Some concentration inequalities that are useful in statistics on point processes.

P. Reynaud-Bouret

CNRS - LJAD University of Nice

Journées MAS, Clermont-Ferrand, 2012

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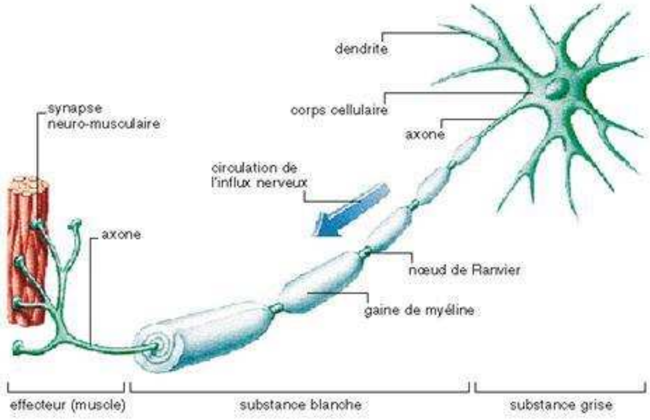
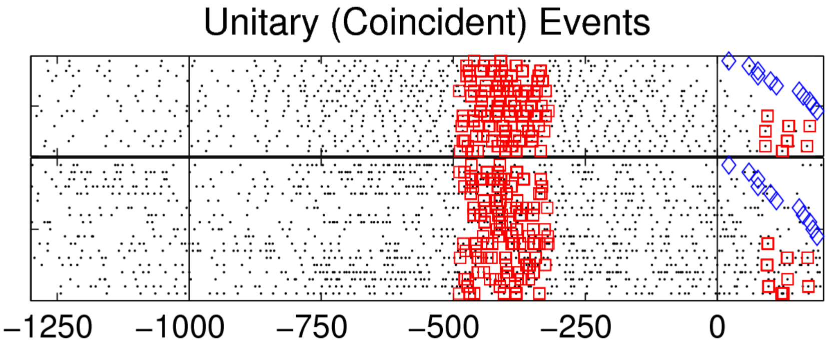
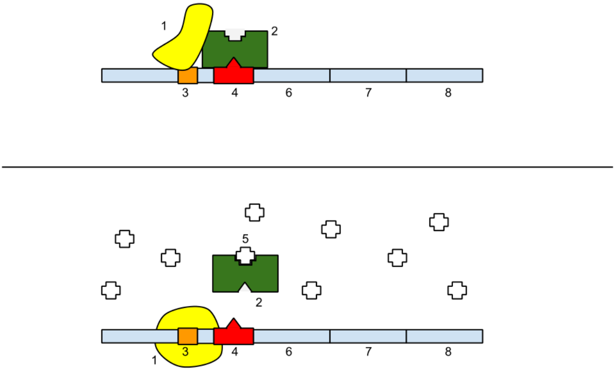
Neuroscience and neuronal unitary activity



Neuronal data and Unitary Events



Genomics and Transcription Regulatory Elements



Point processes and Poisson processes

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Usually $d\ell = \lambda(t)dt$, $\lambda(t)$ is the intensity, if constant $\rightarrow$ homogeneous

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- Are the processes identically distributed ? $\rightarrow$ Two-sample tests
- Are they dependent ? $\rightarrow$ Independence tests
- Can we detect it locally ? $\rightarrow$ multiple "adaptive" testing problems ...
- Where are the poor or rich regions ? $\rightarrow$ Non parametric estimation

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If two motifs are part of a common biological process, the distance $\simeq$ fixed $\rightarrow$ favored or avoided distances (Gusto, Schbath (2005))	When recorded, a fixed delay between spikes hints for a functional/physical link.

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NB2 : $(N_t - \int_0^t \lambda(s)ds)_t$ is a martingale.

