas}(\Omega) = 0}} \mathrm{clco}(g([x + \delta B] \setminus \Omega)).
$$

However, as shown by Ryan [24] and by Coron [11], Brockett's condition continues to hold for this solution concept, so that the nonholonomic integrator (for example) cannot be stabilized by a discontinuous feedback in the Filippov sense.

In [6] it was shown that any globally asymptotically controllable system is stabilizable by a (possibly discontinuous) feedback if the trajectory $x(\cdot)$ associated to the feedback is defined in a natural way that involves discretizing the control law (closed-loop system sampling) in a manner similar to [19]. We proceed now to describe this concept, which is the one used in this article.

Let $\pi = \{t_i\}_{i \geq 0}$ be a partition of $[0, \infty)$, by which we mean a countable, strictly increasing sequence t_i with $t_0 = 0$ such that $t_i \to \infty$ as $i \to \infty$. The *diameter* of π, denoted $\mathrm{diam}(\pi)$, is defined as $sup_{i \geq 0}(t_{i+1} - t_i)$. Given an initial condition x_0, the π-trajectory $x(\cdot)$ corresponding to π and an arbitrary feedback law $k : \mathbb{R}^n \to \mathcal{U}$ is defined in a step-by-step fashion as follows. Between t_0 and t_1, x is a classical solution of the differential equation

$$
\dot{x}(t) = f(x(t), k(x_0)), \quad x(0) = x_0, \quad t_0 \leq t \leq t_1.
$$

(Of course in general we do not have uniqueness of the solution, nor is there necessarily even one solution, although the latter will be ruled out by the feedback constructed in section 1, which will preclude blow up of the solution in finite time.) We then set $x_1 := x(t_1)$ and restart the system with control value $k(x_1)$:

$$
\dot{x}(t) = f(x(t), k(x_1)), \quad x(t_1) = x_1, \quad t_1 \leq t \leq t_2,
$$

and so on in this fashion. The resulting trajectory x is a physically meaningful one that corresponds to a natural sampling procedure and piecewise constant controls; the smaller diam(π), the greater the sampling rate. Since our results are couched in term of π-trajectories, the issue of defining a solution concept for discontinuous differential equations is effectively sidestepped. Our approach will lead to precise estimates of how small the step size diam(π) must be for a prescribed stabilization tolerance to ensue, and of the resulting stabilization time, in terms of the given data.

The next major point to address concerns the nonsmoothness of the Lyapunov function V. An early and important result of Artstein [1] implies in particular that the nonholonomic integrator fails to admit a smooth V (see [7] for related results). It has been shown by Sontag [25], however, that globally asymptotically controllable systems always admit a continuous Lyapunov function V satisfying the following nonsmooth version of the infinitesimal decrease condition :

$$\inf_{u \in \mathcal{U}} DV(x; f(x, u)) \leq -W(x) < 0, x \neq 0, \tag{3}$$

where the lower Dini derivate DV is defined by

$$DV(x; v) := \liminf_{\substack{t \downarrow 0 \\ v' \to v}} \frac{V(x + tv') - V(x)}{t}. \tag{4}$$

Among the several important ways in which the theory of nonsmooth analysis intervenes in this article is that of asserting the equivalence to (3) of another, and for our purposes more useful, form of the infinitesimal decrease condition :

$$\inf_{v \in \mathrm{co} f(x, \mathcal{U})} \langle v, \zeta \rangle \leq -W(x) < 0 \quad \forall x \neq 0, \forall \zeta \in \partial_P V(x). \tag{5}$$

Here $\partial_P V(x)$ refers to the *proximal subdifferential* of V at x (which may very well be empty); ζ belongs to $\partial_P V(x)$ iff there exists σ and $\eta > 0$ such that

$$V(y) - V(x) + \sigma \|y - x\|^2 \geq \langle \zeta, y - x \rangle \quad \forall y \in B(x, \eta).$$

The equivalence of (3) and (5) is a consequence of Subbotin's Theorem (see for example [9], our principal source for the theory of nonsmooth analysis).

The essential reason for which proximal calculus is well-suited to our approach is because of its relation to metric projection onto sets, upon which is based the "proximal aiming" method that we employ. The crux is this : when $x(t_i) = x$ lies outside a level set $S = S(c) := \{V \leq c\}$ and admits closest point (or projection) s in S, then $x - s$ is a "proximal normal" to S at s, and for some $\lambda > 0$ we have $\lambda(x - s) \in \partial_P V(x)$. Then (5) can be invoked at s to find a suitable value of the control u which moves the state toward S, in the sense that the Euclidian distance d_S decreases at a certain positive rate Δ :

$$d_S(x(t)) - d_S(x(t_i)) \leq -\Delta(t - t_i), \quad \forall t_i \leq t \leq t_{i+1},$$

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provided $x(t_i)$ is close enough to S to start with, and provided diam(π) is small enough. A sequence of such feedbacks is amalgamated in the first section to produce the stabilizing feedback $k(\cdot)$ that is sought.

The approach as presented in §1 requires the Lyapunov function V to be Lipschitz (in the zone under consideration). It is not known whether a globally asymptotically controllable system must admit a suitably Lipschitz V, but that is not quite the setting in which we work. Instead, Theorem 1 derives *finite-time* stabilizability , but to a close *approximation* of some inner level set $S(a)$, as a consequence of the supposed existence of a Lipschitz Lyapunov function. In contrast, [6] obtains asymptotic stabilizability to the origin (the case $S(a) = \{0\}$), and the proof uses Moreau-Yosida inf convolution to produce a Lipschitz Lyapunov function as an intermediate step. This methodology is also employed in [10], in a differential game setting. The direct use of a Lipschitz Lyapunov function, when it is possible, leads to a far more transparent feedback construction, and has the important consequence of yielding robustness, as we discuss presently. The fact that under mild assumptions, suitable Lipschitz functions exist leading to practical stabilization to any required tolerance, is proven in section 2.

Ledyaev and Sontag [21] have recently proved that there is a close relationship between the issues of "how regular a Lyapunov function does the system admit" and "how robust a stabilizing feedback does the system admit". Consider for example a perturbed equation $\dot{x} = f(x, k(x + p))$, where p represents a measurement error. Full robustness of the feedback k is taken to mean that for any ϵ, there is a $\delta > 0$ such that whenever the perturbation $p(t)$ satisfies $\|p(t)\| \le \delta$ for all t, then stabilization to the ϵ-ball takes place. This seems to have been studied by Hermes [17]. Then [21] asserts that the system admits a fully robust stabilizing feedback iff it admits a smooth (C^1 or C^∞) Lyapunov function. Thus the nonholonomic integrator, which *can* be stabilized by a discontinuous feedback (in view of [6]), does not admit a fully robust stabilizing feedback. It appears that Hermes [16] was the first to raise the question of robustness of discontinuous feedback with respect to measurement error.

The above concept of full robustness, unrelated as it is to the system sampling method that we employ, is not the one discussed in this article. Instead, we introduce a type of *relative robustness* in which we require the size of the measurement error to be limited as a function of the maximum step size δ of the underlying partition. This step size δ must still be small enough (for stabilization), but at the same time the individual steps must be big enough to preclude a possible chattering phenomenom, even in the presence of small errors. This consideration, which leads us to specify "reasonably uniform" sampling in section 3, appears to be new in this context. The term "reasonably uniform" is taken here to mean that the following hold :

$$\frac{\delta}{2} \le t_{i+1} - t_i \le \delta \quad \forall i \ge 0,$$

although it is possible to replace the factor $\frac{1}{2}$ by any constant in $(0, 1)$.

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To conclude with the nonholonomic integrator, then, it turns out that the system does admit a stabilizing feedback to within any prescribed tolerance r, in the sense that we will have $\|x(t)\| \leq r$ for all $t \geq T$, whenever x is a π-trajectory, where π is a reasonably uniform partition whose diameter is sufficiently small, and whenever measurement and external error do not exceed a critical level related to the diameter (or sampling rate).

