Some geometric perspectives on the Zimmer program

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Vincent Pecastaing

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après avis des rapporteurs :

Thierry	BARBOT	-	Université d'Avignon
Bertrand	DEROIN	-	CNRS & Université de Cergy
David	FISHER	-	Rice University

devant le jury composé de :

Thierry	BARBOT	-	Université d'Avignon
Bertrand	DEROIN	-	CNRS & Université de Cergy
Sorin	DUMITRESCU	-	Université Côte d'Azur
David	FISHER	-	Rice University
Charles	FRANCES	-	Université de Strasbourg
Fanny	KASSEL	-	CNRS & IHÉS
Karin	MELNICK	-	Université du Luxembourg

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Contents

Chapter 1

Introduction

I present in this text the content and the context of my research from my PhD thesis to my current activities, as well as some future directions that I plan to investigate. During this whole period, R. Zimmer's works and vision have been very influential on the development of my research activities, and the Zimmer program stands naturally as a unifying theme.

This introduction aims to give a brief overview of the area and of my contributions.

1.1 General context: Transformation groups of geometric structures

The guiding principle of Klein's Erlangen program was to unify all the geometries which had emerged during the first half of the 19th century via the point of view of group actions, characterizing a geometry via its group of transformations.

This vision is still in force in the modern approach: understanding the symmetries of certain families of geometric structures, and measuring to what extent a geometry is determined by the dynamics of its group of transformations, are in the continuation of Klein's program.

1.1.1 Geometric structures on manifolds

According to Klein's point of view, a geometry is a certain homogeneous space $\mathbf{X} = G/H$, where we will assume G to be a Lie group and H a closed Lie subgroup. A certain number of variations around this model geometry have been intensively investigated. A famous one was introduced by Ehresmann, and consists in associating to an abstract manifold M an atlas of $\mathbf{X}$-valued charts whose coordinate changes are restrictions of elements of G. This notion of M being locally $\mathbf{X}$ is mainly a topological one, and relates closely to representations of $\pi_1(M)$ into G.

In a series of fundamental works, É. Cartan introduced progressively a geometric notion which is the infinitesimal analogue of a homogeneous space $\mathbf{X} = G/H$, related to the so-called method of moving frames. Remarkably, it formalizes the idea of conformal or projective connection, and is a central tool in conformal and projective geometry for this reason, compensating the lack of natural linear connection. The notion was ultimately generalized by Ehresmann. Essentially, a Cartan geometry reproduces the differential-geometric properies of the principal fibration $G \to G/H$, when G/H is replaced by an arbitrary manifold M with the same dimension as G/H. The geometry is therefore defined by an H-principal fibration $\mathcal{G} \to M$, called the Cartan bundle, and a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, called the Cartan connection, which plays the role of the Maurer-Cartan form of G.

The obstruction for such geometries to be locally equivalent to G/H is given by the *curvature 2-form* $\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$, which produces a certain number of useful invariants. Flat Cartan geometries are then the same as Ehresmann's $(G, \mathbf{X})$-structures.

More widely, we can think of a geometric structure on a manifold M as an object which lives in some principal bundle over M.

1.1.2 Transformation groups

The idea of understanding geometric transformations which come in family arose shortly after the Erlangen program started, with Lie's famous concept of *continuous groups of transformations*. Among many contributions, in his *Theorie der Transformationsgruppen* paper, he classified germs of (what is since then called) smooth Lie group actions on low-dimensional manifolds.

Closely related to these groups of transformations, another famous problem was the *equivalence problem*, essentially seeking to produce a complete set of intrinsic infinitesimal invariants for a given family of geometries. It was notably investigated by É. Cartan in the early 1900's, while working on the existence of biholomorphisms mapping a real hypersurface of $\mathbf{C}^2$ to another. This led him a couple decades later to the notion of Cartan connections mentioned above, unifying large families of geometries in this synthetic concept.

As was first observed in Riemannian geometry by Myers and Steenrod in 1939, it can happen that the full group of transformations of a geometric structure has itself a finite dimensional geometry, *i.e.* is a Lie group. It is then very appealing to investigate the relationship between those two: Which Lie group can arise as the automorphisms group of a certain geometric manifold? And conversely, which geometric manifolds can admit a non trivial action of a certain Lie group?

1.1.3 Rigidity of geometric structures

This suggests in particular to investigate the interplay between the infinitesimal geometric invariants and the *dynamics* of a group action preserving the geometry. A celebrated result in Riemannian geometry was the resolution by Ferrand, Obata and Schoen of a conjecture of Lichnerowicz, which ultimately converged to the following.

Theorem (Ferrand-Obata-Schoen). *Let (M, g) be a Riemannian n-manifold. If the group of conformal transformations of $(M, [g])$ acts non-properly, then $(M, [g])$ is conformally equivalent to the round sphere $\mathbf{S}^n$ or the Euclidean space $\mathbf{E}^n$.*

For closed Riemann surfaces, the statement follows from Poincaré-Koebe's uniformization theorem. In dimension greater than 2, it can be interpreted as a remarkable manifestation of *rigidity* of the conformal class. Intuitively, this means that the pseudo-group of local conformal maps of a Riemannian manifold always is finite dimensional.

While analogues of Ferrand-Obata-Schoen's theorem fail quickly to be true for most general rigid structures on non-compact manifolds, it seems reasonable to expect a persisting phenomenon for compact manifolds. D'Ambra and Gromov suggested in the late 1980's that *large groups* of transformations of rigid geometric structures on compact manifolds might be understandable, which they referred to as a "vague general conjecture". Defining what *large* means in this context is certainly part of the question.

1.1.4 Super-rigidity of higher-rank semi-simple Lie groups

Semi-simple Lie groups, and their discrete subgroups, known for their strong linear rigidity properties, stand as natural candidates for this program.

Historically, rigidity of linear representations of lattices in semi-simple Lie groups arose in the 1960's with various contributions around local and global rigidity of such discrete groups. Mostow's rigidity theorem, later extended in the higher-rank case by Margulis, controls embeddings $\rho : \Gamma \to H$ of a lattice Γ of a semi-simple Lie group G into another simple Lie group H. These deep results were later generalized by Zimmer in the early 1980's to ergodic, probability measure preserving G-actions or Γ-actions on probability spaces. The result is called *super-rigidity of cocycles*. Contrarily to actions of amenable groups, these semi-simple Lie groups were proven to have much more rigid actions, leading Zimmer to conjecture that actions of such groups are combinations of standard algebraic actions. This is essentially the framework of the *Zimmer program*.

In the same period, Gromov introduced in [Gro88] the notion of *rigid geometric structure* on manifolds, extending Cartan's notion of finite type G-structures. He proved impressive general results in which orbits of the pseudo-group of local automorphisms of a rigid structure are essentially described as level sets of certain natural equivariant maps with range in algebraic manifolds.

1.1.5 Pseudo-Riemannian actions and other unimodular actions

Group actions on finite volume pseudo-Riemannian manifolds were a first natural field of application of Zimmer-Gromov's philosophy.

Connected Lie group actions. In [Zim86c], Zimmer used essentially Borel's density theorem to prove an embedding theorem (see Theorem 3.3 below) valid for any probability measure preserving action of a non-compact simple Lie group G, possibly of rank 1, on a geometric structure. The conclusion gives an embedding of the Lie algebra of G into the Lie algebra of the group defining the type of the geometric structure. Hence, a general algebraic obstruction to actions of simple Lie groups on finite volume geometric structures, from which he derived striking consequences for Lie group actions by isometries on compact Lorentzian manifolds.

In fact, these conclusions extend optimally to semi-simple Lie group actions on finite volume pseudo-Riemannian manifolds. This is maybe not a surprise because the homogeneous model action $G \curvearrowright G/\Gamma$, for Γ a lattice of G, preserve systematically a pseudo-Riemannian metric of finite volume on G/Γ induced by the Killing metric.

The geometric considerations on compact Lorentzian manifolds, first obtained by Zimmer, culminated a decade later when Adams, Stuck [AS97a, AS97b], and independently, Zeghib [Zeg98a, Zeg98b], obtained the full classification, up to local isomorphism, of the identity component of the isometry group of compact Lorentzian manifolds. See Theorem 3.5 below.

In this classification, up to finite cover, $\mathrm{PSL}_2(\mathbf{R})$ is the only non-compact simple Lie group which can act non-trivially and isometrically on a compact Lorentzian manifold. Importantly, it acts faithfully by isometries on compact quotients $\mathrm{PSL}_2(\mathbf{R})/\Gamma$ endowed with the Killing metric, which are special cases of compact quotients of the Anti-de Sitter 3-space $\mathbf{AdS}^3$, the Lorentzian analogue of the hyperbolic 3-space. Gromov proved in [Gro88] that in fact, any action of $\mathrm{PSL}_2(\mathbf{R})$ on a closed Lorentzian manifold is covered by an action on a warped product $\mathbf{AdS}^3 {}_{\omega}\!\times N$, where N is a Riemannian manifolds.

Discrete group actions. Actions of higher-rank lattices on unimodular H-structures were also strongly constrained by Theorem F of [Zim86c], assuming the H-structure to be *of finite-type*. This includes again pseudo-Riemannian isometric actions or actions $\Gamma \curvearrowright (M, \nabla, \omega)$ preserving both a linear connection and a volume form, but does not work for a Γ-action on a compact manifold preserving a symplectic form, an almost complex structure or just a volume form.

A longstanding problem in the field, Zimmer's conjecture in the volume-preserving case, has precisely been to drop the finite-type assumption in this theorem. Recent important breakthrough by Brown, Fisher and Hurtado [BFH22, BFH20, BFH21] proved this conjecture in a great number of cases. I refer to [WMZ98, Fis11, Fis17] for a much more detailed history of the subject.

A very concise summary of the present state of the art is that more and more optimal obstructions to lattices actions on compact manifolds have been established. A still widely open area concerns now geometric and dynamical descriptions of lattices actions close to the critical dimension. This is supported by various results such as *local rigidity* of standard model actions, for both volume-preserving actions (*e.g.* $\mathrm{SL}_n(\mathbf{Z})$ acting on the n-torus) and non volume-preserving actions (*e.g.* actions on boundaries). An influential result in the area due to Katok and Spatzier ([KS97]) establishes rigidity for hyperbolic actions of abelian groups $\mathbf{R}^k \times \mathbf{Z}^\ell$, and became central in the Zimmer program. An important Katok-Spatzier conjecture states that actions of abelian groups with a dense set of Anosov elements and without rank 1 factor are smoothly conjugate to algebraic actions (see the introduction of [SV23]). This relates to higher-rank lattices actions via the action of an $\mathbf{R}$-split Cartan subgroup on the suspension space. However, it was remarkably observed in [KL96] that there exist analytic *exotic* volume-preserving actions of $\mathrm{SL}_n(\mathbf{Z})$ on closed n-manifolds, which are not conjugate to any algebraic action. Katok-Lewis examples are nonetheless conjugate to the linear action on the n-torus when restricted to an *invariant open-dense subset*. It is now believed that geometric conclusions follow modulo such restrictions.

For more general discrete groups, additional results were proved for isometric actions on Lorentzian manifolds. It was observed in [Zim86c] that any discrete group with property (T), not necessarily a higher-rank lattice, acting by isometries on a compact Lorentzian manifold must also preserve a Riemannian metric. This ruled out non-trivial isometric actions of lattices in $\mathrm{Sp}(1,n)$ or $F_{4(-20)}$. More recently, Frances extended in [Fra21] this fact to lattices in $\mathrm{SU}(1,n)$ by different methods. This is optimal since $O(1,n)_{\mathbf{Z}}$ acts on the flat Lorentzian torus $(\mathbf{T}^{n+1}, -\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n+1}^2)$.

1.1.6 Non-unimodular Lie group actions

An essential point in Zimmer's proof of the embedding theorem, or the super-rigidity of cocycles theorem, is the existence of a finite measure invariant under the group action. For non-amenable groups, this assumption is a strong requirement and excludes lots of important examples, typically actions on flag manifolds $G \curvearrowright G/P$, for which generally the conclusions of these theorems are false. It is therefore natural to examine actions on geometric structures which *do not* fall into the range of application of these theorems, and see if some rigidity phenomena persist or not.

Nevo-Zimmer's measurable projective factor. Analogously to cocyle super-rigidity in the measure-preserving case, Nevo and Zimmer obtained in [NZ99] general structure result for stationary actions of a semi-simple Lie group G on a probablity space (X,ν). Under a certain mixing assumption, they showed that if G does not preserve ν, then the action fibers measurably over a non-trivial unique flag manifold G/P. Similarly to the measure-preserving case, the main question is whether or not this fibration can be made more regular in geometrized situation. This was later refined in [NZ09] when the action preserves a rigid geometric structure (with other dynamical assumptions), the outcome was a *smooth* projective factor, but in restriction to an open-dense subset.

Bound on the real-rank. If a conclusion as strong as an embedding is not true, Zimmer nonetheless proved in [Zim87b] that for an action of a semisimple Lie group G on a compact manifold, preserving an H-structure, for an arbitrary real algebraic group H, the real-ranks verify $\mathrm{rk}_{\mathbf{R}}\, G \leqslant \mathrm{rk}_{\mathbf{R}}\, H$. Later, Bader and Nevo considered in [BN02] the case of conformal pseudo-Riemannian actions, as well as conformal symplectic actions, *i.e.* $H = \mathbf{R}^* \times O(p,q)$ or $H = \mathbf{R}^* \times \mathrm{Sp}_{2n}(\mathbf{R})$. These two geometries have in common to be defined by a field of non-degenerate bilinear forms on the tangent bundle, which is preserved up to a smooth factor by the G-action. They obtained that if equality holds in the inequality between the real-ranks, then G is locally isomorphic to an orthogonal group (in the pseudo-Riemannian case) or to $\mathrm{SL}_3(\mathbf{R})$ (in the symplectic case), and that if the action is moreover *minimal*, then the manifold is globally equivalent to a certain flag manifold of G. The minimality assumption was then removed in [FZ05]. Finally, Bader, Frances and Melnick proved a version of Zimmer's embedding for Cartan geometries in [BFM09], and derived various restrictions for general Lie group actions (see Section 3.1.7).

1.1.7 Generalizations of Lichnerowicz conjecture

Going back a bit, the general program suggested by D'Ambra and Gromov in [DG91] was in part motivated by Ferrand-Obata's proof of Lichnerowicz conjecture in conformal Riemannian geometry. The result was later generalized by Schoen to strictly pseudo-convex CR structures [Sch95] and finally by Frances to rank 1 parabolic geometries [Fra07]. The phenomenon was that if no reductive "sub-geometry" is preserved, then the manifold is equivalent to the model geometry.

In conformal geometry, this assumption is called *essentiality* (see Definition 3.5) and means that no conformal metric is invariant. However, it turned out that essentiality is much less restrictive for higher signatures (*e.g.* [KR97, Fra05]). For large enough signatures, essentiality moreover authorizes higher-rank simple Lie group actions on manifolds which are not even locally equivalent to the model space ([Fra15]). However, for Lorentzian conformal structures on compact manifolds, a remaining problem is still unsolved. The literature refers to it as Lorentzian Lichnerowicz conjecture.

Conjecture (Conj. 1)**.** *Let* $(M,[g])$ *be a compact manifold endowed with a Lorentzian conformal class. If* $\mathrm{Conf}(M,[g])$ *does not preserve any metric in the conformal class, then* $(M,[g])$ *is conformally flat.*

A weaker (still challenging) form of this conjecture ask the same question when the *identity* component $\mathrm{Conf}(M,[g])_0$ is essential, as was initially asked by Lichnerowicz and solved by Obata. Examples of self-similar Lorentzian metrics studied by Alekseevsky in [Ale85] imply that this problem necessarily deals with the global dynamics of essential conformal groups.

Several analogous problems were also investigated (see [Mel21] for a recent survey). Notably, Matveev proved in [Mat07] an analogue of [Oba71] for the projective group of a Riemannian manifold (see also [Zeg16] and the references therein).

1.1.8 Non volume-preserving discrete group actions

An additional evidence for rigidity of non-unimodular actions of lattices is a result of Katok Spatzier proved in ([KS97]) asserting that for Γ a cocompact lattice in a higher-rank semi-simple Lie group G, then, for every parabolic subgroup $P < G$, the action $\Gamma \curvearrowright G/P$ is locally $\mathcal{C}^\infty$-rigid.

In a more geometric context, Iozzi had proved earlier in [Ioz92] that given a differentiable action $G \to \mathrm{Diff}(M)$, if some lattice $\Gamma < G$ preserves a rigid geometric structure on M, then so does the whole of G. The delicate point is that proving that a Γ-action extends to G, even within the diffeomorphism group, is quite a difficult problem in itself.

In recent advances, Brown, Rodriguez-Hertz and Wang proved in [BRHW22] a stronger version of Nevo-Zimmer's theorem, for general lattices actions on low-dimensional manifolds. This is based on an existence result of Γ-invariant measures proved in the same paper, which was key in the proof of Zimmer's conjecture by Brown, Fisher and Hurtado.

1.2 Organization of the manuscript and personal contributions

Plan

This memoir is globally organized as follows.

In Chapter 2, I briefly setup some conventions and notations, and recall definitions of geometric structures which will be used in the manuscript.

In Chapter 3, I discuss actions of connected Lie groups on geometric structures and describe the results of [Pec17, Pec18] (Section 3.2), [Pec19] (Section 3.3.1), [Pec23] (Section 3.4) on conformal pseudo-Riemannian actions of Lie groups.

Chapter 4 is organized around actions of lattices and contains the results of [Pec20] (Section 4.3), [Pec24] (Section 4.4) for conformal and projective actions of higher-rank lattices.

Chapter 5 is centered on [MP22], a joint work with Karin Melnick on conformal groups of simply-connected Lorentzian manifolds. This is a generalization of a theorem of D'Ambra, and our proof starts similarly by applying Gromov's stratification theorem. The latter, in the context of our proof, can be recovered via the main results of [Pec16], which are detailed in the same chapter in Section 5.2.2.

I give now a brief overview of the main results, following this organization.

1.2.1 Levi factor of the conformal group of a closed Lorentzian manifold

Chapter 3 starts by contextualizing semi-simple Lie group actions on geometric structures in the framework of super-rigidity. Another famous result in this setting is Zimmer's embedding theorem. As an illustration, Zimmer proved that up to covers, $\mathrm{SL}_2(\mathbf{R})$ is the only non-compact simple Lie group that can act by isometries of a closed Lorentzian manifold. Beyond semi-simple isometric Lie group actions, the classification of the full isometry group of a closed Lorentzian manifold by Adams-Stuck-Zeghib was a major achievement in the field, and invited more generally to analyze the algebraic structure of automorphisms groups of similar geometries.

The first contribution presented in Chapter 3 considers compact Lorentzian manifolds with conformal actions of semi-simple Lie groups. The presence of such algebraic structures in the conformal group forces the geometry to be locally conformally flat, supporting Conjecture 1 recalled above. As we will see, a major difference with isometric actions, and certainly a very interesting aspect, is the lack of natural invariant volume form.

Theorem ([Pec17, Pec18], Th. A, Cor. 2)**.** *Let $(M^n, [g])$, $n \geqslant 3$, be a closed manifold endowed with a conformal Lorentzian structure. Let G be a connected semi-simple Lie* *group. If G acts conformally and essentially on $(M,[g])$, then $[g]$ is conformally flat, i.e. near every point, there are coordinates in which the metric reads*

$$g = e^{\sigma}(-\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \cdots + \mathrm{d}x_n^2),$$

for some smooth function σ.

In particular, G is locally isomorphic to an immersed Lie subgroup of $O(2,n)$.

Remark that essentiality forces G to be non-compact. That G locally embeds into $O(2,n)$ is an optimal conclusion, since $\mathrm{PO}(2,n)$ is the conformal group of the *model space* of conformal Lorentzian geometry, the Einstein universe $\mathbf{Ein}^{1,n-1} \simeq (\mathbf{S}^1 \times \mathbf{S}^{n-1}, [-\mathrm{d}t^2 \oplus g_{\mathbf{S}^{n-1}}])/\mathbf{Z}_2$, where $g_{\mathbf{S}^k}$ refers to the Riemannian metric of sectional curvature $+1$ on the k-sphere.

As will be detailed in this chapter, essentiality of G can be characterized via the existence of low-dimensional closed orbits in the closure of any G-orbit. Two proofs for this result are detailed:

- The first, [Pec17], is inspired by the proof of Gromov's centralizer theorem, and is under an additional analyticity assumption. At that time, Melnick had just published [Mel11], a version for Cartan geometries of a theorem of Gromov on local integration of isometric jets, called "Frobenius' theorem". It seemed to have nice geometric consequences for actions of semi-simple Lie groups. We will see how it can be implemented in this non-unimodular context to provide existence of *additional* local Killing vector fields.

- The second, [Pec18], removes the analyticity assumption. Using a different strategy, the proof shows that there always exist *closed G-orbits*, in the neighborhood of which some element g_0 acts similarly to the local conformal vector field exhibited in the analytic proof. The new ingredient used to find these closed orbits was the local stable manifold of the flow associated to an hyperbolic one-parameter subgroup $\{h^t\} < G$ to prove that it must have a periodic orbit.

1.2.2 Extension to higher signatures

The next results show that similar phenomena happen in more general pseudo-Riemannian actions of semi-simple Lie groups. After several anterior investigations [Zim87b, BN02, FZ05], conformal actions of simple Lie groups whose real rank is *small* compared to the signature were still poorly understood.

[Pec19] is a first step in a larger project, and addresses the question of metric signatures on which a given simple Lie group of rank 1 can act conformally and non-trivially. It is somehow the opposite situation compared to anterior results, which considered groups with maximal possible real-rank. The question was essentially for $\mathrm{Sp}(1,k)$ and $F_{4(-20)}$.

Theorem ([Pec19] Th B). *Let $(M^n,[g])$ be a closed pseudo-Riemannian manifold of signature (p,q), with $n \geqslant 3$. Suppose that there exists $\rho : \mathrm{Sp}(1,k) \to \mathrm{Conf}(M,[g])$ a conformal action with discrete kernel. Then,*

1. $\min(p,q) \geqslant 3$;
2. *If* $\min(p,q) = 3$*, then* $(M,[g])$ *is conformally flat. Moreover, any minimal, compact,* G*-invariant subset of* M *is a compact orbit conformally equivalent to* $\mathbf{Ein}^{3,3k-1}$*, on which* $\mathrm{Sp}(1,k)$ *acts via a Fefferman fibration.*

The Einstein universe $\mathbf{Ein}^{p,q}$ is the natural extension in signature (p,q) of its Lorentzian version seen above.

Hence, under a minimality assumption, this result recovers similarly to [BN02] homogeneity of the conformal structure. The action moreover respects a principal fibration $\mathrm{Sp}(1) \to \mathbf{Ein}^{3,3k-1} \to \mathrm{Sp}(1,k)/P$ over the boundary, hence give explicitly a smooth projective factor.

The case of $F_{4(-20)}$ conformal actions is a bit more curious, and behaves differently. It was still possible to obtain an optimal lower-bound for the metric index, as in the previous theorem (Theorem C). The geometry, however, remains more mysterious.

1.2.3 Radical of the conformal group of a closed Lorentzian manifold

Chapter 3 finishes with recent contributions prepublished in [Pec23]. The global perspectives are: 1) obtain a classification, up to local isomorphism, of the identity component $\mathrm{Conf}(M,[g])_0$ of the conformal group of closed Lorentzian manifolds (M,g), extending Adams-Stuck-Zeghib theorem, 2) obtain general essentiality criteria for Lie group actions, and 3) prove conformal flatness in as many contexts as possible, in view of Lorentzian Lichenrowicz conjecture. Naturally, these three directions interact with each others.

A standard strategy to understand an abstract Lie group such as $\mathrm{Conf}(M,[g])_0$ is to consider a Levi decomposition $S.R$, where R denotes the solvable radical, and $S \simeq \mathrm{Conf}(M,[g])_0/R$ a Levi factor. This factor is classified by Theorem A cited above, and Conjecture 1 is true if S is non-compact. Essentiality of S is also characterized in terms of stabilizers. Ideally, it would be nice to obtain similar descriptions for the action of the radical R.

We will see that most questions are now reduced to R either abelian, or locally isomorphic to an $\mathbf{R}$-split semi-direct product $\mathbf{R} \ltimes \mathbf{R}^k$.

Theorem ([Pec23], Th. D)**.** *Let* $(M^n,[g])$*,* $n \geqslant 3$*, be a compact manifold endowed with a Lorentzian conformal structure, and let* R *be a connected, solvable Lie subgroup of* $\mathrm{Conf}(M,[g])$*. If* R *is essential, then there exists a Lie algebra embedding* $\mathfrak{r} \hookrightarrow \mathfrak{so}(2,n)$*.*

This embedding provides an optimal obstruction, again because $\mathbf{Ein}^{1,n-1} = \mathrm{PO}(2,n)/P$, where P is the stabilizer of a null-line. When the Levi factor is compact (which can be assumed by Theorem A), essentiality of the whole identity component is reduced to that of its *nilradical.* This means that if the nilradical preserves a conformal metric, then so does $\mathrm{Conf}(M,[g])_0$. This is a consequence of Theorem E.

Theorem ([Pec23], Cor. 3)**.** *Let* G *the identity component of* $\mathrm{Conf}(M,[g])$*. Let* $R \lhd G$ *be its solvable radical and let* $N \lhd R$ *be the nilradical. If* G/R *is compact, then* G *is essential if and only if* N *is essential.*

In particular, if the identity component $\mathrm{Conf}(M,[g])_0$ is essential, then some nilpotent subgroup acts essentially. Theorem F which considers essential nilpotent Lie group actions, completing results of [FM10]. We will see that it implies in particular:

Corollary ([Pec23], Cor. 4)**.** *Let (M,g) be a closed real-analytic Lorentzian manifold and suppose that $G = \mathrm{Conf}(M,[g])_0$ is essential. If the nilradical N of G is non-abelian, then (M,g) is conformally flat.*

This finally reduces Conjecture 1 (in the weak form and the analytic case) to the case where the nilradical is abelian, and the Levi factor is compact. Further technical considerations also eliminate many possibilities for R and it seems plausible that the conjecture can be reduced to the case where $\mathrm{Conf}(M,[g])_0 \simeq K \ltimes \mathbf{R}^d$, for K a compact group. The methods of [MP22] described in Chapter 5 will probably be helpful in the future for this situation of an abelian essential action.

1.2.4 Actions of lattices

Chapter 4 starts with a brief survey of the origins of the Zimmer program. In particular, some measurable conclusions derived from cocycle super-rigidity are cited to motivate the conjectures. In Section 4.2, background materials for stating existence results for Γ-invariant measures from [BRHW22] are introduced. In Section 4.2.4, I explain how results on *non-existence* of Γ-invariant measures for certain actions on geometric structures can be obtained, extending the approach of [Pec20] in conformal geometry.

We will then see how the main results of [BN02] and [FZ05] (discussed in the chapter on Lie groups) can be extended to actions of lattices. The following result synthesises the main theorems of [Pec20] and [Pec24], and current works in finalization in the Lorentzian case and non-uniform setting.

Theorem ([Pec20, Pec24] Th. G)**.** *Let $(M,[g])$ be a compact manifold of dimension at least 3 endowed with a conformal structure of signature (p,q), with $1 \leqslant p \leqslant q$. Let Γ be a lattice in a simple Lie group G with $\mathrm{rk}_{\mathbf{R}}\, G \geqslant 2$ and finite center. Let $\rho : \Gamma \to \mathrm{Conf}(M,[g])$ be a conformal action such that $\rho(\Gamma)$ is not relatively compact. Then,*

1. $\mathrm{rk}_{\mathbf{R}}\, G \leqslant p+1$;
2. *If $\mathrm{rk}_{\mathbf{R}}\, G = p+1$, then $\widetilde{M}$ is conformally diffeomorphic to $\widetilde{\mathbf{Ein}}^{p,q}$.*
   (a) *If $p > 1$, then $|\pi_1(M)| \leqslant 2$ and M is either $\widetilde{\mathbf{Ein}}^{p,q}$ or its projective model.*
   (b) *If $p = 1$, then $\pi_1(M)$ is virtually infinite cyclic, and up to finite index, it is sent by the holonomy homomorphism into $\mathcal{Z} \times O(n)$, where $\mathcal{Z}$ refers to the center of $\widetilde{O}(2,n)$ and $O(n)$ to the lift to $\widetilde{O}(2,n)$ of the $O(n)$ factor of the maximal compact of $O(2,n)$.*
3. *If $\mathrm{rk}_{\mathbf{R}}\, G = p+1$, then $G \simeq_{\mathrm{loc}} O(p+1,k+1)$, with $p \leqslant k \leqslant q$, and the Γ-action almost extends to a G-action.*

As explained in this chapter, the proof relies mainly on the fact that an unbounded Γ-action *cannot* preserve any finite measure, and combined with the main result of [BRHW22], this implies that the Lyapunov functionals of the A-action on a natural auxiliary space M^α have to satisfy certain combinatorics, ruled by the algebraic structure of the parabolic subgroup $P < \mathrm{PO}(2,n)$ stabilizing a null-line. This information on the Lyapunov spectrum can be re-transcribed into the Γ-action and provides enough dynamical information to prove local conformal flatness of M. The final globalizing step amounts to understand Γ-actions on compact manifolds with $(\mathbf{G},\mathbf{X})$-structures, with $\mathbf{G} = \mathrm{PO}(p+1,q+1)$ and $\mathbf{X} = \mathbf{Ein}^{p,q}$. It relies on the incidence relations between *Minkowski patches* in $\mathbf{X}$.

