Singularities of 2θ-divisors in the Jacobian

C. Pauly E. Previato [1]

Abstract

We consider the linear system $|2\Theta_0|$ of second order theta functions over the Jacobian JC of a non-hyperelliptic curve C. A result by J. Fay says that a divisor $D \in |2\Theta_0|$ contains the origin $\mathcal{O} \in JC$ with multiplicity 4 if and only if D contains the surface $C - C = \{\mathcal{O}(p-q); p,q \in C\} \subset JC$. In this paper we generalize Fay's result and some previous work by R. C. Gunning. More precisely, we describe the relationship between divisors containing $\mathcal{O}$ with multiplicity 6, divisors containing the fourfold $C_2 - C_2 = \{\mathcal{O}(p+q-r-s); p,q,r,s \in C\}$, and divisors singular along $C-C$, using the third exterior product of the canonical space and the space of quadrics containing the canonical curve. Moreover we show that some of these spaces are equal to the linear span of Brill-Noether loci in the moduli space of semi-stable rank 2 vector bundles with canonical determinant over C, which can be embedded in $|2\Theta_0|$.

1 Introduction

Let C be a smooth, connected, projective non-hyperelliptic curve of genus $g \geq 3$ over the complex numbers and let $\mathrm{Pic}^d(C)$ be the connected component of its Picard variety parametrizing degree d line bundles, for $d \in \mathbb{Z}$. The variety $\mathrm{Pic}^{g-1}(C)$ carries a naturally defined divisor, the Riemann theta divisor Θ, whose support consists of line bundles that have nonzero global sections. Translating Θ by a theta characteristic, we get a symmetric theta divisor, denoted Θ_0, on the Jacobian variety of C, $JC := \mathrm{Pic}^0(C)$. Our principal object of study are the linear systems of 2θ-divisors $|2\Theta_0|$ over JC and $|2\Theta|$ over $\mathrm{Pic}^{g-1}(C)$, their linear subspaces and subvarieties. One of the features of these linear systems is the canonical duality, called Wirtinger duality ([Mu] p. 335), which we will use somewhat implicitely throughout this paper

$$w : |2\Theta|^* \cong |2\Theta_0|. \tag{1.1}$$

Because of (1.1) we can view the Kummer variety $\mathrm{Kum} := JC/\pm$ as a subvariety of $|2\Theta|$ and many classical aspects of its projective geometry, such as existence of trisecants, tangent cones at its singular points [F] [Gu1], can be expressed in terms of 2θ-functions. The starting point of our investigations of 2θ-divisors is the following remarkable equivalence which was observed by J. Fay (see e.g. [We] Prop. 4.8, [Gu1] Cor. 1)

$$\mathrm{mult}_0(D) \geq 4 \iff C - C \subset D \qquad \forall D \in |2\Theta_0| \tag{1.2}$$

where the surface $C - C$ denotes the image of the difference map $\phi_1 : C \times C \to JC$, which sends a pair of points (p,q) to the line bundle $\mathcal{O}(p-q)$. Motivated by (1.2) van Geemen and van der Geer [vGvdG] introduce the subseries $\mathbb{P}\Gamma_{00} \subset |2\Theta_0|$ consisting of 2θ-divisors having multiplicity at least 4 at the origin and formulate a number of Schottky-type conjectures, some of which have been proved ([We] and [I1]).

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We can reformulate (1.2) more geometrically. Let $\langle C - C\rangle \subset |2\Theta|$ be the linear span of the image of $C - C$ in $|2\Theta|$ and let $\mathbb{T}_0 \subset |2\Theta|$ be the embedded tangent space at the singular point $\mathcal{O}$ to the Kummer variety. Then ([vGI] Lemma 1.5 and [We] Prop. 4.8) these linear projective spaces coincide

$$\mathbb{T}_0 = \langle C - C\rangle.$$

Note that their polar space in $|2\Theta|^*$ ($\cong |2\Theta_0|$) is $\mathbb{P}\Gamma_{00}$. More precisely, if we denote by $\Gamma_0 \subset H^0(JC, \mathcal{O}(2\Theta_0))$ the hyperplane of 2θ-divisors containing $\mathcal{O}$, we get an isomorphism by restricting divisors to $C - C$

$$\Gamma_0/\Gamma_{00} \xrightarrow{\sim} \mathrm{Sym}^2 H^0(K). \tag{1.3}$$

Having the equivalence (1.2) in mind, we can ask whether there exist relations between higher order derivatives of 2θ-functions at the origin $\mathcal{O}$ and natural subvarieties of JC. Working in an analytic set-up, Gunning ([Gu1] section 8, [Gu2] section 9) establishes some linear relations between vectors of 2θ-functions. Inspired by Gunning's previous work, we will compare the following subseries of $\mathbb{P}\Gamma_{00}$, using algebraic methods

$$\mathbb{P}\Gamma_{11} = \{D \in \mathbb{P}\Gamma_{00} \mid C_2 - C_2 \subset D\} \tag{1.4}$$

$$\mathbb{P}\Gamma_{000} = \{D \in \mathbb{P}\Gamma_{00} \mid \mathrm{mult}_0(D) \geq 6\} \tag{1.5}$$

The fourfold $C_2 - C_2$ is defined to be the image of the difference map $\phi_2 : C_2 \times C_2 \to JC$, which maps a 4-tuple $(p + q, r + s)$ to the line bundle $\mathcal{O}(p + q - r - s)$ and C_2 is the second symmetric product of the curve. Moreover we will be naturally led to consider the subseries of 2θ-divisors which are singular along the surface $C - C$, i.e.

$$\mathbb{P}\Gamma_{00}^{(2)} = \{D \in \mathbb{P}\Gamma_{00} \mid \mathrm{mult}_{p-q}(D) \geq 2 \;\; \forall p, q \in C\}. \tag{1.6}$$

We observe that the subseries $\mathbb{P}\Gamma_{00}, \mathbb{P}\Gamma_{11}, \mathbb{P}\Gamma_{00}^{(2)}$ are closely related to the geometry of the moduli space $\mathcal{SU}_C(2, K)$ of semi-stable rank 2 vector bundles with fixed canonical determinant. The morphism (section 2)

$$D : \mathcal{SU}_C(2, K) \longrightarrow |2\Theta_0| \qquad E \longmapsto D(E) = \{\xi \in JC \mid h^0(E \otimes \xi) > 0\}$$

was recently shown [BV] [vGI] to be an embedding. Hence we may view $\mathcal{SU}_C(2, K)$ as a subvariety of $|2\Theta_0|$. Of considerable interest are the Brill-Noether loci $\mathcal{W}(n) \subset \mathcal{SU}_C(2, K)$ for $n \geq 1$ defined by

$$\mathcal{W}(n) = \{[E] \in \mathcal{SU}_C(2, K) \mid h^0(E) = n \text{ and } E \text{ is globally generated}\}.$$

These loci (more precisely their closure) have been extensively studied in connection with Fano threefolds [M1], [M2], [M3] and for their own sake [OPP]. A simple argument now shows that one has the following implications

$$[E] \in \mathcal{W}(2) \text{ (resp. } \mathcal{W}(3), \mathcal{W}(4), \mathcal{W}(5)) \qquad \Rightarrow \qquad D(E) \in \mathbb{P}\Gamma_0 \text{ (resp. } \mathbb{P}\Gamma_{00}, \mathbb{P}\Gamma_{00}^{(2)}, \mathbb{P}\Gamma_{11})$$

We see that we get two filtrations which are related under the map D, one given by the dimension of the space of sections of $E \in \mathcal{SU}_C(2, K)$, the other given by 2θ-divisors containing certain subschemes of JC. As a consequence of our results we see that the Brill-Noether loci $\mathcal{W}(n)$ for $n = 2, 3, 4$ linearly span the corresponding subseries. We expect this to hold for $n = 5$ too.

In the next two theorems we describe the first two quotients of the filtration

$$\Gamma_{11} \subset \Gamma_{00}^{(2)} \subset \Gamma_{00} \subset \Gamma_0,$$

the last one being given in (1.3). Let $\langle \mathrm{Sing}\Theta\rangle$ be the linear span of the image of the singular locus $\mathrm{Sing}\,\Theta \subset \mathrm{Pic}^{g-1}(C)$ under the morphism into $|2\Theta_0|$.

1.1. Theorem. *For any non-hyperelliptic curve*

(1) there exists a canonical isomorphism (up to a scalar)

$$\Gamma_{00}/\Gamma_{00}^{(2)} \xrightarrow{\sim} \Lambda^3 H^0(K)$$

(2) we have an equality among subspaces of $\mathbb{P}\Gamma_{00}$

$$\langle \mathrm{Sing}\Theta \rangle = \mathbb{P}\Gamma_{00}^{(2)}.$$

The method used in the proof of this theorem (section 4) has been developed in a recent paper by van Geemen and Izadi [vGI] and the key point are the incidence relations (section 2.4) between two families of stable rank 2 vector bundles with fixed trivial (resp. canonical) determinant. One of these families of bundles is related to the gradient of the 2θ functions along the surface $C-C$, the other family is the Brill-Noether locus $\mathcal{W}(3)$. In section 2.5 we describe the relationship between these bundles and the objects discussed in [vGI], which are related to the embedded tangent space at the origin to $\mathcal{SU}_C(2,K)$. We also need (section 3) some relations between vectors of second order theta functions, which one derives from Fay's trisecant formula and its generalizations [Gu1].

Let $I(2)$ (resp. $I(4)$) be the space of quadrics (resp. quartics) in canonical space $|K|^*$ containing the canonical curve.

1.2. Theorem. *For any non-trigonal curve, there exists a canonical isomorphism*

$$\Gamma_{00}^{(2)}/\Gamma_{11} \xrightarrow{\sim} \mathrm{Sym}^2 I(2). \tag{1.7}$$

The harder statement in Theorem 1.2 is the surjectivity of the map in (1.7). The proof uses essentially two ideas: first, we can give an explicit basis of quadrics in $I(2)$ of rank less than or equal to 6 (Petri's quadrics, section 5.1) and secondly, we can construct out of such a quadric a rank 2 vector bundle in $\mathcal{W}(4)$. This construction [BV] is recalled in section 5.2 and generalized in section 8. As a corollary of Theorem 1.2 (section 5.4), we get another proof of a theorem by M. Green, saying that the projectivized tangent cones to Θ at double points span $I(2)$.

The subspace Γ_{000} is of a different nature and rank 2 vector bundles turn out to be of no help in studying it. We gather our results in the next theorem.

1.3. Theorem. *For any non-hyperelliptic curve, we have the following inclusions*

$$\Gamma_{11} \subset \Gamma_{000} \subset \Gamma_{00}^{(2)}. \tag{1.8}$$

The quotient of the first two spaces is isomorphic to the kernel of the multiplication map m

$$\Gamma_{000}/\Gamma_{11} \cong \ker m : \mathrm{Sym}^2 I(2) \longrightarrow I(4). \tag{1.9}$$

The proof of Theorem 1.3 is more in the spirit of Gunning's previous work and uses only linear relations between vectors of second order theta functions. The inclusions (1.8) were proposed as plausible in [Gu1] p.70. Except for a few cases (section 6.2) we are unable to deduce the dimension of Γ_{000} from (1.9).

In section 7 we give the version of Theorem 1.2 for trigonal curves.

We observe that the vector bundle constructions used in the proofs of Theorems 1.1 and 1.2 can be seen as examples of a global construction (section 8) which relates a bundle in $\mathcal{W}(n)$ to the geometry of the canonical curve.

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Notation.