1   A feedback construction

For a given function $V : \mathbb{R}^n \longmapsto (-\infty, \infty]$, we shall deal frequently with the sublevel sets $S(r)$ defined as follows :

$$S(r) := \{x \in \mathbb{R}^n : V(r) \leq r\}.$$

In addition, the following sets are considered :

$$S(a,b) := \{x \in \mathbb{R}^n : a \leq V(x) \leq b\}.$$

Let a and b be two given numbers with $a < b$. The following hypotheses are made concerning the function V and the system function f :

(H1) V is lower semicontinuous, $S(b) \neq \emptyset$, and for some $\eta > 0$, V is Lipschitz of rank L_V on $S(a,b) + \eta B$:

$$|V(x) - V(y)| \leq L_V \|x - y\| \quad \forall (x,y) \in S(a,b) + \eta B.$$

(H2) $\exists \delta_1 \in (0, b-a)$ and $\delta_2 > 0$ such that

$$S(a + \delta_1) + \delta_2 B \subset S(b).$$

(H3) $f(x,u)$ is continuous on $S(b) + \eta B$ as a function of x for each $u \in \mathcal{U}$, and $\exists m > 0$ such that

$$\|f(x,u\| \leq m \quad \forall x \in S(b) + \eta B, \forall u \in \mathcal{U}.$$

(H4) f is Lipschitz in x of rank L_f on $S(a,b) + \eta B$:

$$\|f(x,u) - f(y,u)\| \leq L_f \|x - y\| \quad \forall x, y \in S(a,b) + \eta B, \forall u \in \mathcal{U}.$$

(H5) There exists $\omega > 0$ such that, for every $x \in S(a,b) + \eta B$, we have

$$\inf_{v \in \mathrm{co} f(x,\mathcal{U})} DV(x;v) \leq -\omega.$$

Remark 1.1 We do not require that f and V be defined except on $S(b) + \eta B$; the Lipschitz conditions on these functions, as well as the Infinitesimal Decrease Condition (H5), are posited only on a neighborhood of $S(a,b)$. No hypotheses are made concerning the abstract set $\mathcal{U}$, nor on the nature of the dependence of f on the control variable.

Theorem 1 *For any $\gamma > 0$ sufficiently small, there exist positive numbers δ, T and a feedback $k : S(b) + \eta B \longmapsto \mathcal{U}$ such that whenever a partition π satisfies $\mathrm{diam}(\pi) < \delta$, then any $\pi-$trajectory $x(\cdot)$ having $x(0) \in S(b) + \gamma B$ satisfies*

$$x(t) \in S(b) + \gamma B \quad \forall t \geq 0,$$

$$x(t) \in S(a) + \gamma B \quad \forall t \geq T.$$

Remark 1.2 Thus we almost recover the conclusion of the "ideal case" discussed in the Introduction, but in approximate terms, with a discontinuous feedback, and for a nonsmooth Lyapunov function satisfying localized hypotheses. The proof is constructive and gives estimates of γ, δ and T in terms of the given data.

Remark 1.3 Note that (H5) is a weaker hypothesis than (3). An important fact used in the proof is that the Infinitesimal Decrease Condition (H5) has a strictly equivalent formulation in proximal terms, as follows :

$$\inf_{u \in \mathcal{U}} \langle \zeta, f(x, u) \rangle \leq -\omega \quad \forall x \in S(a, b) + \eta B, \forall \zeta \in \partial_P V(x).$$

Here, $\partial_P V$ is the *proximal subdifferential* of V [9]. The equivalence of the formulations is nontrivial, and is a consequence of Subbotin's Theorem [9, Theorem 3.4.2].

The proof of Theorem 1 is based upon defining a feedback control, via projections. The first two lemmas below guarantee that the projections lie in the set where the hypotheses are active.

Lemma 1.1 *Let ϵ lie in $[0, \delta_1]$ and suppose that x is a point in the set*

$$[S(a + \epsilon) + \min\{\delta_2, \eta\}B] \setminus S(a + \epsilon).$$

Then $x \in S(a, b)$, and if $s \in \mathrm{proj}(x, S(a + \epsilon))$, then $s \in S(a, b) + \eta B$.

Proof   Since we have $S(a + \delta_1) + \delta_2 B$ contained in $S(b)$ by hypothesis, it follows that x lies in $S(b)$. Since x does not belong to $S(a + \epsilon)$, we deduce $x \in S(a, b)$. Finally, we have

$$\|s - x\| < \min\{\delta_2, \eta\} \leq \eta,$$

whence $s \in S(a, b) + \eta B$.□

Lemma 1.2 *Let $0 < \gamma < \eta/2$, and suppose that for some r' and r with $a \leq r' < r \leq b$ we have $x \in [S(r) + \gamma B] \setminus [S(r') + \gamma B]$. Then $x \in S(a, b) + \gamma B$, and if $s \in \mathrm{proj}(x, S(r))$, then $s \in S(a, b) + \eta B$.*

Proof   There exists $y \in S(r)$ having $\|y - x\| < \gamma$. Since x does not belong to $S(r') + \gamma B$, we have $V(y) > r'$ necessarily. Thus $y \in S(a, b)$ and $x \in S(a.b) + \gamma B$. Finally, we note $\|x - s\| < \gamma$, whence $\|y - s\| < 2\gamma$ and s lies in $S(a, b) + \eta B$. □

The next "solvability" result is central to our approach. The notation u_+ stands for $\max\{u, 0\}$.

Lemma 1.3 *For any $r \in [a, b]$, for any $x \in S(a, b)$, we have*

$$d(x, S(r)) \leq \frac{m}{\omega}(V(x) - r)_+.$$

Proof We shall invoke results (and terminology) from [9] to give a short proof of this result, whose proof from first principles would be lengthy.

We define a lower semicontinuous function $g : \mathbb{R}^n \longmapsto [0, \infty]$ as follows :
$g(x) := (V(x) - r)_+ + I_{S(b)}(x)$, where $I_{S(b)}(\cdot)$ is the indicator function of the set $S(b)$. At any point x in the open set $C := \{y : g(y) > 0\}$ at which g is finite, we have $x \in S(a, b)$, and the infinitesimal decrease condition implies that

$$\inf\{Dg(x; v) : v \in \mathrm{co}f(x, \mathcal{U})\} \leq -\omega.$$

It follows from this that for any $\epsilon > 0$, for any $x \in C$, for any $\zeta \in \partial_P g(x)$, there exists $u \in \mathcal{U}$ such that

$$\langle \zeta, f(x, u) \rangle \leq -\omega + \epsilon.$$

Since $\|f(x, u)\| \leq m$ and since $\epsilon > 0$ is arbitrary, we derive $\|\zeta\| \geq \frac{\omega}{m}$. This verifies the hypothesis of the Solvability Theorem [9, Theorem 3.3.1] (applied with $V = \Omega = \mathbb{R}^n$), whose conclusion is precisely the desired one since $S(r) = \{x : g(x) = 0\}$. $\square$

We now proceed to fix $\gamma > 0$ such that

$$\gamma < \min\left\{\delta_1, \frac{\eta}{2}, \frac{\omega}{12 L_f L_V}\right\}, \tag{6}$$

and we define

$$\beta := \min\left\{\delta_1, \frac{(b-a)}{2}, \frac{\gamma \omega}{4m}\right\}. \tag{7}$$

Let N be the first integer such that

$$b - N\beta > a \geq b - (N+1)\beta.$$

Note that $N \geq 1$ since $\beta < b - a$. We proceed to define certain sets $\Omega_i (i = 0, 1, \cdots, N+1)$ that lie at the heart of our construction.

For $0 \leq i \leq N - 1$, we set

$$\Omega_i := [S(b - i\beta) + \gamma B] \setminus [S(b - (i+1)\beta) + \gamma B];$$

for $i = N$ we set

$$\Omega_N := [S(b - N\beta) + \gamma B] \setminus [S(b - N\beta) + \frac{\gamma}{4}B];$$

and finally, we define $\Omega_{N+1} := S(b - N\beta) + \frac{\gamma}{4}B$.

We now gather some facts about these sets.

Lemma 1.4

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(a) *The Ω_i are disjoint, and Ω_i is contained in $S(a,b) + \gamma B$ for $i \le N$.*

(b) $\bigcup_{i=0}^{N+1} \Omega_i = S(b) + \gamma B.$

(c) *If $x \in \Omega_i$ for some $i \in \{0, 1, \cdots, N\}$ and $s \in \text{proj}(x, S(b - i\beta))$, then $s \in S(a,b) + \eta B.$*

(d) $S(b - i\beta) + \frac{\gamma}{4} B \subset S(b - (i+1)\beta)) + \gamma B \quad (i = 0, 1, \cdots, N-1).$

(e) *For every $i \in \{0, 1, \cdots, N\}, \forall x \in \Omega_i$, we have $\frac{\gamma}{4} \le d(x, S(b - i\beta)) < \gamma.$*

(f) $S(b - N\beta) + \frac{\gamma}{2} B \subset S(a) + \gamma B.$

Proof

(a): That the Ω_i are disjoint is evident; that they lie in $S(a,b) + \gamma B$ for $i \le N$ follows from Lemma 1.2 for $i < N$ and from Lemma 1.1 for $i = N$ (recall that $b - N\beta - a \le \beta \le \delta_1$ and $\gamma < \delta_2$).

(b): Evident.

(c): Direct from Lemma 1.2 ($i < N$) or Lemma 1.1 ($i = N$).