The same strategy can be implemented for different geometric structures. A first example was provided in [Pec24] for *projective actions.*

Theorem ([Pec24], Th. I)**.** *Let Γ be a lattice in a simple Lie group G of real-rank $n \geqslant 2$ and finite center. Let M be a compact n-manifold on which Γ acts by preserving a projective class of linear connections $[\nabla]$.*

If the action is infinite, then $(M,[\nabla])$ is projectively equivalent to either $\mathbf{R}P^n$ or $\mathbf{S}^n$, endowed with their standard projective structures.

In Section 4.4, further natural future directions are discussed. Inexistence of finite Γ-invariant measures, general control on the Lyapunov spectrum in terms of the model geometry are *a priori* good starting points to various generalizations, including actions on compact manifold preserving parabolic geometries (extending Theorem 1.5 of [BFM09] to lattices actions) as well as elliptic H-structures (extension of [Can04, CZ12] to non-Kähler complex manifolds for instance).

1.2.5 Conformal groups of compact simply-connected Lorentzian manifolds

In this final chapter 5, the main result of an article obtained in collaboration with K. Melnick is presented.

Theorem ([MP22], Th. J)**.** *Let $(M,[g])$ be a real-analytic, compact, simply-connected conformal Lorentzian structure. Then, its conformal group* $\mathrm{Conf}(M,g)$ *is compact.*

It generalizes to the conformal setting the exact same statement for the *isometry group*, proved by D'Ambra in [D'A88]. If compact Riemannian manifolds all have compact isometry group by Myers-Steenrod theorem, a plethora of compact pseudo-Riemannian manifolds have non-compact isometry groups. D'Ambra's theorem shows that in Lorentzian signature, under an analyticity assumption, all these manifolds must have infinite fundamental group. As we will see, this relates to a result of Gromov on representations of the fundamental group of compact unimodular analytic rigid geometric structures admitting large group actions.

D'Ambra's theorem is proper to Lorentzian geometry, and so is ours. But in our situation, the reason for that is the geometry of the model space $\mathbf{Ein}^{1,n-1}$, which *forbids* conformal flatness of M as in Theorem J, and explains maybe more conceptually this Lorentzian specificity. So, all preceding results related to Conjecture 1 indicated that the conformal group cannot be large. The question was then to eliminate every non-compact case.

The proof starts like the proof of D'Ambra, and uses heavily Gromov's stratification theorem (see Chapter 2). We will see that the difficulty resides in the fact that no invariant volume may exist. Zeghib's fundational works on foliations in Lorentzian geometry attached to isometric dynamics already provided an alternative proof of D'Ambra's theorem. The idea was that if $\mathrm{Conf}(M,[g])_0$ is non-compact, then we can exhibit foliations by hypersurfaces attached to some well chosen diverging sequences of conformal maps. As it frequently happens, the foliation was only defined in restriction to an open-dense subset and a technical point was to extend leaves to a singular locus where the group has compact orbits to finally deduce a contradiction, because all leaves must accumulate to this small singular subset.

This approach looks promising for further investigations of conformal essential actions of higher-rank abelian Lie groups, which are the remaining cases for Conjecture 1. Notably, the same approach was used in a recent proof by Frances and Melnick [FM21] of the weak conjecture in dimension 3 and with analytic regularity.

1.2.6 Elementary proof of Frobenius' theorem

Gromov's stratification theorem is intensively used in the proof of Theorem J. It follows from another result of Gromov's, which he called "Frobenius theorem", a result of local integration of isometric r-jets. This theorem was also used in the first analytic proof of Theorem A obtained during my PhD. I obtained in [Pec16] a new elementary proof of "Frobenius theorem" for Cartan geometries, and other natural rigid geometric structures. The same approach provided a generalization to Cartan geometries of a theorem of Singer characterizing local homogeneity of Riemannian manifolds at infinitesimal scale. See Section 5.2.2 for more details.

Theorem ([Pec16] Th. K)**.** *Let $(M,\mathcal{G},\omega)$ be a Cartan geometry with model space $(\mathbf{G},\mathbf{H})$, such that $\mathrm{Ad}_{\mathfrak{g}}(\mathbf{H})$ is an algebraic subgroup of $\mathrm{GL}(\mathfrak{g})$, and let $\phi : \mathcal{G} \to W$ be an equivariant map, where W is a vector space with an action of $\mathrm{Ad}_{\mathfrak{g}}(\mathbf{H})$.*

Then, there exists an open dense subset $\Omega \subset M$, and an $\mathbf{H}$-equivariant map $\psi : \mathcal{G} \to V$ where V is a finite dimensional vector space with a linear action of $\mathrm{Ad}(\mathbf{H})$, such that for any $b \in \mathcal{G}$ projecting in Ω and $u \in T_b\mathcal{G}$, the following are equivalent.

1. *There exists a local Killing vector field X of $(M,\mathcal{G},\omega)$ defined on a neighborhood of b and such that $X(b) = u$.*
2. *$(\mathcal{L}_u\psi)(b) = 0$.*

The map ψ encodes essentially the curvature map and its covariant derivatives up to order $\dim \mathbf{G}$. The advantage of having such a (more or less) concrete map which detects local Killing fields can be used in various situations to exhibit local symmetries, and can be widely used (for instance to prove local homogeneity). A natural one would be to consider higher-rank lattices actions on unimodular Cartan geometries, and see if a version of Gromov's centralizer theorem could be deduce. This relates to questions raised in Chapter 4.

Denoting the derivatives of the curvature by κ^i, the generalization of Singer's theorem reads as follows.

Theorem ([Pec16], Th. L)**.** *Let* $(M, \mathcal{G}, \omega)$ *be a Cartan geometry modeled on* $(\mathbf{G}, \mathbf{H})$ *and let* $r = \dim \mathbf{H}$*. If the maps* $\kappa, \kappa^1, \ldots, \kappa^r$ *all have range in a single* $\mathbf{H}$*-orbit, then the Cartan geometry is locally homogeneous, i.e. its pseudo-group of local automorphisms acts transitively.*

As a byproduct, an elementary proof for the open-dense theorem of Gromov ([Gro88]) in the case of Cartan geometries follows from this generalization of Singer's theorem.

1.3 Publications list

Publications postérieures à la thèse

1. [Pec24] Projective an conformal closed manifolds with a higher-rank lattice action

   Mathematische Annalen, **388**(2024), pp. 939-968.

2. [MP22] The conformal group of compact simply-connected Lorentzian manifolds (avec K. Melnick)

   Journal of the American Mathematical Society, **35**(2022), pp. 81-122.

3. [Pec20] Conformal actions higher-rank lattices on compact pseudo-Riemannian manifolds

   Geometric and Functional Analysis **30**(2020), pp. 955-987.

4. [Pec19] Conformal actions of real-rank 1 simple Lie groups on pseudo-Riemannian manifolds

   Transformation Groups, **24**(2019), no. 4, pp. 1213-1239.

5. [Pec18] Lorentzian manifolds with a conformal action of $\mathrm{SL}(2, \mathbf{R})$

   Commentarii Mathematici Helvetici **93**(2018), no 2, pp. 401-439.

Publications issues de la thèse

1. [Pec17] Essential conformal actions of $\mathrm{PSL}(2, \mathbf{R})$ on real-analytic compact Lorentz manifolds

   Geometriae Dedicata **188** (1), 171-194 (June 2017).

2. [Pec16] On two theorems about local automorphisms of geometric structures
Annales de l'Institut Fourier **66**(1), 175-208 (2016).

Prépublication

1. [Pec23] Conformal actions of solvable Lie groups on closed Lorentzian manifolds
Preprint arXiv:2307.05436 (07/2023).

Chapter 2

Preliminaries

2.1 A couple conventions

Real-rank and restricted roots. Let $\mathfrak{g}$ be a semi-simple real Lie algebra with no compact factor. An **R-split Cartan subalgebra** is a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}$ whose adjoint action $\mathrm{ad}(\mathfrak{a})$ is **R**-diagonalizable. The **real-rank** $\mathrm{rk}_{\mathbf{R}}\, \mathfrak{g}$ of $\mathfrak{g}$ is the common dimension of all its **R**-split Cartan subalgebras. The set of its **restricted roots** will be denoted by Σ, the **restricted root-space** associated to $\lambda \in \Sigma$ will be denoted by $\mathfrak{g}_\lambda$, $\mathfrak{m}$ will be the compact part of the centralizer of $\mathfrak{a}$, so that the restricted root space decomposition writes $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda$.

Standing notation. Most of the time, I will reserve the letter G for a semi-simple Lie group without compact factors (often of real-rank larger than or equal to 2) which will play the role of the acting group, or also when a lattice of G acts on a geometric manifold. For model spaces of Cartan geometry for instance, I will prefer to use the bold font $\mathbf{G}$, $\mathbf{P}$ etc.., H for the structural group of a principal fiber bundle, etc..

Implicit convention. If a Lie group acts by diffeomorphisms of a compact manifold, its Lie algebra identifies with a Lie algebra of complete vector fields, the vector field corresponding to $X \in \mathfrak{g}$ being, by definition, the infinitesimal generator of $\{e^{-tX}\}_{t \in \mathbf{R}}$. This identification will be used each time a Lie group action arises.

Lattices. An **R**-split Cartan subgroup of G is a connected Lie subgroup $A < G$ tangent to an **R**-split Cartan subalgebra at the identity. The real-rank of G is the real-rank of $\mathfrak{g}$. "G has higher-rank" means $\mathrm{rk}_{\mathbf{R}}\, G \geqslant 2$. A lattice $\Gamma < G$ is a discrete subgroup such that $\mathrm{vol}(G/\Gamma) < \infty$, where vol refers to the Haar measure on G/Γ. A lattice Γ is said to be uniform if G/Γ is compact. A lattice is said to be irreducible if the projection of Γ on a non-trivial factor of G (if any) is dense.

Pseudo-Riemannian metrics. A pseudo-Riemannian metric g of signature (p,q) on a manifold M is a smooth assignment of quadratic forms of signature (p,q) on tangent spaces of M. Here, p refers to the dimension of maximal negative-definite subspaces, and q to the dimension of maximal positive-definite subspaces. A **conformal class** of signature (p, q) is an equivalence class of pseudo-Riemannian metric for the relation $g \sim g'$ if and only if $g' = e^{\sigma} g$ for some $\sigma \in \mathcal{C}^{\infty}(M)$. A metric is said to be **Lorentzian** if it has signature $(1, n-1)$.

Principal fibrations. Let H be a Lie group. A principal fiber bundle with structural group H (or an H-principal fibration) consists of a manifold P with a smooth, free and proper H-action, by convention *on the right*, so that the quotient $M := P/H$ has a smooth manifold structure. It is a fiber bundle $\pi : P \to M$, whose fibers are parametrized by a free right-action of H. Given two H-principal fibrations $\pi_1 : P_1 \to M_1$ and $\pi_2 : P_2 \to M_2$, an isomorphism is a diffeomorphism $F : P_1 \to P_2$ verifying $F(u_1.h) = F(u_1).h$ for all $u_1 \in P_1$ and $h \in H$. In particular, F induces a diffeomorphism $f : M_1 \to M_2$ such that $\pi_2 \circ F = f \circ \pi_1$. If the fibrations are the same, all these isomorphisms form a group, called the automorphisms group of the principal H-bundle.

2.2 Geometric structures on differentiable manifolds

The aim is to define a general notion of geometric structures on manifolds which will encompass all classical examples. Of course, there are several approaches. I will discuss three major cases: G-structures, Cartan geometries, and Gromov's A-rigid geometric structures.

Although it is not clear for the second family in general, the idea will systematically be to define a geometric structure ϕ on a manifold M in local charts and require that there exists $r \geqslant 1$ such that the "value" of ϕ at x only depends on the r-th jet at x of the chart and transforms equivariantly when the chart changes. To sum up, the starting point is an equivariant map ϕ from the bundle of r-frames with values in some space W on which the structural group $D^r(n)$ acts naturally.

I will denote by $J^r_x(f)$ the r-th jet at a point x of a smooth map between differentiable manifolds, essentially its Taylor expansion at order r at x.

Definition 2.1. Given a smooth manifold M, and an integer $r \geqslant 1$, its r-frames bundle is defined as $\mathcal{F}^r(M) = \{J^r_0(\psi),\ x \in M,\ \psi : (\mathbf{R}^n, 0) \to (U, x) \text{ local chart at } x\}$. Denote by $D^r(n) = \{J^r_0(f),\ f : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0) \text{ germ of diffeomorphism}\}$.

The natural projection $\pi : \mathcal{F}^r(M) \to M$ defines a $D^r(n)$-principal fibration, with the action $J^r_0(\psi).J^r_0(f) := J_0(\psi \circ f)$. For $r = 1$, $D^1(n)$ identifies with $\mathrm{GL}(\mathbf{R}^n)$ and a 1-frame at x is the same as a linear isomorphism $\mathbf{R}^n \to T_x M$, which can be seen as the data of a basis of vector space.

Automorphisms group. The group of all automorphisms of a geometric structure may itself have a remarkable geometric structure, and turn out to be a *Lie transformation group*. Certainly, one of the most famous results in this area is a theorem in Riemannian geometry due to Myers and Steenrod.

Theorem 2.1 ([MS39])**.** *Let* (M^n, g) *be a Riemannian manifold. Then, the isometry group* $\mathrm{Isom}(M, g)$ *has a unique Lie group structure, compatible with the compact-open topology, such that its action on* M *is smooth.*

Furthermore, if M *is compact, then* $\mathrm{Isom}(M, g)$ *is compact.*

Finite dimensionality of the isometry group is a manifestation of *rigidity* of the geometric structure. This phenomenon has been systematized by Gromov in a very influential contribution [Gro88]. Compactness, however, is proper to Riemannian geometry and to the natural length distance, preserved by all isometries. This distance disappears once the metric tensor is allowed to be non positive definite for instance.

2.2.1 G-structures

Definition 2.2. Given a smooth n-manifold M, an integer $r \geqslant 1$ and a Lie subgroup $G < D^r(n)$, a G-structure on M is the data of a principal reduction $P \subset \mathcal{F}^r(M)$ with structural group G.

Equivalently, a G-structure is the same as a section $\sigma : M \to \mathcal{F}^r(M)/G$, or to a $D^r(n)$-equivariant map $\phi : \mathcal{F}^r(M) \to D^r(n)/G$.

Example 1.

1. Let g be a pseudo-Riemannian metric on M. Then, it is the same as an $O(p, q)$-structure on M (hence defined on the bundle of 1-frames). The corresponding equivariant map being $\phi^g : u \in \mathcal{F}^1(M) \mapsto (g_{\pi(u)}(u_i, u_j))_{ij} \in \mathcal{S}^{p,q}$, where a 1-frame is seen as a basis $(u_1, \ldots, u_n)$ of $T_{\pi(u)}M$, and $\mathcal{S}^{p,q}$ is the $\mathrm{GL}_n(\mathbf{R})$-homogeneous space of symmetric matrices of signature (p, q), whose isotropy is a conjugate of $O(p, q)$.

2. Let $[g] = \{e^\sigma g,\ \sigma \in \mathcal{C}^\infty(M)\}$ be a conformal class of signature (p, q). The post-composition $p^+ \circ \phi^g : \mathcal{F}^1(M) \to \mathbb{P}^+(\mathcal{S}^{p,q}) = \{\text{rays of quadratic forms}\}$ is independent of the metric g in the conformal class, and defines a $\mathrm{GL}_n(\mathbf{R})$-equivariant map $\mathcal{F}^1(M) \to \mathrm{GL}_n(\mathbf{R})/\mathrm{CO}(p, q)$, where $\mathrm{CO}(p, q) = \mathbf{R}_{>0} \times O(p, q)$. So, $[g]$ is the same as a $\mathrm{CO}(p, q)$-structure on M.

3. A bit more technically, we can observe that a linear connection ∇ on M is the same as a $\mathrm{GL}_n(\mathbf{R})$-reduction of $\mathcal{F}^2(M)$, the map being essentially $J^2_0(\psi) \mapsto (\Gamma^k_{i,j}) \in \mathbf{R}^{n^3}$, where $\Gamma^k_{i,j}$ denote the Christoffel symbols of ∇ in the chart ψ, which only depends on the 2-jet of ψ.

Definition 2.3. Let $P_M \to M$ and $P_N \to N$ be two G-structures of order r on two n-manifolds M and N. Then, an isomorphism of G-structure between M and N is a diffeomorphism $f : M \to N$ such that the differential action $J^r(f)$ sends P_M into P_N.

In particular, given a group Γ, a Γ-action on M is said to be by automorphisms of the G-structure if it lifts to an action by principal bundle automorphisms of $\pi_M : P_M \to M$.

Of course, we recover in this way isometric/conformal/affine actions in the previous examples.

Prolongation procedure. Start with a G-structure of order 1 on M. An important step in Cartan's equivalence method is the so-called prolongation procedure (see [Ste61], [Slo96], or [Kob95]). It consists in associating to the G-structure $P \to M$ a higher-order structure, which we can see as a $G^{(1)}$-structure on the total space P itself, where $G^{(1)}$ is a Lie group algebraically defined by G. Automorphisms of $P \to M$ then canonically lift to automorphisms of this new structure $P^1 \to P$. The procedure can then be iterated, and gives rise to a tower of prolongations $P^{k+1} \to P^k \to \cdots \to P \to M$, where $P^{k+1} \to P^k$ has structural group $G^{(k)}$ which can be explicitly computed. An automorphism of $P \to M$ lifts to every stage of this tower.

Definition 2.4. A G-structure is said to be of *finite type* if there exists an integer $k \geqslant 1$ such that $G^{(k)} = \{e\}$.

See [Kob95] for details on this definition and examples. For instance, $O(p,q)$-structures are of finite order 1, $\mathrm{GL}_n(\mathbf{C})$-structures are of infinite order, $\mathrm{CO}(p,q)$-structure are of finite order (equal to 2) if and only if $p+q \geqslant 3$.

In particular, when the structure is of finite type k, automorphisms of G-structures of finite type act on the last stage $P^{(k)} \to P^{(k-1)}$ by preserving an $\{e\}$-structure, *i.e.* a global frame field on $P^{(k-1)}$, the most *rigid* geometric structure. This leads to the proof of the following.

Theorem 2.2. *Let $\pi : P \to M$ be a G-structure of finite type k. Then, the automorphisms group* $\mathrm{Aut}(P \to M)$ *has a unique Lie group structure, compatible with the compact-open topology, such that its action on M is smooth. Moreover, its dimension is bounded above by* $\dim M + \dim \mathfrak{g} + \cdots + \dim \mathfrak{g}^{(k)}$.

Elliptic G-structures. Another important family of G-structures are given by linear subgroups $G < \mathrm{GL}_n(\mathbf{R})$ whose Lie algebra $\mathfrak{g}$ contains no matrix of rank 1. These are called *elliptic*. Certainly, the most emblematic case is $G = \mathrm{GL}_n(\mathbf{C}) < \mathrm{GL}_{2n}(\mathbf{R})$, and corresponds to an almost-complex structure J on an even-dimensional manifold.

A remarkable property is that even-though elliptic G-structures are not of finite type in general, nor rigid in Gromov's sense, a *global* rigidity phenomenon happens.

Theorem 2.3 ([Och66])**.** *Let M be a compact manifold and let $P \to M$ be an elliptic G-structure. Then, its automorphisms group is a Lie transformation group.*

The proof in this non-locally rigid situation relies on the fact that the Lie algebra of (globally defined) Killing fields of the G-structure satisfies a certain system of elliptic PDE's, which forces it to be finite dimensional.

In particular, the group of biholomorphisms of a compact almost-complex manifold is a Lie transformation group (this was first proved in [BKW63]). Contrarily to Theorem 2.2, it is crucial to require compactness of M.

2.2.2 Cartan geometries

[Sha97, ČS09] are standard references. See also these recent notes of McKay [McK23].

Let G be a Lie group, H a closed subgroup, and $X = G/H$. Let M be a differentiable manifold with the same dimension as X. A Cartan geometry on M is a notion which can be thought as a curved version of the homogeneous space X, similarly to the idea that a Riemannian manifold is a curved version of the Euclidean space. It mimics the infinitesimal properties of the principal fibration $G \to X$.

Definition 2.5. A Cartan geometry on M, with model space (G, H), is the data of an H-principal fibration $\pi : \mathcal{G} \to M$, called the Cartan bundle, and a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, called the Cartan connection, such that:

1. For all $b \in \mathcal{G}$, $\omega_b : T_b\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism.
2. For all $A \in \mathfrak{h}$ and $b \in \mathcal{G}$, $\omega_b \left(\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} b.e^{tA} \right) = A$.
3. For all $h \in H$, $(R_h)^*\omega = \mathrm{Ad}(h^{-1})\omega$.

An isomorphism between two Cartan geometries $(M, \mathcal{G}_M, \omega_M)$ and $(N, \mathcal{G}_N, \omega_N)$ is a diffeomorphism $f : M \to N$, which can be lifted to a principal bundle automorphism $F : \mathcal{G}_M \to \mathcal{G}_N$ such that $F^*\omega_N = \omega_M$. Under a standard assumption on the model space, such lifts are uniquely determined, and we can consider without ambiguity isomorphisms to be defined either between the base manifolds or between the Cartan bundles.

The notion of Cartan geometry encompasses most classical notions of geometric structures in differential geometry.

Examples. Among others (I skip normalizing conditions on the Cartan connection):

1. A pseudo-Riemannian metric of signature (p, q) is a Cartan geometry with model space $X = \mathbf{R}^{p,q}$ with the transitive isometric action of $O(p, q) \ltimes \mathbf{R}^{p+q}$.
2. A linear connection on an n-manifold is a Cartan geometry with model space $X = \mathbf{R}^n$, with the transitive affine action of $\mathrm{GL}_n(\mathbf{R}) \ltimes \mathbf{R}^n$.
3. A pseudo-Riemannian conformal class $[g]$ of signature (p, q), with $p + q \geqslant 3$, is a Cartan geometry with model space $X = \mathbf{Ein}^{p,q}$, with the transitive action of $\mathrm{PO}(p + 1, q + 1)$. See **ref** for the definition, it is the analogue of the standard conformal n-sphere $\mathbf{S}^n$ with the action of the Möbius group $\mathrm{PO}(1, n + 1)$.

These geometries carry a notion of *curvature*, which coincides with the Riemann curvature tensor in the cases of linear connections or pseudo-Riemannian metrics. In general, the curvature is the 2-form on $\mathcal{G}$ defined by $\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$. A Cartan geometry is said to be flat if $\Omega = 0$, which means that the geometry is *locally isomorphic to the model space*. Hence, flat Cartan geometries are (G, X)-structures.

As for finite-type G-structures or Gromov's rigid A-structures, Cartan geometries are locally rigid, because the lift of any local automorphism preserves the Cartan connection, which defines a global framing on $T\mathcal{G}$. In particular, the automorphism group of a Cartan geometry modeled on (G, H) is a Lie transformation group, with dimension bounded above by $\dim G$. However, unless the curvature vanishes, and apart from this dimension bound, the Lie group structure has *a priori* no reason to be related to G.

2.2.3 Gromov's A-rigid structures

The notion of geometric structure in Gromov's sense extends the definition of G-structures. In this setting, a geometric structure on an n-manifold M is an equivariant map

$$\phi : \mathcal{F}^r(M) \to W,$$

where W is a manifold with an action of $D^r(n)$, but the image of ϕ is not required to lie in a single $D^r(n)$-orbit. Similarly to G-structures, an isomorphism between (M_1, ϕ_1) and (M_2, ϕ_2) is a diffeomorphism f such that $\phi_1 = \phi_2 \circ J^r(f)$.

Gromov developed in [Gro88] an impressive arsenal of results describing geometric structures from various perspectives (topological properties, orbits of the pseudo-group of local automorphisms, infinitesimal automorphisms..), under two main assumptions: 1) W is a smooth real variety on which $D^r(n)$ acts algebraically, 2) the geometric structure is rigid.

This notion of rigidity is an infinitesimal property, and its formal definition needs to consider "isometric r-jets". For the sake of readability, we can retain the following.

Intuitive definition: For $k \geqslant 1$, a geometric structure is said to be *k-rigid* if any local automorphism is determined by its k-jet $J^k_{x_0}(f)$ at any given point $x_0 \in M$. In particular, the Lie algebra of local Killing fields defined on any open subset is *finite dimensional*.

The general good definition builds on this idea but at infinitesimal scale.

Among many others, below are a few consequences of this theory.

1. Open-dense theorem. This theorem states that if the pseudo-group of local automorphisms a rigid geometric A-structure admits a dense orbit, then this orbit is open, *i.e.* the geometric structure is *locally homogeneous* when restricted to an open-dense subset. A famous application was found in [BFL93] in the classification of contact Anosov flows with smooth stable and unstable foliations. An elementary proof of the open-dense theorem for Cartan geometries and other geometric structures is established in [Pec16].

2. Representation of the π_1. This result states that in the analytic case, if a non-compact simple Lie group G acts on a rigid A-structure on a compact manifold M, by preserving a volume density, then there exists a linear representation of $\pi_1(M)$ whose Zariski closure contains a copy of G. Hence provides topological obstructions to simple Lie group actions.

3. Stratification by $\mathrm{Aut}^{\mathrm{loc}}$-orbits. Again in the analytic case, for a rigid A-structure on a compact manifold M, this theorem gives an analytic stratification $M = \Omega_1 \cup \cdots \cup \Omega_k$, invariant under local automorphisms, such that in each stratum, $\mathrm{Aut}^{\mathrm{loc}}$-orbits coincide locally with level sets of a map with constant rank. This is key in D'Ambra's theorem [D'A88], which has recently been extended by Melnick and me to the conformal setting [MP22].

Chapter 3

Lie group actions on conformal structures

In this chapter, I mainly discuss connected Lie group actions on pseudo-Riemannian manifolds.

3.1 History and motivations

A strong indication that Lie group actions by automorphisms of geometric structures shall be understandable is Zimmer's cocycle super-rigidity theorem, an extension of Margulis' super-rigidity theorem to ergodic actions of higher-rank semi-simple Lie groups. In fact, Margulis' theorem itself can be interpreted as a structure theorem for certain actions of semi-simple Lie groups. Hence this super-rigidity phenomena suggest that, to some extent, higher-rank semi-simple Lie group actions can be understood.

Throughout all this section, unless otherwise stated, G will always denote a connected, real, semi-simple Lie group without compact factor, with finite center and with real-rank at least 2. Also, Γ will denote an irreducible lattice subgroup of G.

3.1.1 Super-rigidity in terms of actions on principal fiber bundles and cocycles

Margulis' super-rigidity theorem works in the general context of product of algebraic groups defined over local fields of characteristic zero. For expository reason, I will stick to the real case, which is enough for the purpose of this text, optimal statements can be found in [Mar91, Zim84a].

Theorem 3.1. *Let G be a real semi-simple Lie group, with finite center and with $\mathrm{rk}_{\mathbf{R}}\, G \geqslant 2$. Let Γ be an irreducible lattice in G. Let H be a non-compact, simple, real algebraic group and let $\rho : \Gamma \to H$ be a group homomorphism whose image is Zariski dense in H. Then, there exists a Lie group homomorphism $\overline{\rho} : G \to H$ such that $\rho = \overline{\rho}|_{\Gamma}$.*

The hypothesis of Zariski density is not very restrictive, as for any lattice Γ and homorphism $\rho : \Gamma \to H$, the Zariski closure of $\rho(\Gamma)$ is always semi-simple (even for other fields than $\mathbf{R}$), [Mar91] Theorem 6.16.

Another (maybe more concrete) form of this theorem says that for any linear finite dimensional representation $\rho : \Gamma \to \mathrm{GL}_d(\mathbf{R})$, there exists a Lie group representation $\overline{\rho} : G \to \mathrm{GL}_d(\mathbf{R})$, a compact subgroup $K < \mathrm{GL}_d(\mathbf{R})$ centralizing $\overline{\rho}(G)$, and a group homomorphism $\rho_K : \Gamma \to K$ such that

$$\rho(\gamma) = \overline{\rho}(\gamma)\rho_K(\gamma)$$

for all $\gamma \in \Gamma$, *i.e.* ρ extends to a Lie group representation up to a compact noise. We will say that ρ *almost extends* to $\overline{\rho}$.

Remark 1. Margulis' theorem in fact also completely describes the compact valued homomorphism ρ_K.

A famous consequence of Theorem 3.1 and its p-adic version is the arithmeticity theorem of Margulis, asserting that all irreducible lattices of a higher-rank semi-simple Lie group are arithmetic (see Theorem 6.1.2 of [Zim84a] and the references therein).

G-actions on H-principal bundles. Let us take a moment to look at this theorem with a more geometric point of view. Fix $\Gamma < G$ a lattice subgroup. Then, the data of a group homomorphism $\rho : \Gamma \to H$ is the same as that of an H-principal bundle $\pi : P \to G/\Gamma$ together with a G-action by principal bundle automorphisms above the natural G action on G/Γ.

To see it, associate to any ρ the principal bundle $P^\rho := (G \times H)/\Gamma$, where Γ acts via $\gamma.(g, h) = (g\gamma, \rho(\gamma^{-1})h)$, on which G acts naturally on the left via the first coordinate. Conversely, to an H-principal fibration $\pi : P \to G/\Gamma$ with a principal G-action, and to any $x \in \pi^{-1}(e\Gamma)$, corresponds $\rho_x : \Gamma \to H$ defined by the relation $\gamma.x = x.\rho_x(\gamma)^{-1}$ for all $\gamma \in \Gamma$. A different choice of x yields a conjugate of ρ_x by an element of H, hence we can associate a conjugacy class of representations $\Gamma \to H$ modulo conjugacy by H. It is then straightforward to exhibit a G-equivariant isomorphism $\varphi : P \to P^{\rho_x}$ such that $\varphi(x) = (e, e).\Gamma$.