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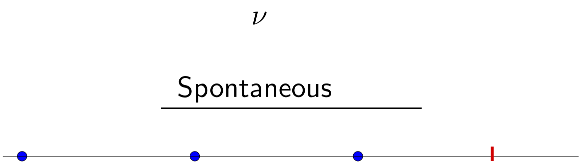

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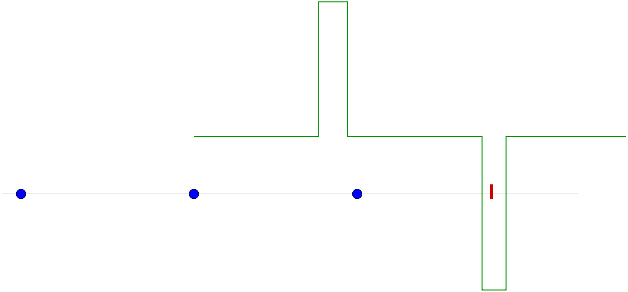
The intensity $\lambda(t)$ is given by

$$\nu \quad + \quad \sum_{T \in N} h(t - T)$$

Spontaneous Self-exciting



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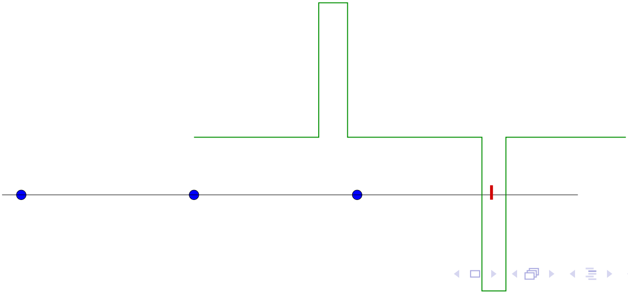
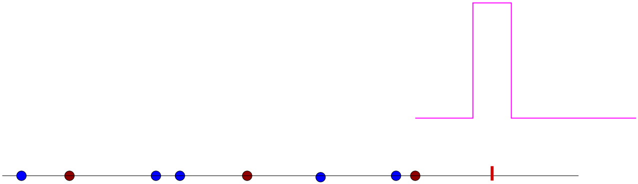


The Hawkes process interaction with itself + an additional interaction

$$\lambda(t) = \nu \quad + \quad \sum_{T \in N} h(t-T) \quad + \quad \sum_{X \in N_2} h_2(t-X)$$

Spontaneous — Self-interaction — Interaction with other type



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Spontaneous Self-interaction Interaction with other type

If h is null and if N_2 is fixed (no reciprocal interaction), then N is a Poisson process given N_2.

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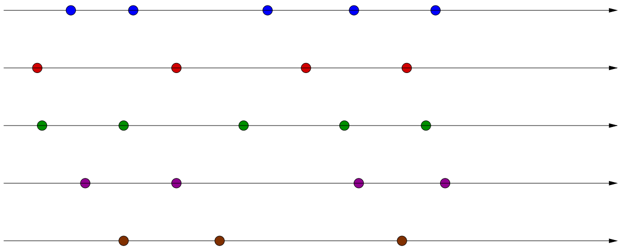
One observes $N^{(1)}, ..., N^{(r)}, ..., N^{(M)}$ processes such that



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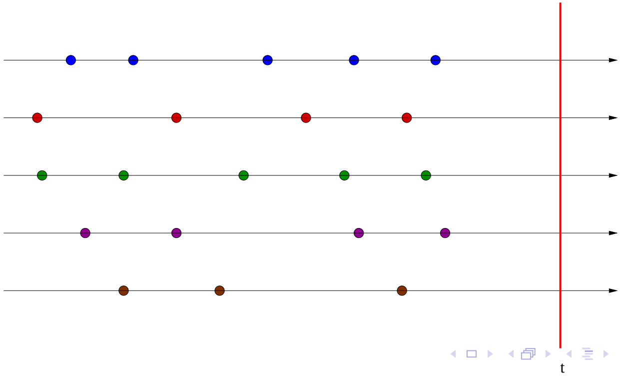

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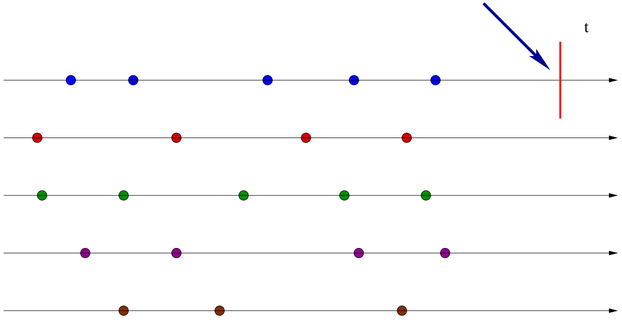