(d): Let x lie in $S(b - i\beta) + \frac{\gamma}{4} B$, and let $s \in S(b - i\beta)$ satisfy $\|x - s\| < \frac{\gamma}{4}$. Then $V(s) \le b - i\beta$, and if $V(s) \le b - (i+1)\beta$ the conclusion is immediate. Otherwise we have
$$V(s) > b - (i+1)\beta > a,$$
so that $s \in S(a,b)$. By lemma 1.3 there exists $y \in S(b - (i+1)\beta)$ such that
$$\|s - y\| \le \frac{m}{\omega}[V(s) - b + (i+1)\beta] \le \frac{m\beta}{\omega} \le \frac{\gamma}{2},$$
in view of (7). Then $\|x - y\| \le \|x - s\| + \|s - y\| < \frac{\gamma}{4} + \frac{\gamma}{2} < \gamma$, which established the desired conclusion.

(e): For $i = N$, this is immediate from the definition of Ω_N; for $i < N$, it is a consequence of (d).

(f): Let x belong to $S(b - N\beta) + \frac{\gamma}{2} B$, and let $s \in S(b - N\beta)$ satisfy $\|x - s\| < \frac{\gamma}{2}$. If $V(s) \le a$, then $x \in S(a) + \gamma B$. Otherwise, s belongs to $S(a,b)$, and lemma 1.3 implies the existence of $y \in S(a)$ such that
$$\|y - s\| \le \frac{m}{\omega}[V(s) - a] \le \frac{m}{\omega}[b - N\beta - a] \le \frac{m\beta}{\omega} \le \frac{\gamma}{2}.$$
But then $\|x - y\| < \gamma$, so again $x \in S(a) + \gamma B$.

$\square$

Lemma 1.5 *Let $x \in \Omega_i (i = 0, 1, \cdots, N)$, and let $s \in \text{proj}(x, S(b - i\beta))$. Then there exists $u \in \mathcal{U}$ such that*
$$\langle x - s, f(s, u) \rangle \le \frac{-\omega}{2L_V} \|x - s\|.$$

Proof   By definition, $x - s$ lies in the proximal normal cone $N^P(x, S(b - i\beta))$. Note that s lies in $S(a,b) + \eta B$ (by Lemma 1.2 for $i < N$, Lemma 1.1 for $i = N$), so that V is Lipschitz of rank L_V in a neighborhood of s. A basic calculus result [9, 1.11.26] yields the existence of $\lambda > 0$ such that $\lambda(x - s) \in \partial_L V(s)$, and necessarily $\lambda \|x - s\| \leq L_V$. In accord with Remark 1.1, there exists $u \in \mathcal{U}$ such that

$$\langle \lambda(x - s), F(s, u) \rangle \leq \frac{-\omega}{2}.$$

The result follows.$\square$

Defining the feedback

We now define a feedback $k(\cdot)$ on $S(b) + \gamma B$ as follows. If $x \in \Omega_i$ for some $i \in \{0, 1, \cdots, N\}$, then we set $k(x) = u$, where u corresponds to x (and a projection s) as in lemma 1.5. There remain the points x in Ω_{N+1} to consider (see Lemma 1.4 (b)). For such x, we define $k(x)$ to be any point in $\mathcal{U}$, for example one which minimizes $\|f(x, u)\|$.

The remainder of the proof consists in establishing that for suitably small mesh size, any π-trajectory generated by $k(\cdot)$ with initial condition in $S(b) + \gamma B$ remains in $S(b) + \gamma B$, enters $S(a) + \gamma B$ within a certain (uniform) time, and then remains in that set subsequently.

We consider countable partitions $\{t_j\}$ such that $t_0 = 0, \lim_{j \to \infty} t_j = \infty$, and such that $t_{j+1} - t_j \leq \delta \quad \forall j \geq 0$, where δ is any positive number satisfying

$$\delta < \min \left\{ \frac{\gamma}{4m}, \frac{\omega}{6mL_f L_V}, \frac{\gamma\omega}{48m^2 L_V}, 1 \right\}. \tag{8}$$

For such a partition, let x_0 be any point in $S(b) + \gamma B$, and let $x(\cdot)$ be a π-trajectory with $x(0) = x_0$. We denote $x(t_j)$ by x_j, and we set $\Delta := \frac{\omega}{60 L_V}$.

Lemma 1.6 *For some $t_j \in \pi$, suppose that $x_j \in \Omega_i, i \in \{0, 1, \cdots, N\}$.*
Then $x(t) \in S(b) + \gamma B$ for $t \in [t_j, t_{j+1}]$, and

$$d(x(t), S(b - i\beta)) \leq d(x_j, S(b - i\beta)) - \Delta(t - t_j) \quad \forall t \in [t_j, t_{j+1}].$$

Proof   We have $x_j \in S(a,b) + \gamma B$ by Lemma 1.4 (a), and $\|\dot{x}(t)\| \leq m$ while $x(t)$ lies in $S(b) + \eta B$. Since $\delta m < \frac{\gamma}{4}$ by (8) and $\gamma < \frac{\eta}{2}$, it follows that $x(t)$ lies in $S(a,b) + \eta B$ for $t \in [t_j, t_{j+1}]$, as does the point s that figures in the definition of $k(x_j)$; this was pointed out in the proof of Lemma 1.5, where we also deduced the inequality

$$\langle x_j - s, f(s, k(x_j)) \rangle \leq \frac{-\omega}{2L_V} \|x_j - s\|. \tag{9}$$

We fix $t \in (t_j, t_{j+1})$ and set $\Psi := \frac{x(t) - s}{\|x(t) - s\|}$.(Note that $x(t) \neq s$, since $\|x_j - s\| \geq \frac{\gamma}{4}$ by Lemma 1.4 (e), and since $\|x(t) - x_j\| < \delta m < \frac{\gamma}{4}$.)
We now observe two inequalities :

$$d(x(t), S(b - i\beta)) \leq \|x(t) - s\| = \langle \Psi, x(t) - s \rangle,$$

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$$d(x_j, S(b - i\beta)) = \|x_j - s\| \geq \langle \Psi, x_j - s \rangle.$$

These together imply

$$
\begin{aligned}
d(x(t), S(b - i\beta)) - d(x_j, S(b - i\beta)) &\leq \langle \Psi, x(t) - x_j \rangle \\
&= \tau \langle \Psi, f_j \rangle, \quad (10)
\end{aligned}
$$

where we introduce the notation $\tau := t - t_j$,

$$x(t) = x_j + \tau f_j, \quad f_j := \frac{1}{\tau} \int_{t_j}^{t} f(x(r), k(x_j)) dr.$$

We also set

$$\hat{f}_j := f(s, k(x_j)) = \frac{1}{\tau} \int_{t_j}^{t} f(s, k(x_j)) dr.$$

Note that $\|f_j - \hat{f}_j\| \leq \frac{1}{\tau} \int_{t_j}^{t} L_f \|x(r) - s\| dr$
(the Lipschitz condition applies because we are in $S(a, b) + \eta B$)

$$
\begin{aligned}
&\leq \frac{1}{\tau} \int_{t_j}^{t} L_f (\|x(r) - x_j\| + \|x_j - s\|) dr \\
&\leq L_f(\tau m + d(x_j, S(b - i\beta))).
\end{aligned}
$$

It follows from this and (9) that we have

$$
\begin{aligned}
\langle x_j - s, f_j \rangle &= \langle x_j - s, \hat{f}_j + f_j - \hat{f}_j \rangle \\
&\leq \frac{-\omega}{2L_V} \|x_j - s\| + L_f d(x_j, S(b - i\beta))\{\tau m + d(x_j, S(b - i\beta))\} \\
&\leq d(x_j, S(b - i\beta))[\frac{-\omega}{2L_V} + L_f \delta m + \gamma L_f] \\
&\quad \text{(since } \tau < \delta \text{ and } d(x_j, S(b - i\beta)) \leq \gamma) \\
&\leq d(x_j, S(b - i\beta))[\frac{-\omega}{2L_V} + \frac{\omega}{6L_V} + \frac{\omega}{6L_V}] \\
&\quad \text{(we have invoked (8) and (6))} \\
&\leq -\frac{\gamma \omega}{24 L_V} \\
&\quad \text{(since } d(x_j, S(b - i\beta)) \geq \frac{\gamma}{4} \text{ by Lemma 1.4 (e)).}
\end{aligned}
$$

We shall use this bound on $\langle x_j - s, f_j \rangle$ to derive one on $\langle x(t) - s, f_j \rangle$ as follows:

$$
\begin{aligned}
\langle x(t) - s, f_j \rangle &= \langle x_j + \tau f_j - s, f_j \rangle \leq \langle x_j - s, f_j \rangle + \tau \|f_j\|^2 \\
&\leq \frac{-\gamma \omega}{24 L_V} + \delta m^2 \leq \frac{-\gamma \omega}{48 L_V} \quad (11)
\end{aligned}
$$