Now that the data of ρ is the same as that of a principal bundle $\pi : P \to G/\Gamma$ with a principal G-action, then how does the content of Margulis super-rigidity theorem translate?

First, the hypothesis of Zariski density of $\rho(\Gamma)$ is equivalent to assuming that there does not exist a non-trivial, G-invariant, principal reduction $P' \subsetneq P$ with structural group a proper algebraic subgroup $H' < H$. More generally, the notion of Zariski closure of $\rho(\Gamma)$ leads to the definition of *algebraic hull* of a G-action on a principal bundle with algebraic structural group, and here we assume the algebraic hull of the action to be the whole structural group H.

Now, a group homomorphism $\rho : \Gamma \to H$ extends to a Lie group homomorphism $\overline{\rho} : G \to H$ if and only if the associated H-principal bundle $P \to G/\Gamma$ admits a global trivialization $P \simeq G/\Gamma \times H$ in which the G-action reads

$$g.(x,h) = (g.x, \overline{\rho}(g)h), \tag{3.1}$$

for all $x \in G/\Gamma$ and $h \in H$.

So finally, Margulis super-rigidity theorem can be rephrased by saying that given an H-principal fibration $P \to G/\Gamma$, on which G acts by automorphisms above the natural action $G \curvearrowright G/\Gamma$, and "algebraically irreducibly" in the sense that no proper algebraic reduction of P is G-invariant, then the G-action can be "diagonalized" in some smooth global trivialization of P as in (3.1).

Theorem 3.1 is sufficiently strong to provide arithmeticity of all irreducible higher-rank lattices. It is therefore very appealing to investigate the consequences of its geometric interpretation.

Super-rigidity of cocycles In the early 1980, Zimmer extended the previous interpretation to a *non-homogeneous* setting, and proved that the same phenomenon persists when the G-action on G/Γ is replaced by an ergodic, measure preserving G-action on a probability space (X,μ). This more general version is formulated in the category of measurable spaces, in which any principal fibration is trivial. Consequently, the analogue of the principal action of G on $P \to G/\Gamma$ are G-actions on a direct product $P = X \times H$ which commute with the right-action of H on P and project to the initial action of G on X. These are all of the form $g.(x,h) = (g.x, c(g,x)h)$ where $c : G \times X \to H$ is a *cocycle* over the G-action on X.

Definition 3.1. Let G act on a Borel space (X,μ) and let H be a real algebraic group. An *H-valued cocycle* over the G-action is a measurable map $c : G \times X \to H$ such that for all $g_1, g_2 \in G$ and μ-almost every $x \in X$,

$$c(g_1 g_2, x) = c(g_1, g_2.x)c(g_2, x). \tag{3.2}$$

The relation (3.2) is called the *cocycle identity*.

Example 2. If X is a smooth manifold on which G acts differentiably, then, given a measurable trivialization $TX \simeq X \times \mathbf{R}^n$, for any $g \in G$ and $x \in X$, the differential $\mathrm{d}_x g$ is identified with an element $c(g,x) \in \mathrm{GL}_n(\mathbf{R})$, and the map $(g,x) \mapsto c(g,x)$ satisfies the cocycle identity by the chain rule.

Cocycles appear in many other natural contexts, some of which being described for instance in these introductive notes of Feres: [Fer02b].

Let (X,μ) be a probability space on which G acts measurably. Given a cocycle $c : G \times X \to H$, we can define a principal bundle action[1] of G on $P := X \times H$ by $g.(x,h) = (g.x, c(g,x)h)$. We have seen that conversely, all principal actions of G on P are of this form.

[1]That is: an action of G on $X \times H$ which projects to the action on X and commutes with the right H-action $(x,h).h' = (x, hh')$.

Analogously to Margulis' theorem, the conclusion of the cocycle super-rigidity theorem is that there exists a special *measurable* trivialization of P in which the cocycle associated to the G-action is of the simplest possible form: $c(g,x) = \rho(g)$ for μ-almost every $x \in X$, where $\rho : G \to H$ is a Lie group homomorphism. A change of trivialization of P is on the form $(x,h) \mapsto (x, \varphi(x)h)$, where $\varphi : X \to H$ is a measurable map. After this change, the cocycle transforms into

$$c'(g,x) = \varphi(g.x)^{-1} c(g,x) \varphi(x). \tag{3.3}$$

Definition 3.2. Two cocycles verifying (3.3) are said to be cohomologous.

We can now state Zimmer's cocycle super-rigidity.

Theorem 3.2. *Let G be a real semi-simple Lie group without compact factor, with finite center, and with $\mathrm{rk}_{\mathbf{R}} G \geqslant 2$. Let (X,μ) be a probability space on which G acts by preserving the measure μ. Let H be a real algebraic group and $c : G \times X \to H$ a measurable cocycle. Then, there exists a smooth homomorphism $\rho : G \to H$, a compact subgroup $K < G$ centralizing $\rho(G)$, and a measurable cocycle $c_K : G \times X \to K$ such that μ-almost everywhere*

$$c(g,x) = \rho(g) c_K(g,x).$$

Remark 2. Instead of group actions on measurable principal bundles, this theorem deals with measurable cocycles over a G-action, which a more efficient point of view, but it is exactly the same notion.

Remark 3. Margulis' super-rigidity theorem is obtained as a consequence of this result, as any group homomorphism $\rho : \Gamma \to H$ yields a cohomoglogy class of cocycles $c_\rho : G \times G/\Gamma \to H$, and $X = G/\Gamma$ is by definition equipped with a finite G-invariant measure.

Remark 4. There is another version of Theorem 3.2, which is the straight analogue of Theorem 3.1: if the algebraic hull of the cocycle is all of H (which corresponds to the Zariski density of $\rho(\Gamma)$ in Margulis' theorem), then the cocycle c_K can be made trivial.

3.1.2 Geometric structures as principal bundles

Many natural and familiar geometric structures can be interpreted as principal fiber bundles. An important class is that of H-structures and offers nice geometric consequences of super-rigidity results. References on the subject can be found in [Kob95, Ste61, CQB04].

Given a smooth n-manifold, we denote by $\mathcal{F}^r(M)$ the bundle of r-frames of M. A point of $\mathcal{F}^r(M)$ is the r-jet at some point $x \in M$ of a germ of local chart $\psi : (\mathbf{R}^n, 0) \to (U, x)$ defined on a neighborhood of x. Hence, $\mathcal{F}^r(M)$ has a natural structure of principal fiber bundle over M, with structural group $D^r = \{J_0^r(f),\ f : (\mathbf{R}^n,0) \to (\mathbf{R}^n,0) \text{ smooth germ}\}$. Note that for $r=1$, $D^r \simeq \mathrm{GL}_n(\mathbf{R})$ and elements of $\mathcal{F}^1(M)$ are linear frames.

Definition 3.3. Let M^n be a smooth n-manifold, $r \geqslant 1$ and $H < D^r$ a Lie subgroup. An H-structure on M is an H-reduction P of $\mathcal{F}^r(M)$. Equivalently, it is a global section $s : M \to \mathcal{F}^r(M)/H$. The integer r is called the order of the structure.

Example 3. For instance, the data of a pseudo-Riemannian metric of signature (p, q) on M is the same as that of an H-structure of order 1, with structure group $H = O(p, q) < \mathrm{GL}_n(\mathbf{R})$. Indeed, by tensoriality, the metric yields a $\mathrm{GL}_n(\mathbf{R})$-equivariant map $\mathcal{F}^1(M) \to \mathrm{GL}_n(\mathbf{R})/O(p, q)$. Various natural geometric structures (such as conformal, almost-complex or almost-symplectic structures) also fall into the setting of H-structures of order 1, see Chapter 1 of [Kob95].

Example 4. Another example is that of linear connections, or more generally projective classes of connections, which are typically non-tensorial, hence need higher order frames to be interpreted as equivariant maps (see [KN64]). For instance, a linear connection is the same as a $\mathrm{GL}_n(\mathbf{R})$-reduction of $\mathcal{F}^2(M)$.

Given two n-manifolds M and N, endowed with H-structures $P_M \to M$ and $P_N \to N$, a (local) isomorphism between them is a (local) diffeomorphism $f : M \to N$ such that $J^r(f)(P_M) \subset P_N$. In particular, the group of automorphisms of the H-structure $P_M \to M$ acts on on P_M by bundle automorphisms. Hence, any action of a group G on a geometric structure which is interpretable as an H-structure can be thought as a G-action on an H-principal fiber bundle over the base manifold, which in turn gives rise to a cohomology class of measurable cocycles $G \times M \to H$.

Therefore, when G is either a higher-rank semi-simple Lie group, or a lattice in such group, and when G preserves a finite measure on M, Theorem 3.2 yields interesting dynamical information. In [Zim84c], a first version of Zimmer's embedding theorem, recalled below, was proved in the higher-rank case as a consequence of super-rigidity of cocycles. Later, Zimmer extended in [Zim86c] this obstruction to Γ-actions on unimodular, *finite type*[2] H-structures. We recall the statement in the next chapter on lattices actions. Let us highlight the finite-type assumption, Zimmer's original conjectures in the volume-preserving case ([Zim87a]) claimed precisely that this assumption can be dropped.

3.1.3 Zimmer's embedding theorem

Definition 3.4. An H-structure on a manifold M is said to be unimodular if the linear part H_ℓ of H is contained in $\mathrm{SL}'_n(\mathbf{R}) = \{g \in \mathrm{GL}_n(\mathbf{R}) \ : \ \det g = \pm 1\}$.

Equivalently, unimodular H-structures are those which define naturally a volume density on the base manifold, and in particular a finite invariant measure when the manifold is compact. For instance, pseudo-Riemannian metrics ($H = O(p, q)$) or symplectic forms ($H = \mathrm{Sp}_{2n}(\mathbf{R})$) define unimodular H-structures, whereas conformal structures or linear connections do not.

Theorem 3.3 ([Zim86c])**.** *Let M^n be a compact manifold endowed with a unimodular H-structure $\pi : P \to M$, with H an algebraic subgroup of $\mathrm{GL}^{(r)}(n)$. Let G be a non-compact simple Lie group acting locally faithfullly on this H-structure.*

[2] In Cartan's sense, see [Kob95]. It essentially means that the structure is rigid, in the sense of having finite dimensional local groups of transformations.

Then, there exists a Lie algebra embedding $\iota : \mathfrak{g} \hookrightarrow \mathfrak{h} \subset \mathfrak{gl}^{(r)}(n)$. *Furthermore, if* $p : \mathfrak{gl}^{(r)}(n) \to \mathfrak{gl}^{(1)}(n) = \mathfrak{gl}_n(\mathbf{R})$ *is the natural projection, the representation* $p \circ \iota : \mathfrak{g} \to \mathfrak{gl}_n(\mathbf{R})$ *contains the adjoint representation of* $\mathfrak{g}$ *as a direct factor.*

The proof relies essentially on a generalization of Borel's density theorem, which asserts that if an algebraic non-compact simple group acts algebraically on a variety, by preserving a finite measure, then it acts trivially on the Zariski closure of the support of the measure.

Remark 5. It is worth noting that contrarily to the analogue statement for lattices stated later (Theorem 4.2), the H-structure here is *not* assumed to be of finite type, the conclusion follows for instance for $\mathrm{Sp}_{2n}(\mathbf{R})$-structures. However, the important point is that it is unimodular, the conclusions are not valid for conformal structures for instance ($H = \mathbf{R}_{>0} \times O(p,q)$ but $\mathrm{SO}(p+1,q+1)$ is the conformal group of $(\mathbf{S}^p \times \mathbf{S}^q, -g_{\mathbf{S}^p} \oplus g_{\mathbf{S}^q})$, and does not embed locally into H).

The following phenomenon is key in the proof of the embedding theorem, and illustrates in a simpler context a method that can be implemented more generally (see [BN02, BFM09]).

Proposition 3.1 ([Zim84c]). *Let* G *be a non-compact simple Lie group action by* $\mathcal{C}^1$ *diffeomorphisms of a manifold* M *and by preserving a finite measure* μ. *Then, for* μ-*almost every* x, x *is either a* G-*fixed point, or the stabilizer* G_x *is discrete.*

Proof. Consider the G-equivariant map $\Phi : x \in M \mapsto \mathfrak{g}_x \in W := \sqcup_{k \leqslant \dim G} \mathrm{Gr}_k(\mathfrak{g})$, where $\mathfrak{g}_x$ refers to the Lie algebra of the stabilizer of x. We have a natural algebraic action of $\mathrm{Ad}(G)$ on every stratum of W, which is a projective variety. The map Φ is G-equivariant, so the push-forward $\nu := \Phi_*\mu$ is $\mathrm{Ad}(G)$ invariant on W. The key idea, which can be fruitfully used in many more sophisticated contexts, is to apply the following.

Lemma 1. *Let* $H \curvearrowright V$ *be a real algebraic group action on a variety* V. *Suppose that* H *preserves a finite measure* μ. *Then, the restriction of the* H-*action to the Zariski closure* $V' := \overline{\mathrm{Supp}\,\mu}$ *has cocompact kernel.*

Applying this to any stratum of W which intersects the support of ν, we deduce that $\mathrm{Ad}(G)$ acts trivially on $\overline{\mathrm{Supp}\,\nu}$. This means that μ-almost every point of M is sent to an ideal of $\mathfrak{g}$, which is exactly the desired conclusion. □

Remark 6. In other contexts, the action can be shown to be locally free almost everywhere. Typically, for isometric actions on finite volume pseudo-Riemannian manifold, the G-action is linearized near any fixed point, and the set of G-fixed points has empty interior by rigidity. So, fixed points form always locally a submanifold with positive codimension. Hence, the set of G-fixed points is nowhere dense and the set of full Lebesgue measure in Proposition 3.1 can be chosen in the complement of G-fixed points.

Extension to Cartan geometries. Theorem 3.3 has been extended to non-simple Lie group actions on Cartan geometries in [BFM09].

Let H be a connected Lie group and let $S < H$ be a subgroup, not required to be closed. Following [Sha99], define the *discompact radical* of S as the largest algebraic subgroup $\overline{S}_d$ in the Zariski closure of $\mathrm{Ad}_{\mathfrak{h}}(S)$ which does not admit any proper, algebraic, normal, cocompact subgroup. For instance, $\overline{H}_d = \mathrm{Ad}_{\mathfrak{h}}(H)$ for $S = H$ an algebraic semi-simple Lie group of non-compact type. The idea is to apply Lemma 1 to some finite measure preserving algebraic action of $\overline{H}_d$ and conclude that it is trivial on the support of the measure.

Let $(M, \mathcal{G}, \omega)$ be a Cartan geometry with an effective model space $(\mathbf{G}, \mathbf{P})$. We denote by $\pi : \mathcal{G} \to M$ the fibration. Let $H < \mathrm{Aut}(M, \mathcal{G}, \omega)$ be a Lie subgroup. The corresponding H-action gives rise to a natural map $\iota : \mathcal{G} \to \mathrm{Mon}(\mathfrak{h}, \mathfrak{g})$, where $\mathrm{Mon}(\mathfrak{h}, \mathfrak{g})$ denotes the variety of injective linear maps from $\mathfrak{h}$ to $\mathfrak{g}$, defined by $\iota(b)(X) = \omega_b(X)$ for all $b \in \mathcal{G}$ and $X \in \mathfrak{h}$. Recall that by effectiveness of the model, we can define without ambiguity lifts to the Cartan bundle of infinitesimal automorphisms of the Cartan geometry and X can be seen as a vector field on M or as a right-P-invariant vector field on $\mathcal{G}$ (see [ČS09]).

Theorem 3.4 ([BFM09]). *Suppose that $\mathrm{Ad}_{\mathfrak{g}}(P)$ is almost algebraic in $\mathrm{GL}(\mathfrak{g})$. If S preserves a finite Borel measure μ on M, then for μ-almost every x, for every $b \in \pi^{-1}(x)$, there exists an algebraic subgroup $\check{S} < \mathrm{Ad}_{\mathfrak{g}}(P)$ such that*

- *for all $\check{p} \in \check{S}$, $\check{p}.\iota_b(\mathfrak{h}) = \iota_b(\mathfrak{h})$,*
- *the induced homomorphism $\check{S} \to \mathrm{GL}(\mathfrak{h})$ is algebraic, with image $\overline{S}_d$.*

In particular, this result can be applied systematically when M is compact and S is amenable, and as corollaries they obtained upper bounds on the real-rank of a semisimple Lie group action (take $S < G$ to be an $\mathbf{R}$-split Cartan subgroup in a semi-simple Lie group H) or the nilpotence degree of a nilpotent Lie group action. In more specific Cartan geometries, the idea can be implemented to provide further restrictions, as in [Pec19], see Theorem B and Theorem C below.

Isometric actions on pseudo-Riemannian manifolds. As an illustration of Theorem 3.3, let us observe that its conclusions are optimal in the case of pseudo-Riemannian isometric actions of simple Lie groups, *i.e.* for $r = 1$ and $H = O(p, q)$.

Let G be a non-compact simple Lie group acting isometrically on a closed pseudo-Riemannian manifold of signature (p, q). Zimmer's result implies that $\mathfrak{g}$ identifies with a Lie subalgebra of $\mathfrak{so}(p, q)$, and that there exists a $\mathfrak{g}$-invariant vector subspace $V \subset \mathbf{R}^{p,q}$ and a linear isomorphism $f : \mathfrak{g} \to V$ such that $f([X, Y]) = X.f(Y)$ for all $X, Y \in \mathfrak{g}$. By simplicity of $\mathfrak{g}$, this subspace V is either totally isotropic, or non-degenerate.

Conversely, given a non-compact simple Lie group G and a pair of non-negative integers (p, q), the existence of an embedding of $\mathfrak{g}$ into $\mathfrak{so}(p, q)$ and of such a subspace V yields that of a compact pseudo-Riemannian manifold of signature (p, q) on which G acts isometrically with discrete kernel. Indeed,

- either V is non-degenerate, and we obtain an $\mathrm{ad}(\mathfrak{g})$-skew-symmetric quadratic form of signature (p', q') on $\mathfrak{g}$, with $p' \leqslant p$ and $q' \leqslant q$, hence a bi-invariant pseudo-Riemannian metric of signature (p', q') on G ; or
- V is totally isotropic, and then $\dim \mathfrak{g} \leqslant \min(p, q)$, and the Killing form of $\mathfrak{g}$ has signature (p', q') with $p' \leqslant p$ and $q' \leqslant q$, and we obtain similarly a bi-invariant metric on G of signature (p', q').

In both cases, G acts isometrically (for instance) on a direct product $G/\Gamma \times \mathbf{T}^{p-p',q-q'}$, where Γ is a cocompact lattice of G and G/Γ is endowed with the G-invariant metric of signature (p', q') exhibited previously.

In conclusion:

Corollary 1. *Given a non-compact, simple Lie group G and a signature (p, q), the following are equivalent:*

1. *There exists a compact pseudo-Riemannian manifold (M, g) of signature (p, q) on which G acts isometrically and locally faithfully.*
2. *There exists a non-degenerate bi-invariant quadratic form*[3] *on $\mathfrak{g}$ of signature (p', q') with $p' \leqslant p$, $q' \leqslant q$, or $q' \leqslant p$ and $p' \leqslant q$.*

In particular, up to local isomorphism, $\mathrm{SL}_2(\mathbf{R})$ is the only simple, non-compact Lie group admitting an isometric action on some closed Lorentzian manifold ([Zim86c], Theorem 4.1). This is because $\mathfrak{sl}_2(\mathbf{R})$ is the only non-compact simple Lie algebra with Lorentzian Killing form.

3.1.4 Isometry groups of closed Lorentzian manifolds

Restrictions on isometric actions of nilpotent Lie groups are also derived in [Zim86c], which suggested that the whole Lie group structure of $\mathrm{Isom}(M, g)$ is understandable, for (M, g) a closed Lorentzian manifold. This was made concrete by several contributions of Adams, Stuck and, independently, Zeghib.

Theorem 3.5 ([AS97a, AS97b, Zeg98a, Zeg98b])**.** *Let (M, g) be a compact Lorentzian manifold and let $G = \mathrm{Isom}(M, g)_0$ denote the identity component of its isometry group. Then, its Lie algebra splits into a direct product $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{k} \oplus \mathfrak{a}$ where $\mathfrak{k}$ is the Lie algebra of a compact semi-simple Lie group, $\mathfrak{a}$ is abelian and $\mathfrak{s}$ is in the following list:*

1. $\mathfrak{sl}_2(\mathbf{R})$
2. $\mathfrak{heis}(2n+1)$, $n \geqslant 1$,
3. *Oscillator algebras, i.e. certain solvable extensions $\mathbf{R} \ltimes \mathfrak{heis}(2n+1)$, $n \geqslant 1$,*

[3]If $\mathfrak{g}$ is non-complex, $\mathrm{ad}(\mathfrak{g})$-skew-symmetric quadratic forms are scalar multiples of the Killing form. For $\mathfrak{g} = \mathfrak{h}^{\mathbf{R}}$ with $\mathfrak{h}$ a simple complex Lie algebra, the real and imaginary parts of the Killing form of $\mathfrak{h}$ span the space of $\mathrm{ad}(\mathfrak{g})$-skew-symmetric quadratic forms on $\mathfrak{g}$.

4. $\{0\}$.

Conversely, for any such Lie algebra $\mathfrak{g}$*, there exists a compact Lorentzian manifold whose isometry group admits* $\mathfrak{g}$ *as Lie algebra.*

See for instance §3.3 of [BGZ19] for definitions of oscillator algebras and their quadratic structures in a broader context. A noticeable point in this result is the absence of the affine algebra $\mathbf{R} \ltimes \mathbf{R}$, which means that an isometric action of the affine group $\mathrm{Aff}(\mathbf{R})$ of the real line always gives rise to a locally faithful action of $\widetilde{\mathrm{SL}}_2(\mathbf{R})$, see Theorem 3.9 below and the discussion.

Ideally, it would be very interesting to obtain similar understanding of automorphisms groups of geometric structures. This was suggested by Gromov with his "vague conjecture" on rigid geometric structures ([DG91] §0.8). With further motivations, this problem has always been driving my research interests.

3.1.5 Warped product structures in the presence of an isometric simple Lie group action

Isometric actions of $\mathrm{SL}_2(\mathbf{R})$ on closed Lorentzian manifolds can be realized as follows. Let $\mathbf{AdS}^3 = (\mathrm{PSL}_2(\mathbf{R}), g_K)$, where g_K refers to the bi-invariant Killing metric. This is an important Lorentzian space form, of sectional curvature -1, whose compact quotients have attracted strong interest , in particular via their relations with surface group representations into $\mathrm{PSL}_2(\mathbf{R})$ (see for instance [BBD+12], although the literature grew significantly afterwards).

The 3-dimensional case has the specificity of having a non-simple isometry group whose identity component is $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$, hence admits various sorts of discrete groups acting properly discontinuously. We are interested here in the most elementary version: quotients by subgroups of the form $\{\mathrm{id}\} \times \Gamma$, where Γ is a uniform lattice in $\mathrm{PSL}_2(\mathbf{R})$. These subgroups are centralized by a whole copy of $\mathrm{PSL}_2(\mathbf{R})$, which consequently acts on the quotient manifold by isometries, providing examples of isometric actions of $\mathrm{PSL}_2(\mathbf{R})$ on closed Lorentzian manifolds.

These examples are in fact the main ingredients to build such actions in general. In [Gro88], §5., it is proved that if $\mathrm{PSL}_2(\mathbf{R})$ acts on a closed Lorentzian manifold (M, g), then the action is locally free *everywhere*[4] , the ambient metric induces a scalar multiple of the $\mathbf{AdS}^3$ metric on all orbits, and that furthermore the orthogonal distribution to $\mathrm{PSL}_2(\mathbf{R})$-orbits is integrable with geodesic leaves. Finally, some cover of M is isometric to a warped product $\mathbf{AdS}^3 {}_\omega\!\times N$, where N is a Riemannian manifold and $\omega : N \to \mathbf{R}_{>0}$.

The same observation is generalized in theorem 5.3.E. of [Gro88] for an isometric action of a simple Lie group G on a closed pseudo-Riemannian manifold (M, g), provided that $\mathrm{rk}_{\mathbf{R}}\, G \geqslant \min(p, q) - \min(p', q')$, where (p, q) is the signature of (M, g) and (p', q') the signature of the Killing form of G. Also, in [QB06] and other related articles, Quirroga-Barranco described closed pseudo-Riemannian manifold with simple Lie group actions, under various assumptions such as topological transitivity.

[4] Recall Proposition 3.1 which only implied local freeness over a subset of full Lebesgue measure.

3.1.6 Induced action over flag manifolds and measurable projective factors

All previous results cover cases of probability-measure-preserving G-actions. Hence, they do not include important examples such as semi-simple Lie group actions on flag manifolds. More generally, natural directions to look at are dynamics on non-unimodular G-structures or parabolic Cartan geometries. Even though such actions are out of range of super-rigidity results, various advances have been established via different techniques including a generalization of Borel's density theorem as well as general structure results of Furstenberg about stationary measures of semi-simple Lie group actions.

A remarkable one is a general structure result up to measurable equivalence due to Nevo and Zimmer, which can be viewed as a counter-part to super-rigidity of cocycles, in the non-measure preserving case. Under a mixing assumption and for higher-rank G, it states that G-actions on probability spaces are measurably equivalent to standard examples of stationary actions built as a fibration over a flag manifold of G.

Example 5. Let $P < G$ be a parabolic subgroup, (X, ν) be a probability space on which P acts measurably and preserves ν. Then, we can form the associated space $Y = G \times_P X = (G \times X)/P$, where as usual, $(g, x) \sim (gp, p^{-1}.x)$. The G-action on the first coordinate induces an action of G on Y which projects to the action on G/P. Given an admissible[5] measure μ on G and a (G, μ)-stationary probability measure ν_0 on G/P, we can associate a (G, μ)-stationary measure on Y.

We reproduce below Nevo-Zimmer's existence result of measurable projective factor for higher-rank semi-simple Lie group actions.

Theorem 3.6 ([NZ99])**.** *Let G be a semi-simple Lie group with finite center, no compact factors and real-rank at least 2. Let μ be an admissible measure on G. Let (X, ν) be a probability space with a measurable μ-stationary G-action. Suppose that the action of the minimal parabolic subgroup P is mixing.*

Then, there is a parabolic subgroup $Q < G$, an ergodic Q-space (X_0, λ), where λ is a Q-invariant probability measure, such that (X, ν) is measurably and G-equivariantly isomorphic to the associated space $G \times_Q X_0$.

In particular, the conclusion says that there exists a measurable, G-equivariant map $\varphi : (X, \nu) \to G/Q$, called a *measurable projective factor.* The G-action on X preserves the measure ν if and only if $Q = G$.

More recently, this result has been reinforced and extended to lattices actions in [BRHW22] and independently in [BH21] within the framework of operator algebras, but the conclusions are always in the measurable category. A natural problem is then:

Question 1. *Starting from a smooth (G, μ)-stationary action on a compact manifold (M, ν), under which conditions the measurable isomorphism of [NZ99] can be made differentiable?*

[5]A probability measure whose support generates G as a semi-group and such that $\mu^{\otimes k}$ is absolutely continuous with respect to Lebesgue measure for some $k \geqslant 1$.

3.1.7 Actions on conformal structures and more general geometric structures

In [Zim87b], Zimmer proved that when a semi-simple Lie group G without compact factor acts on an H-structure $P \to M$ on a compact n-manifold M, with $H < \mathrm{GL}_n^{(r)}(\mathbf{R})$ algebraic, we always have the upper bound $\mathrm{rk}_{\mathbf{R}}\, G \leqslant \mathrm{rk}_{\mathbf{R}}\, H$. Remark that no invariant volume is assumed, so this applies to conformal actions for instance. Further works built on this observation, this is detailed later in Section 4.4.1 in the chapter on lattices actions.

Let us cite for now the following result by Bader, Frances and Melnick for actions on parabolic Cartan geometries ([BFM09], Theorem 1.5).

Theorem 3.7. *Let $(M, \mathcal{G}, \omega)$ be a Cartan geometry on a compact manifold M, modeled on a flag manifold $\mathbf{G}/\mathbf{P}$. Let $H < \mathrm{Aut}(M, \mathcal{G}, \omega)$ be a connected Lie subgroup. Let $\mathfrak{g}$ denote the Lie algebra of $\mathbf{G}$. Then,*

1. $\mathrm{rk}_{\mathbf{R}}(\mathrm{Ad}(H)) \leqslant \mathrm{rk}_{\mathbf{R}}(\mathbf{G})$ *;*
2. *If $\mathrm{rk}_{\mathbf{R}}(\mathrm{Ad}(H)) = \mathrm{rk}_{\mathbf{R}}(\mathbf{G})$, then M is isomorphic, as a Cartan geometry, to some quotient $\Gamma \setminus \widetilde{\mathbf{X}}$, for $\Gamma < \widetilde{\mathbf{G}}$.*

Specializing to semi-simple H, the first bound is in concordance with [Zim87b], and the geometric conclusions extend [BN02] and [FZ05]. Hence, actions of semi-simple Lie groups of *maximal real-rank* were completely understood, but the general problem of lower-rank actions was still unclear after these works.