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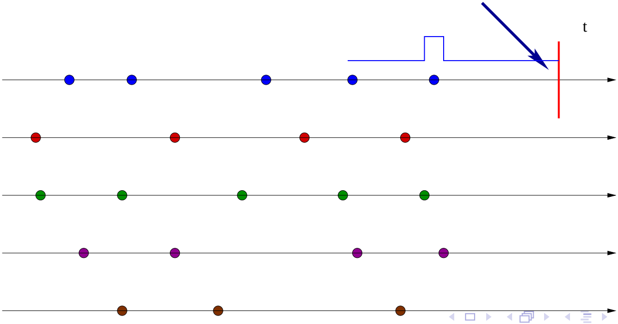

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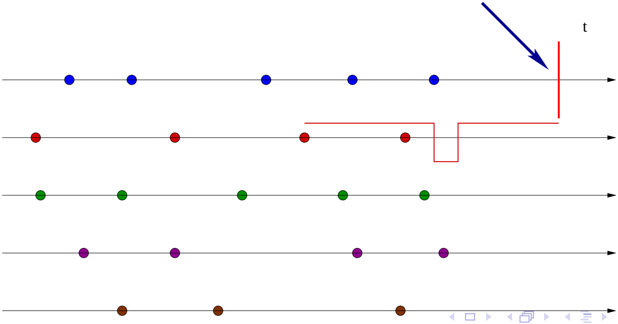

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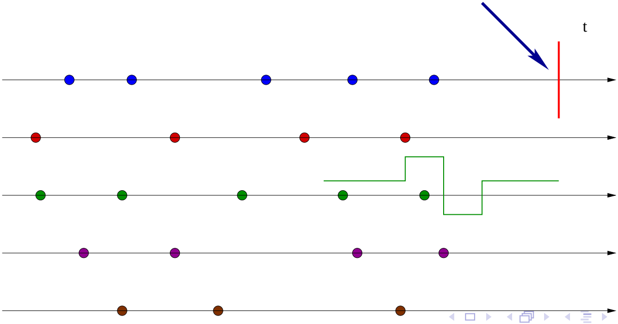

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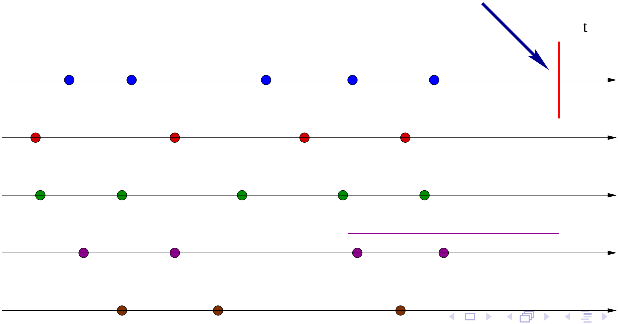

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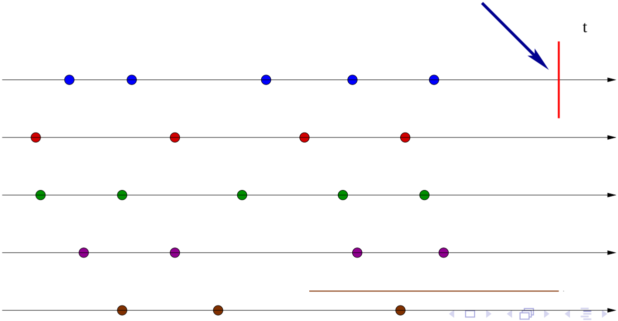

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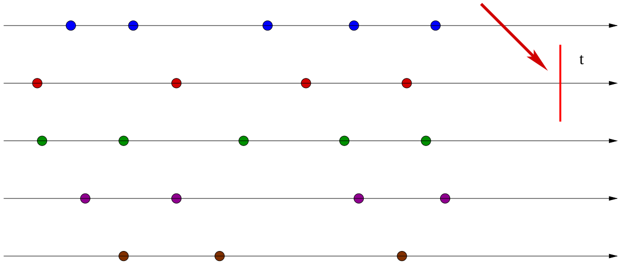

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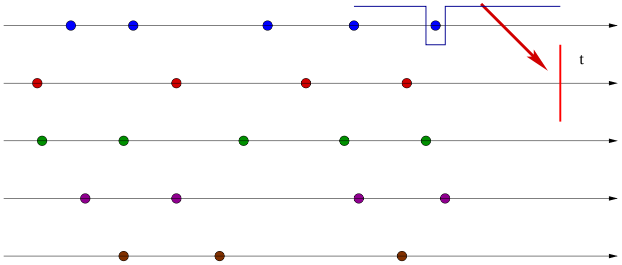