(in light of (8)).
We also have

$$
\begin{aligned}
\|x(t) - s\| = \|x_j + \tau f_j - s\| &\leq \|x_j - s\| + \tau \|f_j\| \leq \gamma + \delta m \\
&< \frac{5\gamma}{4} (\text{by (8)}).
\end{aligned}
$$

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Combining this with (11) we arrive at

$$\langle \Psi, f_j \rangle = \langle \frac{x(t) - s}{\|x(t) - s\|}, f_j \rangle \leq \frac{-\gamma \omega}{48 L_V} / \frac{5\gamma}{4} = -\Delta.$$

Together with (10), this gives the inequality asserted by the Lemma. Since this inequality evidently implies

$$d(x(t), S(b - i\beta)) < \gamma,$$

it also follows that $x(t) \in S(b) + \gamma B$. $\square$

Lemma 1.7 *If $x_j \in \Omega_i$ where $0 \leq i \leq N$, then x_{j+1} lies in $\Omega_{i'}$ for some $i' \geq i$.*

Proof Since $x_j \in \Omega_i$, we have $d(x_j, S(b - i\beta)) < \gamma$, and (by Lemma 1.6) $d(x_{j+1}, S(b - i\beta)) < \gamma$. Now let $1 \leq k < j$. Since $S(b - i\beta) \subset S(b - (k+1)\beta)$, we deduce $d(x_{j+1}, S(b - (k+1)\beta)) < \gamma$. But then $x_{j+1} \notin \Omega_k$ by definition of Ω_k. Since $x_{j+1} \in S(b) + \gamma B$ by Lemma 1.6, we must have $x_{j+1} \in \Omega_{i'}$ for some $i' \geq i$, in view of Lemma 1.4 (b).$\square$

Lemma 1.8 *If $x(\tau) \in \Omega_{N+1}$ for some $\tau \in \pi$, then $x(t) \in S(a) + \gamma B$ $\forall t \geq \tau$.*

Proof We know that $x(\tau)$ lies in the interior of $S(a) + \gamma B$ by Lemma 1.4 (f). For $t > \tau$, as long as $d(x(t), S(b - N\beta))$ does not attain or exceed $\frac{\gamma}{2}$, then $x(t)$ remains in $S(a) + \gamma B$. Thus $\|\dot{x}(t)\|$ remains bounded by m and no blow-up occurs (i.e., $x(t)$ is well-defined).
It suffices therefore to prove that the continuous function $g(t) := d(x(t), S(b - N\beta))$ does not become greater than or equal to $\frac{\gamma}{2}$ for some $t_0 > \tau$. We have $g(\tau) \leq \frac{\gamma}{4}$.
Note that $S(b - N\beta) + \gamma B$ lies in $S(b) + \gamma B$, so that the bound $\|\dot{x}\| \leq m$ continues to apply for some time after any positive point t_0 at which $g(t)$ becomes equal to $\frac{\gamma}{2}$. Thus $g(t)$ is locally well-defined. Yet Lemma 1.6 (for $i = N$) shows that g is decreasing when $g(t)$ lies between $\frac{\gamma}{4}$ and γ. The conclusion is that $g(t) \leq \frac{\gamma}{2}$ $\forall t > \tau$ (or in fact that $g(t)$ never even exceeds $\frac{\gamma}{4}$).$\square$
    One last lemma and the proof is complete.

Lemma 1.9 *Let $T := (1 + \frac{b-a}{\beta})(1 + \frac{45\gamma L_V}{\omega})$. Then*

$$x(t) \in S(a) + \gamma B \quad \forall t \geq T.$$

Proof In view of Lemma 1.8 and Lemma 1.4 (f), it suffices to prove that there is a node $\tau \in \pi$ with $\tau \leq T$ for which $x(\tau) \in \Omega_{N+1}$. Note that $x(0)$ belongs to some $\Omega_i (0 \leq i \leq N + 1)$ by Lemma 1.4 (b); if $i = N + 1$ we are done, so assume $i \leq N$. Since $\delta < 1$ by (8), there is a node τ_1 lying in the open interval $(\sigma, \sigma + 1)$, where $\sigma := \frac{3\gamma}{4\Delta}$. By Lemma 1.7, $x(\tau_1)$ belongs either to Ω_i or to $\Omega_{i'}$ for some $i' > i$. In the former case, it follows that $x(t)$ lies in Ω_i for every node $t \in \pi$ lying between 0 and τ_1, and the inequality of Lemma 1.6 applies to give

$$\begin{aligned}
d(x(\tau_1), S(b - i\beta)) &\leq& d(x(0), S(b - i\beta)) - \Delta \tau_1 \\
&<& \gamma - \frac{3\gamma}{4} = \frac{\gamma}{4}.
\end{aligned}$$

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However, the left side is no less than $\frac{\gamma}{4}$ by Lemma 1.4 (e). This contradiction shows that, in fact, $x(\tau_1)$ must belong to some $\Omega_{i'}$ for an index $i' > i$. The same argument, beginning now at $(\tau_1, x(\tau_1))$, yields the existence of a node $\tau_2 \in \pi$ with $\tau_2 \leq 2\sigma + 2$ such that $x(\tau_2)$ belongs to $\Omega_{i''}$, where $i'' > i'$. Continuing in this manner, we find that (since there are at most $N + 1$ steps as above prior to landing in Ω_{N+1}), there is a node $\tau \in \pi$ with $\tau \leq (N + 1)(\sigma + 1)$ such that $x(\tau) \in \Omega_{N+1}$. But $N < \frac{b-a}{\beta}$ implies that the T defined in the statement of the Lemma is greater than $(N + 1)(\sigma + 1)$. $\square$

2   Construction of a Lyapunov function

We show in this section thst under reasonable assumptions, there always exist Lyapunov functions having the properties required for the feedback construction of the preceding section, and giving rise to practical feedback stabilization of arbitrarily prescribed range. While the result below seems new and the approach to proving it has some novel features, there is a familiar heuristic at work : the Lyapunov function is contructed as the value function associated with a parametrized family of optimal control problems.

The function $f(x, u)$ describing the dynamics is supposed in this section to satisfy much the same regularity conditions as before. Specifically, we require that for any bounded subset S of $\mathbb{R}^n$, there exist constants $m = m(S)$ and $L = L(S)$ such that

$$\|f(x, u)\| \leq m \quad \forall x \in S, \forall u \in \mathcal{U}$$

$$\|f(x, u) - f(y, u)\| \leq L\|x - y\| \quad \forall x, y \in S, \forall u \in \mathcal{U}.$$

(As before, $\mathcal{U}$ is just an abstract set, and no hypotheses are made concerning the nature of the dependence of f on u.)

In addition, we require controllability to the ball of radius r_0 via relaxed trajectories. Let us now proceed to make this precise. We are given $r_0 \geq 0$, and we define a multifunction Γ on $\mathbb{R}^n$ by

$$\Gamma(x) := \text{clco}\{f(x, u) : u \in \mathcal{U}\}.$$

By "trajectory" we mean an absolutely continuous function $x(\cdot)$ on an interval $[0, T]$ such that

$$\dot{x}(t) \in \Gamma(x(t)) \quad a.e. \quad t \in [0, T].$$

Given $\alpha \in \mathbb{R}^n$, we define $T_0(\cdot)$ as the least time required for a trajectory to go from α to the closed ball $r_0 \bar{B}$:

$$T_0(\alpha) := \inf\{T \geq 0 : x(\cdot) \text{ is a trajectory on } [0, T], x(0) = \alpha, \|x(T)\| \leq r_0\}.$$

the controllability hypothesis that we make is that every α admits a trajectory steering it to $r_0 \bar{B}$ in finite time, a time which goes to zero as α approaches $r_0 \bar{B}$. Equivalently :

$$\text{(CH)} \qquad T_0(\alpha) < \infty \quad \forall \alpha \in \mathbb{R}^n, \text{and} \lim_{\|\alpha\| \downarrow r_0} T_0(\alpha) = 0.$$

Theorem 2 *For any pair r and R with $R > r > r_0$, there exist a function V and numbers a, b, γ, η such that*

$$S(a) + \gamma B \subset B(0, r) \subset \bar{B}(0, R) \subset S(b) + \gamma B,$$

and such that V satisfies all the hypotheses of Theorem 1; in addition $S(b)$ is compact. Thus there is a feedback defined on a neighborhood of $\bar{B}(0, R)$ which stabilizes the state to $B(0, r)$ in the sense of Theorem 1.