3.2 $\mathrm{SL}_2(\mathbf{R})$-actions on conformal Lorentzian structures

Let (M, g) be a pseudo-Riemannian manifold. The conformal class of g is $[g] = \{e^{\sigma} g,\ \sigma \in \mathcal{C}^{\infty}(M)\}$, and a diffeomorphism $f : M \to M$ is said to be conformal if it preserves $[g]$, setwise. We call $(M, [g])$ a conformal structure and denote by $\mathrm{Conf}(M, [g])$ its group of conformal diffeomorphisms.

Of course, for any $g' \in [g]$, $\mathrm{Isom}(M, g') < \mathrm{Conf}(M, [g])$. Legitimately, we can ask: when does there exist conformal diffeomorphisms which are not isometries of some conformal metric?

Conformal structures admitting such conformal maps are called *essential* (see Definition 3.5). Let us recall a famous theorem of Ferrand, Obata and Schoen characterizing essential Riemannian conformal structures [Oba71, LF71, LF76, Fer96, Sch95]. In the compact case, [Oba71] solved a conjecture of Lichnerowicz about the essentiality of the identity component $\mathrm{Conf}(M, [g])_0$. Later, [LF71, LF76] extended Obata's result to the whole of $\mathrm{Conf}(M, [g])$, still for M compact. Finally, in [Fer96, Sch95] the result is ultimately extended to non-compact Riemannian manifolds.

Theorem 3.8 (Ferrand-Obata-Schoen)**.** *Let (M^n, g), $n \geqslant 2$, be a Riemannian manifold. If $\mathrm{Conf}(M, [g])$ is essential, then $(M, [g])$ is conformally diffeomorphic to either $\mathbf{S}^n$ or $\mathbf{R}^n$ with their standard Euclidean structures.*

This result generated a great amount of work in various directions, see [Mel21] for a survey. Noticeably, the result can, partly, be explained in dimension greater than 2 by the *rigidity* of conformal structures. It led Gromov to conjecture in [Gro88] that more generally, every rigid geometric structure on which a "large" group G acts by automorphisms are classifiable.

It turned out that the situation is certainly more complicated than what Gromov expected, even for very concrete geometric structures. Various works, including [KR97, Fra05, Fra15], showed that for pseudo-Riemannian signatures, no analogous statement can reasonably be expected.

However, in Lorentzian signature, although infinitely many distinct compact topologies, on which infinitely many distinct conformal structures are essential, all known examples are conformally flat. And a problem remains open, called the *Lorentzian Lichnerowicz conjecture*:

Conjecture 1. *Let $(M, [g])$ be a Lorentzian conformal structure on a compact manifold of dimension at least* 3*. If it is essential, then it is conformally flat.*

A weaker (still challenging) form of this conjecture ask the same question when the *identity* component $\mathrm{Conf}(M, [g])_0$ is essential, as was initially asked by Lichnerowicz.

Recall that conformal structures are all conformally flat in dimension 2. Importantly, if true, a proof of this conjecture will necessarily be based on the *global* dynamics of groups acting essentially and conformally on M, as it is locally false: for instance, Alekseevsky exhibited in [Ale85] infinitely many non-conformally flat, Lorentzian metrics g on $\mathbf{R}^n$ admitting a conformal vector field X, which is essential[6] on arbitrarily small neighborhoods of its singularities.

This global geometrico-dynamical problem is another important source of motivation for my investigations.

3.2.1 Some examples of closed Lorentzian manifolds with large conformal group

Let us first review some examples of compact Lorentzian manifolds with large essential conformal groups. Tori or $\mathbf{AdS}^3$-manifolds described in Section 3.1.5 are natural constructions, but the identity component of their conformal group are all inessential (this follows from Proposition 3.2).

1. *Einstein Universe.* A central object in conformal Lorentzian geometry is the Lorentzian Einstein Universe $\mathbf{Ein}^{1,n-1}$. It can be defined as the conformal boundary of the $(n+1)$-dimensional anti de Sitter space $\mathbf{AdS}^{n+1}$, just as the Möbius sphere $(\mathbf{S}^n, [g_{\mathrm{can}}])$ is the conformal boundary of the real hyperbolic space $\mathbf{H}^{n+1}_{\mathbf{R}}$.
   As a manifold, $\mathbf{Ein}^{1,n-1}$ is the smooth projective quadric $\mathbb{P}(\{q_{2,n} = 0\}) \subset \mathbf{R}P^{n+1}$ where, for $x \in \mathbf{R}^{n+2}$,
$$q_{2,n}(x) = -x_0^2 - x_1^2 + x_2^2 + \cdots + x_{n+1}^2.$$

[6] Meaning that $\mathcal{L}_X g' \neq 0$, for any $g' \in [g]$.

The degenerate metric induced by $q_{2,n}$ on the isotropic cone $\{q_{2,n} = 0\}$ descends to a Lorentzian conformal class on $\mathbf{Ein}^{1,n-1}$, conformally invariant under $\mathrm{PO}(2, n)$. When $n \geqslant 3$, a first manifestation of rigidity of conformal structures is that every local conformal map of $\mathbf{Ein}^{1,n-1}$ is obtained in this way. This is Liouville's theorem (see [CK83]). It is doubly covered by $(\mathbf{S}^1 \times \mathbf{S}^{n-1}, [-\mathrm{d}t^2 \oplus g_{\mathbf{S}^{n-1}}])$, hence has universal cover $\mathbf{R} \times \mathbf{S}^{n-1}$ with the lift of the conformal class.

Another point of view is that it is the *conformal compactification* of Lorentzian spaces of constant sectional curvature. In particular, Minkowski space-time $\mathbf{R}^{1,n-1}$ is conformally equivalent to an open-dense subset of $\mathbf{Ein}^{1,n-1}$, unique up to the $\mathrm{PO}(2, n)$-action. Such open subsets are called *Minkowski patches* and the conformal diffeomorphisms with $\mathbf{R}^{1,n-1}$ *stereographic projections*, similarly to the Euclidean case. A sensible divergence is the nature of infinity: in the Euclidean case, it consists in a single point (the pole of projection), in the Lorentzian case, it is a light-cone emanating from a point.

Importantly in what follows, as a $\mathrm{PO}(2, n)$-space, $\mathbf{Ein}^{1,n-1}$ is a flag manifold $\mathrm{PO}(2, n)/P$, where P is the stabilizer of a null-line. By a generalization of a theorem of É. Cartan, a general conformal Lorentzian structure $(M, [g])$ is a normalized Cartan geometry with model space $(\mathrm{PO}(2, n), \mathbf{Ein}^{1,n-1})$, hence is a special case of *parabolic geometry* ([ČS09]).

Finally, let us mention that this construction extends analogously to any pseudo-Riemannian signature (p, q).

2. *Lorentzian Hopf manifolds.* Consider the pointed Minkowski space $\mathbf{R}^{1,n-1} \setminus \{0\}$. For $\lambda > 1$, the infinite cyclic group $\langle h_\lambda \rangle$ generated by the homothetie $h_\lambda(x) = \lambda x$ acts freely properly discontinuously and conformally, hence the flat metric of $\mathbf{R}^{1,n-1}$ induces a conformally flat metric on $\langle h_\lambda \rangle \setminus (\mathbf{R}^{1,n-1} \setminus \{0\})$. The conformal group $\mathbf{R}^* \times O(1, n-1)$ descends to the quotient, and induces an action of $\mathbf{S}^1 \times O(1, n-1)$. This construction can be twisted by modifying the $\mathbf{Z}$-action of h_λ, and will reduce accordingly the size of the conformal group at the quotient.

   The projection of the null-cone is a single, compact, degenerate $O(1, n-1)$-orbit diffeomorphic to $\mathbf{S}^{n-2} \times \mathbf{S}^1$. All other $O(1, n-1)$-orbits are non-compact, and all their accumulation points are on the compact orbit.

3. *Kleinian Lorentzian examples.* In [Fra05], Frances constructed for every $g \geqslant 2$, a compact manifold, diffeomorphic to $(\mathbf{S}^1 \times \mathbf{S}^{n-2})^{\sharp g} \times \mathbf{S}^1$, endowed with a conformally flat Lorentzian structure $[g]$, and with a conformal essential action of $\mathrm{SL}_2(\mathbf{R})$. These are obtained by modding out an open domain of $\mathbf{Ein}^{1,n-1}$ by a Lorentzian Schottky group.

   Using Ehresmann-Thurston's principle, he proved that these structures can be deformed in such a way that they still admit a one-parameter subgroup $\{\varphi^t\}$ of esseêntial conformal maps.

3.2.2 Discreteness of stabilizers as a criterion for inessentiality

Recall that a general property of isometric actions of non-compact simple Lie groups is that they are locally free over an open-dense subset. For Lorentzian metric, they act locally freely on the whole manifold, it is the first step to obtain the local product structure described in Section 3.1.5.

An isometric action being a special case of conformal action, a very natural question is to detect when a conformal action can be reduced to an isometric one. For instance, for (M, g) a Riemannian manifold, a closed subgroup $H < \mathrm{Conf}(M, [g])$ fixes a metric in the conformal class if and only if it acts properly on M.

This is far from being completely understood in general, so general criteria for *essentiality* are interesting problems in themselves, or for further use. Let us remind:

Definition 3.5. Let (M, g) be a pseudo-Riemannian manifold. Let $H < \mathrm{Conf}(M, [g])$ be a subgroup. Then, H is said to be *inessential* if there exists a metric $g' \in [g]$ such that $H < \mathrm{Isom}(M, g')$. Otherwise, H is said to be *essential*.

Remark 7. The action of H is inessential if and only if it preserves a volume-density on M.

Below is a question that arises quickly after taking this definition.

Question 2. *Let $H_1 < H_2 < \mathrm{Conf}(M, [g])$ be subgroups. Suppose that H_1 has non compact closure*[7] *in $\mathrm{Conf}(M, [g])$ and that H_2 is essential. Then, is H_1 also essential ? If not, under which criteria (algebraic, dynamical..) can we be sure that it is?*

For instance, suppose that $H_2 = \mathrm{SL}_2(\mathbf{R})$ acts conformally and essentially on M. Does any non-elliptic one-parameter subgroup also act essentially?

The first results about conformal actions of simple Lie groups that I present are the following criteria of essentiality. By Theorem 3.5, any conformal action, on a closed Lorentzian manifold, of a non-compact simple Lie group not locally isomorphic to $\mathrm{SL}_2(\mathbf{R})$ must be essential. For Lie groups locally isomorphic to $\mathrm{SL}_2(\mathbf{R})$, we have the following, proved in [Pec18].

Proposition 3.2 ([Pec18])**.** *Let $(M, [g])$ be a closed Lorentzian conformal structure and let $S < \mathrm{Conf}(M, [g])$ be an immersed Lie subgroup locally isomorphic to $\mathrm{SL}_2(\mathbf{R})$. Then, the following are equivalent:*

1. *S acts locally freely everywhere.*
2. *S is inessential.*
3. *$\mathrm{Conf}(M, [g])_0$ is inessential.*

The implication 2. $\Rightarrow$ 1. was already know as recalled before. The non-trivial part in 2. $\iff$ 3. is that if S preserves a metric in the conformal class, then so does all of the identity component of $\mathrm{Conf}(M, [g])$. Hence, we get a concrete criterion to answer Question 2 for $H_1 = \mathrm{SL}_2(\mathbf{R})$ and $H_2 = \mathrm{Conf}(M, [g])_0$: if H_2 is essential, then H_1 is essential and it admits an orbit of dimension less or equal than 2.

[7] Otherwise, an elementary averaging argument provides an H_1-invariant metric.

3.2.3 Conformal flatness in the presence of an essential semi-simple Lie group

The main contribution of the work of my PhD and my first postdoc is the following local conformal flatness result for conformal semi-simple Lie group actions on closed Lorentzian manifolds. It gives a rigidity result for actions of $\mathrm{SL}_2(\mathbf{R})$ on conformal Lorentzian structure, completing Gromov's local product structure for isometric actions of $\mathrm{SL}_2(\mathbf{R})$ recalled in Section 3.1.5.

Theorem A ([Pec17, Pec18]). *Let $(M^n, [g])$, $n \geqslant 3$, be a closed manifold endowed with a conformal Lorentzian structure. Let G be a connected semi-simple Lie group. If G acts conformally and essentially on $(M, [g])$, then $[g]$ is conformally flat, i.e. near every point, there are coordinates in which the metric reads*

$$g = e^{\sigma}(-\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \cdots + \mathrm{d}x_n^2),$$

for some smooth function σ.

This result is not far from being optimal. As shown in Example 6 below, there are essential actions of $\mathrm{PSL}_2(\mathbf{R})$ on some *non-conformally flat, real-analytic* closed pseudo-Riemannian manifolds of signature $(3, n)$, $n \geqslant 4$. However, the question of an extension of Theorem A to signature $(2, n)$ is still open to my knowledge.

Levi factor of the conformal group

By Proposition 3.2, an equivalent formulation of this result is that given a closed Lorentzian manifold (M, g) of dimension at least 3, if the Levi factor of $\mathrm{Conf}(M, g)_0$ is non-compact, then the Lorentzian Lichnerowicz conjecture (Conjecture 1) is true.

From this local conclusion, we derive a conformal extension of theorem 4.1 of [Zim86c], where a non-compact simple Lie group acting isometrically on a closed Lorentzian manifold is proved to be necessary a finite cover of $\mathrm{PSL}_2(\mathbf{R}) \simeq O(1, 2)_0$.

We obtain here more possibilities for the Levi factor of $\mathrm{Conf}(M, [g])_0$.

Corollary 2. *Let (M^n, g), $n \geqslant 3$, be a compact Lorentzian manifold and let G be a connected semi-simple Lie group of non-compact type. If G acts conformally on $(M, [g])$, then G is locally isomorphic to a Lie group of the form $S \times K$, where K is a compact semi-simple Lie group and S is isomorphic to a semi-simple Lie subgroup of non-compact type of $O(2, n)$, i.e. S is locally isomorphic to one of the following :*

- $O(1, k)$, $2 \leqslant k \leqslant n$;
- $\mathrm{SU}(1, k)$, $1 \leqslant k \leqslant n/2$;
- $O(2, k)$, $k \leqslant n$;
- $O(1, k) \times O(1, k')$, *with* $k + k' \leqslant \max(n, 4)$.

Proof. A Lie algebra of conformal vector fields of an open subset of $\mathbf{R}^{1,n-1}$ is isomorphic to a Lie subalgebra of $\mathfrak{so}(2,n)$. □

Remark 8. Conversely, for any S in the previous list, there exists a compact quotient of $\widetilde{\mathbf{Ein}}^{1,n-1}$ whose conformal group is locally isomorphic to $S \times K$, for some compact Lie group K.

Remark 9. This corollary is recovered in [Pec19] without using conformal flatness of $(M,[g])$ by using ideas similar to [BN02]. Let G simple and non-compact act conformally on a closed Lorentzian manifold. Considering tangent spaces to G-orbits, we obtain a G-equivariant map $M \to \mathbb{P}(S^2\mathfrak{g}^*)$, similar to the map used in the proof of Proposition 3.2. Considering an Iwasawa decomposition $G = KAN$, and an AN-invariant probability measure on M, Lemma 1 gives existence of points at which AN is "virtually" in the stabilizer, *i.e.* geometrically, everything happens like if it was, although it could not be. From this, enough algebraic constraints on G follows to conclude that it must be in the list of Corollary 2. The approach is implementable in higher signature, and is started for rank 1 actions in [Pec19].

I gave two proofs to Theorem A. The first corresponds to the work of my PhD and is under the additional assumption of analyticity of the conformal structure. The second proof, a year later, extended the result to the general smooth category. It needed to overcome a significant difficulty compared to the first analytic proof: a major advantage of working with rigid real-analytic structures is that certain general results of Gromov on their local automorphisms become much stronger than in the $\mathcal{C}^\infty$ case, and as explained below, it seems that Gromov's results do not fit well with smooth, non-volume preserving actions of Lie groups. Recently, Frances made an intensive use in [Fra20] of methods related to Frobenius' theorem (see below) for smooth *isometric* actions of discrete groups on closed Lorentzian 3-manifolds, but again, in his context, the invariant volume is crucial.

Proof of Theorem A in the real-analytic setting ([Pec17])

This approach is similar to the proof of Gromov's centralizer theorem (see for instance [CQB03]), a key step in the proof of Gromov's representation of the fundamental group, see Theorem 5.2 below. In the context of a volume preserving action of a non-compact simple Lie group G preserving a rigid analytic geometric structure on a compact manifold M, the centralizer theorem guarantees the existence, for Lebesgue-almost every $x \in M$, of a Lie algebra $\mathfrak{h}$ of local Killing vector fields defined on a neighborhood U of x, centralizing every vector field of $\mathfrak{g}$, and such that $T_x(G.x) \subset \{X_x,\ X \in \mathfrak{h}\}$. Typically, if $M = G/\Gamma$ and the structure is the pseudo-Riemannian metric defined by the Killing form, then G acts *locally* on the right on M, in the sense that left-invariant vector fields of G define a Lie algebra of local Killing vector fields at the neighborhood of every point, which centralizes the Lie algebra of (globally defined) right-invariant vector fields.

The main idea of the analytic proof of Theorem A is that the technical tool used for exhibiting this "new" Lie algebra of local symmetries works for any rigid geometric structure, and that it would probably yield additional dynamical information to the G-action. The difference in this conformal context is the lack of a G-invariant volume, or even of a G-invariant finite measure, and classical proofs of Gromov's centralizer theorem weren't adaptable. The proof was reduced to actions of Lie group locally isomorphic to $\mathrm{SL}_2(\mathbf{R})$ by Proposition 3.2, so let $G = \mathrm{SL}_2(\mathbf{R})$ to simplify.

The main ingredient is the so-called "Frobenius' theorem", dealing with the question of local integration of infinitesimal isometries (§1. of [Gro88], see also [Ben97]). Melnick proved a version for Cartan geometries of this theorem in the analytic case (see Section 5.2.2 below for more details). Let $\pi : \mathcal{G} \to M$ denote the Cartan bundle associated to the analytic conformal structure $[g]$. Theorem 3.11 of [Mel11] gives the existence of an equivariant map $\phi : \mathcal{G} \to V$, where V is some vector space with an action of the structural group, such that for any $b \in \mathcal{G}$ and $u \in T_b\mathcal{G}$, $u.\phi(b) = 0$ if and only if there exists a local Killing vector field X of $\mathcal{G}$ defined near b and such that $X(b) = u$. By definition, this vector field projects to a local conformal vector field $\overline{X}$ defined near $\pi(b)$, and is (theoretically but not explicitly) completely determined by $X(b)$, which is in some sense its 2-jet at $\pi(b)$.

The strategy, first used by Melnick in her proof of Gromov's centralizer theorem for Cartan geometries in [Mel11], was to modify Bader-Frances-Melnick's proof of Zimmer's embedding theorem for Cartan geometries [BFM09], and include the map ϕ in their machinery. Considering $S < G$ the affine group, and a finite S-invariant measure μ on M, Furstenberg's generalization of Borel's density theorem [Fur76] implies the existence, at some point b projecting into the support of μ, of a tangent direction $u \in T_b\mathcal{G}$ annihilating the map ϕ, and provided also algebraic information on $\omega_b(u) \in \mathfrak{so}(2, n)$, where ω is the Cartan connection.

If the measure is chosen to be supported in a special S-invariant compact subset, the outcome was a local conformal vector field X, vanishing at $x = \pi(b)$, which is locally contracting, and whose flow is defined for all positive times and conjugate to $\mathrm{diag}(1, e^{-t}, \ldots, e^{-t}, e^{-2t})$. This local conformal vector fields is *not contained in the initial* $\mathrm{SL}_2(\mathbf{R})$*-action*, and vanishing of the Weyl tensor follows from considering both its dynamics and the original action of $\mathrm{SL}_2(\mathbf{R})$.

Proof of Theorem A in the general case ([Pec18])

The conclusions of Frobenius' theorem are true only over an *open-dense subset* for a smooth, rigid structure defined on a compact manifold (an elementary proof for Cartan geometries in the smooth case is performed in [Pec16]). This subset is in fact proved to be the *whole manifold* if the structure is assumed to be analytic. The main advantage of working in the real-analytic case was this specific point.

Removing this exceptional set with empty interior makes a big difference, especially for actions which do not preserve a volume. Their orbits often accumulate to a singular locus for the action (*e.g.* a low-dimensional closed orbit or something more complicated). So, if Frobenius' theorem does not work in some closed subset $F \subset M$ with empty interior, then it could unfortunately be that all G-orbits accumulate to F, a real difficulty for the dynamical approached which I used, since every finite S-invariant measure could be supported in F. If a measure with non-empty interior support was invariant, then the methods would extend but this is almost assuming the action inessential.

So, it seemed necessary to find a different angle to remove the analyticity assumption. Let G still denote $\mathrm{SL}_2(\mathbf{R})$. By Proposition 3.2, it follows that there always exist orbits of dimension 1 or 2. Technical works, combined with Frances-Melnick results [FM13] on normal forms of conformal vector fields, gave conformal flatness of the neighborhood of any 1-dimensional orbit. The main problem, where Frobenius theorem was used in the analytic case, was to prove that if the closure of a 2-dimensional orbit $G.x$ does not contain fixed points nor 1-dimensional orbits, then for some point $y \in \overline{G.x}$, $G.y$ is a closed, 2-dimensional degenerate orbit diffeomorphic to a 2-torus.

Considering the action of a hyperbolic one-parameter subgroup $\{h^t\} < G$, this reduced to proving that the corresponding conformal flow has a periodic orbit in some compact invariant subsets where it has no singularity. After that, the first return map h^{t_0} played the role of the local conformal vector field in the analytic proof.

The technical point to get the existence of a closed orbit of h^t was to show that for an ergodic measure supported in the compact subset in question, all its Lyapunov exponents are non-zero and with the same sign, thanks to the special form of its conformal distortion. Pesin's local stable manifold (see Theorem 4.5) is then a small piece of hypersurface, transverse to the direction of the flow. Using Poincaré's recurrence theorem, periodicity follows from a standard fixed point result in the stable manifold.

3.3 Extensions to higher signatures

Corollary 2 raises the general question of classification of semi-simple Lie group actions on compact pseudo-Riemannian manifolds. For a closed conformal structure $(M, [g])$ of signature (p, q), with $\dim M \geqslant 3$, if G is a semi-simple Lie group without compact factor acting conformally on M, then $\mathrm{rk}_{\mathbf{R}}\, G \leqslant \min(p, q) + 1$, and that if equality holds then M is a quotient of the universal cover of $\mathbf{Ein}^{p,q}$ by a discrete subgroup of conformal maps acting freely properly discontinuously, and in particular G is locally isomorphic to a Lie subgroup of $\mathrm{SO}(p + 1, q + 1)$. For G simple, this was proved in [BN02] (the geometric conclusion under a minimality assumption on the action) and [FZ05] (which removed the assumption). The semi-simple case follows from Theorem 1.5 of [BFM09].

Even though a conformal pseudo-Riemannian structure is rigid, assuming the existence of an essential non-compact simple Lie group of conformal transformations is not necessarily enough to force the geometry to be even locally equivalent to the model space $\mathbf{Ein}^{p,q}$, as the following example given in [Fra15] shows.

Example 6. The metric $\mathrm{d}x_1\mathrm{d}x_2 + \mathrm{d}x_3\mathrm{d}x_4 + x_3^2\mathrm{d}x_1^2 + g_{\mathbf{R}^{p-2,q-2}}$ on $\mathbf{R}^{p+q}$ is conformally invariant under any matrix of the form $\varphi_\lambda = \mathrm{diag}(e^{-\alpha+2\beta}, e^{3\alpha}, e^{2\alpha-\beta}, e^{3\beta}, e^{\alpha+\beta}, \ldots, e^{\alpha+\beta})$, and for any choice of $\lambda = (\alpha, \beta)$ such that $\alpha < \beta < \alpha/2 < 0$, the group $\langle \varphi_\lambda \rangle$ acts properly discontinuoulsy on $\mathbf{R}^{p+q} \setminus \{0\}$, yielding a compact conformal structure M_λ, diffeomorphic to $\mathbf{S}^1 \times \mathbf{S}^{p+q-1}$. This structure is non-conformally flat. Any diffeomorphism of $\mathbf{R}^{p+q}$ fixing the origin, which is conformal with respect to the above metric and normalizes φ_λ

descends to a conformal transformation of M_λ. In particular, $O(p-2,q-2)$ acts conformally on M_λ. Moreover, if the action was inessential, by Proposition 3.1, then it would be locally free over an open-dense subset of M, which is not the case here, proving that this is an essential conformal action of $O(p-2,q-2)$ on a compact, non-conformally flat, real-analytic pseudo-Riemannian manifold of signature (p,q).

Hence, there are quite large orthogonal groups (whose rank can go up to the maximal rank minus 3) which preserve a whole family of non-conformally flat compact structures of a given signature, and this seems to indicate that a general understanding of conformal actions of semi-simple Lie groups is out of reach.

3.3.1 Minimal metric index

Nonetheless, another natural approach is to determine, given a semi-simple Lie group without compact factor G (or any group in general), what is the *minimal metric index* k_G of a compact conformal structure on which G can act non-trivially. By metric index, we mean the integer $\min(p,q)$ where (p,q) denotes the signature of the metric, hence the dimension of its maximal isotropic subspaces. We note first that the adjoint representation of G always yields a conformal action of G on $\mathbf{Ein}^{p,q}$, where $(p+1,q+1)$ is the signature of the Killing form. Hence, $k_G \leqslant \min(\dim K, \dim G - \dim K) - 1$, where K denotes a maximal compact subgroup of G.

From this perspective, the results of [BN02, FZ05] mean that for a semi-simple Lie group G of real-rank r, the minimal metric index k_G is greater than or equal to $r-1$, and that when equality holds, the manifold is some quotient of $\widetilde{\mathbf{Ein}}^{p,q}$ with $\min(p,q) = r-1$.

In particular, if G is not locally isomorphic to a Lie subgroup of some $O(r,s)$ with $s \geqslant r$, then $k_G > r-1$ and it is interesting to determine this value. The adjoint representation gives an easily determined upper bound for k_G, and more generally, $k_G \leqslant k_G^0$, where

$$k_G^0 = \min\{k > 0 \mid \exists \ell \geqslant k,\ \exists \rho : \mathfrak{g} \hookrightarrow \mathfrak{so}(k+1,\ell+1)\}.$$

Comparably to Zimmer's conjectures, we can ask whether or not this inequality can be strict:

Question 3. *Does there exist a compact conformal structure* $(M,[g])$ *of signature* (p,q)*, with* $\min(p,q) < k_G^0$*, on which* G *acts conformally, with discrete kernel?*

Expectedly, the answer should be no, *i.e.* $k_G = k_G^0$. If it turns out to be the case, the next question is naturally: what can be said on the geometry of a conformal structure $(M,[g])$ of optimal metric index k_G on which G acts conformally? Is it necessarily, up to finite cover, conformal to some $\mathbf{Ein}^{p,q}$?

In any event, if non-homogeneous, the orbit structure of the G-action on M would certainly be interesting to analyze. In particular, it follows from Zimmer's cocycle super-rigidity that G cannot preserve any finite measure on M (recall that $\min(p,q) = k_G$). Therefore, considering a minimal compact G-invariant subset, [NZ02] can provide the existence of a *non-trivial* measurable projective factor.

I started investigating this question in [Pec19], where it is addressed for rank 1 simple Lie groups, somehow the opposite situation of [BN02, FZ05].

For $G = SO(1,k)$, $k \geqslant 2$, we of course have $k_G = 0$ since it acts on the Möbius sphere. For $G = \mathrm{SU}(1,k)$, $k \geqslant 2$, we have $k_G = 1 = k_G^0$ since $\mathrm{SU}(1,k) \hookrightarrow \mathrm{SO}(2,2k)$ and by Ferrand-Obata-Schoen's theorem, it cannot act conformally on a closed Riemannian manifold. For $\mathrm{Sp}(1,k)$ and $F_{4(-20)}$, the problem was less clear.

Theorem B. *Let $(M^n,[g])$ be a closed pseudo-Riemannian manifold of signature (p,q), with $n \geqslant 3$. Suppose that there exists $\rho : \mathrm{Sp}(1,k) \to \mathrm{Conf}(M,[g])$ a conformal action with discrete kernel. Then,*

1. $\min(p,q) \geqslant 3$ *;*
2. *If $\min(p,q) = 3$, then $(M,[g])$ is conformally flat. Moreover, any minimal, compact, G-invariant subset of M is a compact orbit conformally equivalent to $\mathbf{Ein}^{3,3k-1}$, on which $\mathrm{Sp}(1,k)$ acts via a Fefferman fibration.*

Remark 10. There is a natural way to embed $\mathfrak{sp}(1,k) \hookrightarrow \mathfrak{so}(4,4k)$, which then produces a *transitive* action of $\mathrm{Sp}(1,k)$ on $\mathbf{Ein}^{3,4k-3}$. This remarkable action is an analogue of the $\mathrm{SU}(1,n)$-action on $\mathbf{Ein}^{1,2n-1}$, seen as the Fefferman fibration over the CR sphere $\mathbf{S}^{2n-1}$. Here, we have a principal fibration $\mathrm{Sp}(1) \to \mathbf{Ein}^{3,4k-3} \to \mathrm{Sp}(1,k)/P$ over the boundary at infinity of the quaternionic hyperbolic space, and the $\mathrm{Sp}(1,k)$-action is by bundle automorphisms.

So, under a minimality assumption, this proves existence of a smooth projective factor $M \to G/P$ (to be compared with Theorem 3.6).