- If X is a vector space or a vector bundle, by X^* we denote its dual.
- K is the canonical line bundle on the curve C.
- For a vector bundle E over C, $H^i(C,E)$ is often abbreviated by $H^i(E)$ and $h^i(E) = \dim H^i(C,E)$.
- C_n is the n-th symmetric product of the curve C.
- $W^r_d(C)$ is the subvariety of $Pic^d(C)$ consisting of line bundles L such that $h^0(L) > r$.
- The canonical curve C_{can} is the image of the embedding $\varphi_K : C \longrightarrow |K|^*$.
- The vector space $I(n)$ is the space of forms in $\mathrm{Sym}^n H^0(K)$ defining degree n hypersurfaces in $|K|^*$ containing C_{can}.
- We denote by Γ^*_* (e.g. $\Gamma_{00}, \Gamma^{(2)}_{00}, \Gamma_{11}, \Gamma_{000}$) a vector subspace of $H^0(JC, \mathcal{O}(2\Theta_0))$ and by $\mathbb{P}\Gamma^*_*$ its projectivization, so $\mathbb{P}\Gamma^*_* \subset |2\Theta_0|$.
- Given a subspace $\mathbb{P}V \subset |2\Theta|$, we denote by $\mathbb{P}V^\perp \subset |2\Theta_0|$ the image under the Wirtinger duality (1.1) of its annihilator.
- If E is a semi-stable vector bundle, then $[E]$ denotes its S-equivalence class.

2 Rank 2 vector bundles

In this section we recall some facts on the geometry of the projective spaces $|2\Theta_0|$ and $|2\Theta|$ and on extensions of line bundles. We use these results on extensions to construct in section 2.2 and 2.3 two families of stable rank 2 bundles over C. Their incidence relations (Prop.2.6) are key to the proof of Theorem 1.1 (section 4).

2.1 Preliminaries on extension spaces

Let $\mathcal{SU}_C(2,\mathcal{O})$ (resp. $\mathcal{SU}_C(2,K)$) be the moduli space of rank 2 vector bundles over C with fixed trivial (resp. canonical) determinant. It can be shown [B1] that the Kummer maps given by the linear system $|2\Theta_0|$ over JC (resp. $|2\Theta|$ over $\mathrm{Pic}^{g-1}(C)$) can be factorized through the moduli space $\mathcal{SU}_C(2,\mathcal{O})$ (resp.$\mathcal{SU}_C(2,K)$). This gives the two following commutative diagrams

$$\begin{array}{ccccccc} JC & \xrightarrow{Kum} & |2\Theta_0|^* & & \mathrm{Pic}^{g-1}(C) & \xrightarrow{Kum} & |2\Theta|^* \\ \downarrow i & & \downarrow w & & \downarrow i & & \downarrow w \\ \mathcal{SU}_C(2,\mathcal{O}) & \xrightarrow{D} & |2\Theta| & & \mathcal{SU}_C(2,K) & \xrightarrow{D} & |2\Theta_0| \end{array} \qquad (2.1)$$

The vertical morphisms i map JC (resp. Pic^{g-1}) to the semi-stable boundary of the moduli space $\mathcal{SU}_C(2,\mathcal{O})$ (resp. $\mathcal{SU}_C(2,K)$) by sending a line bundle ξ to the split bundle $\xi \oplus \xi^{-1}$ (resp. $\xi \oplus K\xi^{-1}$). The rightmost morphism D associates to a semi-stable rank 2 bundle E with canonical determinant a divisor $D(E)$, whose support is given by

$$D(E) = \{\xi \in JC \mid h^0(E \otimes \xi) > 0\} \qquad (2.2)$$

The definition of $D(E)$ for E with trivial determinant is obtained by replacing JC by $\mathrm{Pic}^{g-1}(C)$. The composite map $w \circ Kum = D \circ i$ in the rightmost diagram (2.1) is the translation morphism

$$\begin{aligned} \iota : \mathrm{Pic}^{g-1}(C) &\longrightarrow |2\Theta_0| \\ \xi &\longmapsto \Theta_\xi + \Theta_{K\xi^{-1}} \end{aligned}$$

where the divisor $\Theta_\xi \subset JC$ is obtained by translating the Riemann theta divisor Θ by $-\xi$. Dually, we get $\iota : JC \longrightarrow |2\Theta|$ by replacing Θ by Θ_0. Note that the symmetric theta divisor Θ_0 depends on the choice of a theta characteristic.

Recently it has been shown that the morphism D is an embedding for any non-hyperelliptic curve ([vGI] Thm 1). Therefore we may view the varieties Kum and $\mathcal{S}U_C(2,\mathcal{O})$ as subvarieties of $|2\Theta|$

$$\mathrm{Kum} \subset \mathcal{S}U_C(2,\mathcal{O}) \subset |2\Theta| = \mathbb{P}^{2^g-1}$$

In this paper all subvarieties of JC will be studied in $|2\Theta|$ via the Kummer map and we often identify semi-stable vector bundles with their image in $|2\Theta|$.

Now we are going to describe some useful subvarieties of $\mathcal{S}U_C(2,\mathcal{O})$, namely extension spaces of line bundles [Ber].

2.1.1 degree 1

Given an $x \in \mathrm{Pic}^1(C)$, let $\mathbb{P}(x) = |Kx^2|^* = \mathbb{P}H^1(C, x^{-2})$. This g-dimensional projective space parametrizes isomorphism classes of extensions

$$0 \longrightarrow x^{-1} \longrightarrow E \longrightarrow x \longrightarrow 0 \tag{2.3}$$

and the composite of the classifying map $\psi : \mathbb{P}(x) \to \mathcal{S}U_C(2,\mathcal{O})$ followed by the embedding $D : \mathcal{S}U_C(2,\mathcal{O}) \to |2\Theta|$ is linear and injective ([B2] lemme 3.6). The linear system $|Kx^2|$ can be used to map the curve into $\mathbb{P}(x)$

$$\varphi = \varphi_{Kx^2} : C \longrightarrow \mathbb{P}(x).$$

The following lemma describes the incidence relations between the extension spaces $\mathbb{P}(x)$, which we view as subspaces of $|2\Theta|$.

2.1. Lemma. *Let $x, y \in \mathrm{Pic}^1(C)$. If $x \otimes y = \mathcal{O}(p+q)$, the intersection $\mathbb{P}(x) \cap \mathbb{P}(y)$ is the common secant line $\overline{pq}$ of the two images of the curve C in either $\mathbb{P}(x)$ or $\mathbb{P}(y)$.*

Proof. The proof is similar to the proof of [OPP] prop.1.2, which is a dual version of this lemma. □

It is shown in [B2] lemme 3.6 that a point in $\mathbb{P}(x)$ represents a stable bundle precisely away from $C_x := \varphi(C)$, while the image of a point $p \in C$ represents the equivalence class of the semi-stable bundle $x(-p) \oplus x^{-1}(p) = i(x(-p))$. The Abel-Jacobi map $t_x : C \to JC$ defined by $p \mapsto x(-p)$ therefore fits in the following commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{t_x} & JC \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle i} \\ \mathbb{P}(x) & \xrightarrow{\psi} & \mathcal{S}U_C(2,\mathcal{O}) \end{array} \tag{2.4}$$

2.1.2 degree 2

Given an $x \in \mathrm{Pic}^2(C)$, we define similarly $\mathbb{P}(x) = |Kx^2|^* = \mathbb{P}H^1(C, x^{-2})$. This $(g+2)$-dimensional projective space parametrizes as in section 2.1.1 isomorphism classes of extensions (2.3) and the composite of the classifying map $\psi : \mathbb{P}(x) \to \mathcal{SU}_C(2, \mathcal{O})$ followed by the embedding $D : \mathcal{SU}_C(2, \mathcal{O}) \to |2\Theta|$ is given by the complete system of quadrics through the embedded curve $\varphi_{Kx^2} : C \hookrightarrow \mathbb{P}(x)$. Thus this composite map is defined away from the curve C. We will use the fact that a secant line $\overline{pq}$ to the curve $C \subset \mathbb{P}(x)$ is contracted by the linear system of quadrics to a point of $\mathcal{SU}_C(2, \mathcal{O}) \subset |2\Theta|$ represented by the split bundle $x(-p-q) \oplus x^{-1}(p+q)$. For the proofs of the preceding results we refer to [Ber] thm 1 and p. 451-452, and also [vGI] section 5.

2.2 The bundles $E(p, q, r)$

In this section we will associate to any three distinct non-collinear points p, q, r on the curve C a unique stable rank 2 bundle $E(p, q, r)$. We say that three points p, q, r are collinear if they are collinear as points on the canonical curve C_{can}. Later (section 2.5) we will indicate how these bundles are related to the objects introduced in [vGI] section 6 in order to study the singular point $[\mathcal{O} \oplus \mathcal{O}] \in \mathcal{SU}_C(2, \mathcal{O}) \subset |2\Theta|$. We consider the line bundle

$$x = \mathcal{O}(p + q - r) \in \mathrm{Pic}^1(C). \tag{2.5}$$

Then we claim that the point $p \in C_x \subset \mathbb{P}(x)$ is a smooth point on the curve C_x: first if we suppose that $h^0(x^2) = 0$, then an elementary computation involving the Riemann-Roch theorem shows that any point on C_x is smooth; secondly if $h^0(x^2) = 1$, then there exist points $u, v \in C$ such that the divisor $2p + 2q$ on C is linearly equivalent to $2r + u + v$ and the map $\varphi : C \to C_x \subset \mathbb{P}(x)$ identifies u with v. Since $\varphi(u) = \varphi(v)$ is the only singular point on C_x, it is still true that p is smooth on C_x.

Therefore we can consider the embedded tangent line $T_p(C_x)$ to the curve $C_x \subset \mathbb{P}(x)$ at the point $p \in C_x$. We recall (1.3) that the subspace $\mathbb{T}_0 \subset |2\Theta|$ equals the span of the image under the Kummer map of the surface $C - C \subset JC$ in $|2\Theta|$. Since the composite map $D \circ \psi : \mathbb{P}(x) \to |2\Theta|$ is a linear embedding, we may view $T_p(C_x)$ as a line in $|2\Theta|$.

2.2. Lemma. $T_p(C_x) \subset \mathbb{T}_0 \iff$ *p, q, r are collinear*

Proof. First we observe that the image of the point p under the Abel-Jacobi map $t_x(p) = \mathcal{O}(q-r)$ lies on the surface $C - C \subset JC$. We can identify the projectivized tangent space to JC at the point $t_x(p) = \mathcal{O}(q - r)$ with canonical space, i.e. we have an isomorphism

$$\mathbb{P}T_{q-r}JC \cong |K|^* \tag{2.6}$$

It is well-known that the projectivized tangent space to the Abel-Jacobi curve $t_x(C) \subset JC$ at the point $t_x(p)$ corresponds to the point $p \in |K|^*$ via (2.6) and the projectivized tangent space to the surface $C - C$ at the point $t_x(p)$ corresponds to the secant line $\overline{qr} \subset |K|^*$ via (2.6). Now using the fact that the Kummer map is an embedding locally near the point $\mathcal{O}(q - r)$ and the commutative diagram (2.4) which identifies the curves $t_x(C)$ and C_x in $|2\Theta|$, we see that if $p \in \overline{qr}$, then the embedded tangent line $T_p(C_x)$ is contained in the embedded tangent space at the point $\mathrm{Kum}(q-r)$ to the Kummer image of $C - C$, which is obviously contained in the span $\langle C - C \rangle = \mathbb{T}_0$.