We begin by defining another multifunction $\tilde{\Gamma}$ (more useful than Γ for being compact-valued) :

$$\tilde{\Gamma}(x) := \text{clco}\{\frac{v}{1 + \|v\|} : v \in \Gamma(x)\}.$$

We set

$$\tilde{T}_0(\alpha) := \inf\{T \geq 0 : x(\cdot) \text{ is a } \tilde{\Gamma} - \text{trajectory on } [0, T], x(0) = \alpha, \|x(T)\| \leq r_0\}.$$

Evidently (or by convention) we have $\tilde{T}_0 = 0$ on $r_0 \bar{B}$.

Lemma 2.1   *(a) $\tilde{\Gamma}$ is locally Lipschitz and has nonempty convex compact values in $\bar{B}(0, 1)$.*

  (b) $\tilde{T}_0(\alpha)$ is finite $\forall \alpha \in \mathbb{R}^n$.

  (c) $\lim_{\|\alpha\| \downarrow r_0} \tilde{T}_0(\alpha) = 0$.

  (d) There exists a positive number ϵ such that whenever $\alpha \in B(0, r_0 + \epsilon)$, and whenever the $\tilde{\Gamma}$-trajectory $x(\cdot)$ has $x(0) = \alpha$ and $\|x(T)\| \leq r_0$ for some $T \leq \tilde{T}_0(\alpha) + \epsilon$, then we have $\|x\|_\infty \leq r_0 + 1$. We can suppose $\epsilon < 1, \epsilon < r - r_0$, and that

$$\sup\{\tilde{T}_0(\alpha) : \alpha \in B(0, r_0 + \epsilon)\} < \frac{(r - r_0)^2}{17(r_0 + 1)}. \tag{12}$$

Proof   We omit the routine proof of (a). For (b), let $\alpha \in \mathbb{R}^n$ be given. By assumption, there is a Γ-trajectory x on an interval $[0, T]$ such that $x(0) = \alpha, \|x(T)\| = r_0$. We set

$$\tilde{T} := \int_0^T (1 + \|\dot{x}(t)\|) dt$$

and we define a function $\tilde{x}$ on $[0, \tilde{T}]$ by

$$\tilde{x}(\tau) := x(t),$$

where $t = t(\tau)$ is determined in $[0, T]$ by

$$\tau = \int_0^t (1 + \|\dot{x}(r)\|) dr$$

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(this change of variables or time scale is known as the Erdmann Transform.)
Then
$$\frac{d\tilde{x}}{d\tau} = \frac{\dot{x}(t)}{1 + \|\dot{x}(t)\|} \in \tilde{\Gamma}(\tilde{x}(\tau)) \quad a.e.,$$

so that $\tilde{x}$ is a $\tilde{\Gamma}$-trajectory. Hence $\tilde{T}_0(\alpha) \le \tilde{T} < \infty$.

We turn now to (c). Let α_i be a sequence for which $\|\alpha_i\|$ decreases to r_0. Then $T_0(\alpha_i) \to 0$ by assumption. Let m be such that $\|f(x, u)\| \le m$ for $(x, u) \in \bar{B}(0, r_0 + 1) \times \mathcal{U}$. Then, as soon as $T_0(\alpha_i)$ is strictly less than $\frac{1}{m}$, there is a Γ-trajectory x_i on an interval $[0, T_i]$ such that

$$x_i(0) = \alpha_i, \|x_i(T_i)\| \le r_0, T_i < \frac{1}{m}, T_i < T_0(\alpha_i) + \frac{1}{i}.$$

It follows that $\|x_i\|_\infty < r_0 + 1$. Now let $\tilde{x}_i$ be the Erdmann Transform of x_i as given above. Then

$$\tilde{T}_0(\alpha_i) \le \tilde{T}_i = \int_0^{T_i} (1 + \|\dot{x}_i(t)\|) dt \le (1 + m)T_i < (1 + m)(T_0(\alpha_i) + \frac{1}{i}).$$

It follows that $\tilde{T}_0(\alpha_i) \to 0$, as required.

We now examine (d). If the assertion is false, there exist a sequence α_i with $\|\alpha_i\| \downarrow r_0$ and corresponding $\tilde{\Gamma}$-trajectories x_i with $x_i(0) = \alpha_i, \|x_i(T_i)\| \le r_0$, such that

$$T_i \le \tilde{T}_0(\alpha_i) + \frac{1}{i}, \|x_i\|_\infty > r_0 + 1.$$

Since $\tilde{T}_0(\alpha_i) \to 0$ by (c), we have $T_i \to 0$. On the other hand, there is a subinterval of $[0, T_i]$ in which $\|x_i\|$ goes from being $r_0 + 1$ to r_0, and since $\|\dot{x}_i(t)\| \le 1$ the length of that subinterval (and hence, T_i) is at least 1. This contradiction establishes the first part of (d); the rest follows immediatly by shrinking ϵ as required, in light of (c). $\square$

We proceed now to define a new multifunction $F(x)$ whose effect is to enlarge the set $\tilde{\Gamma}(x)$ for $\|x\| < r_0 + \epsilon$. We set

$$F(x) := \begin{cases} \tilde{\Gamma}(x) & \text{for } \|x\| \ge r_0 + \epsilon \\ \tilde{\Gamma}(x) + 2[\frac{r_0 + \epsilon - \|x\|}{\epsilon}]\bar{B} & \text{for } r_0 \le \|x\| \le r_0 + \epsilon \\ \tilde{\Gamma}(x) + 2\bar{B} & \text{for } \|x\| \le r_0 \end{cases}$$

Having done this, we define a value function $V(\cdot)$ on $\mathbb{R}^n$ in terms of the trajectories of F as follows :

$$V(\alpha) := \inf\{\int_0^T \|x(t)\| dt : T \ge 0, x(0) = \alpha, \dot{x} \in F(x) a.e., \|x(T)\| \le r_0\}.$$

We stress that T is a choice variable here, in this free time problem.

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Lemma 2.2 *(a) F is compact and convex-valued, uniformly bounded and locally Lipschitz.*

(b) $V(\cdot)$ is finite-valued and lower semicontinuous, and the infinum defining $V(\alpha)$ is attained for every α.

(c) $V(\alpha) > 0$ iff $\|\alpha\| > r_0$, and $\lim_{\|\alpha\| \downarrow r_0} V(\alpha) = 0$.

(d) The sublevel sets $S(b) := \{x : V(x) \leq b\}$ of V are compact.

Proof The assertions of (a) are immediate. Since $F(x)$ is uniformly bounded, the attainment and the lower semicontinuity asserted in (b) follow from standard "compactness of trajectories" arguments; see [9, Chapter 4] for details. The first assertion of (c) is clear, and the other one stems from Lemma 2.1 as follows. Let $\alpha \in B(0, r_0 + \epsilon)$, and let the $\tilde{\Gamma}$-trajectory x satisfy $x(0) = \alpha, \|x(T)\| \leq r_0$, and $T \leq \tilde{T}_0(\alpha) + \delta$, for some $\delta \in (0, \epsilon)$.
Then $\|x\|_\infty \leq r_0 + 1$, and we deduce

$$V(\alpha) \leq \int_0^T \|x(t)\| dt \leq (\tilde{T}_0(\alpha) + \delta)(r_0 + 1).$$

Since $\tilde{T}_0(\alpha) \downarrow 0$ as $\|\alpha\| \downarrow r_0$, (c) follows. Finally we turn to (d). If $\|\alpha\| > r_0 + 1 + \epsilon$, then the time required for a trajectory to go from $\|x\| = \alpha$ to $\|x\| = r_0 + \epsilon$ is at least $\alpha - r_0 - \epsilon$. But then $V(\alpha) \geq (r_0 + \epsilon)(\|\alpha\| - r_0 - \epsilon)$. This established (d). $\square$

The next step invokes Hamiltonian conditions for optimal control, and uses the lower Hamiltonian h associated with F :

$$h(x, p) := \min\{\langle p, v \rangle : v \in F(x)\}.$$

Lemma 2.3 *Let $\zeta \in \partial_P V(\alpha)$, where $\|\alpha\| > r_0$. Let x be a trajectory solving the problem that defines $V(\alpha)$, with associated time T. Then there exists an absolutely continuous function p on $[0, T]$ such that*

$$
\begin{align}
\left(-\dot{p} - \frac{x}{\|x\|}, \dot{x}\right) &\in \partial_C h(x, p) \quad a.e. \quad t \in [0, T] \tag{13} \\
p(0) &= \zeta \tag{14} \\
h(x(t), p(t)) + \|x(t)\| &= 0 \quad \forall t \in [0, T]. \tag{15}
\end{align}
$$

Proof By definition of $\partial_P V(\alpha)$, we have for some $\sigma \geq 0$ and for all α' near α,

$$V(\alpha') + \sigma \|\alpha' - \alpha\|^2 - \langle \zeta, \alpha' \rangle \geq -\langle \zeta, \alpha \rangle.$$