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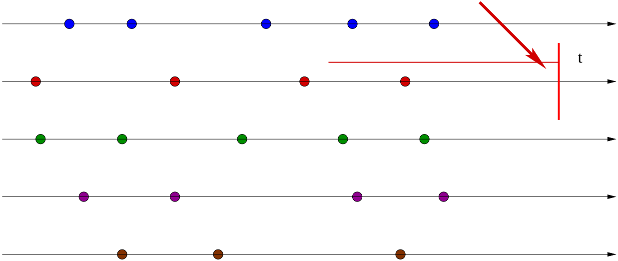

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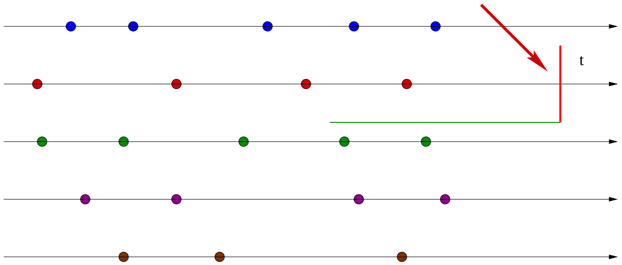

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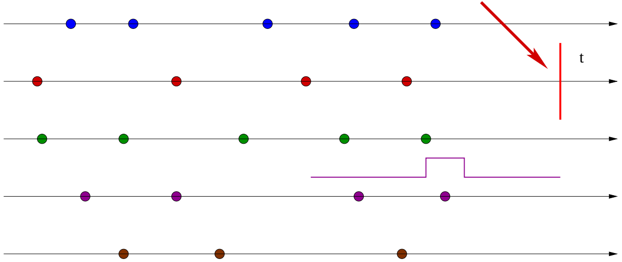

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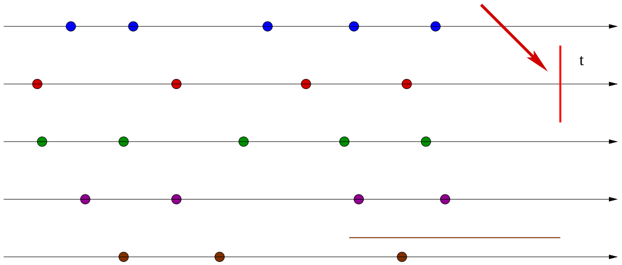

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Link with graphical model of local independence (see Didelez (2008))

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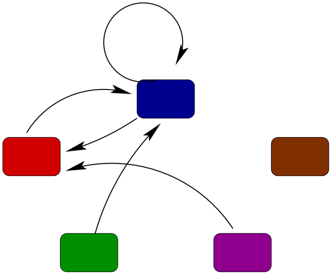

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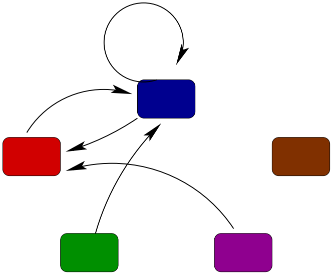

Hence we need a sparse adaptive estimation (functions, support of the functions) !

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- Here, conditionally to the total number of points is n, points behave under H_0 as a n uniform iid sample $\to$ easy access to quantile

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"High" = quantile under H_0.
Problem = we don't know which coefficients $\rightarrow$ aggregation of tests.

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- $t^{(n)}_{m,\alpha}$ the $1 - \alpha$ quantile of the conditional distribution.

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- refined .... for simulation (possible to guarantee equality in the level)

Need of concentration ?

For λ in H_1, Error of 2nd kind $=$
$\mathbb{P}_\lambda(\forall m \in \mathcal{M}, T_m \leq t^{(N)}_{m,\alpha_m}) \leq \mathbb{P}_\lambda(T_m \leq t^{(N)}_{m,\alpha_m})$ for all m in $\mathcal{M}$.

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$\rightarrow$ how $t^{(N)}_{m,\alpha}$ depends on α ?

- if there is exponential decay, possible to aggregate $|\mathcal{M}|$ without losing much more than a logarithmic term
- Hence methods powerful against "ugly" alternatives (such as weak Besov spaces) and usually minimax if well done ...