Conversely, by [I2] Théorème A, we know that the projectivized tangent spaces to divisors in $\mathbb{P}\Gamma_{00} = \mathbb{T}_0^{\perp}$ at the point $\mathcal{O}(q - r)$ cut out, via (2.6), the secant line $\overline{qr} \subset |K|^*$. Therefore, if $T_p(C_x) \subset \mathbb{T}_0$ the projectivized tangent line at the point $t_x(p)$ to the Abel-Jacobi curve $t_x(C)$ is forced to lie on $\overline{qr}$, hence $p \in \overline{qr}$. □

We now introduce the line bundles

$$y = \mathcal{O}(p + r - q) \qquad\qquad z = \mathcal{O}(q + r - p) \tag{2.7}$$

2.3. Proposition. *For any distinct non-collinear points p, q, r, there exists a unique stable bundle $E := E(p, q, r) \in \mathcal{S}U_C(2, \mathcal{O})$ containing the three line bundles x^{-1}, y^{-1}, z^{-1}. Moreover $D(E) \notin \mathbb{T}_0$.*

Proof. Since $x \otimes y = \mathcal{O}(2p)$, we see from lemma 2.1 that the intersection of the two projective spaces $\mathbb{P}(x)$ and $\mathbb{P}(y)$ is the tangent line $T_p(C_x)$, which coincides with the tangent line $T_p(C_y)$. Hence

$$T_p(C_x) = T_p(C_y) = \mathbb{P}(x) \cap \mathbb{P}(y). \tag{2.8}$$

Any bundle whose extension class lies in (2.8), but not on the curve C_x or C_y, is stable and contains the line bundles x^{-1} and y^{-1}. Similarly by lemma 2.1, we see that

$$T_q(C_x) = T_q(C_z) = \mathbb{P}(x) \cap \mathbb{P}(z) \qquad T_r(C_z) = T_r(C_y) = \mathbb{P}(z) \cap \mathbb{P}(y)$$

By the Riemann-Roch theorem we have

$$h^0(Kx^2(-2p - 2q)) = h^0(K(-2r)) = g - 2 \tag{2.9}$$

hence the two tangent lines $T_p(C_x)$ and $T_q(C_x)$ are contained in a projective plane and are distinct. So they intersect in a unique point $T_p(C_x) \cap T_q(C_x)$. Repeating this argument with the remaining pairs of tangent lines in $\mathbb{P}(y)$ and $\mathbb{P}(z)$, we obtain that there exists a unique extension class in the triple intersection of the three distinct tangent lines. In other words

$$[E(p, q, r)] := \mathbb{P}(x) \cap \mathbb{P}(y) \cap \mathbb{P}(z) \tag{2.10}$$

Obviously the bundle $E(p, q, r)$ contains the three line bundles x^{-1}, y^{-1}, z^{-1}. To conclude the proof, we need to check that the bundle $E(p, q, r)$ is stable, i.e. that the extension class (2.10) does not lie on the curve C_x (or C_y or C_z). Suppose that the contrary holds: by the Riemann-Roch argument as in (2.9) there exists a point $s \in C$ such that $h^0(K(-2r - s)) = g - 2$, so $h^0(2r + s) = 2$ and C would be trigonal with trigonal divisor $2r + s$. Then by the independence of the construction from the ordering of the points, one also deduces that C is trigonal with trigonal divisors $2p + s$ and $2q + s$. Since C is non-hyperelliptic, there are at most one (for $g \geq 5$) or two (for $g = 4$) trigonal series, so this is impossible. Finally, by lemma 2.2 the tangent line $T_p(C_x)$ (resp. $T_q(C_x)$) meets $\mathbb{T}_0$ only at the point p (resp. q) of C_x. Since, as noted before, the intersection point $T_p(C_x) \cap T_q(C_x) = [E(p, q, r)]$ does not lie on C_x, $D(E) \notin \mathbb{T}_0$. □

2.4. Proposition. *If $a \in \mathrm{Sing}\,\Theta$, then $h^0(E(p, q, r) \otimes a) > 0$.*

Proof. We write $E = E(p, q, r)$ and take x as in (2.5). We tensor the exact sequence (2.3) with a and take cohomology

$$0 \longrightarrow H^0(x^{-1} \otimes a) \longrightarrow H^0(E \otimes a) \longrightarrow H^0(x \otimes a) \xrightarrow{\delta} H^1(x^{-1} \otimes a) \tag{2.11}$$

If $h^0(x^{-1} \otimes a) > 0$, we are done. So we can assume that $h^0(x^{-1} \otimes a) = 0$. By Serre duality, Riemann-Roch and the fact that $\det E = \mathcal{O}$ implies that E is self-dual, we have $h^0(E \otimes a) = h^0(E \otimes a')$ with $a' = K \otimes a^{-1}$, so we can also assume that $h^0(x^{-1} \otimes a') = 0$. But by Riemann-Roch this is equivalent to $h^0(x \otimes a) = h^1(x^{-1} \otimes a') = 1$. Finally we see from the exact sequence (2.11) that

$h^0(E \otimes a) > 0$ if and only if the coboundary map δ is zero, i.e. the one-dimensional image of the multiplication map

$$H^0(x \otimes a) \otimes H^0(x \otimes a') \longrightarrow H^0(Kx^2) \tag{2.12}$$

is contained in the hyperplane of $H^0(Kx^2)$ corresponding to the extension class of the bundle E and which is, by definition, the linear span of the two subspaces $H^0(Kx^2(-2p))$ and $H^0(Kx^2(-2q))$. Next we observe that multiplication by a section of $H^0(\mathcal{O}(p+q))$ maps injectively $H^0(a(-r)) \hookrightarrow H^0(x \otimes a)$ and, since $a \in \mathrm{Sing}\,\Theta$ by hypothesis, $h^0(a) \geq 2$, so $h^0(a(-r)) \geq 1$, hence both spaces have dimension one and the injection is an isomorphism. Therefore the unique global section of $x \otimes a$ vanishes at p and q. Since the same holds for $x \otimes a'$, the image of the map (2.12) is contained in $H^0(Kx^2(-2p-2q))$ and we are done. □

2.3 The bundles E_W

In this section we introduce the second family of stable rank 2 vector bundles. We recall their construction from [vGI] section 4. Let $Gr(3, H^0(K))$ be the Grassmannian of 3-planes in $H^0(K)$ and $\mathcal{W}(3)$ the locus of stable bundles E in $\mathcal{SU}_C(2, K)$ that are generated by global sections and for which $h^0(E) = 3$. We will define a rational map

$$Gr(3, H^0(K)) \xrightarrow{\beta} \mathcal{W}(3).$$

For a generic 3-plane $W \subset H^0(K)$, it can be shown that the multiplication map

$$W \otimes H^0(K) \to H^0(K^2) \quad \text{is surjective.} \tag{2.13}$$

Then the dual E_W^* of the bundle $\beta(W) = E_W$ is defined by the exact sequence

$$0 \longrightarrow E_W^* \longrightarrow W \otimes \mathcal{O}_C \xrightarrow{ev} K \longrightarrow 0 \tag{2.14}$$

where the last map is the evaluation map of global sections. Then condition (2.13) implies that $E_W \in \mathcal{W}(3)$. Conversely, we can associate to a bundle $E \in \mathcal{W}(3)$ a 3-plane W_E, i.e. we have an *inverse* map

$$\mathcal{W}(3) \xrightarrow{\alpha} Gr(3, H^0(K)).$$

To define $\alpha(E) = W_E$, we consider the exact sequence which we get from the evaluation map of global sections of E

$$0 \longrightarrow K^{-1} \xrightarrow{i} H^0(E) \otimes \mathcal{O}_C \xrightarrow{ev} E \longrightarrow 0$$

then the dualized exact sequence induces an injective linear map on global sections

$$H^0(i^*) \ : \ H^0(E)^* \longrightarrow H^0(K)$$

and we let $W_E = \mathrm{im}\ H^0(i^*)$. Moreover, under the natural duality $H^0(E)^* \cong \Lambda^2 H^0(E)$, W_E coincides with the image of the exterior product map $\Lambda^2 H^0(E) \to H^0(K)$.

2.5. Remark. Since E_W is generated by global sections we can define a map

$$\begin{array}{rcl} C & \longrightarrow & \mathbb{P}H^0(E_W) = \mathbb{P}^2 \\ p & \longmapsto & s_p := \ker\,(H^0(E_W) \xrightarrow{ev} E_{W|p}) \end{array}$$

which associates to a point p the unique global section s_p of E_W vanishing at p. This map coincides with the canonical map $\varphi_K : C \hookrightarrow |K|^*$ followed by the projection with center $\mathbb{P}W^\perp \subset |K|^*$.

2.4 Incidence relations

In the previous sections we have constructed two families of bundles i.e. $E(p,q,r)$ (section 2.2) and E_W (section 2.3); each gives 2θ-divisors under the D-maps in $|2\Theta|$ and $|2\Theta_0|$. These spaces are dual to each other (1.1) and it is therefore useful to determine their incidence relations. We will denote by $H_W \in |2\Theta|^*$ the hyperplane in $|2\Theta|$ corresponding under (1.1) to $D(E_W) \in |2\Theta_0|$ and by $p\wedge q\wedge r \in \mathbb{P}\Lambda^3 H^0(K)^*$ (resp. $\mathbb{P}\Lambda^3 W \in \mathbb{P}\Lambda^3 H^0(K)$) the Plücker image of the linear 3-plane in $H^0(K)^*$ spanned by the 3 points $p,q,r \in C_{can}$ (resp. the 3-plane W in $H^0(K)$). Therefore we may view $\mathbb{P}\Lambda^3 W$ as a hyperplane in the dual Plücker space $\mathbb{P}\Lambda^3 H^0(K)^*$.

2.6. Proposition. *For any distinct non-collinear points p,q,r and any 3-plane $W \subset H^0(K)$ such that $h^0(E_W)=3$, we have the following equivalence*

$$D(E(p,q,r)) \in H_W \iff p\wedge q\wedge r \in \mathbb{P}\Lambda^3 W.$$

Proof. We write $E=E(p,q,r)$ and take x as in (2.5). By [B2] the first condition $D(E)\in H_W$ is equivalent to $h^0(E_W\otimes E)>0$. Using remark 2.5, we see that the second condition is equivalent to the three sections $s_p, s_q, s_r \in \mathbb{P}H^0(E_W)=\mathbb{P}^2$ being collinear.

We tensor the exact sequence (2.3) with E_W and take cohomology

$$0 \longrightarrow H^0(E_W\otimes x^{-1}) \longrightarrow H^0(E_W\otimes E) \longrightarrow H^0(E_W\otimes x) \xrightarrow{\delta(\epsilon)} H^1(E_W\otimes x^{-1}) \qquad (2.15)$$

Let us first consider the case when $h^0(E_W\otimes x^{-1})>0$. Then $h^0(E_W\otimes E)>0$, and after tensoring the defining exact sequence of the bundle E_W (2.14) by Kx^{-1}, taking cohomology and using the fact that $E_W^*\otimes K = E_W$, we obtain

$$H^0(E_W\otimes x^{-1}) = \ker\,(W\otimes H^0(Kx^{-1}) \xrightarrow{ev} H^0(K^2x^{-1})) \qquad (2.16)$$

For dimensional reasons, we have $H^0(Kx^{-1}) = H^0(K(-p-q))$. Furthermore, if dim $W\cap H^0(K(-p-q))=1$, then the kernel of the map (2.16) is zero. Hence dim $W\cap H^0(K(-p-q))\geq 2$, which implies that dim $W\cap H^0(K(-p-q-r))\geq 1$. This last equality means that the center $\mathbb{P}W^\perp \subset |K|^*$ of the projection $|K|^* \longrightarrow \mathbb{P}H^0(E_W)=\mathbb{P}^2$ intersects the 2-plane spanned by p,q,r in $|K|^*$. So we are done.