Let x' be a trajectory near x (in the L^∞ norm), put $\alpha' = x'(0)$ and $\alpha = x(0)$ and rearrange to derive that $x'(\cdot) = x(\cdot)$ solves locally the problem of minimizing

$$\int_0^{T'} \|x'(t)\| dt - \langle \zeta, x'(0) \rangle + \sigma \|x'(0) - x(0)\|^2$$

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over the trajectories x' for F satisfying $\|x'(T')\| \leq r_0$. (Here T' and $x'(0)$ are free.) We apply the corollary of Theorem 3.6.1 of [5] (in reverse time) to deduce the existence of an absolutely continuous function q on $[0, T]$ satisfying

$$(-\dot{q}, \dot{x}) \in \partial_C[H(x, q) - \|x\|](x, q) \quad a.e. \quad t \in [0, T], \quad (16)$$
$$q(0) = -\zeta, \quad (17)$$
$$H(x(t), q(t)) - \|x(t)\| = 0, \quad t \in [0, T], \quad (18)$$

where $H(x, p)$ is the function $-h(x, -p)$ and ∂_C denotes the generalized gradient. The Hamiltonian inclusion above implies

$$(\dot{q} - \frac{x}{\|x\|}, \dot{x}) \in \partial_C h(x, -q) \quad a.e.$$

Now putting $p := -q$ gives the conclusions the form asserted in the statement of the lemma.□

Lemma 2.4 *For any constant $c > 0$, there is a constant M_c with the following property. If $\alpha \in S(c)$ and if the trajectory x on $[0, T]$ attains the infimum defining $V(\alpha)$, then $\|x\|_\infty \leq M_c, T \leq M_c$.*

Proof If $\|x\|_\infty > c + r_0 + 1$, then the time required for $\|x\|$ to attain the value $r_0 + 1$ exceeds c (since $\|\dot{x}\| \leq 1$). But then $V(\alpha) \geq (1 + r_0)c > c$. This shows that $\|x\|_\infty$ is bounded by $c + r_0 + 1$. By Lemma 2.1(c), $\lim_{\|\alpha\| \downarrow r_0} \tilde{T}_0(\alpha) = 0$. So there exists $\rho > r_0$ such that $\forall x \in B(0, \rho), |\tilde{T}_0(x)| \leq 1$. Now for $\|\alpha\| \geq \rho$, let τ_ρ denote the first time that $\|x(t)\|$ attains the value ρ. Then $V(\alpha) \geq \rho\tau_\rho$, whence $\tau_\rho \leq \frac{c}{\rho}$ for $\alpha \in S(c)$.
We deduce that

$$\tilde{T}_0(\alpha) \leq \tau_\rho + 1 \quad \text{(by construction of } \rho\text{)}$$
$$\leq \frac{c}{\rho} + 1.$$

If $\|\alpha\| < \rho$, the same bound evidently holds.
It suffices now to set

$$M_c := \max\{\frac{c}{\rho} + 1, c + r_0 + 1\}.$$

□

Lemma 2.5 *V is locally Lipschitz on $\mathbb{R}^n$.*

Proof We prove first that V is locally Lipschitz on the open set $\{V > 0\} = \text{comp}(\bar{B}(0, r_0))$. Let α_0 belong to this set, take any $\delta > 0$ such that $\delta < \|\alpha_0\| - r_0$, and any element $\zeta \in \partial_P V(\alpha)$, where

$$\|\alpha - \alpha_0\| < \delta, V(\alpha) \leq V(\alpha_0) + \delta =: c. \quad (19)$$

The conclusions of Lemma 2.4 are available for any trajectory solving the $V(\alpha)$ problem. If K is a Lipschitz constant for F on the ball $B(0, M_c + 1)$ (where M_c comes from Lemma 2.4), then the Hamiltonian inclusion (13) implies

$$\|\dot{p}\| \leq K\|p\| + 1. \tag{20}$$

The condition (15) at $t = T$ gives $-\|p(T)\| \geq -r_0$ since $\|x(T)\| = r_0$, and since $F(x(T)) = \mathrm{clco}\,f(x(T), \mathcal{U}) + 2\bar{B} \supset \bar{B}$. Thus $\|p(T)\| \leq r_0$. This, together with (20) and Gronwall's Lemma, leads to

$$
\begin{aligned}
\|\zeta\| = \|p(0)\| &\leq e^{KT}\|p(T)\| + \int_0^T e^{K(T-s)} ds \\
&\leq r_0 e^{KT} + \frac{e^{KT} - 1}{K} \\
&\leq r_0 e^{KM_c} + \frac{e^{KM_c} - 1}{K},
\end{aligned}
$$

since $T \leq M_c$ by Lemma 2.4. This establishes a uniform bound on elements of $\partial_P V(\alpha)$ whenever α satisfies (19), which proves that V is Lipschitz on a neighborhood of α_0 [9, 1.11.11]. Thus V is locally Lipschitz on the set where it is strictly positive.

There is a neighborhood N of $\bar{B}(0, r_0)$ on which V is bounded above, in view of Lemma 2.2(c). The argument above therefore yields a bound L on elements of $\partial_P V(\alpha)$ for all $\alpha \in N \setminus \bar{B}(0, r_0)$, so that V is uniformly Lipschitz of rank L on $\alpha \in N \setminus \bar{B}(0, r_0)$ by [9, Theorem 1.7.3]. Of course, $V = 0$ on $\bar{B}(0, r_0)$. That V is Lipschitz on N, and hence locally Lipschitz on $\mathbb{R}^n$, now follows.$\square$

Lemma 2.6

$$\sup\{V(\alpha) : \alpha \in B(0, r_0 + \epsilon)\} < \inf\left\{V(\alpha) : \|\alpha\| \geq \frac{r_1 - r_0}{2}\right\}$$

Proof Let $\|\alpha\| < r_0 + \epsilon$, fix $\delta \in (0, \epsilon)$ and let the trajectory x on $[0, T]$ satisfy $x(0) = \alpha, \|x(T)\| \leq r_0, T < \tilde{T}_0(\alpha) + \delta$. Then by Lemma 2.1 we have $\|x\|_\infty \leq r_0 + 1$ and so

$$V(\alpha) \leq \int_0^T \|x(t)\| dt \leq (\tilde{T}_0(\alpha) + \delta)(r_0 + 1).$$

We derive $V(\alpha) \leq (r_0 + 1)\tilde{T}_0(\alpha)$, and (from (12))

$$V(\alpha) < \frac{(r - r_0)^2}{17}.$$

Now let $\|\alpha\| \geq \frac{r - r_0}{2}$, and let x solve the problem defining $V(\alpha)$. There is an interval of length at least $\frac{r - r_0}{4}$ during which $\|x(t)\| \geq \frac{r - r_0}{4}$ (since $\|\dot{x}\| \leq 1$), whence

$$V(\alpha) > \frac{(r - r_0)^2}{16}.$$

The result follows.$\square$

Lemma 2.7 *There exist positive numbers a, b, η with $a < b$ such that*

$$S(a) + \eta B \subset B(0, r) \subset \bar{B}(0, R) \subset S(b)$$

and

$$S(a, b) + \eta B \subset \{\alpha : r_0 + \epsilon < \|\alpha\|\}.$$

Proof   Pick a number $a > 0$ lying between the two quantities in the statement of Lemma 2.6. Then evidently

$$S(a) \subset B(0, \frac{r - r_0}{2}), \quad \text{cl}(\text{comp} S(a)) \subset \text{comp} \bar{B}(0, r_0 + \epsilon),$$

whence $S(a) + \eta B \subset B(0, r)$ for $\eta > 0$ suitable small. It also follows that the compact set $\text{cl} S(a, b)$ is contained in the open set $\{\alpha : r_0 + \epsilon < \|\alpha\|\}$. By shrinking η further if necessary, this will be true of $\text{cl} S(a, b) + \eta B$ as well. Finally, any b suitably large will satisfy $\bar{B}(0, R) \subset S(b)$, since V is bounded on bounded sets.$\square$

Lemma 2.8 *The Infinitesimal Decrease Condition (H5) of §holds on $S(a, b) + \eta B$, with $\omega := -r_0 - \epsilon$.*

Proof   As remarked in §1, it suffices to show that for any $\alpha \in S(a, b) + \eta B$, for any $\zeta \in \partial_P V(\alpha)$, one has :

$$\inf\{\langle \zeta, f(\alpha, u) \rangle : u \in \mathcal{U}\} \leq -r_0 - \epsilon. \tag{21}$$

Let x be a trajectory solving the problem defining $V(\alpha)$.
Then, by Lemma 2.3, we have (at $t = 0$) :

$$h(\alpha, \zeta) + \|\alpha\| = 0.$$

Since $\|\alpha\| > r_0 + \epsilon$ by Lemma 2.7, we have

$$F(\alpha) = \tilde{\Gamma}(\alpha),$$

so that the preceding equality yields, for any $\delta > 0$, the existence of some element $v \in \Gamma(\alpha)$ such that

$$\langle \zeta, \frac{v}{1 + \|v\|} \rangle \leq -\|\alpha\| + \delta < -(r_0 + \epsilon) + \delta.$$

For δ small enough, the right side is negative, whence

$$\langle \zeta, v \rangle < -(r_0 + \epsilon) + \delta.$$

Given that $\Gamma(x) := \text{clco} f(x, \mathcal{U})$, this yields the existence of $u \in \mathcal{U}$ for which

$$\langle \zeta, f(\alpha, u) \rangle < -(r_0 + \epsilon) + 2\delta.$$

Since δ is arbitrarily small, (21) ensues.$\square$

   Since $S(b)$ is compact, f is Lipschitz in x and bounded on $S(b) + \eta B$, in accord with hypotheses (H3) (H4) of section 1. When the level sets are compact, (H2) always holds. The verification of this is the last property to confirm.