Theorem C. *Let $(M^n,[g])$ be a closed pseudo-Riemannian manifold of signature (p,q), with $n \geqslant 3$. Suppose that there exists $\rho : F_{4(-20)} \to \mathrm{Conf}(M,[g])$ a conformal action with discrete kernel. Then, $\min(p,q) \geqslant 9$.*

Remark 11. There exists a realization of the Lie algebra $\mathfrak{f}_{4(-20)}$ in $\mathfrak{so}(10,16)$, hence a locally faithful action of $F_{4(-20)}$ on $\mathbf{Ein}^{9,15}$.

The proofs in these theorems, as explained in Remark 9, build on [BN02]. Particularly encouraging is a comparison with Theorem 1 of [BN02]. It states that there exists a point x at which T_xM contains a totally isotropic of dimension $\mathrm{rk}_{\mathbf{R}}\, G - 1$. The rest of the proof essentially uses maximality of this isotropic subspace when $\mathrm{rk}_{\mathbf{R}}\, G = \min(p,q) + 1$.

In [Pec19], for G of real-rank 1 and restricted-root system of type $(BC)_1$ (*i.e.* $\{\pm\alpha, \pm 2\alpha\}$), it is proved that a point $x \in M$, T_xM contains a totally isotropic subspace of dimension $\dim \mathfrak{g}_{2\alpha}$[8], which readily gives the lower bound $\min(p,q) \geqslant \dim \mathfrak{g}_{2\alpha}$. This was optimal, except for $F_{4(-20)}$. It was a bit curious since this dimension relates naturally to the division algebra (complex, quaternionic, octonionic) over which the hyperbolic space admitting G/P as boundary is defined, so looked as very good candidate. But apparently, no analogue of the Fefferman fibration exist for the octonionic case.Technical

[8] *i.e.* of dimension 1 for $\mathrm{SU}(1,k)$, 3 for $\mathrm{Sp}(1,k)$ and 7 for $F_{4(-20)}$.

works at the Lie algebra level allowed to prove that the metric index has to be in fact at least 9, but the geometry of the critical case is still mysterious to me.

The next step is of course to continue this investigation for every semi-simple Lie group, and obtain an understanding similar to Corollary 1. For these types of Lie groups, we have seen that the initial control on the rank can be promoted to a control on the dimension of a certain restricted root space $\mathfrak{g}_{2\alpha}$. A more general relation seems plausible and very interesting.

3.4 The solvable radical of the conformal group of closed Lorentzian manifolds

The automorphisms group of a *rigid* geometric structure has a natural Lie group structure, and a standard problem is to determine, conversely, which Lie group can arise this way. Describing the identity component of these groups is already a challenging problem in itself, and seems more tractable in a first attempt.

The initial motivations from super-rigidity theorems naturally lead to consider semi-simple Lie group actions, hence to understand the "semi-simple part" of the identity component of these automorphisms groups. It is nevertheless tempting to consider the "solvable part" of the automorphisms groups of geometric structures, and see if similar phenomenon occur.

3.4.1 Brief review

Isometry groups of closed Lorentzian manifolds were originally seen as very suitable candidates for such program. In [Zim86c], it is proved that for (M, g) a closed Lorentzian manifolds, the nilradical of $\mathrm{Isom}(M, g)$ is at most 2-step nilpotent. Further works led ultimately to the full classification in [AS97a, AS97b, Zeg98a], up to local isomorphism, of the identity component $\mathrm{Isom}(M, g)$ of a general compact Lorentzian manifold (M, g). This was recalled in Section 3.1.4.

Let G be the identity component of $\mathrm{Isom}(M, g)$. As any Lie algebra, its Lie algebra decomposes into a semi-direct product $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ where $\mathfrak{r}$ is the solvable radical of $\mathfrak{g}$ and $\mathfrak{s}$ is a semi-simple Lie subalgebra, *i.e.* a section of $\mathfrak{g} \to \mathfrak{g}/\mathfrak{r}$. This is the Levi decomposition of $\mathfrak{g}$. Then, the structure results of Adams-Stuck and Zeghib read:

- The semi-direct product is direct.
- If $\mathfrak{s}$ is non-compact[9], then $\mathfrak{s} \simeq \mathfrak{sl}_2(\mathbf{R}) \oplus \mathfrak{k}$ where $\mathfrak{k}$ is semi-simple of compact type, and $\mathfrak{r}$ is abelian.
- If $\mathfrak{r}$ is non-abelian, then $\mathfrak{r} \simeq \mathfrak{heis}(2d+1) \oplus \mathbf{R}^k$ or $\mathfrak{r} \simeq (\mathbf{R} \ltimes \mathfrak{heis}(2d+1)) \oplus \mathbf{R}^k$.

[9]This shortcut means that any Lie group, for instance $\mathrm{Aut}(\mathfrak{s})$, admitting $\mathfrak{s}$ as a Lie algebra is non-compact, or equivalently that the Killing form of $\mathfrak{s}$ is not negative definite.

It follows from their work that if a solvable Lie group G acts isometrically on a closed Lorentzian manifold, then $\mathfrak{g}$ is isomorphic to a direct product $\mathbf{R}^k \oplus \mathfrak{g}_0$, where $\mathfrak{g}_0$ is either an Heisenberg Lie algebra $\mathfrak{heis}(2d+1)$, an oscillator Lie algebra $\mathbf{R} \ltimes \mathfrak{heis}(2d+1)$ or the affine Lie algebra of the line $\mathfrak{aff}(\mathbf{R})$.

Concerning the last case of the affine algebra, in [AS97b] and [Zeg98a], the authors established the following remarkable rigidity result:

Theorem 3.9. *Suppose that the affine Lie group* $\mathrm{Aff}(\mathbf{R})$ *acts by isometries on a closed Lorentzian manifold* (M,g). *Then, its action extends to an* $\widetilde{\mathrm{SL}}_2(\mathbf{R})$ *action, up to a central factor, i.e. there exists a locally faithful, isometric action of* $\widetilde{\mathrm{SL}}_2(\mathbf{R})$ *and an isometric action of* $\mathbf{R}$ *(possibly trivial) such that the initial action comes from an embedding* $\mathrm{Aff}(\mathbf{R}) \hookrightarrow \widetilde{\mathrm{SL}}_2(\mathbf{R}) \times \mathbf{R}$.

This result gives an echo to a theorem of Ghys [Ghy85] asserting that if a closed 3-manifold admits a (say smooth) locally-free action of $\mathrm{Aff}(\mathbf{R})$, preserving a $\mathcal{C}^0$ volume-form, then the action is smoothly conjugate to an homogeneous action of $\mathrm{Aff}(\mathbf{R})$ on G/Γ, where G is either $\widetilde{\mathrm{SL}}_2(\mathbf{R})$ or SOL. In [Asa12], Asaoka classified locally-free actions of $\mathrm{Aff}(\mathbf{R})$ and obtained as corollary that the invariant volume cannot be removed in Ghys' result, as there exist non-homogeneous, locally free actions actions of $\mathrm{Aff}(\mathbf{R})$. By analogy, it can be asked what remains from Theorem 3.9 for conformal essential action of the affine group, which do not preserve a volume form neither.

For higher signatures, a wide range of possibilities are open for solvable Lie group actions and it seems quite difficult to achieve a description as sharp as in the Lorentzian case. Nonetheless, several advances have been established in the homogeneous case, for transitive isometric actions of solvable Lie groups on pseudo-Riemannian manifolds of finite volume, see [BGZ19] and references therein.

3.4.2 Embedding result for the radical

Let $(M,[g])$ be a compact Lorentzian structure. Similarly to what has been exposed above, a strategy to understand $\mathrm{Conf}(M,[g])$ is to consider a Levi decomposition $\mathfrak{s} \ltimes \mathfrak{r}$ of its Lie algebra, classify the possibilities for $\mathfrak{s}$ and $\mathfrak{r}$, and finally reconstruct the semi-direct product.

Corollary 2 classifies the possible Levi factors. For the radical R of $\mathrm{Conf}(M,[g])_0$, the following result for general solvable Lie group actions, combined with Theorem E below, gives an optimal obstruction.

Theorem D ([Pec23])**.** *Let* $(M^n,[g])$, $n \geqslant 3$, *be a compact manifold endowed with a Lorentzian conformal structure, and let* R *be a connected, solvable Lie subgroup of* $\mathrm{Conf}(M,[g])$. *If* R *is essential, then there exists a Lie algebra embedding* $\mathfrak{r} \hookrightarrow \mathfrak{so}(2,n)$.

Recall that $\mathrm{PO}(2,n) = \mathrm{Conf}(\mathbf{Ein}^{1,n-1})$, so every Lie subgroup of $\mathrm{PO}(2,n)$ acts faithfully and conformally on at least one closed Lorentzian n-manifold. This result supports Conjecture 1 because if true, the latter would imply theorem as an immediate consequence (just as Corollary 2 is a consequence of Theorem A).

Applying Theorem D to the solvable radical R of $\mathrm{Conf}(M,[g])_0$, and Theorem E below, we conclude that for any closed Lorentzian manifold (M,g), if $\mathrm{Conf}(M,[g])_0$ is essential, then, up to local isomorphism, it is a semi-direct product of a compact semi-simple Lie group with an immersed subgroup of $O(2,n)$.

Any semi-simple Lie subgroup of $O(2,n)$ can be realized, up to local isomorphism, as the conformal group of some quotient $\Gamma \setminus \widetilde{\mathbf{Ein}}^{1,n-1}$. This raises the following problem (to be compared with Theorem 3.9).

Question 4. *Which solvable Lie subgroups of $O(2,n)$ can be realized as the radical of the conformal group of some closed Lorentzian n-manifold?*

For instance, for any $k \leqslant n-2$, the Heisenberg group H_{2k+1} of dimension $2k+1$ embeds in $O(2,n)$, but to my knowledge, it is not clear if we can construct compact Lorentzian manifolds with a conformal group of the form $K \ltimes H_{2k+1}$ for some compact semi-simple Lie group K.

3.4.3 Criteria for essentiality

Question 2 can be addressed in the present situation of solvable Lie group actions. Recall that it asked, given two conformal groups $H_1 < H_2$, if the existence of an H_1-invariant metric in the conformal class implies that of an H_2-invariant metric.

Theorem E ([Pec23])**.** *Let (M,g) be a closed Lorentzian manifold and let R be a solvable Lie subgroup of $\mathrm{Conf}(M,[g])$. Let N be the nilradical of R. If N is inessential, then so is R.*

Furthermore, when N is non-abelian, then its essentiality is characterized by that of N_k, the last non-zero term of its lower-central series.

Corollary 3 ([Pec23])**.** *Let G the identity component of $\mathrm{Conf}(M,[g])$. Let $R \lhd G$ be its solvable radical and let $N \lhd R$ be the nilradical. If G/R is compact, then G is essential if and only if N is essential.*

Lorentzian manifolds for which G/R is non-compact and G is essential are conformally flat by [Pec18]. The fact that their holonomy centralizes a non-compact simple Lie subgroup of $\mathrm{PO}(2,n) = \mathrm{Conf}(\mathbf{Ein}^{1,n-1})$ seems to be an indication that they are classifiable up to conformal equivalence, justifying our assumption.

3.4.4 Nilpotent Lie group actions

By Corollary 3, if the identity component $\mathrm{Conf}(M,[g])_0$ is essential and has compact Levi factor, then some nilpotent subgroup of $\mathrm{Conf}(M,[g])_0$ is also essential.

Frances and Melnick proved in [FM10] that if a nilpotent Lie group N, with nilpotence degree k, acts conformally on a closed pseudo-Riemannian manifold of signature (p,q), then $k \leqslant 2\min(p,q)+1$ and that if equality holds, the manifold is a quotient of the universal cover of $\mathbf{Ein}^{p,q}$ by a cyclic group. Apart from this result, to the best of my knowledge, most results about solvable Lie group actions in pseudo-Riemannian geometry concern *isometric actions* and not much is known about their conformal essential actions, even in Lorentzian signature (see nonetheless [BDRZ23] in the general homogeneous case and the announced results therein).

The last result from [Pec23] I would like to cite deals, in Lorentzian signature, with nilpotent Lie groups with no restriction on the nilpotence degree. The convention used below is that for a nilpotent Lie algebra $\mathfrak{n}$, its nilpotence degree is the smallest integer $d \geqslant 1$ such that $\mathfrak{n}_d = \{0\}$, where we denote by $\{\mathfrak{n}_i\}_{i\geqslant 1}$ the lower central series of $\mathfrak{n}$.

Theorem F. *Let H be a connected nilpotent real Lie group of nilpotence degree $k+1$ and let (M^n, g), $n \geqslant 3$, be a compact Lorentzian manifold. Let H act locally faithfully by conformal transformations of M. Then, we have the following.*

1. *Assume that H is abelian.*
   (a) *Then H acts locally freely on an open-dense subset of M, hence $\dim H \leqslant n$.*
   (b) *If H is essential, then it admits either a fixed point, or an isotropic 1-dimensional orbit.*
   (c) *If $\dim H = n$ or $H \simeq \mathbf{R}^{n-1}$ and if H acts faithfully and essentially, then an open subset of M is conformally flat.*
2. *If H is non-abelian, then it is inessential if and only if H_k acts locally freely.*
3. *If H is non-abelian and essential, then an open subset of M is conformally flat. Precisely, $\mathfrak{h}$ has nilpotence degree $k \leqslant 3$, $\dim \mathfrak{h}_k = 1$ and if $X \in \mathfrak{h}_k \setminus \{0\}$, then X has a singularity of order 2.*

Remark 12. By [FM10], the result is new for $k = 1$ or 2. The proof is independent.

Note that we recover inessentiality criteria in terms of discreteness of stabilizers, similarly to what was observed for $\mathrm{SL}_2(\mathbf{R})$-actions, except for actions of $\mathbf{R}$. For a simple reason:

Example 7. Consider the 3-dimensional Hopf manifold $(M,[g]) = \langle 2\,\mathrm{id}\rangle \setminus (\mathbf{R}^{1,2} \setminus \{0\})$. Let $\{u^t\}$ be a unipotent one-parameter subgroup of $O(1,2)$ and let $\{k^t\}$ be the one induced by the homothetic flow on $\mathbf{R}^{1,2}$ (it factorizes into an $\mathbf{S}^1$-action). Then, the commutative product $\{u^t k^t\}$ is an essential conformal flow with no singularity.

For most solvable Lie groups, acting locally freely on a compact Lorentzian conformal structure is a sufficient condition for inessentiality (combine Theorem E and F). The main remaining case is for actions of the affine group.

Question 5. *Suppose that the affine group $\mathrm{Aff}(\mathbf{R})$ acts locally freely and conformally on a closed Lorentzian manifold. Is the action inessential?*

Remark that if yes, then it would extend (up to a central factor) to an action of $\widetilde{\mathrm{SL}}_2(\mathbf{R})$ by Theorem 3.9.

Concerning Conjecture 1, Frances and Melnick recently proved in [FM21] that a compact, real-analytic, Lorentzian 3-manifold such that $\mathrm{Conf}(M,[g])_0$ is essential is conformally flat. In particular, analyticity reduces the proof to conformal flatness of an open subset.

Under the same analyticity assumption, we deduce from what precedes the following in arbitrary dimension.

Corollary 4. *Let (M,g) be a closed real-analytic Lorentzian manifold and suppose that $G = \mathrm{Conf}(M,[g])_0$ is essential. If the nilradical N of G is non-abelian, then (M,g) is conformally flat.*

Somehow, this suggests that Conjecture 1 is probably reducible to the abelian case, *i.e.* that either $(M,[g])$ is conformally flat, or its conformal group is locally isomorphic to a semi-direct product $K \ltimes \mathbf{R}^k$, with K a compact group. The main problem would be to prove that if $\mathrm{Conf}(M,[g])$ contains an essential subgroup isomorphic to a semi-direct product $\mathbf{R} \ltimes_{\varphi} \mathbf{R}^k$, with $\varphi(\mathbf{R})$ non relatively compact in $\mathrm{GL}(\mathbf{R}^k)$, then $(M,[g])$ is conformally flat. If the Jordan decomposition of $\varphi(\mathbf{R})$ has non-trivial unipotent component, then known methods apply. So, we are essentially left with $\mathbf{R}$-split semi-direct products, with, once more, the affine group as the ultimate case to handle.

Chapter 4

Actions of lattices and the Zimmer program

4.1 A brief history of the Zimmer program

4.1.1 Cocycle super-rigidity and its measurable conclusions

Zimmer's conjectures emerged in the early 1980's and were initially mainly motivated, and supported, by cocycle super-rigidity. This super-rigidity result itself arose when Zimmer worked on the problem of measurable orbit equivalence of semi-simple Lie groups, which also can be interpreted in terms of cocycles [Zim80]. Contrarily to amenable group actions which were proven to be all orbit equivalent shortly before (by results of Dye and Connes-Feldman-Weiss), super-rigidity of cocycles proved that two ergodic, probability measure preserving, free actions of higher-rank semisimple Lie groups are orbit equivalent if and only if the groups are isomorphic and the actions automorphically conjugate.

As recalled in Section 3.1.1, cocycle super-rigidity can be viewed as a structure result for semi-simple Lie group actions on principal fiber bundles. It extends to probability measure preserving actions of lattices via a standard procedure called induction (see Definition 4.2 below), and modulo ergodic assumptions on the action, strong dynamical information are derived for the initial action of the lattice. The introduction of [FM03] gives an account of the different variations by various people (including Lewis, Lifchitz, Margulis, Stuck, Venkatarama and Zimmer) around super-rigidity which preceded their generalization. The version of Fisher-Margulis has the advantage of removing the assumption on the algebraic hull of the action, the price to pay being to have a conclusion modulo compact-valued cocycles.

As before, we formulate below a restricted version of cocycle super-rigidity to the real setting. Let (X, μ) be a probability space and $\Gamma \curvearrowright X$ be a probability measure preserving action. Let H be a real algebraic group and let $c : \Gamma \times X \to H$ be a measurable cocycle. An integrability condition is needed in the result, in order to apply Oseledet's theorem. As in the definition of Lyapunov exponents, it is automatic if c is obtained from a $\mathcal{C}^1$ action of Γ on a compact manifold.

Definition 4.1. A cocycle $c : \Gamma \times X \to H$ is said to be integrable if for any finite set $S \subset \Gamma$, the function $\{x \in X \mapsto \sup_{\gamma \in S} \|c(\gamma, x)\|\}$ is in $L^1(X, \mu)$.

Theorem 4.1 ([FM03])**.** *Let G be a semi-simple Lie group without compact factor, finite center, and real-rank at least 2. Let $\Gamma < G$ be an irreducible lattice and let H be a real algebraic group. Let (X, μ) be a probability space and $\Gamma \curvearrowright X$ an ergodic, probability measure preserving action. Let $c : \Gamma \times X \to H$ be an integrable cocycle.*

Then, there exists a Lie group homomorphism $\rho : G \to H$, a compact subgroup $K \subset H$ which centralizes $\rho(G)$, and a K-valued cocycle $c_K : \Gamma \times X \to H$ such that for μ-almost every $x \in X$ and for all $\gamma \in \Gamma$,

$$c(\gamma, x) = \rho(\gamma) c_K(\gamma, x).$$

Theorem 4.1 implies that for an action of Γ on an H-principal bundle P over a probability measure preserving smooth action of Γ on (M, μ), for say M a compact manifold, there exists a measurable global section $\sigma : M \to P$, and ρ, K, c_K as in the theorem, such that $\gamma.\sigma(x) = \sigma(\gamma.x).\rho(\gamma)c_K(\gamma, x)$ for all $\gamma \in \Gamma$ and for μ-almost every $x \in X$.

Recall that for a differentiable action of Γ on an n-manifold M, any measurable frame field $\sigma : M \to \mathcal{F}^1(M)$ defines a cocycle $c : \Gamma \times M \to \mathrm{GL}_n(\mathbf{R})$ called the *derivative cocycle.* More generally, geometric actions of Γ on M are by definition those which can be lifted to some action by principal bundle automorphisms, and similarly, any global framing of the principal bundle defining the geometry gives rise to an H-valued cocycle, for H the structural group.

Despite of the apparent "flexibility" of its measurable conclusions, for various differentiable actions of Γ, super-rigidity of cocycles has several "rigid applications", all of them saying that dynamical properties of the action of Γ are ruled by a certain linear representation of the Lie algebra $\mathfrak{g}$.

Let us quote:

Corollary 5. *Let $\Gamma \curvearrowright (M, \mu)$ be a smooth, ergodic, probability measure preserving action of Γ on a compact n-manifold M.*

Then, there exists a Lie group homomorphism $\rho : \widetilde{G} \to \mathrm{GL}_n(\mathbf{R})$ such that for every $\gamma \in \Gamma$, the Lyapunov spectrum of γ, seen as a diffeomorphism of M, is $\{\log |\lambda_k|,\ k = 1 \dots n\}$, where $\lambda_1, \dots, \lambda_n$ are the complex eigenvalues of $\rho(\widetilde{\gamma})$ with multiplicities, for any $\widetilde{\gamma}$ projecting to γ.

Interestingly, if G is large compare to $\dim M$, in the sense that every Lie group homomorphism $\widetilde{G} \to \mathrm{GL}_n(\mathbf{R})$ is trivial, then for every Γ-invariant probability measure μ, the Lyapunov exponents of all elements of Γ are zero. In fact:

Corollary 6. *Suppose that Γ acts on an H-structure $P \to M$ and preserves a probability measure μ on the base. If $\mathfrak{g}$ does not embed into $\mathfrak{h}$, then there exists a Γ-invariant, measurable, Riemannian metric on M.*

For instance, for $\Gamma = \mathrm{SL}_n(\mathbf{Z})$, any probability measure preserving $\mathcal{C}^1$-action of Γ on a compact $(n-1)$-manifold automatically preserves a measurable Riemannian metric, because there is no non-trivial homomorphism $\mathfrak{sl}_n(\mathbf{R}) \to \mathfrak{gl}_{n-1}(\mathbf{R})$.

Another interesting observation, with a similar flavor, follows from [Zim90] for co-compact Γ, later generalized to the non-uniform case, which we do not detail here but says essentially that the algebraic hull of a lattice is reductive with compact center (this relates with Fisher-Margulis' version). Considering the action of Γ on the bundle of unimodular 2-frames, whose structural group is of the form $\mathrm{SL}_n(\mathbf{R}) \ltimes U$, with U a unipotent group, it follows:

Corollary 7. *Suppose that Γ acts smoothly on a closed n-manifold M and preserves a volume form ω. Then there exists a Γ-invariant measurable connection.*

Recall that a linear connection on a manifold M is an H-structure of order 2 (see Section 2.2.1), hence a section of a certain fiber bundle over M. A measurable connection is then a measurable section in this point of view.

4.1.2 Zimmer's conjectures

Somehow, the main purpose of the Zimmer program is to upgrade the conclusions that can be derived from his machinery from measurable to differentiable. For instance, if it is proven that the Γ-invariant Riemannian metric in Corollary 6 is in fact regular enough, then it means that the action $\alpha : \Gamma \to \mathrm{Diff}(M^n)$ has in fact range into a *compact Lie subgroup* of dimension at most $\frac{n(n+1)}{2}$ by Myers-Steenrod Theorem, from which finiteness of the action follows by Margulis' super-rigidity.

The idea of starting with a measurable object produced by a group action and prove that it is in fact much more regular is at the core of many proofs of rigidity results, and gives more motivation to Zimmer's conjectures. The proof of Margulis' super-rigidity notably starts with the construction a Γ-equivariant measurable map from the Furstenberg boundary to some projective space on which Γ acts, and use it to build the smooth extension of the homomorphism $\Gamma \to H$.

As originally formulated (see [Zim87a, Zim86a]), the conjectures dealt with *volume preserving* actions of lattices, by analogy with the homogeneous setting: G/Γ with the Haar measure is replaced by a (M, ω) with M compact and ω a volume density.

Conjecture 2. *Let Γ be a lattice in a semi-simple Lie group G with finite center and all of whose simple factors are of real-rank at least 2. Let M be a compact manifold. Let M be a compact n-manifold and let $P \to M$ be an H-structure of ordrer 1, with $H < \mathrm{GL}_n(\mathbf{R})$ algebraic. Suppose that Γ acts on M by preserving both a volume density and the H-structure.*

Then,

- *Either there exists a Lie algebra embedding $\mathfrak{g} \to \mathfrak{h}$;*
- *Or Γ preserves a smooth Riemannian metric on M.*

As a special case, for $G = \mathrm{SL}_d(\mathbf{R})$ and $H = \mathrm{SL}_n(\mathbf{R})$, the condition $\mathfrak{g} \hookrightarrow \mathfrak{h}$ means $d \leqslant n$, so the conjecture sounds more concrete.

Conjecture 3. *Let Γ be a lattice in $\mathrm{SL}_d(\mathbf{R})$ with $d \geqslant 3$. Let M be a compact n-manifold and ω a volume form. If $d > n$, then any volume preserving action $\Gamma \to \mathrm{Diff}_\omega(M)$ has finite image.*

Remark 13. For $\Gamma = \mathrm{SL}_n(\mathbf{Z})$, the action on the n-torus $\mathbf{R}^n/\mathbf{Z}^n$ has infinite image and preserves the Euclidean volume.

Remark 14. When $\mathfrak{g}$ does not embed into $\mathfrak{h}$, Conjecture 2 first would give that the action has range into $\mathrm{Isom}(M, g) < \mathrm{Diff}(M)$ for some Riemannian metric g. The latter is compact and has dimension at most $n(n+1)/2$ by Myers-Steenrod theorem, where $n = \dim M$. In some cases, Margulis' theorem implies that furthermore any homomorphism Γ into such compact groups must have finite image, explaining the stronger conclusion in Conjecture 3.

For other groups than $\mathrm{SL}_d(\mathbf{R})$, the idea is the same: the conjecture predicts that for every semi-simple Lie group G, there exists an explicitly determined bound $d_{\mathrm{vol}}(G)$ such that for any compact manifold M with $\dim M < d_{\mathrm{vol}}(G)$, any volume-preserving action $\Gamma \to \mathrm{Diff}_\omega(M)$ transits through a compact Lie group action (see for instance §4.5. of [Fis11] or §2.4. of [Can17] for details).

4.1.3 Extension of the conjecture by Farb and Shalen

In [FS99], Farb and Shalen obtained partial results for real-analytic actions of some higher-rank lattices on surfaces and some closed 3-manifolds. Instead of applying arguments from ergodic theory, they rather used topological considerations, including for instance Lefschetz fixed-point theorem. Hence, the invariant volume form was no longer a crucial ingredient, and it led them to extend Conjecture 2.

Their conjecture predicted that for another explicitly computable[1] integer $d(G) < d_{\mathrm{vol}}(G)$, for any compact manifold M with $\dim M < d(G)$, any smooth action $\Gamma \to \mathrm{Diff}(M)$ preserves a Riemannian metric.

Although this was not conjectured by Zimmer, the literature refers to it also as Zimmer's non-volume preserving conjecture. In the case $G = \mathrm{SL}_n(\mathbf{R})$, $d(G) = n-1$ and the conjecture is that there does not exist a compact manifold of dimension $\leqslant n-2$ on which a lattice $\Gamma < G$ acts with infinite image. Note that this is sharp, since the whole $\mathrm{SL}_n(\mathbf{R})$ acts on $\mathbf{R}P^{n-1}$.

4.1.4 Advances on the conjectures and related problems until 2016

A first important result which supported the volume-preserving conjecture was this general geometric result proved in [Zim86c].

[1] Except for $\mathfrak{g} = \mathfrak{so}^*(2n)$, $d(G) = \min\{\dim G/P,\ P \text{ strict parabolic subgroup}\}$.

Theorem 4.2. *Let M be a closed manifold endowed with a finite type H-structure $\pi : P \to M$, with H an algebraic subgroup of D^r for some $r \geqslant 1$. Let G be a real semi-simple Lie group without compact factor, with finite center and $\mathrm{rk}_{\mathbf{R}}\, G \geqslant 2$. Let Γ be a lattice of G.*

Let $\rho : \Gamma \to \mathrm{Aut}(P)$ be an action of Γ by automorphisms of the H-structure. Suppose that the Γ preserves a volume density on M. Then,

- *either there is a non-trivial homomorphism $\mathfrak{g} \to \mathfrak{h}$;*
- *or $\rho(\Gamma)$ has compact closure in $\mathrm{Aut}(P)$.*

As a consequence, it follows that if $\Gamma = \mathrm{SL}_m(\mathbf{Z})$, $m \geqslant 3$, and if Γ acts differentiably on a closed n-manifold M, by preserving both an affine connection and a volume density, and if $m > n$, then the action factorizes through a finite group action, *i.e.* is trivial up to passing to a finite index subgroup ([Zim86b]).

It also implies that if Γ acts isometrically, with unbounded image, on a closed pseudo-Riemannian manifold (M, g) of signature (p, q), then $\mathfrak{g} \hookrightarrow \mathfrak{so}(p, q)$. As a consequence, for any closed *Lorentzian* manifold (M^n, g), any isometric action $\mathrm{SL}_3(\mathbf{Z}) \to \mathrm{Isom}(M, g)$ has finite image, independently of n.

Several advances were obtained in the 1990's to prove inexistence of Γ-actions in dimension 1 and 2. In [Wit94], Witte proved that any action by *homeomorphism* $\Gamma \to \mathrm{Homeo}(\mathbf{S}^1)$ has finite image, but under the restrictive assumption $\mathrm{rk}_{\mathbf{Q}}\, \Gamma \geqslant 2$. After proving that the action has a fixed point, the statement was equivalent to prove non-left-orderability of Γ. Later, Ghys [Ghy99], and independently Burger and Monod [BM02], extended Witte's result to any irreducible lattice Γ, provided that the action is by $\mathcal{C}^1$ diffeomorphisms. Ghys approach relies essentially on Thurston's stability theorem, whereas Burger-Monod's proof is based on bounded cohomology. As already mentioned, Farb and Shalen [FS99] obtained inexistence results for "2-big" lattices actions by real-analytic diffeomorphisms on surfaces and certain cases of analytic actions on homology 3-spheres. In [FH03], Franks and Handel obtained a proof of the volume-preserving Conjecture 3 in the case of $\mathrm{SL}_3(\mathbf{Z})$ (and other discrete almost simple groups) acting analytically on closed oriented surfaces.