Now we can assume that $h^0(E_W\otimes x^{-1})=0$, or equivalently by Riemann-Roch $h^0(E_W\otimes x)=2$. We are going to choose a basis $\{s,t\}$ of the 2-dimensional space $H^0(E_W\otimes x)$. Let σ be a global section of the line bundle $\mathcal{O}(p+q)$ and let $\bar{s}_p$ (resp. $\bar{s}_q, \bar{s}_r$) be generators of the one-dimensional vector spaces $H^0(E_W(-p))$ (resp. $H^0(E_W(-q)), H^0(E_W(-r))$). Then multiplication by σ maps injectively $H^0(E_W)$ into the 4-dimensional space $H^0(E_W(p+q))$. Similarly $H^0(E_W\otimes x)$ can be seen as a subspace of $H^0(E_W(p+q))$. These two subspaces intersect in a line, generated by the section $\sigma\cdot\bar{s}_r$. We choose $s=\sigma\cdot\bar{s}_r$ and t any section in $H^0(E_w\otimes x)$ with $t\notin H^0(E_W)$. Then the exterior product $\bar{s}_p\wedge t$ is a section of $H^0(K(q-r))$, which for dimensional reasons is equal to $H^0(K(-r))$, so $\bar{s}_p\wedge t$ vanishes at q. Similarly $\bar{s}_q\wedge t$ vanishes at p. We also note that $\bar{s}_r\wedge t\in H^0(K(p+q-2r))$. Next we observe that the coboundary map $\delta(\epsilon)$ (2.15), which depends on the extension class $\epsilon\in\mathbb{P}(x)$ of the bundle E, is skew-symmetric (note that Serre duality gives an isomorphism $H^1(E_W\otimes x^{-1})\cong H^0(E_W\otimes x)^*$). Therefore, if $\{s,t\}$ is a basis of $H^0(E_W\otimes x)$, we have

$$\begin{aligned} h^0(E_W\otimes E)>0 &\iff \det\,(\delta(\epsilon))=0 \\ &\iff s\wedge t\in H_\epsilon\subset H^0(Kx^2) \end{aligned} \qquad (2.17)$$

where H_ϵ is the hyperplane in $H^0(Kx^2)$ defined by the extension class $\epsilon \in \mathbb{P}(x) = |Kx^2|^*$. By definition H_ϵ is the linear span of the two subspaces $H^0(Kx^2(-2p))$ and $H^0(Kx^2(-2q))$, from which it easily follows that

$$H_\epsilon \cap H^0(Kx^2(-p-q)) = H^0(Kx^2(-2p-2q))$$

Since $s \wedge t = \sigma \cdot (\bar{s}_r \wedge t)$ this equality shows that the last condition of (2.17) is equivalent to

$$\bar{s}_r \wedge t \in H^0(Kx^2(-2p-2q)) = H^0(K(-2r)) \quad \Longleftrightarrow \quad \bar{s}_r \wedge t \text{ vanishes at p and q.}$$

Since $t \notin H^0(E_W)$ we can assume e.g. that $t(q) \neq 0$. Suppose now that $(\bar{s}_r \wedge t)(q) = 0$. Since we always have $(\bar{s}_p \wedge t)(q) = 0$ we can conclude that $\bar{s}_p(q) \wedge \bar{s}_r(q) = 0$ which implies that the three sections s_p, s_q, s_r cannot be linearly independent (otherwise they would generate E_W at q). Conversely if s_p, s_q, s_r are linearly dependent, one easily shows by the same argument that $\bar{s}_r \wedge t$ vanishes at p and q, and we are done. □

2.7. Remark. The incidence relations were first proved in [vGI] prop.6.5 for slightly different objects (see section 2.5). Working with the bundles $E(p,q,r)$ instead of the projective spaces $\mathbb{P}^4_{p,q,r}$ simplifies the proof somewhat.

2.5 The bundles $E(p,q,r)$ and the tangent space to $\mathcal{SU}_C(2,\mathcal{O})$ at the origin

In this section we will show how the bundles $E(p,q,r)$ are related to the projective spaces $\mathbb{P}^4_{p,q,r}$ introduced by van Geemen and Izadi to study the tangent space to the moduli $\mathcal{SU}_C(2,\mathcal{O})$ at the origin. However these facts will not be used in this paper.

Consider the projection $\mathbf{P}$ with center $\mathbb{T}_0$

$$\mathbf{P} : |2\Theta| \longrightarrow |2\Theta|_{\mathbb{T}_0}. \tag{2.18}$$

Here $|2\Theta|_{\mathbb{T}_0}$ denotes the quotient space $|2\Theta|/\mathbb{T}_0$, with abuse of notation. In section 2.2 we have associated to any triple of distinct non-collinear points p,q,r the point $D(E(p,q,r)) \in |2\Theta|$. By proposition 2.3, $D(E(p,q,r)) \notin \mathbb{T}_0$, so we get a well-defined point $\mathbf{P} \circ D(E(p,q,r)) \in |2\Theta|_{\mathbb{T}_0}$. In [vGI] section 6 the authors associate to the same data a 4-dimensional projective space $\mathbb{P}^4_{p,q,r} \subset |2\Theta|$. Their construction goes as follows:

We take $\zeta = \mathcal{O}(p+q) \in \mathrm{Pic}^2(C)$ and consider in the extension space $\mathbb{P}(\zeta)$ (see 2.1.2) the 3-dimensional subspace $\langle 2p+2q+r \rangle$ spanned by the embedded tangent lines at p and q to $C \hookrightarrow \mathbb{P}(\zeta)$ and the point r. Then the restricted linear system of quadrics through C determines a rational map

$$\psi : \langle 2p+2q+r \rangle \longrightarrow |2\Theta|.$$

The image of ψ is a cubic threefold in $\mathbb{P}^4_{p,q,r} \subset |2\Theta|$ which is singular at the origin $D([\mathcal{O} \oplus \mathcal{O}])$of the moduli space $\mathcal{SU}_C(2,\mathcal{O})$. Furthermore if p,q,r are non-collinear, $\dim \mathbb{T}_0 \cap \mathbb{P}^4_{p,q,r} = 3$, hence the projective space $\mathbb{P}^4_{p,q,r}$ is contracted by the projection $\mathbf{P}$ to a point in $|2\Theta|_{\mathbb{T}_0}$.

2.8. Lemma. *For any distinct non-collinear points p,q,r*

$$\mathbf{P} \circ D(E(p,q,r)) = \mathbf{P}(\mathbb{P}^4_{p,q,r}) \in |2\Theta|_{\mathbb{T}_0}$$

Proof. We just need to show that there exists an extension class $\epsilon \in \langle 2p+2q+r \rangle \subset \mathbb{P}(\zeta)$ such that its associated vector bundle $E(\epsilon)$ satisfies the relation

$$D(E(\epsilon)) = D(E(p,q,r)) \text{ modulo } \mathbb{T}_0$$

We recall that $E(p,q,r)$ has been characterized in Prop. 2.3. by the fact that it contains the three line bundles x^{-1}, y^{-1}, z^{-1}. We also recall ([LN] Prop. 1.1) that a bundle $E(\epsilon)$ fitting in the sequence

$$0 \longrightarrow \zeta^{-1} \longrightarrow E(\epsilon) \longrightarrow \zeta \longrightarrow 0$$

contains $\zeta(-2q-r) = z^{-1}$ if and only if $\epsilon \in \langle 2q+r \rangle$, the 2-dimensional subspace spanned by the point r and the embedded tangent line at q. Similarly $E(\epsilon)$ contains $\zeta(-2p-r) = y^{-1}$ if and only if $\epsilon \in \langle 2p+r \rangle$. Therefore we may take $\epsilon \in \langle 2q+r \rangle \cap \langle 2p+r \rangle \neq \emptyset$, so that $E(\epsilon)$ contains y^{-1} and z^{-1}. Such extension classes ϵ are parametrized by $\mathbb{P}(y) \cap \mathbb{P}(z)$, which is (lemma 2.1) the tangent line $T_r(C_z) = T_r(C_y)$ at the point $\mathrm{Kum}(\mathcal{O}(p-q)) \in \mathbb{T}_0$. So $\mathbf{P} \circ D(E(\epsilon)) = \mathbf{P} \circ D(E(p,q,r))$ for any such ϵ, and we are done. □

2.9. Remark. Let us also mention that Gunning's vectors $\xi(-,-,-)$, which will be introduced in the next section, will give a third description of the same point in the projective space $|2\Theta|_{\mathbb{T}_0}$.

3 Gunning's results on second order theta functions

Before we are going to prove Thm 1.1, we will need some information on the upper bound of the dimension of the span of the set of points $\mathbf{P} \circ D(E(p,q,r)) \in |2\Theta|_{\mathbb{T}_0}$ when p,q,r allowed to vary on the curve. This result (Cor. 3.2) will be deduced from Gunning's work, which we recall for the reader's convenience. We also include (Prop. 3.3) some identities which we will use in section 6 to prove Thm 1.3. So in this section we recall some classical theory of theta functions seen as holomorphic quasi-periodic functions on $\mathbb{C}^g$, as well as some results by Gunning on the gradient and Hessian of 2θ-functions along the surface $C-C$. We refer to Fay's book [F] and to [Gu1] for a detailed exposition.

Let $\mathcal{C}$ be the universal covering space of the curve C. We choose a base point $z_0 \in \mathcal{C}$ and a symplectic set of generators of $H_1(C,\mathbb{Z})$ and call the corresponding canonical basis of Abelian differentials $\omega_1, \ldots, \omega_g \in H^0(C,K)$; these can be thought of as holomorphic differential 1-forms on $\mathcal{C}$ invariant under the group Γ of covering tranformations acting on $\mathcal{C}$. We construct from these data the period matrix Ω. The associated Abelian integrals

$$w_j(z) = \int_{z_0}^{z} \omega_j$$

are holomorphic functions on $\mathcal{C}$ and are the coordinate functions for a map $w : \mathcal{C} \longrightarrow \mathbb{C}^g$. Let Λ be the lattice in $\mathbb{C}^g$ defined by the period matrix Ω, then $JC = \mathbb{C}^g/\Lambda$ and we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{w} & \mathbb{C}^g \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ \mathcal{C}/\Gamma = C & \xrightarrow{t_x} & JC \end{array}$$

The horizontal map t_x is the Abel-Jacobi map (see 2.1.1) with the line bundle $x = \mathcal{O}(p)$ and $p = \pi(z_0)$. Both vertical arrows are quotient maps of the group actions of Γ (acting on $\mathcal{C}$) and Λ (acting on $\mathbb{C}^g$). Sections of the line bundle $\mathcal{O}(2\Theta_0)$ over JC correspond to the classical second order theta functions. A basis of the space $H^0(JC, \mathcal{O}(2\Theta_0))$ is given by the holomorphic functions on $\mathbb{C}^g$

$$\theta_2\begin{bmatrix}\nu\\0\end{bmatrix}(w,\Omega) = \theta\begin{bmatrix}\nu\\0\end{bmatrix}(2w, 2\Omega) \quad \text{for } \nu \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g \tag{3.1}$$

where the right-hand side is obtained from the first order theta function with characteristics $\left[\begin{smallmatrix}\nu\\0\end{smallmatrix}\right]$

$$\theta\begin{bmatrix}\nu\\0\end{bmatrix}(w,\Omega) = \sum_{n\in\mathbb{Z}^g} \exp 2\pi i \left[\frac{1}{2}{}^t(n+\nu)\Omega(n+\nu) + (n+\nu)w\right]$$

The functions (3.1) are the coordinate functions of a holomorphic map

$$\begin{array}{rcl}\theta_2 : \mathbb{C}^g & \longrightarrow & \mathbb{C}^{2^g} \\ w & \longmapsto & (\ldots, \theta_2\left[\begin{smallmatrix}\nu\\0\end{smallmatrix}\right](w,\Omega), \ldots)\end{array} \tag{3.2}$$

Using this basis, we identify (via the Wirtinger duality (1.1)) $\mathbb{P}(\mathbb{C}^{2^g}) = |2\Theta|$, so that the map (3.2) coincides (take quotient by the lattice Λ) with the Kummer map $JC \to |2\Theta|$.

Next we introduce the *prime form* (see [Gu1] formula (22)) $q(z_1, z_2)$, which is a holomorphic function on $\mathcal{C}\times\mathcal{C}$ with a simple zero along the subvariety $z_1 = Tz_2$ for all covering transformations T and vanishing nowhere else. Moreover, $q(z_1, z_2) = -q(z_2, z_1)$. Alternatively the function q is, up to a multiplicative constant, the pull-back to $\mathcal{C}\times\mathcal{C}$ of the unique global section having as zero scheme the diagonal in $C \times C$.

We use canonical coordinates on the universal covering $\mathcal{C}$ (see section 6 [Gu1]) and we denote by w_j' the derivative of the holomorphic function w_j with respect to the canonical coordinates. To a point $a \in \mathcal{C}$ we associate the differential operator

$$D_a = \sum_{j=1}^{g} w_j'(a)\frac{\partial}{\partial w_j}.$$

This operator corresponds, up to multiplication by a scalar, to the unique translation-invariant vector field over JC, which has as tangent vector at the origin $\mathcal{O} \in JC$ the tangent direction at $\mathcal{O}$ to the curve C, where C is embedded in JC by $q \mapsto \mathcal{O}(q-p)$ with $p = \pi(a) \in C$.