Lemma 2.9 *Hypothesis (H2) holds.*

Proof If (H2) fails, then there exist sequences $\alpha_i \in \mathbb{R}^n, \epsilon_i \downarrow 0$, and $u_i \in B(0,1)$ such that
$$V(\alpha_i) \leq a + \epsilon_i \text{ and } V(\alpha_i + u_i) > b.$$

Since $S(b)$ is compact, we can suppose by passing to a subsequence that $\alpha_i \to \alpha_0$. Then $V(\alpha_0) \geq b > a \geq V(\alpha_0)$, a contradiction. $\square$

The setting of Theorem 1 is established, and Theorem 2 is proved.

3 Robustness

We prove in this section that the feedback constructed in §1 is robust with respect to small measurement error and persistent external disturbance, in a precise sense that requires two stipulations. The first is that the measurement error must not exceed in order of magnitude the step size of the underlying discretization, a condition which appears to be rather natural. The second requirement is perhaps more surprising, and surfaces from the nature of the feedback construction. It dictates that each step be "big enough" (while continuing to be "small enough") so as to counteract the measurement error by means of the attractive effect inherent in the construction. Thus the partitions used to discretize the effect of the control are taken to be "reasonably uniform".
Our perturbed system is modeled by
$$\dot{x} = f(x, k(x + p)) + q,$$

where the external disturbance $q : [0, \infty) \longmapsto \mathbb{R}^n$ is a bounded measurable function :
$$\|q(t)\| \leq E_q \quad t \geq 0 \quad a.e.$$

Given a partition $\pi = \{t_i\}_{i \geq 0}$ of $[0, \infty)$ and the initial condition x_0, the resulting π-trajectory of our perturbed system is defined by successively solving the differential equation
$$\dot{x}(t) = f(x(t), k(x(t_i) + p_i)) + q(t), \quad t \in [t_i, t_{i+1}],$$

with $x(0) = x_0$. The continuous function $x(t)$ is the real state of the system, while the sequence $\{x(t_i) + p_i\}$ corresponds to the inexact measurements used to select control values.

Theorem 3 *The feedback $k : S(b) + \gamma B \to \mathcal{U}$ constructed in Theorem 1 is robust in the following sense. There exist positive numbers δ_0, T and E_q such that, for every $\delta \in (0, \delta_0)$ there exists $E_p(\delta) > 0$ having the following property : for any partition $\pi = \{t_i\}_{i \geq 0}$ having*
$$\frac{\delta}{2} \leq t_{i+1} - t_i \leq \delta \quad i \geq 0,$$

where $0 < \delta < \delta_0$, for any set of measurement errors $\{p_i\}_{i \geq 0}$ having

$$\|p_i\| \leq E_p \quad i \geq 0,$$

for any initial condition x_0 such that $x_0 + p_0 \in S(b) + \gamma B$, for any disturbance q having $\|q\|_\infty \leq E_q$, the resulting π-trajectory x satisfies

$$x(t_i) + p_i \in S(b) + \gamma B \quad \forall i \geq 0,$$

$$x(t) \in S(b) + 2\gamma B \quad \forall t \geq 0,$$

$$x(t) \in S(a) + \gamma B \quad \forall t \geq T.$$

Remarks 3.1   (a) Note that unlike T and E_q, the maximum admissible measurement error E_p depends on δ. Note also that (in contrast to Theorem 1) $x(t)$ may not lie in $S(b) + \gamma B$ for all t, although for large it must do so. (For example, we do not require $x(0) \in S(b) + \gamma B$.)
However, the "observed values" of the state, namely the values $x(t_i) + p_i (i \geq 0)$, all fall in $S(b) + \gamma B$ the domain of definition of k.

(b) Certain other kinds of error, for example a disturbance $d(\cdot)$ entering into the dynamics in the form $\dot{x} = f(x, k(x) + d)$, can be reduced to that of external disturbance by positing suitable continuity of f in the control variable.

(c) The maximum admissible disturbance measure E_q will be seen to be proportional to $\frac{\omega}{L_V}$. This has a natural physical meaning, as can easily be seen in the case of smooth V and a continuous feedback $k(x)$ such that $\langle \nabla V(x), f(x, k(x)) \rangle \leq -W(x)$.
Then we see that the perturbed system

$$\dot{x} = f(x, k(x)) + q$$

is stabilized by k if $\|q\|_\infty < \frac{W(x)}{\|\nabla V(x)\|}$ for every x, a bound akin to that involving $\frac{\omega}{L_V}$.

We adapt the proof of Theorem 1, whose first five lemmas hold with no change whatever, as does the definition of $k(\cdot)$. Recall that γ, β and N were introduced (earlier; see (6) (7)). We now define our upper bound for δ :

$$\delta_0 \quad := \quad \min\left\{\frac{6\gamma L_V}{\omega}, 1, \frac{\gamma\omega}{24 L_V (m + \frac{\omega}{6L_V} + 1)^2}, \frac{\gamma}{20m}\right\} \tag{22}$$

$$E_q \quad := \quad \frac{\omega}{6L_V} \tag{23}$$

$$T \quad := \quad \left(1 + \frac{b-a}{\beta}\right)\left(1 + \frac{81\gamma L_V}{\omega}\right) \tag{24}$$

(this T differs slightly from the one in Theorem 1) and we let $\pi = \{t_i\}_{i\geq 0}$ be a partition as described i the statement of Theorem 3, with corresponding measurement errors $\{p_i\}_{i\geq 0}$ having $\|p_i\| \leq E_p$ for some $E_p > 0$ satisfying

$$E_p < \min\{\frac{3\gamma}{80}, \delta, \frac{\delta\omega}{432L_V}\}. \tag{25}$$

We also admit any disturbance $q(\cdot)$ for which $\|q\|_\infty \leq E_q$, and we take x_0 such that $x_0 + p_0 \in S(b) + \gamma B$. We shall show that the corresponding π-trajectory has the required properties. We introduce the notation

$$x_i := x(t_i), y_i := x_i + p_i$$

for the actual and the measures space state values at time t_i, and proceed to develop modified versions of the fourlast lemmas figuring in the proof of Theorem 1. We set $\tilde{\Delta} := \frac{\omega}{108L_V}$.

Lemma 3.1 *For some $t_j \in \pi$, suppose that $y_j \in \Omega_i$, $i \in \{0, 1, \cdots, N\}$. Then*

$$x(t) \in S(b) + 2\gamma B \subset S(b) + \eta B, \quad t_j \leq t \leq t_{j+1}$$

$$y_j \in S(b) + \gamma B, y_{j+1} \in S(b) + \gamma B,$$

$$d(y_{j+1}, S(b - i\beta)) \leq d(y_j, S(b - i\beta)) - \tilde{\Delta}(t_{j+1} - t_j).$$