4.1.5 Katok-Lewis examples and invariant geometric structures

If Zimmer's conjectures are true, then it is natural to consider actions at the critical dimension. In the case $\Gamma = \mathrm{SL}_n(\mathbf{Z})$, the action on the n-torus has optimal dimension among volume-preserving actions. This action is known to be locally rigid, as a special case of [FM03]. However, there exist volume-preserving *exotic actions* of $\mathrm{SL}_n(\mathbf{Z})$ on closed n-manifold which are not homeomorphic to a torus. The first construction is due to Katok-Lewis [KL96], later extended by Benveniste. It consists in blowing-up the origin in $\mathbf{R}^n/\mathbf{Z}^n$, which is an $\mathrm{SL}_n(\mathbf{Z})$-fixed point. Benveniste and Fisher [BF05] later observed that these actions do not preserve any rigid geometric structure. More recently, Fisher and Melnick [FM22] constructed new examples of exotic $\mathrm{SL}_n(\mathbf{Z})$-actions.

However, they do preserve a rigid geometric structure(s) in the complement of the exceptional divisor in Katok-Lewis' example, namely the flat connection. An open problem is the following.

Question 6. *Let* $\mathrm{SL}_n(\mathbf{Z})$ *act on a closed* n*-manifold by volume-preserving diffeomorphisms. Does there exist an* $\mathrm{SL}_n(\mathbf{Z})$*-invariant rigid-geometric structure defined on an open-dense subset? Similarly for any volume-preserving action of a higher-rank lattice at the critical dimension.*

Notably, this problem is supported by a topological version of cocycle super-rigidity due to Feres and Labourie [FL98] (see also [Lab98] for the context).

In particular, this raises the question of understanding Γ-actions on (compact or non-compact) manifold which preserve a finite volume and a rigid geometric structure.

Recall Corollary 7 which gives a *measurable* Γ-invariant connection. A natural problem which has been investigated in the volume preserving case was to assume this connection to be smooth. Contributions of Zimmer, Feres, Goetze and finally Zeghib led to the following geometric result.

Theorem 4.3 ([Zim86b, Fer92, Goe94, Zeg97])**.** *Let* $n \geqslant 3$ *and* $\Gamma < \mathrm{SL}_n(\mathbf{R})$ *be a lattice. Suppose that* Γ *acts on a compact* n*-manifold* M *and preserves a linear connection* ∇ *and a volume form* ω*. If the action is infinite, then* Γ *is a finite-index subgroup of* $\mathrm{SL}_n(\mathbf{Z})$*, and up to affine covering, the action is the linear action* $\Gamma \curvearrowright \mathbf{T}^n$*.*

Zeghib's article goes a step further and also characterizes non-trivial actions in dimension $n+1$, as being of the form $\mathbf{T}^n \times \mathbf{S}^1$, with a trivial action on the second factor, always up to a finite covering.

Theorem I below gives a non-unimodular version of this theorem, when the action preserves a projective class of connections in dimension $n-1$.

4.1.6 Recent advances on Zimmer's conjectures

In a series of articles [BFH22, BFH20, BFH21], the first being prepublished in 2016, Brown, Fisher and Hurtado made spectacular progress on both volume and non-volume preserving conjectures. Their methods apply in all situations and give lower bounds on the dimension of a compact manifold on which Γ acts with infinite image. These bounds are the bounds announced by the conjecture in the non-volume preserving case and when G is $\mathbf{R}$-split. In particular, it proves Conjecture 3, and its non-volume preserving version, *in any dimension*.

The methods of their articles build on several techniques. One is strong property (T) of Lafforgue, which provides at the end existence of a Γ-invariant differentiable Riemannian metric, essentially via a fixed-point argument in appropriate spaces of jets of metrics. To apply this, they needed to prove that the Γ-action has sub-exponential growth of derivatives, and for that they used recent results by Brown, Rodriguez-Hertz and Wang [BRHW22] - also prepublished in 2016 - which guarantee the existence of finite Γ-invariant measures in various dynamical configurations.

This point strongly interested me for its possible applications to conformal geometry. It is detailed in the next section.

Let us mention, to conclude this very quick and biased survey of the field, a recent result by Deroin and Hurtado [DH20] which extends [Ghy99, BM02] to actions by homeomorphisms, and relies on similar methods for constructing invariant measures on an auxiliary compact space with a Γ-action , called the almost periodic space.

4.2 Recent tools from smooth ergodic theory

Let Γ be a lattice in a semi-simple Lie group G, with no compact factors and finite center. Suppose that $\alpha : \Gamma \to \mathrm{Diff}(M)$ is a smooth action on a closed manifold M. To α corresponds an associated fiber bundle $M^\alpha \to G/\Gamma$, whose total space is $M^\alpha = (G \times M)/\Gamma$, where Γ acts on the right via $(g, x).\gamma = (g\gamma, \gamma^{-1}.x)$, and whose projection is the map induced by the projection on the first factor.

By construction, G acts on M^α via left translations on the first coordinate. This action is smooth, locally free and fiber-preserving. Moreover, G-orbits are transverse to the fibers, hence they define a flat, G-invariant connection on M^α.

Definition 4.2. Given a Γ-action α on a manifold M, the fibration $M^\alpha \to G/\Gamma$ is called the *suspension space* of α and the G-action on M^α is called the *induced action.*

Some properties of the Γ-action on M are reflected in the induced G-action. Importantly:

Proposition 4.1 ([NZ99], Lem. 6.1)**.** *Let α be a Γ-action on a compact manifold M. Then, Γ preserves a finite measure μ on M if and only if G preserves a finite measure ν on M^α.*

For the direct implication, given a Γ-invariant measure μ on M, we obtain a G-invariant family of finite measures $\{\mu_{g\Gamma}\}$ on M^α such that $\mu_{g\Gamma}$ is supported on the fiber of $g\Gamma$, for every $g\Gamma \in G/\Gamma$. Therefore, $\nu = \int_{G/\Gamma} \mu_{g\Gamma}\, \mathrm{dvol}(g\Gamma)$ is finite and G-invariant. For the converse, starting from a G-invariant finite measure ν on M^α, ν can be desintegrated along the fibration $M^\alpha \to G/\Gamma$ and with respect to the Haar measure. The measures on the fibers then yield a Γ-invariant measure on M, essentially because of the uniqueness of the desintegration.

Also, Γ acts ergodically on M with respect to μ if and only if G acts ergodically on M^α with respect to ν.

4.2.1 Lyapunov exponents and higher-rank Oseledets theorem

Let $f : M \to M$ be a smooth map of a manifold M, and suppose that f preserves a finite ergodic measure μ. Oseledets' Multiplicative Ergodic Theorem (see for instance [Fil19, Boc, Wil17]), guarantees that if $(\log \|\mathrm{d}_x f\|)^+ \in L^1(M, \mu)$ and $(\log \|\mathrm{d}_x f^{-1}\|)^+ \in L^1(M, \mu)$, then there exists a set Λ of full measure, measurable distributions $\{0\} = E_0 \subsetneq$

$E_1 \subsetneq \ldots \subsetneq E_r = T_xM$ defined over Λ and numbers $\chi_1 < \cdots < \chi_r$ such that for all $1 \leqslant k \leqslant r$, and for all $v \in E_k \setminus E_{k-1}$, we have

$$\frac{1}{n} \log |\mathrm{d}_x f^n.v| \underset{+\infty}{\longrightarrow} \chi_k. \tag{4.1}$$

For $f \in \mathrm{Diff}(M)$, combining the preceding result for f and f^{-1}, the filtration becomes a measurable splitting $TM = E'_1 \oplus \cdots \oplus E'_r$ over Λ such that (4.1) holds for every $x \in \Lambda$ and $v \in E'_k(x)$ as $n \to \pm\infty$.

This result has a generalization when the probability measure preserving transformations or flows of vector fields come in commutative families, which gives a higher rank version (see for instance [Hu93] or §3.6.1 of [BP07]). The result is a simultaneous Oseledets splitting of TM for which Lyapunov exponents vary *linearly*. We will apply in fact the theorem to the restriction of the differential action to a subbundle of TM, so I give the following formulation.

Theorem 4.4. *Let M be a manifold on which a connected abelian group $A \simeq \mathbf{R}^k$ acts differentiably, and let μ be an A-invariant, A-ergodic probability measure. Let $E \to M$ be a vector bundle and suppose that the A-action lifts to an action by bundle automorphisms of E. Suppose that for any compact subset $K \subset A$, the function $x \mapsto \sup_{a\in K} |\log(\|\mathrm{d}_x a\|)|$ is μ-integrable.*

Then, there exist:

1. *a measurable set $\Lambda \subset M$ of μ-measure 1,*

2. *a finite set of linear forms $\chi_1, \ldots, \chi_r \in \mathfrak{a}^*$,*

3. *and a measurable, A-invariant splitting $E = E_1 \oplus \cdots \oplus E_r$ defined over Λ,*

such that for any Riemannian norm $\|.\|$ on E and for every $x \in \Lambda$ and every $v \in E_i(x) \setminus \{0\}$,

$$\frac{1}{|X|}(\log \|e^X.v\| - \chi_i(X)) \underset{\substack{|X|\to\infty \\ X\in\mathfrak{a}}}{\longrightarrow} 0,$$

and

$$\frac{1}{|X|}(\log |\det \mathrm{Jac}_x(e^X)| - \sum_{1\leqslant i\leqslant r} \chi_i(X) \dim E_i(x)) \underset{\substack{|X|\to\infty \\ X\in\mathfrak{a}}}{\longrightarrow} 0,$$

where $\mathrm{Jac}_x(e^X)$ denotes the matrix of $e^X : E(x) \to E(e^X.x)$ with respect to some bounded measurable frame field of E.

Given a linear A-action on the vector bundle E, and a finite measure μ satisfying the hypothesis of Theorem 4.4, we will call $\chi_1, \ldots, \chi_r$ the *Lyapunov functionals* of μ.

4.2.2 Pesin's stable manifold theorem and Ledrappier-Young formula

Lyapunov exponents give "tangential" asymptotic estimates on the action of the diffeomorphism f. This information passes in fact locally to its action on the manifold, and importantly relates to the metric entropy of the f-invariant measure.

Let $f : M \to M$ be a $\mathcal{C}^{1+\alpha}$ diffeomorphism of a compact manifold and let μ be an ergodic f-invariant probability measure. Suppose that f has a negative Lyapunov exponent and let $\chi_1 < \cdots \chi_i < 0 \leqslant \chi_{i+1} < \cdots < \chi_r$ be the Lyapunov exponents of f. Let $\Lambda \subset M$ be the full measure subset where (4.1) holds. Equip M with an arbitrary Riemannian metric and let d be the associated length distance.

I state below a light version of Pesin's Stable Manifold Theorem. Important estimates on the size of local stable manifolds, as well as a lower subexponential bound on their decay along future f-orbits, are skipped and I only focus on what will be used later. Many references contain proofs of this central result of differentiable dynamics, in its full strength, among which [BP07] and [FHY83].

Theorem 4.5. *Let $\rho \in]\chi_i, 0[$. Then, there exist a full measure subset $\Lambda_\rho \subset \Lambda$, and for every $x \in \Lambda_\rho$, an embedded disc $W_s^{loc,\rho}(x)$ containing x, such that $T_x W_s^{loc,\rho}(x) = \bigoplus_{k \leqslant i} E_k(x)$ is the strongly stable subspace at x and a constant $C(x) > 0$ such that for all $y, z \in W_s^{loc,\rho}(x)$, and $n \geqslant 0$,*

$$d(f^n(y), f^n(z)) \leqslant C(x) d(y,z) e^{\rho n}. \tag{4.2}$$

Moreover, in an a priori smaller disc at x, any point y such that $d(f^n(y), f^n(x)) \leqslant Cd(y,x)e^{\rho n}$, for a certain constant C, in fact belongs to $W_s^{loc,\rho}(x)$.

Definition 4.3. For any $x \in \Lambda_\rho$, the embedded disc $W_s^{loc,\rho}(x)$ is called the *local stable manifold* of f at x.

Margulis-Ruelle inequality relates the *metric entropy* of a finite f-invariant measure μ and its *positive* Lyapunov exponents and reads:

$$h_\mu(f) \leqslant \sum_{\lambda_k > 0} \lambda_k \dim E_k(x). \tag{4.3}$$

A major result measures the default of equality in Margulis-Ruelle's inequality with the Hausdorff dimension of the marginal measures obtained by desintegrating μ along unstable manifolds. It is Ledrappier-Young's formula ([LY85a, LY85b]). It is a fundamental tool in the proof of Theorem 4.6 from [BRHW22] and restated below.

4.2.3 A-invariant measures with maximal entropy

Consider a higher-rank uniform[2] lattice $\Gamma < G$ acting by, say $\mathcal{C}^2$, diffeomorphisms of a compact manifold. Let $\pi : M^\alpha \to G/\Gamma$ denote the suspension space of this action. Fix

[2]To avoid technicality, I use this restriction which makes the total space of the suspension compact, so that we avoid various technical problems (*e.g.* the integrability condition in Oseledets' theorem). But no significant difficulty is hidden.

$A < G$ an $\mathbf{R}$-split Cartan subgroup, let $\Sigma \subset \mathfrak{a}^*$ denote the restricted-roots of $\mathfrak{a}$ and let $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda$ be the corresponding restricted root space decomposition.

Recall that the orbits of the locally free action of G on M^α are everywhere transverse to the fibers, hence they define a smooth, G-invariant splitting $TM^\alpha = H \oplus F$ where for all $x_\alpha \in M^\alpha$, H_{x_α} is the tangent space to $G.x_\alpha$ and F_{x_α} the tangent space to the fiber $\pi^{-1}(\pi(x_\alpha))$. For $V \subset \mathfrak{g}$ a vector subspace and $x_\alpha \in M^\alpha$, denote by $V(x_\alpha) = \{X_{x_\alpha},\ X \in V\}$[3].

Let μ be a finite A-invariant, A-ergodic measure on M^α. A part of the conclusions of Theorem 4.4 is directly predictable, independently of μ: the splitting

$$\mathfrak{g}(x^\alpha) = \mathfrak{g}_0(x^\alpha) \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda(x^\alpha)$$

diagonalizes the action of A on the horizontal distribution H since any $g \in A$ commutes with vector fields generated by elements of $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ and if $X \in \mathfrak{g}_\lambda$ and if $g = e^{X_0}$, then $\mathrm{d}_{x^\alpha} g.X_{x^\alpha} = e^{\lambda(X_0)} X_{gx^\alpha}$. In particular, for any A-invariant probability measure μ, the Lyapunov spectrum of any $g \in A$ with respect to μ is completely known in the horizontal direction and does not depend on μ.

In contrast, the central question is to undestand its vertical part where all the dynamics of Γ is encoded. So, let us apply Theorem 4.4 to the vertical subbundle $F \to M^\alpha$. We call *vertical Lyapunov functionals* of μ the Lyapunov functionals obtained in this way.

Definition 4.4. Let $\{\chi_1, \dots, \chi_r\} \subset \mathfrak{a}^*$ be the vertical Lyapunov functionals of μ. A restricted root $\lambda \in \Sigma$ is said to be *μ-resonant* if there exists $1 \leqslant i \leqslant r$ and $c > 0$ such that $\lambda = c\chi_i$.

Theorem 4.6 ([BRHW22])**.** *Let μ be an A-invariant, A-ergodic measure on M^α such that $\pi_*\mu$ is the Haar measure on G/Γ.*

Let $\lambda \in \Sigma$ be a restricted root. If λ is not μ-resonant, then μ is G_λ-invariant.

The heuristic for obtaining G-invariance of μ is then very natural: let G_μ denote the stabilizer of μ in G. Then, G_μ contains A, and every restricted root-space G_λ for $\lambda \notin \cup_{i=1}^r \mathbf{R}_{>0}.\chi_i$. Now, a standard algebraic property states that a *strict* Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, containing $\mathfrak{a}$, cannot contain more than a certain explicit number of restricted root-spaces. For instance, in $\mathfrak{g} = \mathfrak{sl}_n(\mathbf{R})$, a Lie subalgebra containing the Cartan subalgebra and strictly more than $(n-1)^2 = n(n-1) - (n-1)$ restricted root-spaces must be equal to $\mathfrak{sl}_n(\mathbf{R})$. More generally, this leads to the idea of minimal resonant codimension introduced in [BRHW22].

Definition 4.5. Let $\mathfrak{g}$ be a semi-simple Lie algebra without compact factors and with restricted root-system Σ. Let

$$\Sigma' = \begin{cases} \Sigma & \text{if } \Sigma \neq (BC)_\ell \\ B_\ell & \text{if } \Sigma = (BC)_\ell. \end{cases}$$

[3] Recall that any $X \in \mathfrak{g}$ is identified with a vector field of M^α, say $\overline{X}$, defined by $\overline{X}_{x_\alpha} = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} e^{tX}.x_\alpha$. I use implicitly this identification $X = \overline{X}$ in all the text.

The minimal resonant codimension $r(\mathfrak{g})$ of $\mathfrak{g}$ is then, denoting by $\mathfrak{g}'$ the real split form of type Σ', the minimum of $\dim(\mathfrak{g}'/\mathfrak{p}')$, when $\mathfrak{p}'$ runs among proper parabolic subalgebras of $\mathfrak{g}'$.

Hence, $r(\mathfrak{g})$ is the smallest possible number of *coarse* restricted roots of Σ that can appear transversally to a strict parabolic subalgebra of $\mathfrak{g}$. We see also that when $\mathfrak{g}$ is split, $r(\mathfrak{g})$ coincides with the minimal dimension of a flag manifold G/P.

An important consequence of Theorem 4.6 above is:

Corollary 8. *Suppose that Γ acts on a compact manifold M and that for a given $\mathbf{R}$-split Cartan subalgebra $A < G$, there exists a finite A-invariant, A-ergodic measure μ on the suspension M^α, projecting to the Haar measure of G/Γ, with strictly less than $r(\mathfrak{g})$ vertical Lyapunov functionals (e.g. when $\dim M < r(\mathfrak{g})$). Then, μ is G-invariant.*

With no more restriction than the differentiability of the action, there is no hope to have a stronger estimates on the number of Lyapunov exponents than a dimensional count. However, if some geometric structure is assumed to be invariant, then more restrictions on these functionals appear and therefore, we can prove existence of finite invariant measures in a broader context.

4.2.4 Non-existence of finite invariant measures for actions on rigid geometric structures

Certain families of geometries do not carry natural volume density, and they do not define finite measures even when the underlying manifold is compact. This is for example the case for conformal structures or linear connections. The lack of finite invariant measure is an additional difficulty in the study of dynamics of group actions on these geometric structures.

In fact, we can go further than this general (which may sound pessimistic) observation, and prove that in the case of actions of semi-simple Lie groups and their lattices, provided that they are big enough, they do not preserve any finite Borel probability measure.

I collect here some useful observations which can be applied in geometric contexts, detailed later in section 4.4.1. The Proposition below is a variation on Proposition 4.1 of [Pec20], which is in the special case of conformal structures, *i.e.* $H = \mathbf{R}_{>0} \times O(p,q)$. Definition of H-structures of finite type is recalled in Section 2.2.1.

Proposition 4.2. *Let M be a compact manifold endowed with an H-structure of finite type, with H a reductive group. Suppose that Γ acts on M by automorphisms of the H-structure and such that the image of the action is unbounded. If $\mathfrak{g}$ does not embed into $\mathfrak{h}$, then there does not exist any finite Γ-invariant measure on M.*

This observation was used by Zimmer in his proof of Theorem F of [Zim86c] for actions on unimodular H-structures of finite type, with a slightly different formulation and more restricted context. The ideas for proving the above statement are the following.

Proof. By definition of a finite type H-structure, the prolongation procedure of the initial H-structure $P \to M$ stabilizes after finitely many steps to an $\{e\}$-structure $P^{k+1} \to P^k$, *i.e.* a global framing on P^k, and for all intermediate index $i \leqslant k$, the Γ action lifts to an action on $P^{i+1} \to P^i$. The key fact is that if Γ preserves a finite measure on P^i, then so does the lift of its action to P^{i+1}. Hence, Γ finally acts on the last stage P^k by preserving a finite measure. But this violates the fact that Γ acts freely and properly on P^k and the hypothesis makes that Γ is closed when realized as a subgroup of the Lie group $\mathrm{Aut}(P^i)$ for any i ([Ste61] Ch. VII, [Kob95] Ch.I, Theorem 5.1, see also [McK23], §.21, for a similar approach in the case of Cartan geometries).

Let us briefly comment on how measures are lifted in the prolongation tower. It is where the assumption on $\mathfrak{g}$ is used: G does not embed locally in any of the structural groups of the principal bundles of the prolongation. So, if Γ preserves a finite measure μ_i on P^i, then cocycle super-rigidity (Theorem 4.1) implies that the action is cohomologuous to a *compact valued* action, *i.e.* that there is a measurable trivialization $P^{i+1} \simeq P^i \times H^i$, a compact subgroup $K < H^i$ such that the action preserves the measurable subbundle $P^i \times K$. Consequently, the Γ-action on P^{i+1} preserves the pull-back μ_{i+1} of $\mu_i \otimes \mathrm{d}k$, where $\mathrm{d}k$ denotes the Haar measure on K. □

A few facts and definitions about Cartan geometries are recalled in Section 2.2.2

Corollary 9. *Let $(M, \mathcal{G}, \omega)$ be a Cartan geometry modeled on a flag manifold $\mathbf{G}/\mathbf{P}$. Suppose Γ acts on M by automorphisms of the Cartan geometry and that G does not embed into $\mathbf{P}$. If Γ preserves a finite measure on M, then the action has compact closure.*

This is basically due to the fact that the Cartan geometry defines an H-structure on M, with $H = \overline{\mathrm{Ad}}_{\mathfrak{g}/\mathfrak{p}}(\mathbf{P}) \subset \mathrm{GL}(\mathfrak{g}/\mathfrak{p})$, isomorphic to the Levi subgroup of $\mathbf{P}$.

Therefore, the non-existence of finite Γ-invariant measures on $(M, \mathcal{G}, \omega)$ is provided by the non-existence of an embedding of G into the Levi subgroup of $\mathbf{P}$, which can be small compared to $\mathbf{G}$, *e.g.* when $\mathbf{P}$ is the Borel subgroup.

Example 8. For conformal structures of signature (p, q) in dimension at least 3, the Levi subgroup of the corresponding parabolic subgroup of $O(p+1, q+1)$ is isomorphic to $\mathbf{R}^* \times O(p, q)$, hence the criterion is that for G not locally isomorphic to a subgroup of $O(p, q)$, any lattice $\Gamma < G$ acting conformally on a pseudo-Riemannian manifold of signature (p, q) cannot preserve a finite measure, unless the action is bounded.

Example 9. When $\mathbf{G} = \mathrm{SL}_n(\mathbf{R})$ and $\mathbf{P}$ is the Borel subgroup, any action of a higher-rank lattice on a compact manifold, that preserves a Cartan geometry modeled on $(\mathbf{G}, \mathbf{P})$ and a finite measure must be bounded, regardless the size of n.

For groups with property (T), similar conclusions follow for actions on certain finite type H-structures.

Proposition 4.3 ([Zim84b], Theorem 10)**.** *Let Γ be a discrete group with property (T) and H a real algebraic group. Let (X, μ) be a probability space on which Γ acts by preserving μ and ergodically. Let $\alpha : \Gamma \times X \to H$ be a measurable cocycle. Then, α is cohomologous to a cocycle β whose image is contained in an algebraic subgroup $H_1 < H$ with property (T).*

Since neither $\mathrm{SO}(1,n)$ nor $\mathrm{SU}(1,n)$ admit a non-compact algebraic subgroup with property (T), it follows for instance that if Γ has property (T) and acts conformally on a closed Lorentzian manifold and preserves a finite measure μ, then the action has compact closure, and similarly for actions on finite type H-structures, with H an algebraic group whose Levi factor is of the form $K \times \mathrm{SO}(1,n)$ or $K \times \mathrm{SU}(1,n)$.

Similarly, the same argument can be applied for Γ-actions on certain Cartan geometries, for instance projective structures on closed surfaces. For these, assuming the action $\Gamma \to \mathrm{Proj}(S,[\nabla])$ faithful and with unbounded image, we get a free and proper action of Γ on the Cartan bundle $\mathcal{G} \to S$, whose structural group is $\mathrm{GL}_2(\mathbf{R}) \ltimes \mathbf{R}^2$, hence does not contain no non-compact algebraic subgroup with property (T). Therefore, the action cannot be probability measure preserving.

In the case where Γ is a lattice in $\mathrm{Sp}(n,1)$ or $F_{4(-20)}$, these observations can be combined with the methods of [BRHW22] and the output is a dynamical information which encourages further investigations for Γ-actions on such rigid geometric structures which do not admit Γ-invariant measures. In particular, if $\{h^t\}$ is a one parameter-subgroup parametrizing the $\mathbf{R}$-split Cartan subgroup, then for any $\{h^t\}$-invariant measure for the induced G-action, the Lyapunov exponents of h^t cannot be all zero. Furthermore, specific restrictions on the Lyapunov spectrum follow from [Kai89] and seems to be a first interesting step into this direction.

4.3 Conformal actions of higher-rank lattices at the critical case

The results of [BRHW22] give existence of finite Γ-invariant measures, provided certain dynamical conditions. Once their result was prepublished, a very natural direction to look at was conformal actions of Γ on pseudo-Riemannian manifolds. Since their approach was key in the non-existence of low-dimensional actions of Γ (proof of Conjecture 3), it seemed plausible that it will give other interesting obstructions in this geometric context.

For instance, a question which was not known to the experts was the existence (or non-existence) of a non-trivial conformal action of Γ on certain compact Lorentzian manifolds. Typically: does there exist a compact Lorentzian manifold on which $\mathrm{SL}_3(\mathbf{Z})$, or a finite index subgroup, acts conformally with finite kernel?

If the action is isometric, then Theorem F of [Zim86c] gives non-existence in any dimension. If the action extends to a conformal action of $\mathrm{SL}_3(\mathbf{R})$, then non-existence is a consequence of [BN02], and also the subsequent developments about conformal Lie group actions. In [Ioz92], Iozzi showed that if there exists a differentiable action of $G \to \mathrm{Diff}(M)$, such that the restriction to Γ preserves the conformal structure (or any rigid structure), then all of G preserves the conformal structure, which would again be impossible. But, this requires to show (if true) that the Γ-action extends to a G-action by diffeomorphisms, which *a priori* does not simplify the question.

4.3.1 The Riemannian case

For positive-definite metrics, the question of existence of a conformal action of a higher-rank lattice is directly answered by Ferrand-Obata's theorem in the compact case, already discussed previously on essentiality problems, which we reproduce below.

Theorem 4.7 ([Oba71, LF71, LF76])**.** *Let $(M, [g])$ be a compact manifold endowed with a Riemannian conformal structure. If $\mathrm{Conf}(M, [g])$ is non-compact, then $(M^n, [g])$ is conformally diffeomorphic to the round sphere $\mathbf{S}^n$.*

The conformal group of $\mathbf{S}^n$ is the Möbius group $\mathrm{PO}(1, n+1)$, fundamentally because $\mathbf{S}^n = \partial_\infty \mathbf{H}^{n+1}_{\mathbf{R}}$ is the conformal boundary of the real hyperbolic space. Hence, any conformal action of a higher-rank lattice Γ on a closed Riemannian manifold either transits through a compact Lie group action, or is a conformal action on $\mathbf{S}^n$, *i.e.* a group homomorphism $\Gamma \to \mathrm{PO}(1, n+1)$, so has compact closure again, by Margulis super-rigidity theorem.

4.3.2 General case: discretization of Bader-Nevo's and Frances-Zeghib's theorems

The pseudo-Riemannian analogue of the Möbius sphere is the conformal boundary of the pseudo-hyperbolic space $\mathbf{H}^{p,q+1}$. The latter is the projectivization of $\{-x_0^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_{p+q+1}^2 < 0\}$, with the metric inherited from $\mathbf{R}^{p+1,q+1}$. It has constant negative sectional curvature and its isometry group is $\mathrm{PO}(p+1, q+1)$, whose action extends to the conformal boundary $\mathbf{Ein}^{p,q}$, the projective quadric $\{-x_0^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_{p+q+1}^2 = 0\}$. The latter comes now with a conformal class of pseudo-Riemannian metric of signature (p, q), which is conformally flat and has $\mathrm{PO}(p+1, q+1)$ as conformal group. As a homogeneous space, $\mathbf{Ein}^{p,q}$ identifies with a flag manifold of $\mathrm{PO}(p+1, q+1)$, with isotropy the maximal parabolic subgroup P isomorphic to the stabilizer of a null line.