Gunning (see [Gu1] formulae (41), (42), (44)) introduces for $a_1, a_2, a_3, a_4 \in \mathcal{C}$ the following vectors in $H^0(2\Theta)/\mathbb{T}_0 = \mathbb{C}^{2^g}/\mathbb{T}_0$. With abuse of notation, we also denote by $\mathbb{T}_0$ the vector subspace of $H^0(2\Theta)$ corresponding to $\mathbb{T}_0 \subset |2\Theta|$ and by $\mathbf{P}$ the linear projection map $\mathbb{C}^{2^g} \longrightarrow \mathbb{C}^{2^g}/\mathbb{T}_0$ (2.18).

$$\xi(a_1, a_2, a_3) = q(a_2, a_3)^{-2}\mathbf{P}D_{a_1}\theta_2(w(a_2 - a_3)) \tag{3.3}$$

$$\sigma(a_1; a_2, a_3; a_4) = \left[\frac{\partial}{\partial a_1}\log\frac{q(a_1,a_2)}{q(a_1,a_3)}\right]\cdot\xi(a_2, a_3, a_4) \tag{3.4}$$

$$\tau(a_1, a_2; a_3, a_4) = q(a_3, a_4)^{-2}\mathbf{P}D_{a_1}D_{a_2}\theta_2(w(a_3 - a_4)) \tag{3.5}$$

Then ξ (resp. τ) defines a holomorphic function on $\mathcal{C}^3$ (resp. $\mathcal{C}^4$) with values in $\mathbb{C}^{2^g}/\mathbb{T}_0$ and σ defines a meromorphic function on $\mathcal{C}^4$ which has as singularities at most simple poles along the loci $a_1 = Ta_2$ and $a_1 = Ta_3$ for all covering transformations $T \in \Gamma$. The functions ξ, σ, τ have the following symmetry properties: ξ is skew-symmetric in a_1, a_2, a_3 (Cor. 4 [Gu1]), σ is symmetric in a_2, a_3 and τ is symmetric in a_1, a_2 and in a_3, a_4. Given four points $a_1, a_2, a_3, a_4 \in \mathcal{C}$ we will let $\sigma_{ijkl} = \sigma(a_i; a_j, a_k; a_l)$, $\tau_{ijkl} = \tau(a_i, a_j; a_k, a_l)$ and $q_{ij} = q(a_i, a_j)$ where i, j, k, l are four indices such that $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then the following statements hold:

3.1. Theorem (Theorem 3 [Gu1]). *There exist vectors $\xi_{jkl} \in \mathbb{C}^{2^g}/\mathbb{T}_0$ which are skew-symmetric in their indices j,k,l and such that*

$$\xi(a_1, a_2, a_3) = \sum_{j,k,l} \xi_{jkl} w'_j(a_1) w'_k(a_2) w'_l(a_3).$$

3.2. Corollary. *The dimension of the linear span in $\mathbb{C}^{2^g}/\mathbb{T}_0$ of the vectors $\xi(a_1, a_2, a_3)$ for a_i varying in $\mathcal{C}$ is at most $\binom{g}{3}$.*

3.3. Proposition (Cor. 5 [Gu1]). *For any points $a_1, a_2, a_3, a_4 \in \mathcal{C}$*

(1) $\frac{1}{2}\mathbf{P} D_{a_1} D_{a_2} D_{a_3} D_{a_4} \theta_2(0) = \tau_{1234} + \tau_{1324} + \tau_{2314} - 2\sigma_{1234} - 2\sigma_{2134} - 2\sigma_{3124}$

(2) $\frac{1}{2}\tau_{1324} + \frac{1}{2}\tau_{1423} = \left(\frac{q_{12}q_{34}}{q_{13}q_{14}q_{23}q_{24}}\right)^2 \mathbf{P}\theta_2(w(a_1 + a_2 - a_3 - a_4)) + \sigma_{1342} - \sigma_{3241} - \sigma_{4231}$

4 Proof of theorem 1.1

We consider the rational map

$$\begin{array}{rcl} \rho : C_3 & \longrightarrow & |2\Theta|_{\mathbb{T}_0} \\ (p, q, r) & \longmapsto & \mathbf{P} \circ D(E(p, q, r)) \end{array}$$

By Lemma 2.2 the rational map ρ is defined away from the triples of collinear points. We will denote by $\mathbb{T} \subset |2\Theta|$ the inverse image under the projection $\mathbf{P}$ (2.18) of the span in $|2\Theta|_{\mathbb{T}_0}$ of the image $\rho(C_3)$. Obviously we have an inclusion $\mathbb{T}_0 \subset \mathbb{T}$. We claim that

$$\mathbb{T}^\perp = \mathbb{P}\Gamma_{00}^{(2)}$$

which we can see as follows. Consider a hyperplane in $|2\Theta|$ which contains $\mathbb{T}_0$ and the point $D(E(p, q, r))$ for (p, q, r) a triple of non-collinear points. Since the point $D(E(p, q, r))$ lies on the embedded tangent line $T_p(C_x)$ at the point $\mathrm{Kum}(q - r) \in \mathbb{T}_0 = \langle C - C \rangle$ to the Abel curve C_x, we see that such a hyperplane is equivalent via (1.1) to a 2θ-function f whose derivative at $\mathrm{Kum}(q-r)$ vanishes in the direction of the tangent line $T_p(C_x)$, i.e. with abuse of notation $D_p f(q - r) = 0$. Now if we vary the point p on the curve, keeping q and r fixed, we see that a hyperplane containing $\mathbb{T}$ determines a 2θ-divisor which is singular at $\mathcal{O}(q - r)$ since the set of tangent directions (i.e the canonical curve) spans the full tangent space at $\mathcal{O}(q - r)$ to the Jacobian. Next, if we vary q, r, we see that $\mathbb{T}^\perp = \mathbb{P}\Gamma_{00}^{(2)}$. Finally we see from the preceding description and the definition of the vectors ξ (3.3) that the point $\mathbf{P} \circ D(E(p, q, r)) \in |2\Theta|_{\mathbb{T}_0}$ corresponds to the vector $\xi(\tilde{p}, \tilde{q}, \tilde{r}) \in \mathbb{C}^{2^g}/\mathbb{T}_0$, if $\tilde{p}, \tilde{q}, \tilde{r} \in \mathcal{C}$ are lying over the points p, q, r.

The rest of the argument coincides with the argument given in [vGI] section 6. We therefore just sketch their proof: the rational map ρ factorizes as follows

$$\begin{array}{ccc} C_3 & \xrightarrow{\rho} & \mathbb{T}/\mathbb{T}_0 \\ \downarrow^{\pi} & & \uparrow^{\gamma} \\ Gr(3, H^0(K)^*) & \xrightarrow{Pl} & \mathbb{P}(\Lambda^3 H^0(K)^*) \end{array} \qquad (4.1)$$

where $\pi(p, q, r)$ is the linear 3-plane in $H^0(K)^*$ spanned by the 3 points p, q, r and Pl is the Plücker embedding. We recall that $p \wedge q \wedge r$ denotes the Plücker image in $\mathbb{P}(\Lambda^3 H^0(K)^*)$ of the 3-plane $\pi(p, q, r)$. By the incidence relations (Prop. 2.6) the support of the pull-back $\rho^* H_W$ equals the set

$$\{(p, q, r) \in C_3 \mid p \wedge q \wedge r \in \mathbb{P}\Lambda^3 W\}.$$

More precisely, we have an equality $\rho^* H_W = (Pl \circ \pi)^*(\mathbb{P}\Lambda^3 W)$ as divisors on C_3. Here we view $\mathbb{P}\Lambda^3 W$ as a hyperplane in the Plücker space $\mathbb{P}\Lambda^3 H^0(K)^*$. Hence we deduce the factorization and the injectivity of the linear map γ. Since dim $(\mathbb{T}/\mathbb{T}_0) \leq \binom{g}{3} - 1$ by Cor. 3.2, the map γ is an isomorphism

$$\gamma : \mathbb{P}\Lambda^3 H^0(K)^* \xrightarrow{\sim} \mathbb{T}/\mathbb{T}_0.$$

Since $\mathbb{T}^\perp = \mathbb{P}\Gamma_{00}^{(2)}$ and $\mathbb{T}_0^\perp = \mathbb{P}\Gamma_{00}$, we have $(\mathbb{T}/\mathbb{T}_0)^* \cong \mathbb{P}(\Gamma_{00}/\Gamma_{00}^{(2)})$. So we get Thm. 1.1 (1), by taking the dual of the isomorphism γ.

To prove Thm. 1.1 (2), we note that we have an inclusion $\langle \mathrm{Sing}\Theta \rangle \subset \mathbb{P}\Gamma_{00}^{(2)}$: by Riemann's singularity theorem, for $\xi \in \mathrm{Sing}\ \Theta$, $h^0(\xi) \geq 2$, which implies that $h^0(\xi(p-q)) \geq 1$ and similarly $h^0(K\xi^{-1}(p-q)) \geq 1$. So we see that

$$\forall p, q \in C \quad \mathcal{O}(p-q) \in \Theta_\xi \cap \Theta_{K\xi^{-1}} \subset \mathrm{Sing}\ (\Theta_\xi + \Theta_{K\xi^{-1}})$$

i.e. the divisor $\Theta_\xi + \Theta_{K\xi^{-1}}$ is singular along $C - C$. Since both spaces have the same dimension (for a computation of dim $\langle \mathrm{Sing}\Theta \rangle$ see [vGI] section 7), we obtain equality.

4.1. Remark. It follows from lemma 2.8 and [vGI] cor 6.10 that the subspace $\mathbb{T} \subset |2\Theta|$ coincides with the embedded tangent space to $\mathcal{SU}_C(2, \mathcal{O})$ at the origin. So we get three descriptions of a subseries of $|2\Theta_0|$

$$\langle \mathrm{Sing}\Theta \rangle = \mathbb{P}\Gamma_{00}^{(2)} = \mathbb{T}^\perp$$

5 Quadrics on canonical space

5.1 Petri's quadrics

We will denote by $\tilde{Q}$ the polarized form of a quadric Q on canonical space $|K|^*$, i.e. $\tilde{Q}$ is the symmetric bilinear form such that $\tilde{Q}(v, v) = Q(v)$, $\forall v \in H^0(K)$.

5.1. Lemma. *We consider $g-2$ points in general position $p_1, \ldots, p_{g-2}$. If a quadric $Q \in I(2)$ is such that*

$$\tilde{Q}(p_i, p_j) = 0 \qquad \forall i, j \in \{1, \ldots, g-2\}$$

then Q is identically zero.

Proof. By the general position theorem ([ACGH] p.109), we know that a general hyperplane $H \subset |K|^*$ meets C in $2g-2$ points any $g-1$ of which are linearly independent. We consider such an H and $g-1$ independent points $q_1, \ldots, q_{g-1}$ in H and suppose that the $g-2$ points $p_1, \ldots, p_{g-2}$ are taken among the $g-1$ residual points of $H \cap C$, i.e

$$H \cap C = \{p_1, \ldots, p_{g-2}, q_1, \ldots, q_{g-1}, q_g\}.$$

It is clear that a general $(g-2)$-tuple $(p_1, \ldots, p_{g-2})$ can be realized in this way.

Now suppose that $\tilde{Q}(p_i, p_j) = 0$, $\forall i, j \in \{1, \ldots, g-2\}$, i.e. Q contains the linear subspace Π spanned by the p_i's. Since Π is a hyperplane in H, any line $\overline{q_i q_j}$, for $1 \leq i < j \leq g$ intersects Π. So $\overline{q_i q_j}$ is entirely contained in Q, since it meets Q in at least three points. Hence we obtain that $\tilde{Q}(q_i, q_j) = 0$, $\forall i, j \in \{1, \ldots, g\}$, i.e. Q contains the hyperplane H. But since Q contains C, it cannot be the union of two hyperplanes, hence Q is identically zero. □

From now on we fix $g-2$ points $p_1, \dots, p_{g-2}$, which are chosen in general position. By the preceding Lemma 5.1, the $\binom{g-2}{2}$ hyperplanes (for $1 \leq i < j \leq g-2$)

$$\mathcal{H}_{ij} = \{Q \in I(2) \mid \tilde{Q}(p_i, p_j) = 0\}$$

are linearly independent in $I(2)^*$, hence they form a basis of $I(2)^*$. Let us denote by $\{Q_{ij}\}$ the corresponding dual basis. The quadrics $Q_{ij} \in I(2)$ are characterized by the properties

$$\begin{aligned} &\tilde{Q}_{ij}(p_\alpha, p_\beta) = 0 \ \text{if} \ \{i,j\} \neq \{\alpha, \beta\} \\ &\tilde{Q}_{ij}(p_i, p_j) \neq 0. \end{aligned} \tag{5.1}$$

This basis of quadrics has been used by K. Petri ([P], see also [ACGH] p.123-135) in his work on the syzygies of the canonical curve. He defines them in a slightly different way:

Choose two additional points p_{g-1}, p_g in general position. For each i, $1 \leq i \leq g$, pick a generator ω_i of the one-dimensional space

$$H^0(K(-\sum_{j=1, j\neq i}^{g} p_j)) = \mathbb{C}\omega_i \tag{5.2}$$

Up to a constant the ω_i's form a dual basis to the points $p_i \in |K|^*$. Then there are constants ([ACGH] p.130) $\lambda_{sij}, \mu_{sij}, b_{ij} \in \mathbb{C}$ such that, if we let

$$\eta_{ij} = \sum_{s=1}^{g-2} \lambda_{sij}\omega_s \qquad \nu_{ij} = \sum_{s=1}^{g-2} \mu_{sij}\omega_s \tag{5.3}$$

the quadratic polynomials

$$R_{ij} = \omega_i\omega_j - \eta_{ij}\omega_{g-1} - \nu_{ij}\omega_g - b_{ij}\omega_{g-1}\omega_g \tag{5.4}$$

all vanish on C. Moreover the R_{ij}'s form a basis of $I(2)$ and, obviously, the rank of the quadric R_{ij} is less than or equal to 6.