Concentration of U-statistics

T_m is a degenerate U-statistics of order 2 under H_0 conditionnally to $N_{tot} = n$, ie it's a

$$U_n = \sum_{i \neq j} g(X_i, X_j),$$

with g symmetric $\mathbb{E}(g(X_i, X_j) | X_j) = 0$.

Theorem

If $\|g\|_\infty \leq A$ *then for all* $u, \varepsilon > 0$

$$\mathbb{P}(U_n \geq 2(1+\varepsilon)^{3/2} C\sqrt{u} + \square_\varepsilon Du + \square_\varepsilon Bu^{3/2} + \square_\varepsilon Au^2) \leq \square e^{-u}$$

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- without constants Giné, Latala, Zinn (2000)
- with constant Houdré, RB (2003) - also Poisson processes
- higher order Adamczak (2006)

Conclusions for testing

- Concentration inequalities are a tool to evaluate the dependency in α of the $1-\alpha$ quantile
- In the upper bound, no need for precise constants or observable quantities
- But dependency of for instance, A, B, C, D in m crucial...
  Best if dimension free or dependency in m as small as possible
  $\rightarrow$ choice of the test statistics and the $\mathcal{M}$'s.

Poisson case

Here again $\lambda(t) = Ls(t)$ with L known ($\to \infty$), s unknown.

Least square contrast

$$\gamma(f) = -\frac{2}{L}\int f(t)dN_t + \int f^2(t)dt$$

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- Let S_m be any finite vectorial subspace with ONB $(\varphi_\lambda, \lambda \in \Lambda_m)$.
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Penalized model selection

$$\hat{m} = \mathrm{argmin}_{m \in \mathcal{M}} \{\gamma(\hat{s}_m) + \mathrm{pen}(m)\}$$

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$$\gamma(f) = -\frac{2}{L}\int f(t)(dN_t - s(t)dt) + \|f - s\|^2 - \|s\|^2.$$

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with $v = \sup_{a \in A} \int_{\mathbb{X}} \psi_a^2(x) d\ell_x$
and $\kappa = 6$, $\kappa(\varepsilon) = 1.25 + 32\varepsilon^{-1}$.

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Application to $\chi(m)$

Corollary (RB 2003)

Let

$$M_m = \sup_{f \in S_m, \|f\|=1} \int_{\mathbb{X}} f^2(x) s(x) dx \quad et \quad B_m = \sup_{f \in S_m, \|f\|=1} \|f\|_\infty.$$

then for all $u, \varepsilon > 0$,

$$\mathbb{P}\left(\chi(m) \geq (1+\varepsilon)\sqrt{\frac{1}{L}\sum_\lambda \int \varphi_\lambda^2(x) s(x) dx} + \sqrt{\frac{2\kappa M_m u}{L}} + \kappa(\varepsilon)\frac{B_m u}{L} \right) \leq e^{-u}.$$

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Let $\{L_m, m \in \mathcal{M}\}$ *tq* $\sum_{m\in\mathcal{M}} e^{-L_m|m|} \leq \Sigma$ *with* $|\Gamma| \leq L(\ln L)^{-2}$.
For all $c > 1$, *if*
$\mathrm{pen}(m) = \frac{c\tilde{M}|m|}{L}(1+\sqrt{2\kappa L_m})^2$ *avec* $\tilde{M} = \sup_{I\in\Gamma} \frac{N_I}{\mu(I)}$,
then

$$\mathbb{E}(\|s - \hat{s}_{\hat{m}}\|^2) \leq \square_c \inf_{m\in\mathcal{M}} \left[\|s - s_m\|^2 + \frac{M|m|}{L}(1+L_m)\right] + \square_{c,\Sigma,M}\frac{1}{L},$$

where $M = \sup_{I\in\Gamma} \frac{\int_I s(x)dx}{\mu(I)}$.

Here constants in the concentration inequalities are crucial $\rightarrow$ penalty.

Counting processes with linear intensities

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where $\Psi_.(.)$ known predictable linear transformation. Functional parameter s unknown.

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$$\Psi_s(t)^{(r)} = \lambda^{(r)}(t) = \nu_r + \sum_{\ell=1}^{M} \int_{-\infty}^{t-} h_\ell^{(r)}(t-u)dN_u^{(\ell)}.$$

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Observation on $[0, T]$.

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  minimal when $\Psi_{f-s}(t) = 0$ a.s., a.e. $\rightarrow f = s$.
- In general, $\frac{1}{T}\int_0^T \Psi_f(t)^2 dt$ is random, true norm only with high probability.