Proof   Note that $y_j \in S(b) + \gamma B$ by Lemma 1.4; it will follow from the last conclusion of the current Lemma that $y_{j+1} \in S(b) + \gamma B$. Also, $\|x_j - y_j\| = \|p_j\| \leq E_p$, together with $\|x(t) - x_j\| \leq \delta m$, yield

$$x(t) \in S(a, b) + \gamma B + (E_p + \delta m)B \subset S(b) + 2\gamma B,$$

since $E_p + \delta m < \gamma$ in view of (22) and (25). Since $2\gamma < \eta$ by (6), this gives $x(t) \in S(b) + \eta B$. By Lemma 1.5 we have

$$\langle y_j - s, f(s, k(y_j))\rangle \leq -\frac{\omega}{2L_V}\|y_j - s\|, \tag{26}$$

where $s \in \text{proj}(y_j, S(b - i\beta))$. Fix $t \in (t_j, t_{j+1})$ and set

$$\Psi := \frac{x(t) - s}{\|x(t) - s\|}.$$

Note that $x(t) \neq s$ since $\|y_j - s\| \geq \frac{\gamma}{4}$ by Lemma 1.4 (e), while

$$\|y_j - x(t)\| \leq \|y_j - x_j\| + \|x_j - x(t)\| < E_p + \delta m < \frac{\gamma}{4},$$

as already noted.
We observe the relations

$$\begin{aligned} d(x(t), S(b - i\beta)) &\leq & \|x(t) - s\| = \langle \Psi, x(t) - s\rangle, \\ d(y_j, S(b - i\beta)) &=& \|y_j - s\| \geq \langle \Psi, y_j - s\rangle, \end{aligned}$$

whence

$$\begin{aligned}
d(x(t), S(b - i\beta)) - d(y_j, S(b - i\beta)) &\leq \langle \Psi, x(t) - y_j \rangle \\
&= \langle \Psi, x_j + \tau(f_j + q_j) - y_j \rangle \\
&\leq \tau \langle \Psi, f_j + q_j \rangle + \|p_j\|, \qquad (27)
\end{aligned}$$

where we have introduced

$$\begin{aligned}
\tau &:= t - t_j, \\
f_j &:= \frac{1}{\tau} \int_{t_j}^{t} f(x(r), k(y_j)) dr, \\
q_j &:= \frac{1}{\tau} \int_{t_j}^{t} q(r) dr.
\end{aligned}$$

We also set $\hat{f}_j := f(s, k(y_j))$; note $\|\hat{f}_j\| \leq m$. We have

$$\begin{aligned}
\|f_j - \hat{f}_j\| &\leq \frac{1}{\tau} \int_{t_j}^{t} L_f \|x(r) - s\| dr \leq \frac{L_f}{\tau} \int_{t_j}^{t} (\|x(r) - x_j\| + \|x_j - s\|) dr \\
&\leq L_f(\delta m + \|x_j - y_j\| + \|y_j - s\|) \\
&\leq L_f(\delta m + E_p + \gamma) \leq \frac{5}{4} L_f \gamma. \qquad (28)
\end{aligned}$$

We deduce

$$\begin{aligned}
\langle x(t) - s, f_j + q_j \rangle &= \langle x_j + \tau(f_j + q_j) - s, f_j + q_j \rangle \\
&= \langle y_j - s - p_j, f_j + q_j \rangle + \tau \|f_j + q_j\|^2 \\
&\leq \langle y_j - s, f_j \rangle + E_p(m + E_q) + \delta(m + E_q)^2 \\
&= \langle y_j - s, \hat{f}_j \rangle + \langle y_j - s, f_j - \hat{f}_j \rangle + (m + E_q)[\delta m + \delta E_q + E_p] \\
&\leq -\frac{\omega}{2L_V} \|y_j - s\| + \frac{5}{4} \|y_j - s\| L_f \gamma + (m + E_q)[\delta m + \delta E_q + E_p] \\
&\qquad \text{(where we have used (26) and (28))} \\
&\leq d(y_j, S(b - i\beta)) \left[-\frac{\omega}{2L_V} + \frac{\omega}{6L_V} \right] + \delta[m + E_q + 1]^2 \\
&\qquad \text{(by (6), and since } E_p < \delta \text{ by (25))} \\
&\leq \frac{\gamma}{4} \left[-\frac{\omega}{3L_V} \right] + \frac{\gamma \omega}{24 L_V} = -\frac{\gamma \omega}{24 L_V}
\end{aligned}$$

(since $d(y_j, S(b - i\beta)) \geq \frac{\gamma}{4}$ by Lemma 1.4, and since $\delta < \delta_0$ defined by (22)).
Note also that

$$\begin{aligned}
\|x(t) - s\| &= \|y_j - p_j + \tau(f_j + q_j) - s\| \\
&\leq \gamma + E_p + \delta(m + E_q) \leq \frac{5\gamma}{4} + \delta E_q < \frac{9\gamma}{4}
\end{aligned}$$

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(note $\delta E_q < \gamma$ because of $\delta < \delta_0$, in view of (22) (23)).
It follows that

$$\langle \Psi, f_j + q_j \rangle = \langle \frac{x(t) - s}{\|x(t) - s\|}, f_j + q_j \rangle \leq -\frac{(\gamma\omega)/(24L_V)}{(9\gamma)/4} = -\frac{\omega}{54L_V}.$$

Substituting into (27) leads to

$$d(x(t), S(b - i\beta)) - d(y_j, S(b - i\beta)) \leq -2\tilde{\Delta}(t - t_j) + E_p.$$

We obtain from this

$$\begin{aligned}
d(y_{j+1}, S(b - i\beta)) - d(y_j, S(b - i\beta)) &\leq& \|y_{j+1} - x(t_{j+1})\| - 2\tilde{\Delta}(t - t_j) + E_p \\
&\leq& -\tilde{\Delta}(t_{j+1} - t_j) + [2E_p - \tilde{\Delta}(t_{j+1} - t_j)] \\
&\leq& -\tilde{\Delta}(t_{j+1} - t_j),
\end{aligned}$$

by (25), and since $t_{j+1} - t_j \geq \frac{\delta}{2}$.$\square$

Lemma 3.2 *If $y_j \in \Omega_i$, where $i \leq N$, then y_{j+1} lies in Ω_k for some $k \geq i$.*

Proof We know that $y_{j+1} \in \Omega_k$ for some k, since y_{j+1} belongs to $S(b) + \gamma B$ by Lemma 3.1. Suppose that $k < i$. We have $d(y_j, S(b - i\beta)) < \gamma$ by definition of Ω_i, and Lemma 3.1 implies $d(y_{j+1}, S(b - i\beta)) < \gamma$. But $S(b - i\beta) \subset S(b - (k+1)\beta)$, so that $d(y_{j+1}, S(b - (k+1)\beta)) < \gamma$. But then $y_{j+1} \notin \Omega_k$ by definition of Ω_k (note that $k \leq N$). This contradiction proves the Lemma.$\square$

Lemma 3.3 *If $\tau \in \pi$ is such that $y(\tau) \in \Omega_{N+1}$, then*

$$x(t) \in S(a) + \gamma B \quad \forall t \geq \tau.$$

Proof We first establish

$$d(y(\tau'), S(b - N\beta)) \leq \frac{2\gamma}{5} \quad \text{for all nodes } \tau' \geq \tau. \tag{29}$$

We consider first $\tau' = \tau + 1$. We have $d(y(\tau), S(b - N\beta)) \leq \frac{\gamma}{4}$, whence

$$\begin{aligned}
d(y(\tau + 1), S(b - N\beta)) &\leq& d(x(\tau + 1), S(b - N\beta)) + E_p \\
&\leq& d(x(\tau), S(b - N\beta)) + E_p + m\delta \\
&\leq& d(y(\tau), S(b - N\beta)) + 2E_p + m\delta \\
&<& \frac{\gamma}{4} + \frac{3\gamma}{20} = \frac{2\gamma}{5}(\text{ by (25)}).
\end{aligned}$$

If $d(y(\tau + 1), S(b - N\beta))$ is in fact $\leq \frac{\gamma}{4}$, then this same argument yields

$$d(y(\tau + 2), S(b - N\beta)) \leq \frac{2\gamma}{5}.$$

If however $d(y(\tau + 1), S(b - N\beta)) > \frac{\gamma}{4}$, then $y(\tau + 1)$ lies in Ω_N by definition, and Lemma 3.1 yields again $d(y(\tau + 2), S(b - N\beta)) < \frac{2\gamma}{5}$. Continuing in this

way, we obtain (29) for all nodes $\tau' \geq \tau$.

We use (29) to argue as follows : let $t \geq \tau$, and let $\tau' \geq \tau$ be a node adjacent to t; then

$$
\begin{aligned}
d(x(t), S(b - N\beta)) &\leq d(x(\tau'), S(b - N\beta)) + \delta m \\
&\leq d(y(\tau'), S(b - N\beta)) + E_p + \delta m \\
&< \frac{2\gamma}{5} + \frac{\gamma}{10} = \frac{\gamma}{2} \text{(by (25))}.
\end{aligned}
$$

This gives $x(t) \in S(a) + \gamma B$ by Lemma 1.4 (f).$\square$

Lemma 3.4 *Let*
$$
T := \left(1 + \frac{b - a}{\beta}\right)\left(1 + \frac{81\gamma L_V}{\omega}\right).
$$

Then $x(t) \in S(a) + \gamma B \quad \forall t \geq T$.

Proof In view of Lemma 3.3, it suffices to prove that some node $\tau \in \pi$ with $\tau \leq T$ is such that $y(\tau) \in \Omega_{N+1}$. The argument is identical to that used to prove Lemma 1.8, with Δ replaced by $\tilde{\Delta}$, and applied to the y_i rather than the x_i.$\square$

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