5.2. Lemma. *For each $1 \leq i < j \leq g-2$, the quadrics R_{ij} satisfy the conditions* (5.1), *hence $R_{ij} = Q_{ij}$.*

Proof. This follows immediately from the definition (5.2) of the ω_i's. □

5.3. Proposition. *If C is neither trigonal nor a smooth plane quintic and the points $p_1, \dots, p_{g-2}$ are in general position, then the quadrics Q_{ij} have the following properties*

$$(i)\ \mathrm{Sing}\, Q_{ij} \cap C = \emptyset \tag{5.5}$$

$$(ii)\ \mathrm{rk}\, Q_{ij} = 5 \ \mathit{or}\ 6. \tag{5.6}$$

Proof. We fix two indices i, j. First we observe that the singular locus $\mathrm{Sing}\, Q_{ij}$ is the annihilator of the linear space

$$\langle \omega_i, \omega_j, \omega_{g-1}, \omega_g, \eta_{ij}, \nu_{ij} \rangle \subset |K|. \tag{5.7}$$

Hence $C \cap \mathrm{Sing} Q_{ij}$ is the base locus of this linear subsystem. In particular $C \cap \mathrm{Sing} Q_{ij}$ is contained in the base locus of $\langle \omega_i, \omega_j, \omega_{g-1}, \omega_g \rangle$, which, by construction, consists of the $g-4$ points (we delete the i-th and j-th point)

$$p_1, \dots, \hat{p}_i, \dots, \hat{p}_j, \dots, p_{g-2}. \tag{5.8}$$

We will denote by D_{ij} the degree $g-4$ divisor with support (5.8) and by $\bar{D}_{ij}$ the linear span of D_{ij} in $|K|^*$. Suppose now that there exists a $(g-2)$-tuple $p_1, \dots, p_{g-2}$ such that (i) holds, then, since (i) is an open condition, (i) holds for a general $(g-2)$-tuple of points. Therefore we will assume that (i) does not hold for all $(g-2)$-tuples, i.e. $\forall p_1, \dots, p_{g-2}$ (in general position), there exists a $p_\alpha \in \operatorname{Sing} Q_{ij}$ for some $\alpha \in \{1, \dots, g-2\}, \alpha \neq i, j$. But, since the quadric Q_{ij} does not depend on the order of the $g-4$ points (5.8), $p_\alpha \in \operatorname{Sing} Q_{ij}$ implies that all $g-4$ points (5.8) are in $\operatorname{Sing} Q_{ij}$. Hence

$$\bar{D}_{ij} \subset \operatorname{Sing} Q_{ij} \tag{5.9}$$

and therefore $\operatorname{rk} Q_{ij} \leq 4$. Since $\omega_i, \omega_j, \omega_{g-1}, \omega_g$ are linearly independent, we have $\operatorname{rk} Q_{ij} = 4$. Hence the inclusion (5.9) is an equality (same dimension).

Consider now the two rulings of the rank 4 quadric Q_{ij}: they cut out on the curve two pencils of divisors

$$\mathbb{P}^1 \subset |L| \qquad \mathbb{P}^1 \subset |M|$$

such that L, M are line bundles satisfying

$$L \otimes M = K(-D_{ij}). \tag{5.10}$$

Therefore for general D_{ij}, we have constructed a pair $(L, M) \in W^1_d(C) \times W^1_{d'}(C)$ satisfying (5.10) with $d + d' = \deg K(-D_{ij}) = g+2$. By Mumford's refinement of Martens' Theorem (see [ACGH] p.192-3), if C is neither trigonal, bi-elliptic, nor a smooth plane quintic, then $\dim W^1_d(C) \leq d-4$ for $4 \leq d \leq g-2$. Hence

$$\dim W^1_d(C) \times W^1_{d'}(C) \leq (d-4) + (d'-4) = g-6 < g-4$$

which contradicts relation (5.10).

In order to prove (i) we need to show that the case C bi-elliptic also leads to a contradiction. Let $\pi : C \longrightarrow E$ be a degree 2 mapping onto an elliptic curve E. Then by [ACGH] p.269, exercise E1, the chords $\overline{pq}$ with $p + q = \pi^* e$ for some $e \in E$ all pass through a common point $a \notin C$. In particular a lies on a chord through all points $p_\alpha \in \operatorname{Sing} Q_{ij}$, hence $a \in Q_{ij}$. Since $C \subset |K|^*$ is non-degenerate, $a \in \operatorname{Sing} Q_{ij}$ and for $p_1, \dots, p_{g-2}$ in general position

$$a \notin \bar{D}_{ij}$$

hence $\operatorname{rk} Q_{ij} \leq 3$, a contradiction.

It remains to show that (ii) holds. We observe that

$$\begin{aligned} \operatorname{rk} Q_{ij} = 4 \quad &\Longleftrightarrow \quad \eta_{ij}, \nu_{ij} \in \langle \omega_i, \omega_j \rangle \\ &\Longleftrightarrow \quad \operatorname{Sing} Q_{ij} = \bar{D}_{ij} \end{aligned}$$

and we conclude as before. □

5.4. Remark. If C is trigonal or a smooth plane quintic, then for all $(g-2)$-tuples $p_1, \dots, p_{g-2}$ (in general position) we have

$$\operatorname{Sing} Q_{ij} \cap C = D_{ij} \qquad \text{and} \qquad \operatorname{rk} Q_{ij} = 4$$

We can give a more precise description of the quadrics Q_{ij} in both cases: it will be enough to exhibit a set of quadrics satisfying the characterizing properties (5.1).

1. C is *trigonal.* Let $|g^1_3|$ be the trigonal pencil and consider the complete linear series of degree $g-1$

$$\xi = g^1_3 + D_{ij}$$

where D_{ij} is as in the proof of Prop. 5.3. Then define Q_ξ to be the cone with vertex $\bar{D}_{ij} = \mathbb{P}^{g-5}$ over the smooth quadric $|\xi|^* \times |K\xi^{-1}|^* = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3 = |K(-D_{ij})|^*$

$$\begin{array}{ccc} Q_\xi & \subset & |K|^* \\ \downarrow & & \downarrow^{pr} \\ \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{m} & \mathbb{P}^3 \end{array} \tag{5.11}$$

where pr is the linear projection with center $\bar{D}_{ij}$ and m is the Segre map. From this description it is clear that the rank 4 quadric Q_ξ satisfies (5.1).

2. C is a *smooth plane quintic* ($g = 6$). Let $|g^2_5|$ be the associated degree 5 linear series. We can write $D_{ij} = p_k + p_l$ for some indices k, l and, as in the trigonal case, we consider the quadric Q_ξ defined by the diagram (5.11) with

$$\xi = g^2_5(-p_k) \qquad \text{and} \qquad K\xi^{-1} = g^2_5(-p_l)$$

Again we easily check that Q_ξ satisfies (5.1).

5.2 Rank 6 quadrics and rank 2 vector bundles

In this section we recall a construction [BV] relating rank 6 quadrics and rank 2 vector bundles. We consider a rank 2 bundle E and a subspace $V \subset H^0(E)$ which satisfy the conditions:

$$\begin{array}{l} \det E = K \\ \dim V = 4 \\ V \text{ generates } E. \end{array} \tag{5.12}$$

We can associate to such a bundle the following commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & Gr(2, V^*) \\ \Big\downarrow \varphi_K & & \Big\downarrow \\ |K|^* & \xrightarrow{\lambda^*} & \mathbb{P}(\wedge^2 V^*) = \mathbb{P}^5 \end{array} \tag{5.13}$$

where γ is the morphism (defined since we have a surjection $\mathcal{O}_C \otimes V \to E$)

$$\gamma : p \longmapsto E^*_p \subset V^* \tag{5.14}$$

and λ the map defined by taking the exterior product of global sections of E

$$\lambda : \wedge^2 V \longrightarrow H^0(\wedge^2 E) = H^0(K). \tag{5.15}$$

The Grassmannian $Gr(2, V^*)$ embedded in $\mathbb{P}^5$ by the Plücker embedding is a smooth quadric. We define $Q_{(E,V)}$ to be the inverse image of this quadric:

$$Q_{(E,V)} = (\lambda^*)^{-1}(Gr(2, V^*)). \tag{5.16}$$

Then $\operatorname{rk} Q_{(E,V)} \leq 6$ and $Q_{(E,V)} \in |I(2)|$. If $h^0(E) = 4$, then $V = H^0(E)$ and we denote the quadric $Q_{(E,V)}$ simply by Q_E. We have the following lemmas:

5.5. Lemma. *For any pair* (E, V) *satisfying conditions* (5.12)*, let* $\tilde{Q}_{(E,V)}$ *be the polar form of the quadric* $Q_{(E,V)}$*. Then*

$$\forall p, q \in C,\ p \neq q\ :\ \tilde{Q}_{(E,V)}(p,q) = 0 \iff V \cap H^0(E(-p-q)) \neq \{0\}.$$

Proof. We consider the dual Grassmannian $Gr(2, V)$. Now $\tilde{Q}_{(E,V)}(p,q) = 0$ means that the line joining the two points corresponding to the two subspaces $H^0(E(-p)) \cap V$ and $H^0(E(-q)) \cap V$ is contained in $Gr(2, V)$. But this is equivalent to $H^0(E(-p)) \cap H^0(E(-q)) \cap V \neq \{0\}$ and we are done. □

5.6. Lemma. *For any pair* (E, V) *satisfying conditions* (5.12)*, we have*

$$\mathrm{rk}\, Q_{(E,V)} \leq 4 \iff E \text{ contains a line subbundle } L \text{ with } \dim H^0(L) \cap V = 2.$$

Proof. This is essentially Prop. 1.11 of [BV]. Note that if E is generated by V, then $\dim H^0(L) \cap V \leq 2$. □

5.7. Remark. We see that the definition (5.16) of the quadric $Q_{(E,V)}$ makes sense even if the bundle E is not generated by global sections in V at a finite number of points. We easily see that E is not generated by V at the point p if and only if $p \in \mathbb{P}\ker \lambda^* \subset \mathrm{Sing}\, Q_{(E,V)}$. Moreover, if $\mathrm{rk}\, Q_{(E,V)} \geq 5$, then we have an equality $\mathbb{P}\ker \lambda^* = \mathrm{Sing}\, Q_{(E,V)}$ (see [BV] (1.9)).