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- Once again

$$\chi(m) = \sup_{\|f\|=1, f \in S_m} \frac{1}{T} \int \Psi_f(t)(dN_t - \Psi_s(t)dt).$$

"Talagrand" type inequality for general counting processes

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Let $\{(H_{a,t})_{t\geq 0}, a \in A\}$ *be a countable family of predictable process*

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Then its compensator exists $(A_t)_{t\geq 0}$, it is positive and non decreasing and

$$\forall 0 \leq t \leq T, \quad Z_t - A_t = \int_0^t \Delta Z(s)(dN_s - \lambda(s)ds),$$

for a predictable $\Delta Z(s)$ st $\Delta Z(s) \leq \sup_{a\in A} H_{a,s}$.

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If the H_a *have values in* $[-b, b]$ *and if* $\int_0^T \sup_{a\in A} H_{a,s}^2 \lambda(s)ds \leq v$ *as, then for all* $u > 0$,

$$\mathbb{P}\left(\sup_{[0,T]}(Z_t - A_t) \geq \sqrt{2vu} + \frac{bu}{3}\right) \leq e^{-u}.$$

And for the χ^2 ...

Let

$$\mathcal{C} = \sum_{\lambda} \int_0^T \frac{\Psi_{\varphi_\lambda}(x)^2}{T^2} \lambda(x) dx,$$

with $\mathcal{C} \leq v$ et $\sum_\lambda \Psi_{\varphi_\lambda}(x)^2 \leq b$ for all $x \in [0, T]$. Then for all $u > 0$,

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- Improvement sometimes possible Baraud (2010) but need of an upper bound on $\sqrt{\mathcal{C}}$.
- Still λ inside, which is in general difficult to estimate $\rightarrow$ usually assume known upper bound.

Concrete Problems due to the concentration...

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- We would like to be closer to the true variance of $\hat{s}_m$ and estimate it without bias.
- Talagrand type inequalities lead us to estimate the supremum of the variances (Poisson) or the variance of the supremum

Poisson process and Thresholding

$$\|\hat{s}_{\hat{m}} - s\|^2 \leq \|s - s_m\|^2 + \mathrm{pen}(m) - 2\delta(s_m - s_{\hat{m}}) + 2\delta(\hat{s}_{\hat{m}} - s_{\hat{m}}) - \mathrm{pen}(\hat{m})$$

- Here there exists a large ONB $(\varphi_\lambda, \lambda \in \Lambda)$ and for $m \subset \Lambda$, $S_m = Span(\varphi_\lambda, \lambda \in m)$

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A general thresholding theorem

Theorem (RB Rivoirard 2010)

Let $\beta = (\beta_\lambda)_{\lambda\in\Lambda}$ *st* $\|\beta\|_{\ell_2} < \infty$ *be unknown. Let us observe* $(\hat{\beta}_\lambda)_{\lambda\in\Gamma}$, *where* $\Gamma \subset \Lambda$ *and* $(\eta_\lambda)_{\lambda\in\Gamma}$.

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(A2) *There exists* $1 < a, b < \infty$ *with* $\frac{1}{a} + \frac{1}{b} = 1$ *and* $G > 0$ *st* $\lambda \in \Gamma$,
$$\left(\mathbb{E}\left[|\hat\beta_\lambda - \beta_\lambda|^{2a}\right]\right)^{\frac{1}{a}} \leq G \max\left(F_\lambda,\ F_\lambda^{\frac{1}{a}}\epsilon^{\frac{1}{b}}\right).$$

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(A3) *there exists* τ *st for all* λ *in* Γ / $F_\lambda < \tau\epsilon$,
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A general thresholding theorem (2)

Theorem (RB Rivoirard 2010)

Then under (A1), (A2), (A3),

$$\mathbb{E}\|\tilde{\beta}-\beta\|_{\ell_2}^2 \leq$$
$$\square_\kappa \mathbb{E} \inf_{m\subset\Gamma} \left\{ \sum_{\lambda\notin m} \beta_\lambda^2 + \sum_{\lambda\in m} (\hat{\beta}_\lambda - \beta_\lambda)^2 + \sum_{\lambda\in m} \eta_\lambda^2 \right\}$$
$$+ \square_{\ldots} \sum_{\lambda\in\Gamma} F_\lambda$$