More concretely, this conformal structure is doubly covered by $(\mathbf{S}^p \times \mathbf{S}^q, [-g_{\mathbf{S}^p} \oplus g_{\mathbf{S}^q}])$, where $g_{\mathbf{S}^k}$ denotes the Riemannian metric of constant curvature $+1$ on the k-sphere. Hence, the universal cover of the projective model of Einstein Universe is

$$\widetilde{\mathbf{Ein}}^{p,q} = \begin{cases} (\mathbf{S}^p \times \mathbf{S}^q, [-g_{\mathbf{S}^p} \oplus g_{\mathbf{S}^q}]) \text{ if } p > 1 \\ (\mathbf{R} \times \mathbf{S}^{n-1}, [-\mathrm{d}t^2 \oplus g_{\mathbf{S}^{n-1}}]) \text{ if } p = 1. \end{cases} \tag{4.4}$$

A natural question within conformal geometry is to obtain a pseudo-Riemannian analogue of Theorem 4.7. Several works have shown that no straight analogue is plausible, see Section 3.2.

The following result was known for the action of the full ambient Lie group, by combining [Zim87b], [BN02] and [FZ05]. In [Pec20] and [Pec24], and with a currently finalizing article[4], I have obtained the extension of their results to lattices actions.

[4]The statement available at the time of this writing assumes Γ to be uniform and $\min(p, q) > 1$.

Theorem G. *Let $(M,[g])$ be a compact manifold of dimension at least 3 endowed with a conformal structure of signature (p,q), with $1 \leqslant p \leqslant q$. Let Γ be a lattice in a simple Lie group G with $\mathrm{rk}_{\mathbf{R}} G \geqslant 2$ and finite center. Let $\rho : \Gamma \to \mathrm{Conf}(M,[g])$ be a conformal action such that $\rho(\Gamma)$ is not relatively compact. Then,*

1. $\mathrm{rk}_{\mathbf{R}} G \leqslant p+1$ *;*
2. *If $\mathrm{rk}_{\mathbf{R}} G = p+1$, then $\widetilde{M}$ is conformally diffeomorphic to $\widetilde{\mathbf{Ein}}^{p,q}$.*
   - *(a) If $p>1$, then $|\pi_1(M)| \leqslant 2$ and M is either $\widetilde{\mathbf{Ein}}^{p,q}$ or its projective model.*
   - *(b) If $p=1$, then $\pi_1(M)$ is virtually infinite cyclic, and up to finite index, it is sent by the holonomy homomorphism into $\mathcal{Z} \times O(n)$, where $\mathcal{Z}$ refers to the center of $\widetilde{O}(2,n)$ and $O(n)$ to the lift to $\widetilde{O}(2,n)$ of the $O(n)$ factor of the maximal compact of $O(2,n)$.*
3. *If $\mathrm{rk}_{\mathbf{R}} G = p+1$, then $G \simeq_{\mathrm{loc}} O(p+1,k+1)$, with $p \leqslant k \leqslant q$, and the Γ-action almost extends to a G-action.*

Recall that by *almost extend*, we mean that there exists a non-trivial Lie group homomorphism $\overline{\rho} : G \to \mathrm{Conf}(M,[g])$, a compact subgroup $K < \mathrm{Conf}(M,[g])$ centralizing $\overline{\rho}(G)$, and an homomorphism $\rho_K : \Gamma \to K$ such that $\rho(\gamma) = \overline{\rho}(\gamma)\rho_K(\gamma)$ for all $\gamma \in \Gamma$.

Remark 15. Recall Example 6, which shows that for there are non-conformally flat compact manifolds of signature (p,q), with essential conformal of $O(p-2,q-2)$, so that in Theorem G, no similar geometric conclusions can be expected if we simply drop the assumption on the rank.

Remark 16. The fact that the action almost extends is an observation made *a posteriori*: the proof does not show it until it is reduced to a straight application of Margulis' super-rigidity theorem.

Recall that in Kobayashi's theorem about Lie transformation groups ([Kob95]), the automorphisms group of a finite type G-structure has a natural Lie group structure but *not necessarily connected*. Hence, it is not possible to apply Margulis' theorem *a priori* to the action $\rho : \Gamma \to \mathrm{Conf}(M,[g])$. It could be for instance that $\mathrm{Conf}(M,[g])$ is itself a discrete group containing Γ. The same goes for general geometric structures (rigid or not), whose automorphism group is known to be a Lie group. Remark that this includes complex structures on *compact* manifolds, or elliptic G-structures more generally (see Section 2.2.1), and the same problem arose in [Can04] and [CZ12] where actions by biholomorphisms on compact Kähler manifolds are considered.

So, for a general, say rigid, geometric structure ϕ on a compact manifold M with a Γ-action by automorphisms $\rho : \Gamma \to \mathrm{Aut}(M,\phi)$, if one wants to apply directly Margulis' super-rigidity, the main question is to know whether $\Gamma \cap \rho^{-1}(\mathrm{Aut}(M,\phi)_0)$ has finite index in Γ or not. Again, it is not clear if this brings any simplification to the initial question. Theorem G implies that it is always the case for conformal actions of lattices of maximal real-rank. But the contrary happens for the *inessential* action of $O(p,q)_{\mathbf{Z}}$ on the flat pseudo-Riemannian torus $(\mathbf{T}^{p+q}, -\mathrm{d}x_1^2 - \cdots - \mathrm{d}x_p^2 + \mathrm{d}x_{p+1}^2 + \cdots + \mathrm{d}x_{p+q}^2)$, whose conformal group reduces to its isometry group $O(p,q)_{\mathbf{Z}} \ltimes \mathbf{T}^{p+q}$.

The extension question. Recall Corollary 1, which implies that $O(p,q)$ does not act isometrically on closed pseudo-Riemannian manifolds of signature (p,q), except when $p+q \leqslant 3$. So, for greater values of p and q, inessentiality of a conformal $O(p,q)_{\mathbf{Z}}$-action is an obstruction to the extension.

Remark 17. In [FM22], Fisher and Melnick observed that volume preserving actions of $\mathrm{SL}_n(\mathbf{Z})$ on a compact n-manifold does not extend to a $\widetilde{\mathrm{SL}}_n(\mathbf{R})$ action.

On this example, the obstruction comes from the fact that only elements of $O(p,q)$ normalizing the $\mathbf{Z}^n$-action on $\mathbf{R}^{p,q}$ descend to the quotient, and only a *local action* of $O(p,q)$ survives at the quotient. Although the question makes sense for general volume-preserving actions, in a first attempt, it seems interesting to work on the following.

Question 7. *Let $\Gamma = O(p,q)_{\mathbf{Z}}$ act by isometries on a closed pseudo-Riemannian manifold M. Does the lifted* [5] *action $\widetilde{\Gamma} \curvearrowright \widetilde{M}$ extend to an isometric action of $O(p,q)$?*

Eventually, an additional analytic assumption would be reasonable, in the same spirit as in the proof of Gromov's centralizer theorem (see the next chapter).

Question 8. *Let $\Gamma < O(p,q)$ be a lattice. Suppose that Γ acts conformally essentially on a closed pseudo-Riemannian of signature (p,q). Does the action extend to $O(p,q)$?*

4.3.3 Organization of the proof of Theorem G

Theorem G is in fact the combination of results from [Pec20, Pec24] and ongoing works for the Lorentzian case $p=1$ and the extension to non-uniform lattices. The proof in the uniform case can globally be divided as follows.

We denote by M^α the suspension space of a conformal action $\alpha : \Gamma \to \mathrm{Conf}(M,[g])$. Let A be a fixed $\mathbf{R}$-split Cartan subgroup of G.

Step 1: Lyapunov functionals of the A-action. Given an A-invariant, A-ergodic finite invariant measure μ on M^α, let $\chi_1,\ldots,\chi_r \in \mathfrak{a}^*$ denote its *vertical* Lyapunov functionals. The fact that Γ preserves the conformal class $[g]$ on M implies that there is a smooth, G-invariant, conformal class on the vertical tangent bundle of the fibration $M^\alpha \to G/\Gamma$. Consequently, the fiberwise dynamics of A is restricted, and Proposition 3.5 of [Pec20] gives that there $r \leqslant 2p+1$ and that there exists $\chi \in \mathfrak{a}^*$ such that, after reordering the functionals, $\chi_i + \chi_{r+1-i} = \chi$ for all $1 \leqslant i \leqslant r$.

Step 2: Upper bound on the rank and further restrictions Assume by contradiction $\ell = \mathrm{rk}_{\mathbf{R}}\, G > p+1$. In particular, G does not embed into $\mathbf{R}_{>0} \times O(p,q)$, the structural group of the H-structure associated to the conformal class $[g]$. Therefore, by Proposition **ref**, since $\alpha(\Gamma)$ is not relatively compact in $\mathrm{Conf}(M,[g])$, Γ does not preserve any finite invariant measure on M, and the same goes for the action of G on M^α

[5] Here, $\widetilde{\Gamma}$ is an extension of Γ by $\pi_1(M)$. It is not clear if the short exact sequence splits, so that Γ itself acts on the universal cover, like in the torus case.

by **ref**. Considering a finite measure on M^α which is invariant by the Borel subgroup containing A, we obtain a finite A-invariant, A-ergodic measure μ projecting to the Haar measure of G/Γ. Since μ *cannot* be G-invariant, Corollary 8 implies that $2p+1 \geqslant r(\mathfrak{g})$, a contradiction for every restricted root systems, except A_ℓ (see Table 1 of [BFH22]). For exceptional root-systems, this gives in fact better estimates, which led to Theorem H.

For G admitting A_ℓ as restricted root system, a variation of Theorem 4.6 can be used (see §4.2.1 of [Pec20]), together with the fact that $\chi_1, \dots, \chi_r$ span a subspace of $\mathfrak{a}^*$ of dimension at most $\lfloor \frac{r}{2} \rfloor + 1$.

Step 3: Uniform contractions at the critical case. For $\mathrm{rk}_R\, G = p+1$, G does not embed into $\mathbf{R}_{>0} \times O(p,q)$ neither. So, similarly to the previous step, no finite G-invariant measure exist on M^α. But, this time, no contradiction follows. Instead, from the fact that the stabilizer G_μ of an A-invariant measure μ is a strict subgroup of G, Theorem 4.6 implies that a big number of Lyapunov fonctionals are μ-resonant, hence must follow a pattern related to the roots.

Combining this with the linear relations they satisfy, it follows that a direction $X \in \mathfrak{a}$ satisfies $\chi_1(X) = \cdots = \chi_r(X) = -1$. Consequently, the local stable manifold of the flow associated to e^{tX} on M^α contains an open subset of the fiber. Considering recurrent points of this flow, we can interpret this information in the Γ-action: there exist an open subset U, $x \in U$, a sequence $\{\gamma_k\} \subset \Gamma$ and $T_k \to +\infty$ such that $\gamma_k U \to \{x\}$ (for the Hausdorff distance), and

$$\frac{1}{T_k} \log \|\gamma_{k*} v\| \to -1, \tag{4.5}$$

for any non-zero $v \in TU$.

This procedure can be repeated for any Borel-invariant measure in M^α, so for any compact Γ-invariant subset K of M, there exist $x \in K$, U a neighborhood of x, $\{\gamma_k\}$ and $\{T_k\}$ satisfying (4.5).

Step 4: Vanishing of the Weyl curvature and existence of maximal charts. By standard geometric arguments, the Weyl tensor of $(M,[g])$ can tolerate such dynamics only if it is identically zero over U. If it was not identically zero over M, we would obtain a contradiction by choosing the measure μ supported in the boundary of the Γ-invariant compact subset $\{x \in M \mid W_x = 0\}$.

So, the manifold is conformally flat when $\mathrm{rk}_{\mathbf{R}}\, G = p+1$: every point has a neighborhood conformally diffeomorphic to an open subset of the flat space $\mathbf{R}^{p,q}$. We can do better: starting with a small open subset U with the dynamical data (4.5), we can "go backward" an consider the $\gamma_k^{-1}.U$ which get uniformly larger and larger, take a limit in some sense, to end up with a neighborhood U_∞ of x which is conformally equivalent to the whole $\mathbf{R}^{p,q}$. This forbids for instance M to be a pseudo-Riemannian torus.

Hence, M is in fact covered by open subsets, conformally equivalent to $\mathbf{R}^{p,q}$. We call such subsets maximal charts.

Step 5: Completeness of the geometric structure This final step is based on the global geometry of the model space $\widetilde{\mathbf{Ein}}^{p,q}$, and this is where the Lorentzian and pseudo-Riemannian case diverge. By a generalization of Liouville's theorem, any subset of $\widetilde{\mathbf{Ein}}^{p,q}$ conformally equivalent to $\mathbf{R}^{p,q}$ is a Minkowski patch, *i.e.* a connected component of the complement of a light-cone in $\widetilde{\mathbf{Ein}}^{p,q}$. The previous step shows that $\widetilde{M}$ is covered by open subsets $\{U_i\}$ such that the developing map $\mathcal{D} : \widetilde{M} \to \widetilde{\mathbf{Ein}}^{p,q}$ is injective in restriction to every U_i and sends it onto a Minkowski patch of $\widetilde{\mathbf{Ein}}^{p,q}$. For $\min(p,q) > 1$, Minkowski patches are arranged very similarly to hemispheres in the projective sphere. Injectivity of $\mathcal{D}$ can then be deduced by combining a certain number of elementary arguments, and the result follows easily after that (see the beginning of §.5 of [Pec24]).

The main difference with $\widetilde{\mathbf{Ein}}^{1,n-1}$ resides in the fact that the complement of a light-cone has infinitely many connected components, whereas we only get two "antipodal" Minkowski patches in higher signatures. So, monodrony phenomena seem technically difficult to rule out when trying to extend injectivity of the developing map from patch to patch.

A different global approach, nonetheless shows that in Lorentzian signature, a conformal structure on a compact manifold satisfying the conclusions of Step 4 is *globally hyperbolic*. Results on globally hyperbolic conformally flat space-times [Sal12, Sma23b, Sma23a] ultimately conclude that $\widetilde{M}$ is conformal to $\widetilde{\mathbf{Ein}}^{1,n-1}$.

4.3.4 Lower bounds for the metric index

Similarly to what has been discussed for Lie group actions in Section 3.3, it is natural to ask if other constraints than a bound on the real-rank can be obtained.

Question 9. *Let Γ be an irreducible lattice in a higher-rank semi-simple Lie group G. Let k_G^0 as defined in Section 3.3.1. Does there exist a compact manifold M, with a conformal structure $[g]$ of signature (p,q), with $\min(p,q) < k_G^0$, and with an unbounded conformal action $\Gamma \to \mathrm{Conf}(M,[g])$?*

Some technical considerations in Step 2 on the structure of the restricted-roots which are transverse to the stabilizer of the A-invariant measure μ (which is known to be a maximal parabolic subgroup of G), combined with the fact that they all have to be positively proportional to exactly one Lyapunov functional, led to the following.

Surprisingly, these considerations provide better estimates only for exceptional root-systems. It seems to be proper to conformal geometry and its model space $(\mathrm{PO}(p+1,q+1),P)$, which dictates the general configuration of Lyapunov functionals. It is plausible that this approach will give interesting conclusions for actions on other parabolic geometries.

Theorem H ([Pec20])**.** *Let $(M,[g])$ be a compact manifold of dimension at least 3 endowed with a conformal structure of signature (p,q). Let Γ be a lattice in a simple Lie group G whose restricted root-system is Σ. Assume that there exists a conformal action $\rho : \Gamma \to \mathrm{Conf}(M,[g])$ such that $\rho(\Gamma)$ is not relatively compact.*

1. *If* $\Sigma = E_6$*, then* $\min(p,q) \geqslant 8$.
2. *If* $\Sigma = E_7$*, then* $\min(p,q) \geqslant 14$.
3. *If* $\Sigma = E_8$*, then* $\min(p,q) \geqslant 28$.
4. *If* $\Sigma = F_4$*, then* $\min(p,q) \geqslant 7$.
5. *If* $\Sigma = G_2$*, then* $\min(p,q) \geqslant 2$.

Remark 18. Furthermore, we can deduce from the proof of Theorem G:

1. If $\Sigma = F_4$, a compact manifold of metric index 7 on which such a lattice acts conformally with unbounded image is conformally equivalent to $\mathbf{Ein}^{p,q}$ or $\widetilde{\mathbf{Ein}}^{p,q}$.
2. If $\Sigma = E_8$, a compact manifold of metric index 28 on which such a lattice acts conformally with unbounded image is conformally equivalent to $\mathbf{Ein}^{p,q}$ or $\widetilde{\mathbf{Ein}}^{p,q}$.

Conformal flatness was established in Section 7 of [Pec20], but it was not pointed out in [Pec24] that the global conclusion follows similarly, although it is immediate: the working assumption in that paper was $\min(p,q) > 1$ and for any compact Γ-invariant subset, the existence of a sequence $\{\gamma_n\} \subset \Gamma$ with an assymptotically uniformly contracting dynamics near a point in this compact subset. That is exactly what is obtained at the end of [Pec20] via elementary considerations on the resonance.

I suspect that these actions do not exist. The problem is purely a question of representation theory: by Margulis' Super-rigidity Theorem, the question is that of an embedding of $\mathfrak{g}$ into $\mathfrak{so}(p+1,q+1)$. For instance, the split form $\mathfrak{f}_{4(4)}$ of $\mathfrak{f}_4$ has restricted root-system F_4 and the lowest degree representation of $\mathfrak{f}_{4(4)}$ is 26-dimensional and preserves a quadratic form of signature $(12,14)$ (see Section 5 of [Pec19]), it seems to indicate that there is no embedding into $\mathfrak{so}(8,N)$ for any N. So far, I do not have the answer.

4.4 Extension to projective actions and other geometric structures

I describe in this section ongoing research activities, which are natural subsequent developments following the results on actions of lattices that have been detailed above.

In [Pec24], it is observed that the proof of the global part of Theorem G in the non-Lorentzian case $\min(p,q) > 1$ applies *verbatim* to a similar context where a lattice in a simple Lie group of real-rank n acts projectively on a compact n-manifold endowed with a linear connection. This is due to a global resemblance between $\mathbf{Ein}^{p,q}$ and $\mathbf{R}P^n$, when $\min(p,q) > 1$. Hence, a global statement for projective actions was another outcome of this approach.

Theorem I ([Pec24]). *Let* Γ *be a lattice in a simple Lie group* G *of real-rank* $n \geqslant 2$ *and finite center. Let* M *be a compact* n*-manifold on which* Γ *acts by preserving a projective class of linear connections* $[\nabla]$.

If the action is infinite, then $(M, [\nabla])$ *is projectively equivalent to either* $\mathbf{R}P^n$ *or* $\mathbf{S}^n$*, endowed with their standard projective structures.*

This result was in fact expected to be true without assuming the existence of an invariant projective structure (for instance [BRHW22] Conjecture 1.1 and 1.8, or Question 4.8 of [Fis11]). Although, in a "non-geometric" setting, exotic volume-preserving actions of $\mathrm{SL}_n(\mathbf{Z})$ at the critical dimension n indicates a wide range of possible topologies for volume-preserving actions (see [KL96, BF05, FM22]), none of these examples can be adapted to compact manifolds of dimension $n-1$.

Recently, Brown, Rodriguez-Hertz and Wang have announced a proof of this result in the general differentiable case, *i.e.* without requiring that the action preserves a projective class of connections.

Nonetheless, the proof of Theorem I indicates that it can be extended to other types of geometric structures, for which either the current state of the art does not suggest an accessible, non-geometric proof ready to be developed, or even no analogue at all can be expected.

4.4.1 General bound on the real-rank

In [Zim87b], Zimmer established that if a simple Lie group G acts non-trivially on a compact manifold by preserving an H-structure, with H algebraic, then $\mathrm{rk}_{\mathbf{R}} G \leqslant \mathrm{rk}_{\mathbf{R}} H$, where the latter is naturally understood as the dimension of maximal $\mathbf{R}$-split tori of H. For instance, if $H = \mathbf{R}^* \times O(p,q)$, which corresponds to conformal actions of G on pseudo-Riemannian manifolds of signature (p,q), it means that $\mathrm{rk}_{\mathbf{R}} G \leqslant \min(p,q)+1$.

This observation was extended later in [BFM09] to G-actions preserving Cartan connections, and interestingly to non-simple Lie groups G (as recalled in Section 3.1.3). The conclusion relates similarly, via an inequality, the algebraic rank of the acting group with the algebraic rank of the structural group of the Cartan bundle. But it does not apply to discrete group actions.

Not yet prepublished and directly based on a paper of Kaimanovich [Kai89], the following result systematizes the control on Lyapunov functionals of a probability measure preserving action of an abelian Lie group action which preserves certain geometric structures. These are obtained by projecting the restricted-roots of a semi-simple Lie algebra to some subspace of the dual of the Cartan subalgebra.

Proposition 4.4. *Let H be a semi-simple Lie group without compact factor. Let $A = \mathbf{R}^k$ act differentiably on a manifold M by preserving a finite measure μ, which is also A-ergodic. Let $c : A \times M \to \mathrm{Ad}(H)$ be a cocycle satisfying the assumptions of Theorem 4.4. Then, there exist an $\mathbf{R}$-split Cartan subalgebra $\mathfrak{a}_{\mathfrak{h}}$ of $\mathfrak{h}$ and a linear map $f : \mathfrak{a} \to \mathfrak{a}_{\mathfrak{h}}$ such that the Lyapunov functionals of c with respect to μ are $\{\lambda \circ f,\ \lambda \in \Sigma\}$, where Σ denotes the set of restricted-roots attached to $\mathfrak{a}_{\mathfrak{h}}$.*

Kaimanovich's paper, later generalized by Karlsson and Margulis [KM99] to CAT(0) spaces, interprets geometrically the conclusions of Oseledets' theorem (see also §3 of [Fil19]). We call *Lyapunov regular* any sequence (g_k) of matrices in $\mathrm{GL}_n(\mathbf{R})$ for which there exists a flag $\{0\} = V_0 \subset \cdots \subset V_r = \mathbf{R}^n$ and numbers $\lambda_1 < \cdots < \lambda_r$ such that $\frac{1}{k}\log|g_k v| \to \lambda_i$ for all $v \in V_i \setminus V_{i-1}$, and such that $\lim \frac{1}{k}\log|\det g_k|$ exists and is finite. In particular, Oseledets' theorem means that under an integrability condition on the generator of the cocycle $c : \mathbf{Z} \times X \to \mathrm{GL}_n(\mathbf{R})$, for almost every x, the sequence $g_k := c(k,x)$ is (forward and backward) Lyapunov regular. In this approach, this regularity is reduced to a geometric property of the sequence of points $p_k = g_k'.o$, with $g_k' \in \mathrm{SL}_n^{\pm}(\mathbf{R})$ a renormalization of g_k and $o \in \mathrm{SL}_n^{\pm}(\mathbf{R})/O(n)$ an origin in the corresponding symmetric space of non-compact type: (g_k) is forward Lyapunov regular if and only if there exists a geodesic ray γ_v emanating from o such that $d(p_k, \gamma_v(k))$ grows sub-linearly.

In this perspective, Proposition 4.4 above means that there exists a totally geodesic flat in H/K, together with a linear parametrization, such that the A-orbit of the origin of this flat stays at sub-linear distance from the flat.

In the direction of a discrete version of [Zim87b]. Suppose that $\Gamma < G$ acts on a compact manifold by preserving an H-structure $P \to M$. Proposition 4.4 can then be applied to the action of an $\mathbf{R}$-split Cartan subgroup $A < G$ on the suspension space M^α of the action. Indeed, the fact that Γ preserves an H-structure on M means that the induced action of G preserves an H-structure on the vertical tangent bundle of the suspension space M^α, hence automatically the linear cocycle associated to the vertical differential action is cohomologuous to an H-valued cocycle $G \times M^\alpha \to H$.

With the global motivation of generalizing [Zim87b] to lattices actions, a first step is the next intermediate result, which does not need the geometric structure to be rigid in any sense (in particular not being of finite type). Ideally, we would like to assume H to be a real algebraic subgroup of $D^r(n)$, for general H-structures.

Theorem 4.8. *Let M be a compact manifold endowed with an H-structure of order 1, $\pi : P \to M$, with $H < \mathrm{GL}_n(\mathbf{R})$ algebraic and isogenous to $T \times H_s$, where H_s is an almost algebraic semi-simple group and T is an algebraic torus. Let $\Gamma < G$ be a uniform lattice in a simple Lie group of real-rank at least 2 and finite center. Suppose that Γ acts on M by preserving the H-structure and that the action is unbounded.*

Then, $\mathrm{rk}_{\mathbf{R}} G \leqslant \mathrm{rk}_{\mathbf{R}} H$, the latter being understood as the dimension of maximal $\mathbf{R}$-split tori of H.

Proof. Here the proof follows more or less directly from that of Theorem 1.1 of [BFH22] in the volume-preserving case. Low dimensionality is used to restrict the number of vertical Lyapunov functionals $\{\chi_1, \ldots, \chi_r\}$ of a carefully chosen A-invariant measure on the suspension space. The volume preserving assumption gives the additional restriction $\chi_1 + \cdots + \chi_r = 0$ which increases by 1 the dimensional lower bound. We can follow exactly the same path but finally use Proposition 4.4 to obtain the restriction in terms of the rank of H. □

4.4.2 Some geometric questions

Applying this result to the case $H = \mathrm{GL}_n(\mathbf{C})$, we obtain that if Γ acts holomorphically on a compact manifold with almost-complex structure of *complex* dimension n, then

$\mathrm{rk}_{\mathbf{R}} G \leqslant n$. This is simply due to the value of the real-rank of $\mathfrak{gl}_n(\mathbf{C})$.

This bound was already known from Cantat ([Can04], see also [CZ12]), in the case of holomorphic actions of lattices on compact *Kähler manifolds*. His conclusion is stronger, and he proves that if the real-rank of G equals n, then the manifold in question is biholomorphic to $\mathbf{C}P^n$. An interesting direction of research would be to generalize or disprove Cantat's and Cantat-Zeghib's geometric results in the setting of holomorphic actions on complex, or even almost-complex manifolds.

Question 10. *Let Γ of real-rank $n \geqslant 2$ act holomorphically on a compact complex n-manifold M. Is M biholomorphic to $\mathbf{C}P^n$? Similarly if M is a $2n$-dimensional compact real manifold with an almost-complex structure.*

Recall that for a compact complex manifold M, the group of biholomorphisms is a Lie group (for global reasons), although a complex structure is not rigid in Gromov's sense. In particular, a closed group of biholomorphisms can *a priori* act non-properly on all the frame bundles $\mathcal{F}^r(M)$, so it is not possible to rule out from the beginning the existence of finite Γ-invariant measures as in Proposition 4.2.

Another corollary concerns actions of lattices actions on *Cartan geometries*.

Corollary 10. *Let $(M, \mathcal{G}, \omega)$ be a compact manifold endowed with a Cartan geometry modeled on a flag manifold $\mathbf{G}/\mathbf{P}$. Suppose that Γ acts on M by automorphisms of the Cartan geometry, with unbouded image. Then,* $\mathrm{rk}_{\mathbf{R}} G \leqslant \mathrm{rk}_{\mathbf{R}} \mathbf{G}$.

Here the proofs is again not long and relies on the fact that any Cartan geometry defines an H-structure, with structure group the adjoint action of $\mathbf{P}$ on $\mathfrak{g}_{\mathbf{X}}/\mathfrak{p}_{\mathbf{X}}$, where $\mathfrak{g}_{\mathbf{X}}$ and $\mathfrak{p}_{\mathbf{X}}$ refer to the Lie algebras of $\mathbf{G}$ and $\mathbf{P}$ respectively. The dimension of $\mathbf{R}$-split tori in this adjoint group is the same as the dimension of those of $\mathbf{G}$. Hence, observing similarly as before that the Γ-invariant Cartan geometry on M defines a G-invariant "fiberwise Cartan geometry" on the suspension space M^α, a routine adaptation yields the conclusion of the corollary.

However, a generalization of Theorem G to general parabolic geometries needs additional ingredients, at the local and global scale.

Question 11. *Let $(M, \mathcal{G}, \omega)$ be a compact manifold endowed with a Cartan geometry modeled on a flag manifold $\mathbf{G}/\mathbf{P}$. Suppose that Γ has real-rank* $\mathrm{rk}_{\mathbf{R}} \mathbf{G}$, *acts on M by automorphisms of the Cartan geometry and with unbouded image. Up to finite covers and quotients, is M isomorphic to the model space $\mathbf{G}/\mathbf{P}$?*

Chapter 5

Conformal extension of D'Ambra's theorem

5.1 Generalities

In the 1980's, D'Ambra proved the following impressive result in Lorentzian geometry.

Theorem 5.1 ([D'A88]). *Let (M,g) be a real-analytic, compact, simply-connected Lorentzian manifold. Then its isometry group* $\mathrm{Isom}(M,g)$ *is compact.*

Let us first give some examples of such manifolds: any compact simply-connected Lie group (*e.g.* $\mathrm{SU}(n)$) can be given an arbitrary left-invariant Lorentzian metric, or more generally, any compact simply-connected manifold M with zero Euler characteristic (an odd dimensional sphere for instance) admits a Lorentzian metric. Warped product of such a manifold with an arbitrary compact simply-connected Riemannian manifold provides additional examples.

If the analyticity assumption is believed to be unnecessary[1], compactness and simple-connectedness are clearly required. Notably, Adams-Stuck-Zeghib's classification result of isometry groups of compact Lorentzian manifolds provides the whole list of non-compact Lie groups which can arise as group of isometries of a compact Lorentzian manifold (see Theorem 3.5 before). The list includes simple, nilpotent and solvable Lie groups. D'Ambra's theorem implies that in the analytic case, if one of these groups act on a closed Lorentzian manifold, then the latter has infinite fundamental group.