The above described construction which associates to the pair (E, V) the quadric $Q_{(E,V)} \in I(2)$ admits an *inverse* construction, i.e. we can recover a bundle E from a general rank 6 quadric: consider a quadric $Q \in I(2)$ satisfying

$$\begin{aligned} r = \mathrm{rk}\, Q = 5 \text{ or } 6 \\ \mathrm{Sing}\, Q \cap C = \emptyset \end{aligned} \tag{5.17}$$

We project away from $\mathrm{Sing}\, Q$

$$\delta : Q \longrightarrow \mathbb{P}^{r-1}.$$

If $r = 6$, the image $\delta(Q)$ is a smooth quadric in $\mathbb{P}^5$ and can be realized as a Grassmannian $Gr(2, 4)$. If $r = 5$, $\delta(Q)$ is a linear section of $Gr(2, 4) \subset \mathbb{P}^5$. We consider the exact sequence over $Gr(2, 4)$

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{O}^4_{Gr} \longrightarrow \bar{\mathcal{U}} \longrightarrow 0$$

where $\mathcal{U}$ (resp. $\bar{\mathcal{U}}$) is the universal subbundle (resp. quotient bundle). Since $\mathrm{Sing}\, Q \cap C = \emptyset$, the restriction of δ to the curve C is everywhere defined and we can consider the two pairs (let $g = \delta_{|C}$)

$$(g^*\mathcal{U}^*, g^* H^0(\mathcal{U}^*)) \qquad (g^*\bar{\mathcal{U}}, g^* H^0(\bar{\mathcal{U}})) \tag{5.18}$$

which satisfy conditions (5.12). The following proposition is proved in [BV] (Prop. (1.18) and (1.19))

5.8. Proposition. *The pairs* (5.18) *are the only pairs defining the quadric* Q*. If* $\mathrm{rk}\, Q = 5$ *then they are isomorphic.*

5.9. Lemma. *Consider a bundle* E *with* $h^0(E) = 4$ *and satisfying* (5.12)*. If* $\mathrm{rk}\, Q_E = 5$ *or*6*, then* E *is stable*

Proof. Suppose that there exists a destabilizing subbundle $L \subset E$, with $\deg L \geq g - 1$. By Lemma 5.6 we have $h^0(L) \leq 1$ and by Riemann-Roch, $h^0(KL^{-1}) \leq h^0(L)$. But then $h^0(E) \leq h^0(L) + h^0(KL^{-1}) \leq 2$, a contradiction. □

5.3 Proof of theorem 1.2

5.3.1 The map in (1.7)

First, we prove the inclusion $\mathbb{P}\Gamma_{11} \subset \mathbb{P}\Gamma_{00}^{(2)}$. Consider the line bundle $\mathcal{O}(q-r) \in C - C$ and a point $p \in C$. The Abel-Jacobi curve $t_x(C) \subset JC$, with $x = \mathcal{O}(p+q-r)$, is contained in the fourfold $C_2 - C_2$. Therefore a hyperplane H in $|2\Theta|$ which contains the Kummer image of $C_2 - C_2$ in $|2\Theta|$ also contains the embedded tangent line $T_{q-r}(C_x)$ at the point $\mathrm{Kum}(\mathcal{O}(q-r))$ to the curve $C_x \subset \mathbb{P}(x) \subset |2\Theta|$ for all triples of points p, q, r. Since the set of tangent directions given by these tangent lines (keeping q, r fixed and varying p) span the projectivized tangent space $\mathbb{P}T_{q-r}JC$, we see that, varying q and r, $\mathbb{T} \subset H$. Since $\mathbb{P}\Gamma_{00}^{(2)} = \mathbb{T}^{\perp}$, we get the inclusion by taking the polar spaces.

Now we consider the difference map

$$\gamma : C^4 \xrightarrow{pr} C_2 \times C_2 \xrightarrow{\phi_2} JC$$

where C^4 is the 4-fold product of the curve and the first arrow pr is the quotient by the transpositions $(1,2)$ and $(3,4)$ acting on C^4. We denote by $\Delta_{i,j}$ the divisor in C^4 consisting of 4-tuples having equal i-th and j-th entry. A straightforward computation shows that

$$\gamma^*\mathcal{O}(2\Theta_0) = \bigotimes_{i=1}^{4} \pi_i^* K(-2\Delta_{1,2} - 2\Delta_{3,4} + 2\Delta_{1,3} + 2\Delta_{1,4} + 2\Delta_{2,3} + 2\Delta_{2,4}) \tag{5.19}$$

Note that the divisor $\Delta_{1,3} + \Delta_{1,4} + \Delta_{2,3} + \Delta_{2,4} \subset C^4$ is invariant under the transpositions $(1,2)$ and $(3,4)$, hence comes from an irreducible divisor in $C_2 \times C_2$, which we call Δ. We also observe that the line bundle $\pi_1^*K \otimes \pi_2^*K(-2\Delta_{1,2})$ over C^2 is invariant under the natural involution, hence comes from a line bundle $\mathcal{M}$ over C_2 and we have a canonical isomorphism

$$H^0(C_2, \mathcal{M}) \cong I(2). \tag{5.20}$$

With this notation we rewrite (5.19) as

$$\phi_2^*(\mathcal{O}(2\Theta_0)) = \pi_1^*\mathcal{M} \otimes \pi_2^*\mathcal{M}(2\Delta). \tag{5.21}$$

Now we want to compute the pull-back of 2θ-divisors vanishing doubly on $C - C$. Let $\mathcal{J}$ be the sheaf of ideals defining the surface $C - C \subset JC$.

5.10. Lemma. *If C is non-trigonal, the inverse image ideal sheaf $\phi_2^{-1}\mathcal{J} \cdot \mathcal{O}_{C_2\times C_2}$ is the invertible sheaf $\mathcal{O}_{C_2\times C_2}(-\Delta)$, hence $\phi_2^*\mathcal{J} = \mathcal{O}_{C_2\times C_2}(-\Delta)$*

Proof. This follows from the observation that the inverse image of $C - C$ under ϕ_2 is isomorphic to the divisor Δ. □

5.11. Remark. If C is trigonal, the inverse image $\phi_2^{-1}(C - C)$ contains, apart from the divisor Δ, a surface which is the image of the morphism

$$\begin{array}{rcl} C \times C & \longrightarrow & C_2 \times C_2 \\ (p,q) & \longmapsto & (g_3^1(-p), g_3^1(-q)) \end{array}$$

where g_3^1 is the trigonal series (unique if $g \geq 5$) and $g_3^1(-p)$ denotes the residual pair of points in the fibre containing p. We deduce that $\phi_2^{-1}\mathcal{J} \cdot \mathcal{O}_{C_2\times C_2} \subset \mathcal{O}_{C_2\times C_2}(-\Delta)$ and that there exists a natural map of $\mathcal{O}_{C_2\times C_2}$-modules $\phi_2^*\mathcal{J} \longrightarrow \mathcal{O}_{C_2\times C_2}(-\Delta)$.

Combining (5.21) and lemma 5.10, we obtain a linear map induced by ϕ_2

$$\phi_2^* : H^0(JC, \mathcal{O}(2\Theta_0) \otimes \mathcal{J}^2) = \Gamma_{00}^{(2)} \longrightarrow H^0(C_2 \times C_2, \pi_1^*\mathcal{M} \otimes \pi_2^*\mathcal{M}) \tag{5.22}$$

This map is equivariant under the natural involutions of JC and $C_2 \times C_2$. Since all second order theta functions are even, the image of ϕ_2^* is contained in the subsapce $\mathrm{Sym}^2 H^0(C_2, \mathcal{M}) \cong \mathrm{Sym}^2 I(2)$, by (5.20). To summarize, we have shown that

$$\Gamma_{11} = \ker\,(\phi_2^* : \Gamma_{00}^{(2)} \longrightarrow \mathrm{Sym}^2 I(2)).$$

5.3.2 Surjectivity of ϕ_2^*

The key point of the proof is the following proposition

5.12. Proposition. *We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{W}(4) & \xrightarrow{D} & \mathbb{P}\Gamma_{00}^{(2)} \\ \big\downarrow{\scriptstyle Q} & & \big\downarrow{\scriptstyle \phi_2^*} \\ |I(2)| & \xrightarrow{Ver} & \mathbb{P}\mathrm{Sym}^2 I(2) \end{array} \tag{5.23}$$

where the notation is as follows:

$\mathcal{W}(4) = \{[E] \in \mathcal{SU}_C(2, K) \mid \dim H^0(E) = 4 \text{ and } E \text{ globally generated}\}$

Q *is the map described in section 5.2*

Ver *is the Veronese map*

D *is the map* (2.2)

Proof. Consider a bundle $E \in \mathcal{W}(4)$. By [L] Prop. V.2., we have an inequality

$$\mathrm{mult}_{p-q}(D(E)) \geq h^0(E(p-q)) \geq 2$$

hence we see that $D(E) \in \mathbb{P}\Gamma_{00}^{(2)}$. To show commutativity, it is enough to check that the zero divisors of the two sections

$$Q_E \otimes Q_E \qquad \phi_2^*(D(E))$$

(regarded as sections of $\pi_1^*\mathcal{M} \otimes \pi_2^*\mathcal{M}$ over $C_2 \times C_2$ (5.22)) coincide as sets. Hence, by Lemma 5.5 and (2.2), it is enough to show the following equivalence: for any four *distinct* points $p, q, r, s \in C$

$$h^0(E(-p-q)) > 0 \text{ or } h^0(E(-r-s)) > 0 \iff h^0(E(p+q-r-s)) > 0$$

The $\Rightarrow$ implication is obvious ($D(E)$ is symmetric). To prove the $\Leftarrow$ implication, we suppose that $h^0(E(-p-q)) = h^0(E(-r-s)) = 0$. Then, by Riemann-Roch and Serre duality, we have $h^0(E(p+q)) = 4$. Since $h^0(E) = 4$, we see that all global sections of $E(p+q)$ vanish at the points p and q. Supposing that there exists a non-zero section φ of $E(p+q-r-s)$, then φ vanishes at p, q, contradicting $h^0(E(-r-s)) = 0$, hence $h^0(E(p+q-r-s)) = 0$. □

From now on, we assume that C is *non-trigonal*. We consider g points $p_1, \ldots, p_g$ in general position and their associated Petri quadrics Q_{ij} for $1 \leq i, j \leq g-2$ (section 5.1), which form a basis of $I(2)$. Then in order to prove surjectivity of ϕ_2^*, it is enough, by Prop. 5.12, to construct a set of $\binom{h+1}{2}$ vector bundles (with $h = \binom{g-2}{2} = \dim I(2)$) in $\mathcal{W}(4)$, which generate linearly $\mathrm{Sym}^2 I(2)$. First we suppose that C is not a smooth plane quintic. We proceed in 3 steps.

Step 1

By Prop. 5.3, the Petri quadrics Q_{ij} satisfy conditions (5.17), so (Prop. 5.8) we can construct for each i, j two pairs (see (5.18)) of bundles $(E_{ij}^{(1)}, V^{(1)})$ and $(E_{ij}^{(2)}, V^{(2)})$, which define the quadric Q_{ij}.

5.13. Lemma. *For general points $p_1, \dots, p_{g-2}$, the bundles $E_{ij}^{(1)}, E_{ij}^{(2)}$ are stable, distinct and $h^0(E_{ij}^{(1)}) = h^0(E_{ij}^{(2)}) = 4$.*

Proof. We can give a different description of the bundles $E_{ij}^{(1)}$ and $E_{ij}^{(2)}$ using extension spaces. Let $D = D_{ij} + p_i$ and consider extensions of the form

$$0 \longrightarrow \mathcal{O}(D) \longrightarrow \mathcal{E}_\varepsilon \longrightarrow \mathcal{O}(K-D) \longrightarrow 0 \qquad (\varepsilon)$$

These extensions are classified by an extension class $\varepsilon \in |2K-2D|^* = \mathbb{P}^{g+2}$. Since $h^0(D) = 1$ and $h^0(K-D) = 3$, we see that $h^0(\mathcal{E}_\varepsilon) = 4$ if and only if $\varepsilon \in \ker m^* = (\operatorname{coker} m)^*$, where m is the multiplication map

$$m : \operatorname{Sym}^2 H^0(K-D) \longrightarrow H^0(2K-2D)$$

which is injective for general points p_i. Note that $\dim \operatorname{coker} m = g-3$. Furthermore, consider a point $p_\alpha \in D_{ij}$ and the multiplication map (which is injective)

$$m_\alpha : H^0(K-D+p_j+p_\alpha) \otimes H^0(K-D-p_j-p_\alpha) \longrightarrow H^0(2K-2D).$$

Then $h^0(\mathcal{E}_\varepsilon(-p_j-p_\alpha)) > 0 \iff \varepsilon \in \ker m^*$. We observe that the image $\operatorname{im} m_\alpha \subset H^0(2K-2D)$ under the canonical surjection $H^0(2K-2D) \longrightarrow \operatorname{coker} m$ is a one-dimensional subspace, which we denote by Z_α. Consider now a hyperplane H in $\operatorname{coker} m$, which contains the linear span of the Z_α for α such that $p_\alpha \in D_{ij}$ (we will see a posteriori that such an H is unique, for dimensional reasons), so that we obtain an extension class $\varepsilon = \varepsilon(H) \in \mathbb{P}(\operatorname{coker} m)^* \subset |2K-2D|^*$. By construction, we have $h^0(\mathcal{E}_\varepsilon) = 4$ and $h^0(\mathcal{E}_\varepsilon(-p_\alpha - p_\beta)) > 0$ if $\{\alpha, \beta\} \neq \{i, j\}$, hence, by (5.1) and lemma 5.5, we get $Q_{\mathcal{E}_\varepsilon} = Q_{ij}$, and $\mathcal{E}_\varepsilon = E_{ij}^{(1)}$.