$\leq \square \mathbb{E} \inf_{m\subset\Gamma} [\|s - s_m\|^2 + \mathrm{pen}(m)] +$ reminder term

Bernstein and variance estimation

For all $u > 0$,

$$\mathbb{P}\left(|\hat{\beta}_\lambda - \beta_\lambda| \geq \sqrt{2uV_\lambda} + \frac{\|\varphi_\lambda\|_\infty u}{3L}\right) \leq 2e^{-u},$$

with $V_\lambda = \frac{1}{L}\int \varphi_\lambda^2(x)s(x)dx$

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with

$$\check{V}_\lambda(u) = \hat{V}_\lambda + \sqrt{2\hat{V}_\lambda \frac{\|\varphi_\lambda\|_\infty^2}{L^2}u} + 3\frac{\|\varphi_\lambda\|_\infty^2}{n^2}u,$$

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$$\check{V}_\lambda(u) = \hat{V}_\lambda + \sqrt{2\hat{V}_\lambda \frac{\|\varphi_\lambda\|_\infty^2}{L^2} u} + 3\frac{\|\varphi_\lambda\|_\infty^2}{n^2} u,$$

where $\hat{V}_\lambda = \frac{1}{L^2}\int \varphi_\lambda^2(x)dN_x$.
Hence

$$\mathbb{P}(|\hat{\beta}_\lambda - \beta_\lambda| > \eta_\lambda(u)) \leq 3e^{-u}$$

with $\eta_\lambda(u) = \sqrt{2u\check{V}_\lambda(u)} + \frac{\|\varphi_\lambda\|_\infty u}{3L}$.

Lasso for other counting processes

Reformulation of the least-square contrast:

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Let Φ be a dictionary of $\mathcal{H}$ and if $\mathbf{a} \in \mathbb{R}^{\Phi}$,

$$f_a = \sum_{\varphi \in \Phi} a_\varphi \varphi.$$

Then

$$\gamma(f) = -2\mathbf{b}^*\mathbf{a} + \mathbf{a}^*\mathbf{G}\mathbf{a}$$

where

- $\mathbf{G}$ is a random observable matrix.
- $\mathbf{b}$ is also a random observable vector.

Lasso criterion

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$$\hat{\mathbf{a}} = \text{argmin}_{\mathbf{a}\in\mathbb{R}^{\Phi}}\{-2\mathbf{b}^*\mathbf{a} + \mathbf{a}^*\mathbf{G}\mathbf{a} + 2\mathbf{d}^*|\mathbf{a}|\}$$

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- Oracle inequality with "high" probability possible....

One of the main probabilistic ingredients

Bernstein type inequality for counting processes

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$\hat{V}^{\mu}_{\tau} = \frac{\mu}{\mu-\phi(\mu)} \int_0^{\tau} H_s^2 dN_s + \frac{b^2 x}{\mu-\phi(\mu)}$, where $\phi(u) = \exp(u) - u - 1$.

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For more details about the Lasso procedure, see V. Rivoirard's talk.

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Sketch of proof (2)

Lemma

Let a, b and x be positive constants and let us consider on $(0, 1/b)$, $g(\xi) = \frac{a\xi}{(1-b\xi)} + \frac{x}{\xi}$. Then $\min_{\xi\in(0,1/b)} g(\xi) = 2\sqrt{ax} + bx$ *and the minimum is achieved in* $\xi(a, b, x) = \frac{xb-\sqrt{ax}}{xb^2-a}$.

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- Then with $\xi(v/2, 1/3, x)$, $\mathbb{P}\left(M_\tau \geq \sqrt{2vx} + x/3 \text{ and } \int_0^\tau H_s^2\lambda(s)ds \leq v \text{ and } \sup_{s\leq\tau}|H_s| \leq 1\right) \leq e^{-x}$.

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  $\leq e^{-x}$.
- But also
  $$\mathbb{P}\Bigg(M_\tau \geq \sqrt{2(1+\varepsilon)\int_0^\tau H_s^2\lambda(s)dsx} + x/3 \text{ and}$$
  $$v(1+\varepsilon)^{-1} \leq \int_0^\tau H_s^2\lambda(s)ds \leq v \text{ and } \sup_{s\leq\tau}|H_s| \leq 1\Bigg) \leq e^{-x}.$$
- Peeling + plug in ...

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- Future work : multiple testing, group Lasso ???

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Thank you !