Remark 19. Recall that a closed Lorentzian manifold with a locally faithful isometric action of $\widetilde{\mathrm{SL}}_2(\mathbf{R})$ is isometrically covered by a warped product of $\widetilde{\mathbf{AdS}}^3 \times_\omega N$ for some Riemannian manifold N by [Gro88]. So the initial manifold has infinite fundamental group. This important example is a basic case of Gromov's representation theorem recalled below. The difficult part in Theorem 5.1 is when $\mathrm{Isom}(M,g)$ is non-compact but has poor algebraic structure compared to one which contains a local copy of $\mathrm{SL}_2(\mathbf{R})$, *e.g.* when $\mathrm{Isom}(M,g)$ is isomorphic to $\mathbf{R}$.

[1] To my knowledge, no proof valid in $\mathcal{C}^\infty$ regularity has been established so far.

5.1.1 Extension to the conformal setting

Many examples of Lorentzian manifolds have essential conformal group: their conformal group does not reduce to the isometry group of any metric in the conformal class (see Definition 3.5). For instance, Lorentzian Hopf manifolds, or Frances' examples of quotients of certain domains of discontinuity in $\mathbf{Ein}^{1,n-1}$ of "Lorentzian Kleinian groups", and some deformations of such structures ([Fra05]), are essential compact Lorentzian manifolds. These examples show a certain abundance of compact essential geometries.

Consequently, it could *a priori* happen that a compact, simply-connected Lorentzian manifold has non-compact conformal group, while having compact isometry group. Remark however that the previous examples are all obtained by modding out a simply-connected open domain Ω of the model space by an infinite group of conformal maps of Ω.

In a joint work with Melnick, we extended D'Ambra's theorem and proved that similarly to the isometric case, in analytic regularity, having infinite fundamental group is a necessary condition to the non-compactness of the conformal group.

Theorem J ([MP22])**.** *Let* $(M,[g])$ *be a real-analytic, compact, simply-connected conformal Lorentzian structure. Then its conformal group* $\mathrm{Conf}(M,g)$ *is compact.*

5.1.2 Relation to Lorentzian Lichnerowicz conjecture

By a standard averaging argument, a compact subgroup of $\mathrm{Conf}(M,[g])$ always preserves a metric in the conformal class. Consequently, Theorem 5.1 implies that for a compact, simply connected, real-analytic Lorentzian manifold (M,g), its conformal group is non-compact if and only if it is essential. Therefore, if true, Lorentzian Lichnerowicz conjecture (Conjecture 1) would imply that if $\mathrm{Conf}(M,[g])$ is non-compact, then $(M,[g])$ is conformally flat. But this is impossible for a very simple reason: the universal cover of $\mathbf{Ein}^{1,n-1}$ is non-compact, and consequently,

Proposition 5.1. *A Lorentzian conformal structure* $[g]$ *on a compact simply connected manifold* M *cannot be conformally flat.*

Proof. The developing map $\mathcal{D} : M \to \widetilde{\mathbf{Ein}}^{1,n-1}$ of a conformally flat structure on $M = \widetilde{M}$ would have to be a covering map by compactness of M, a contradiction. □

Thus, Theorem J is equivalent to the Lorentzian Lichnerowicz conjecture in the special case of real-analytic Lorentzian manifolds with finite fundamental group. This interpretation of the result is in fact leading globally our proof, since we use intensively the conflict between Proposition 5.1 and the curvature restrictions following from the existence of a diverging sequence $\{g_n\} \subset \mathrm{Conf}(M,[g])$.

Remark that among all signatures, Lorentzian structures are characterized by non-compactness of their model space $\widetilde{\mathbf{Ein}}^{1,n-1}$ (see Section 3.2.1 and 4.3.2). In signature (p,q) with $\min(p,q) > 1$, the conformal group of $\widetilde{\mathbf{Ein}}^{p,q} = (\mathbf{S}^p \times \mathbf{S}^q, [-g_{\mathbf{S}^p} \oplus g_{\mathbf{S}^q}])$ is isomorphic to $O(p+1,q+1)$. So, contrarily to the isometric case, it is much simpler to observe that Theorem J is purely Lorentzian, and it explains more naturally the reason in terms of model spaces.

5.1.3 General topological obstruction in the volume-preserving case: Gromov's representation of the fundamental group theorem.

In Theorem 5.1, modulo an extra (or believed to be extra) analyticity assumption, D'Ambra proved that for a compact Lorentzian manifold, the non-compactness of the isometry group forces the underlying topology to have infinite fundamental group. In fact, her theorem gives an echo to a more general theorem on which several authors contributed ([Gro88, DG91, FZ02, Zim89, CQB03] among others), and motivates the investigation of automorphisms groups of unimodular geometric structures with finite fundamental group. The formulation below is, as customary in this memoir, a weakened version of the optimal results.

Theorem 5.2 ([Gro88])**.** *Let G be a non-compact simple Lie group which acts by automorphisms of a real-analytic rigid geometric structure (M, ϕ). Suppose that the G-action also preserves a volume form and is ergodic with respect to Lebesgue's measure class.*

Then, there exists a finite-dimensional representation $\rho : \pi_1(M) \to \mathrm{GL}_d(\mathbf{R})$ such that the Zariski closure $\overline{\rho(\pi_1(M))}^Z$ contains a Lie subgroup locally isomorphic to G.

Consequently, under the assumptions of the previous theorem, the fundamental group of M cannot be amenable because the Zariski closure of an amenable group is still amenable ([Zim84a], Prop. 4.1.15). And even when the fundamental group is non amenable, this result says somehow that it has to be at least "as big as G". For example, a straight application of Margulis' super rigidity implies that $G = \mathrm{SL}_4(\mathbf{R})$ cannot act on a compact manifold whose fundamental group is isomorphic to a lattice in $\mathrm{SL}_3(\mathbf{R})$ and by preserving a real-analytic unimodular rigid geometric structure (*e.g.* by preserving an analytic volume form and a linear connection).

However, these general observations only apply to non-compact simple Lie group actions (and their lattices by [FZ02]) and not for non-semisimple algebraic structures. In a way, [D'A88] establishes an extension of these obstructions to any non-compact group action for Lorentzian metrics, but it is also proved that these are singular among pseudo-Riemannian metrics. Examples of real-analytic, pseudo-Riemannian metrics of signature $(7, 2)$ on $\mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^3$ with non-compact isometry group are constructed in her paper, so Theorem 5.2 does not seem to extend to non-semisimple G. Remarkably, the only reason D'Ambra's proof does not extend to general pseudo-Riemannian metrics is a very elementary result on linear isometries (see Lemma 2 below).

5.2 Gromov's results on rigid geometric structures

An important aspect of D'Ambra's proof is the use that it makes of Gromov's theory of rigid geometric structures. Many of these results work in great generality, including conformal geometry. So, we started investigating what remains from D'Ambra's approach in our case.

Remark that in dimension 2, because a closed surface with Euler characteristic 0 is not simply connected, Theorem J is relevant in dimension at least 3, so the conformal class defines a rigid geometric structure.

5.2.1 Gromov's stratification theorem and D'Ambra's trick

Recall that we take for definition of geometric structure on M an equivariant map $\phi : \mathcal{F}^r(M) \to W$, where W is a manifold acted upon by the structural group D^r of the r-frame bundle. A very convenient property of Gromov's rigid structures is that if ϕ^1 is rigid and ϕ^2 is not, but both are of algebraic type, then their disjoint union $(\phi_1, \phi_2) : \mathcal{F}^{\max(r_1,r_2)}(M) \to W_1 \times W_2$ is a rigid geometric structure of algebraic type. A very interesting idea was to consider, on the simply connected Lorentzian manifold she's considering, a maximal family $\{X_1, \ldots, X_k\}$ of pairwise commuting Killing vector fields, and use that the disjoint union $\phi = (g, X_1, \ldots, X_k)$ of the metric *and* this family of vector fields is rigid. The same can be done when g is replaced by its conformal class.

Because the family is chosen to be maximal abelian, the identity component of $\operatorname{Aut}(M, \phi) = \{f \in \operatorname{Isom}(M, g) \mid f^*X_i = X_i,\ i = 1, \ldots, k\}$ is the Lie subgroup $A < \operatorname{Isom}(M, g)$ tangent to the abelian Lie algebra spanned by $X_1, \ldots, X_k$.

Realizing this maximal abelian subgroup A *as* the identity component of the automorphism group of a rigid geometric structure as very strong consequences on its orbit structure, precisely because of the analytic regularity and simple-connectedness.

The following result is [Gro88], § 3.4. and 3.5.

Theorem 5.3. *Let (M, ϕ) be a compact, simply-connected manifold with an analytic rigid geometric structure of algebraic type. Let G be its automorphisms group. Then, there exists a stratification $M = \Omega_1 \cup \cdots \cup \Omega_k$, with Ω_i open and dense in $\cup_{j \geqslant i} \Omega_j$, and for all i, a G-invariant map of constant rank $\phi_i : \Omega_i \to W_i$ such that for all $x \in \Omega_i$, the G-orbit of x coincides with the connected component of $f^{-1}(f(x))$ that contains x.*

Proof. Let us very briefly sketch the proof for (M, ϕ) a Cartan geometry of model space $(\mathbf{G}, \mathbf{H})$ such that $\operatorname{Ad}_{\mathfrak{g}}(\mathbf{H})$ is algebraic (a detailed proof is given in [Mel11], Theorem 4.1). Theorem K below is valid over all of M by analyticity and compactness of M. Consequently, the curvature map and its covariant derivatives up to order $\dim \mathbf{G}$ give rise to an $\mathbf{H}$-equivariant map $\alpha : \mathcal{G} \to V$ such that for all $b \in \mathcal{G}$ and $u \in T_b\mathcal{G}$, $(u.\alpha)(b) = 0$ if and only if there exists a local Killing field X of $(M, \mathcal{G}, \omega)$ defined on a neighborhood of b and such that $X(b) = u$. By analyticity and simple-connectedness, X extends to a (complete) Killing field of the Cartan geometry. Therefore, level sets of α locally coincide with G-orbits on the total space $\mathcal{G}$. Finally, because the action of $\operatorname{Ad}_{\mathfrak{g}}(\mathbf{H})$ on V is algebraic, Rosenlicht stratification theorem gives an algebraic stratification $V = V_1 \cup \cdots \cup V_k$ such that for all i, the projection $p_i : V_i \to V_i / \operatorname{Ad}_{\mathfrak{g}}(\mathbf{H})$ is a submersion onto a smooth algebraic variety. Finally, taking $\Omega_i := \pi(\alpha^{-1}(V_i))$, where π denotes the projection of the Cartan bundle, we get a constant rank map $\phi_i : \Omega_i \to V_i / \operatorname{Ad}_{\mathfrak{g}}(\mathbf{H})$, such that $\phi_i \circ \pi = p_i \circ \alpha$, and whose level sets coincide locally with G-orbits on M. □

Corollary 11. *In the same context, we have the following:*

1. *G has finitely many connected components.*
2. *For all $x \in M$, the stabilizer G_x has finitely many connected components.*
3. *For all $x \in M$, there exists a compact G-orbit in the closure of $G.x$.*
4. *For all $x \in M$, $G.x$ is locally closed and the orbit closure $\overline{G.x}$ is semi-analytic and locally connected.*

A key feature, which fails to be true if analyticity is removed [2], is the following classic extension property of local Killing vector fields. This seems to be a major difficulty for extending D'Ambra's proof to non-analytic Lorentzian manifolds.

Proposition 5.2 ([Amo79]). *Let $P \to M$ be a real-analytic, finite-type G-structure on a simply connected manifold M. If X is a local Killing field (defined only on an open subset), then X extends to a globally defined Killing field of* $\mathrm{Aut}(P \to M)$.

5.2.2 Integration of local isometries of Cartan geometries: an elementary proof of Frobenius's theorem

Gromov's stratification theorem can be proved in a simplified context, compared to his very general A-rigid geometric structures. It is a consequence of Theorem K below, in the case of a Cartan geometry with an additional structure, which is enough for the proof of Theorem J.

I present here one of the main contributions of [Pec16] which gave an elementary proof of the "Frobenius theorem" stated in [Gro88], §1. In this work of my PhD, the target space W of the additional structure ϕ is assumed to be a quasi-projective variety over $\mathbf{R}$ to authorize examples such as fields of k-planes (W being a Grassmanian), but here I restrict to W simply being a vector space, which is what we need to understand D'Ambra's approach. Let $(M, \mathcal{G}, \omega)$ be a Cartan geometry with model space $(\mathbf{G}, \mathbf{H})$ on a manifold M, with $\mathrm{Ad}_{\mathfrak{g}}(\mathbf{H})$ algebraic. Let W be a vector space with an algebraic action of $\mathrm{Ad}_{\mathfrak{g}}(\mathbf{H})$ and let $\phi : \mathcal{G} \to W$ be a P-equivariant map.

Definition 5.1. A local Killing vector field of $(M, \mathcal{G}, \omega, \phi)$ is a vector field X defined on an open subset U of $\mathcal{G}$, and such that $\mathcal{L}_X \omega = 0$ and $\mathcal{L}_X \phi = 0$. A global Killing vector field is a local Killing vector field defined on $U = \mathcal{G}$.

Since ω trivializes the tangent bundle $T\mathcal{G}$, the curvature 2-form $\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$ gives rise to a $\mathbf{H}$-equivariant map $\kappa : \mathcal{G} \to \mathrm{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$, called the curvature map. For the same reason, given a vector space valued, $\mathbf{H}$-equivariant map $F : \mathcal{G} \to E$, its differential $\mathrm{d}F$ identifies, in the trivialization, with an $\mathbf{H}$-equivariant map $\mathcal{G} \to E \oplus \mathrm{Hom}(\mathfrak{g}, E)$. The differenciation can therefore be iterated, and the result is still an equivariant map

[2] Pick for instance a smooth Riemannian metric on $\mathbf{R}^n$ which is flat on a small ball and generic (*i.e.* with no local isometry at all) elsewhere.

with values in a vector space, which gets bigger and bigger. For $\mathcal{G} = \mathcal{F}^1(M)$ the linear frame bundle and ω defined by a linear connection form, tensors of type (r, s) on M are equivariant maps from $\mathcal{G}$ to vector spaces $((\mathbf{R}^n)^*)^{\otimes r} \otimes (\mathbf{R}^n)^{\otimes s}$ and this differentiation corresponds to taking the covariant derivative.

In the same context, and for $r \geqslant 1$, we define $E = \mathrm{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g}) \oplus W$, $\kappa_\phi = (\kappa, \phi) : \mathcal{G} \to E$ and $\kappa_\phi^r : \mathcal{G} \to E \oplus \mathrm{Hom}(\mathfrak{g}, E) \oplus \cdots \oplus \mathrm{Hom}(\otimes^r \mathfrak{g}, E)$ the $\mathbf{H}$-equivariant map obtained after having differentiated r times κ_ϕ.

Definition 5.2. For any $r \geqslant 1$ and $b \in \mathcal{G}$, we call *Killing generator of order r* any $u \in T_b\mathcal{G}$ such that $(u.\kappa_\phi^r)(b) = 0$. We denote by $\mathrm{Kill}_\phi^r(b) \subset T_b\mathcal{G}$ the subspace of Killing generators of order r of the disjoint union of the Cartan geometry $(M, \mathcal{G}, \omega)$ and the equivariant map ϕ.

Automatically, if X is a local Killing vector field defined on an open subset U of $\mathcal{G}$, then for any $b \in U$, $X(b)$ is a Killing generator at any order $r \geqslant 1$. Frobenius' theorem gives the converse statement, for $r = \dim \mathbf{G}$, but in restriction to an *open-dense subset*, called the *integrability locus*. The following is Theorem 4.19 of [Pec16].

Theorem K. *Let $(M, \mathcal{G}, \omega)$ be a Cartan geometry with model space $(\mathbf{G}, \mathbf{H})$, such that $\mathrm{Ad}_\mathfrak{g}(\mathbf{H})$ is an algebraic subgroup of $\mathrm{GL}(\mathfrak{g})$, and let $\phi : \mathcal{G} \to W$ be an equivariant map, where W is a vector space with an action of $\mathrm{Ad}_\mathfrak{g}(\mathbf{H})$.*

Then, there exists an open dense subset $\Omega \subset M$ such that for any $b \in \mathcal{G}$ projecting to Ω and any $u \in \mathrm{Kill}_\phi^r(b)$, there exists a local Killing vector field X defined on a neighborhood of b and such that $X(b) = u$.

For $(M, \mathcal{G}, \omega)$ real-analytic and M compact, there exists an integer r' (depending a priori on the geometric structure) such that the conclusion above is valid with $\Omega = M$.

Proof. The proof reduces easily to the most elementary case: when the Cartan geometry is in fact the data of a global frame field $(X_1, \ldots, X_n)$ of M, *i.e.* an $\{e\}$-structure. Contrarily to Gromov's approach which uses partial differential relations, the construction of X is significantly simplified and is performed by applying Frobenius' integrability theorem of involutive distributions of k-planes.

Starting with a Killing generator of order r, and considering Ω the open-dense subset of M where the maps $\kappa_\phi, \kappa_\phi^1, \ldots, \kappa_\phi^r$ have locally constant rank, the idea is to build locally the graph of X in $\mathbf{R}^n \times \mathbf{R}^n$ by showing that an n-dimensional distribution to which it must be tangent is integrable when restricted to the subbundle of TM formed of Killing generators of order r. The core of this is a technical property of Killing generators very similar to an argument of Nomizu in Riemannian geometry ([Nom60]), which was extended by Melnick to Cartan geometries in [Mel11]. □

In a similar vein, [Pec16] also extends to Cartan geometries a theorem of Singer which was proved for Riemannian manifolds. In [Sin60], Singer gave necessary and sufficient conditions to local homogeneity of a Riemannian manifold (M^n, g) in terms of "constancy" of the Riemann curvature tensor R and its covariant derivatives $\nabla^r R$ up to order $r \leqslant \frac{n(n-1)}{2}$.

The following result extends Singer's theorem.

Theorem L ([Pec16])**.** *Let* $(M, \mathcal{G}, \omega)$ *be a Cartan geometry modeled on* $(\mathbf{G}, \mathbf{H})$ *and let* $r = \dim \mathbf{H}$. *If the maps* $\kappa, \kappa^1, \ldots, \kappa^r$ *all have range in a single* $\mathbf{H}$*-orbit, then the Cartan geometry is locally homogeneous, i.e. its pseudo-group of local automorphisms acts transitively.*

Recall that a Riemannian metric on M^n defines a Cartan geometry with model space $(O(n) \ltimes \mathbf{R}^n, O(n))$ whose local automorphisms are local isometries of the metric. Hence, Theorem L contains Singer's original theorem and extends it naturally to other geometries. This includes pseudo-Riemannian metrics, recovering an extension by Podesta-Spiro, but also new situation such as a torsion free linear connection[3] for which the same statement as Singer's works, but with order n^2, since the corresponding model space is $(\mathrm{GL}_n(\mathbf{R}) \ltimes \mathbf{R}^n, \mathrm{GL}_n(\mathbf{R}))$.

5.2.3 End of proof for the isometry group

We get back to the proof of Theorem 5.1 and (M, g) denotes a compact, simply-connected manifold with an analytic Lorentzian metric. Recall that any connected maximal abelian subgroup $A < \mathrm{Isom}(M, g)$ can be realized as the automorphims group of the disjoint union $g \cup \{X_1, \ldots, X_k\}$, where $X_1, \ldots, X_k$ are Killing vector fields spanning the Lie algebra of A.

As for any differentiable action, there exists an A-invariant, open-dense subset Ω_A on which A-orbits have locally constant dimension. A characteristic property of Lorentzian signature is that in fact, A acts locally freely on this open subset. It follows from the very elementary

Lemma 2. *Let* (V, q) *be a real finite dimensional vector space endowed with a non-degenerate quadratic form of signature* $(1, n-1)$. *Let* $W \subset V$ *be a vector subspace and* f *a linear isometry of* V. *If* f *acts trivially on both* W *and* V/W, *then* $f = \mathrm{id}$.

The geometric consequence is that given a Lorentzian manifold (N, h), a one-parameter group of isometries $\{\phi^t\}$ and a foliation $\mathcal{F}$ of N, if ϕ^t fixes a point $x \in N$, preserves every leaf of $\mathcal{F}$ and acts trivially in restriction to $\mathcal{F}_x$ and trivially on the transversal, then $\phi^t = \mathrm{id}$. Again, this is proper to Lorentzian geometry.

In the present situation, A being abelian, if a point $x \in \Omega_A$ was fixed by a one parameter subgroup $\{\phi^t\} < A$, then ϕ^t would act trivially on $T_x(A.x)$ and on $T_xM/T_x(A.x)$. By Lemma 2, $\mathrm{d}_x\phi^t = \mathrm{id}$, so $\phi^t = \mathrm{id}$ because the metric is 1-rigid.

Now, A-acts locally freely on the open-dense $\Omega_1 \cap \Omega_A$, where Ω_1 is the first stratum given by Theorem 5.3. By Corollary 11, the stabilizer of a point in $\Omega_1 \cap \Omega_A$ must be finite. Desintegrating the invariant volume density defined by the metric along the fibration $\phi_1 : \Omega_1 \to W$, we obtain that Lebesgue-every orbit $A.x$ has a finite A-invariant measure. So finally, A itself has finite Haar measure, hence is a torus.

The conclusion is then a Lie theoretic property: A Lie group G all of whose maximal abelian connected subgroups are tori is itself compact.

[3] This situation was partially covered by an article of Opozda in the analytic case, but additional algebraic assumptions were required.

5.3 The conformal case

The conformal case can be initiated similarly. We pick a maximal abelian subgroup $H < \mathrm{Conf}(M,[g])$ and want to prove compactness of H. All general properties of rigid structures cited above are valid. Moreover, normal forms of conformal vector field near a singularity of order 2 established in [FM13] allow to prove that H also acts locally freely over an open-dense domain.

The missing ingredient is of course the H-invariant volume. It is possible[4] that every H-invariant measure on M is supported on the complement of the open-dense subset on which H acts locally freely, so the final desintegration argument is no longer valid.

In a first attempt, we wanted to combine strong information on the dynamics of maximal abelian subgroups provided by Theorem 5.3 and several results on normal forms of conformal vector fields due to Frances and Melnick [FM13]. It was successful when the maximal abelian subgroup H has no compact subgroup, but a difficult problem came up when considering a mixed situation $\mathbf{T}^k \times \mathbf{R}^\ell$.

It was counter-intuitive, because I had always perceived the presence of additional symmetries as a simplification in a mathematical problem. Hence, being able to solve the problem for the case $H = \mathbf{R}^\ell$ but not $\mathbf{T}^k \times \mathbf{R}^\ell$ was a bit surprising.

The theoretical problem was that the information Gromov's stratification theorem provides is about the orbit structure of the *whole automorphism group* of a rigid analytic structure (M,ϕ), but a priori smaller subgroups are not constrained and their orbits could accumulate more complicatedly than what is predicted for $\mathrm{Aut}(M,\phi)$-orbits.

5.3.1 An instructive example of $(\mathbf{S}^1 \times \mathbf{R})$-action

Consider the Lorentzian Hopf manifold $M = (\mathbf{R}^{1,2} \setminus \{0\})/\langle 2\,\mathrm{id}\rangle$. Let $\{u^t\} \subset O(1,2)$ be a unipotent one-parameter subgroup, and let $\mathbf{S}^1 \curvearrowright M$ be the action induced by the homothetic flow of $\mathbf{R}^{1,2}$. Combining both actions, we obtain a conformal action of the cylinder $H = \mathbf{S}^1 \times \mathbf{R}$ on M. The latter is maximal abelian in the conformal group of M.

Of course, M is not simply-connected, but as a matter of fact, the H-action on M satisfies *all the conclusions* of Theorem 5.3 and are arranged very similarly to an algebraic action. The orbits of $\{u^t\}$ however do not. They all accumulate by spiraling to their circles of fixed points, which are $\mathbf{S}^1$-orbits.

Hence, even though the action of this cylinder on this Hopf manifold satisfies all the conclusions Gromov's theory would have predicted if the manifold was simply-connected, it does not follow logically from these properties that $\{u^t\}$ (or any copy of $\mathbf{R}$ in the cylinder) behaves the same way. So additional ingredients were necessary.

5.3.2 Strategy of proof for Theorem J

Recall that we picked a maximal connected abelian subgroup $H \simeq \mathbf{T}^r \times \mathbf{R}^\ell < \mathrm{Conf}(M,[g])$. The proof splits mainly into two parts: in the first, we assume that H

[4] And it is very likely that such a phenomenon occurs in conformal dynamics. For instance in the example of Section 5.3.1.

admits fixed points, in the second we assume it does not. In both cases, we use repeatedly the following.

1. No open subset of $(M, [g])$ is conformally flat (analyticity and Proposition 5.1).
2. A conformal vector field of an analytic, non-conformally flat Lorentzian manifold is locally linearizable near its singularities when they exist ([FM13]).
3. A compact simply-connected manifold does not admit real-analytic codimension 1 foliations (a theorem of Haefliger [Hae58]).

I sketch below the strategy in the simpler case $H = \mathbf{S}^1 \times \mathbf{R}$.

Presence of fixed point By Corollary 11, every H-orbit has a compact H-orbit in its closure. If a non-closed H-orbit contains a fixed point in its closure, then we are essentially reduced to the case $H \simeq \mathbf{R}$. By 2., the H action is locally linearizable near any of its fixed points, and falls into two categories:

- either H is locally conjugate to $A \times L$ acting on $\mathbf{R}^{1,n-1}$, where $A \subset \mathbf{R}_{>0} \times O(1, n-1)$ is an $\mathbf{R}$-split one parameter subgroup, and $L \subset O(1, n-1)$ is a copy of $\mathbf{S}^1$ in the maximal compact subgroup ;
- or H is locally conjugate to $U \times L$ acting on $\mathbf{R}^{1,n-1}$, where $U < O(1, n-1)$ is a unipotent one parameter subgroup, and $L \subset O(1, n-1)$ is a copy of $\mathbf{S}^1$ in the maximal compact subgroup ;

In both cases, there are points near the singularity fixed by $\mathbf{S}^1$ but not by $\mathbf{R}$. Their H-orbit coincides with the $\mathbf{R}$-orbit, so necessarily contain a singularity of the corresponding flow in their closure by Corollary 11, which again is either of hyperbolic linear or unipotent linear. This orbit is a connected component of a fiber $f^{-1}(f(x))$, for $f : U_i \to W_i$ given in the stratification theorem. We then prove that it cannot be unipotent thanks to the last point of Corollary 11, and that if it is hyperbolic either an open subset must be conformally flat, or there exists a non-singular, globally defined, isotropic conformal vector field X whose orthogonal distribution $X^\perp$ is integrable. The first case contradicts 1., and the second case contradicts 3.

Absence of fixed point. The next, more technical situation deals with the case of an H-action without fixed point. We know that H-orbits always contain closed H-orbits in their closure, so this implies that $\mathbf{S}^1$ acts locally freely and that for a certain decomposition $H = \mathbf{S}^1 \times \mathbf{R}$, an H-orbit converges to a circle $\mathbf{S}^1.y$ which is fixed pointwise by the $\mathbf{R}$-action. These fixed points are either all linear unipotent or all linear hyperbolic. The delicate situation is when they are all unipotent, *i.e.* when we the local picture near the $\mathbf{S}^1$-orbit is very similar to the example of Section 5.3.1.

The origin of the difficulty resides notably in the fact that in the Hopf manifold case, the dynamics of the unipotent one-parameter subgroup $\{u^t\}$ near any of its singularity is "locally volume preserving" but globally essential. By this, I mean that it is not possible to see its essentiality by looking at the local picture of the conformal vector field near its singularities, but rather by considering the whole spiraling dynamics around the circle of fixed points, which always escape local linearization boxes of the flow.

The idea was then to consider times $(t_n) \to +\infty$ such that $\phi^{t_n}(x) \to y$ and analyse the dynamics of $g_n := \phi^{t_n}$ near $\mathbf{S}^1.y$. The interpretation of $[g]$ in terms of Cartan geometry allows to reduce this question to that of the algebraic description of a sequence $\{p_n\} \subset P < O(2,n)$, where P denotes the maximal parabolic isomorphic to the stabilizer of a null-line. This sequence is called the *holonomy sequence* of $\{g_n\}$ at y, a notion introduced by Frances in [Fra12]. Using the information we have on the H-orbit at y, technical algebraic considerations led to three possible asymptotic for p_k via its Cartan decomposition KA^+K.

Dynamical foliations by hypersurfaces overs an open-dense subset, end of proof. In each case, it follows that over an open-dense subset Ω, the curvature map $\kappa : \mathcal{G} \to E$ takes values in certain sub-modules, an observation which allowed to prove the integrability over Ω of a dynamical distribution associated with $\{g_n\}$ called the approximately stable distribution. This gives rise to an analytic foliation by degenerate hypersurfaces over the open-dense subset Ω, and paves the way for a final contradiction by considering 1-dimensional H-orbits accumulated by leaves of the foliation.

5.3.3 Application of these methods to broader contexts

In [FM21], Frances and Melnick proved the Lorentzian Lichnerowicz conjecture in the 3-dimensional case, under an analyticity assumption. Precisely, if $(M^3,[g])$ is a compact 3-manifold with an analytic Lorentzian conformal class, and if the identity component $\mathrm{Conf}(M,[g])_0$ does not preserve any conformal metric, then $(M,[g])$ is conformally flat. Their proof relates to the approach exposed above. They use similarly the stratification of Aut^{loc}-orbits for the rigid structure defined by the disjoint union of $[g]$ and an essential conformal vector field X, though these are not necessarily orbits of the full automorphism group. Nonetheless, they arrive to a special case where the local action integrates into a glocal action of a cylinder $\mathbf{S}^1 \times \mathbf{R}$. Most of the arguments above are then applied.

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