The other bundle $E_{ij}^{(2)}$ defining the quadric Q_{ij} is constructed in the same way using the divisor $D' = D_{ij} + p_j$ (instead of D). Then we have $E_{ij}^{(1)} \neq E_{ij}^{(2)}$. Indeed, an isomorphism $E_{ij}^{(1)} \to E_{ij}^{(2)}$ would imply the existence of a nonzero section of $\mathcal{H}om(\mathcal{O}(D), \mathcal{O}(K-D')) = \mathcal{O}(K-D-D')$, but then the points $p_1, \dots, p_{g-2}$ are not in general position.

Finally, stability follows from Lemma 5.9 □

We deduce from this lemma and Prop. 5.12 that $Q_{ij} \otimes Q_{ij} \in \operatorname{im} \phi_2^*$.

Step 2

Consider three distinct indices i, j, k. Then all quadrics of the pencil

$$(\lambda Q_{ij} + \mu Q_{ik})_{\lambda, \mu \in \mathbb{C}} \qquad (5.24)$$

have rank less than or equal to 6. This follows from expression (5.4) of Petri's quadrics, namely

$$\lambda Q_{ij} + \mu Q_{ik} = \omega_i(\lambda \omega_j + \mu \omega_k) - \bar{\eta} \omega_{g-1} - \bar{\nu} \omega_g - \bar{b} \omega_{g-1} \omega_g$$

with $\bar{\eta} = \lambda \eta_{ij} + \mu \eta_{ik}, \bar{\nu} = \lambda \nu_{ij} + \mu \nu_{ik}, \bar{b} = \lambda b_{ij} + \mu b_{ik}$. Now a general element of the pencil (5.24) satisfies conditions (5.17), since these are open conditions and are satisfied by Q_{ij} and Q_{ik}. Again by openness and Lemma 5.13, it follows that the two bundles associated with such a general quadric have 4 sections and are stable. Let us pick such a bundle E_{ijk} defining the quadric $\lambda_0 Q_{ij} + \mu_0 Q_{ik}$, for $\lambda_0, \mu_0 \neq 0$. Then we have in $\operatorname{Sym}^2 I(2)$

$$\phi_2^* D(E_{ijk}) = \lambda_0^2 Q_{ij} \otimes Q_{ij} + \mu_0^2 Q_{ik} \otimes Q_{ik} + 2\lambda_0 \mu_0 Q_{ij} \otimes Q_{ik}$$

hence, $Q_{ij} \otimes Q_{ik} \in \operatorname{im} \phi_2^*$.

Step 3

Consider four distinct indices i, j, k, l. Then all quadrics of the 2-dimensional family $\mathcal{F}_{(ij)(kl)} = \mathbb{P}^1 \times \mathbb{P}^1$ (here $(\lambda, \lambda'), (\mu, \mu')$ are a set of homogeneous coordinates) are given by an expression:

$$\mu\lambda Q_{ik} + \mu\lambda' Q_{il} + \mu'\lambda Q_{ik} + \mu'\lambda' Q_{jl} =$$
$$(\mu\omega_i + \mu'\omega_j)(\lambda\omega_k + \lambda'\omega_l) - \bar{\eta}\omega_{g-1} - \bar{\nu}\omega_g - \bar{b}\omega_{g-1}\omega_g,$$

where $\bar{\eta}, \bar{\nu}, \bar{b}$ depend on $(\lambda, \lambda'), (\mu, \mu')$, see (5.4). The same holds for the two families obtained by permuting indices

$$\mathcal{F}_{(ik)(il)} \ : \ \mu\lambda Q_{ij} + \mu\lambda' Q_{il} + \mu'\lambda Q_{jk} + \mu'\lambda' Q_{kl}$$
$$\mathcal{F}_{(il)(kj)} \ : \ \mu\lambda Q_{ik} + \mu\lambda' Q_{ij} + \mu'\lambda Q_{lk} + \mu'\lambda' Q_{jl}$$

As in step 2, we see that a general member of these 3 families satisfies (5.17) and we can pick 3 stable vector bundles $E_{(ij)(kl)}, E_{(ik)(jl)}, E_{(il)(kj)}$ with 4 sections defining the 3 quadrics in these families with coordinates $(\lambda_0, \lambda_0')(\mu_0, \mu_0')$ for some $\lambda_0, \lambda_0', \mu_0, \mu_0' \neq 0$. Now we can write in $\mathrm{Sym}^2 I(2)$

$$\phi_2^* D(E_{(ij)(kl)}) = 2\mu_0\mu_0'\lambda_0\lambda_0'(Q_{ik} \otimes Q_{jl} + Q_{il} \otimes Q_{jk}) + \alpha$$
$$\phi_2^* D(E_{(ik)(jl)}) = 2\mu_0\mu_0'\lambda_0\lambda_0'(Q_{ij} \otimes Q_{kl} + Q_{il} \otimes Q_{jk}) + \beta$$
$$\phi_2^* D(E_{(il)(kj)}) = 2\mu_0\mu_0'\lambda_0\lambda_0'(Q_{ik} \otimes Q_{jl} + Q_{ij} \otimes Q_{kl}) + \gamma$$

for some $\alpha, \beta, \gamma \in \mathrm{im}\ \phi_2^*$ (see step 1 and 2). But these linear equations immediately imply that the three symmetric tensors $Q_{ij} \otimes Q_{kl}, Q_{ik} \otimes Q_{jl}, Q_{il} \otimes Q_{jk} \in \mathrm{im}\ \phi_2^*$ and we are done.

To complete the proof we need to consider the case when C is a smooth plane quintic ($g = 6$). We will show that the map Q in diagram (5.23) is dominant. By [AH] Prop. 3.2, the locus of rank 4 quadrics is a cubic hypersurface in $|I(2)| = \mathbb{P}^5$ and a general quadric has rank 6 (i.e. is smooth). Consider any smooth quadric $Q \in |I(2)|$ and one of the associated pairs (E, V) defining Q (5.18). It will be sufficient to show that $h^0(E) = 4$, hence $E \in \mathcal{W}(4)$. By [OPP] Thm. 8.1 (3), $h^0(E) \geq 5$ if and only if E is an extension of the form

$$0 \longrightarrow g_5^2 \longrightarrow E \longrightarrow g_5^2 \longrightarrow 0.$$

From this we see that if $h^0(E) = 5$, $\dim V \cap H^0(g_5^2) \geq 2$, hence by Lemma 5.6 $\mathrm{rk}\, Q \leq 4$ and if $h^0(E) = 6$, then $E = g_5^2 \oplus g_5^2$ and we can also conclude that $\mathrm{rk}\, Q \leq 4$, contradicting Q smooth.

5.4 Another proof of a theorem by M. Green

As a consequence of Thms. 1.1 and 1.2 we get another proof of the following theorem (in the case of non-trigonal curves) due to M. Green ([Gr], see also [SV])

5.14. Theorem. *For C non-trigonal, the projectivized tangent cones to Θ at double points span $I(2)$.*

Proof. For all $\xi \in \mathrm{Sing}\, \Theta$ with $h^0(\xi) = 2$, the split bundle $E = \xi \oplus K\xi^{-1} \in \mathcal{W}(4)$. Then the associated quadric Q_E has rank less than or equal to 4 and can be described (in the generic case when $\mathrm{rk}\, Q_E = 4$) as a cone over the smooth quadric $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}V^* = \mathbb{P}^3$ (as in diagram (5.11)) where V is the image of the injective multiplication map

$$H^0(\xi) \otimes H^0(K\xi^{-1}) \longrightarrow H^0(K).$$

Then Q_E is the projectivized tangent cone at the double point $\xi \in \mathrm{Sing}\, \Theta$.

Suppose now that the span of the image under Q of $\mathrm{Sing}\, \Theta$ in $|I(2)|$ is degenerate. By Prop. 5.12 and Thm. 1.1 (2), we see that the image of ϕ_2^* in $\mathbb{P}\mathrm{Sym}^2 I(2)$ is also degenerate, contradicting Thm. 1.2. $\square$

5.15. Remark. In fact, we proved a little bit more: the image $Q(\operatorname{Sing}\Theta) \subset |I(2)|$ is not contained in a quadric.

6 The space Γ_{000}

6.1 Proof of theorem 1.3

In this section we regard 2θ-functions as holomorphic functions on $\mathbb{C}^g$. For the proof of Theorem 1.3 we need the following lemma, which we will deduce from Gunning's results (Prop.3.3).

6.1. Lemma. *The following statements are equivalent*

(1) $f \in \Gamma_{000}$

(2) For all $a_1, a_2, a_3, a_4 \in \mathcal{C}$

$$(q_{12}q_{34})^4 f(w(a_1+a_2-a_3-a_4)) + (q_{14}q_{23})^4 f(w(a_1+a_4-a_2-a_3)) + (q_{13}q_{24})^4 f(w(a_1+a_3-a_2-a_4)) = 0$$

Proof. We will derive identity (2) from Prop. 3.3. First we observe that the left-hand side $\mathbf{P}D_{a_1}D_{a_2}D_{a_3}D_{a_4}\theta_2(0)$ of Prop. 3.3 (1) is symmetric in the four variables. In particular it is symmetric in a_2, a_4, which leads to the equality

$$\tau_{1234} + \tau_{2314} = \tau_{1432} + \tau_{4312} + 2\sigma_{1234} + 2\sigma_{2134} + 2\sigma_{3124} - 2\sigma_{1432} - 2\sigma_{4132} - 2\sigma_{3142} \tag{6.1}$$

Combining (6.1) and prop.3.3 (1) we can write

$$\frac{1}{2}\mathbf{P}D_{a_1}D_{a_2}D_{a_3}D_{a_4}\theta_2(0) = \tau_{1324} + \tau_{1432} + \tau_{4312} + \text{some } \sigma's$$

Using Prop. 3.3 (2) and the two relations obtained from it by interchanging a_1 with a_3 and a_1 with a_4, we can write

$$\frac{1}{2}\mathbf{P}D_{a_1}D_{a_2}D_{a_3}D_{a_4}\theta_2(0) = \left(\frac{q_{12}q_{34}}{q_{13}q_{14}q_{23}q_{24}}\right)^2 \mathbf{P}\theta_2(w(a_1+a_2-a_3-a_4)) +$$
$$\left(\frac{q_{32}q_{14}}{q_{13}q_{34}q_{21}q_{24}}\right)^2 \mathbf{P}\theta_2(w(a_1+a_4-a_3-a_2)) + \left(\frac{q_{24}q_{31}}{q_{43}q_{14}q_{23}q_{21}}\right)^2 \mathbf{P}\theta_2(w(a_1+a_3-a_2-a_4)) - X$$

where X is the following sum of σ's

$$(\sigma_{1243} + \sigma_{1234} + \sigma_{1432}) + (\sigma_{3142} + \sigma_{3241} + \sigma_{3214}) + (\sigma_{4132} + \sigma_{4213} + \sigma_{4231})$$

But the three terms within each pair of parentheses add up to zero (use the definition of σ (3.4) and the fact that ξ is skew-symmetric in its variables), hence $X = 0$. Taking duals we obtain the equivalence stated in the lemma. □

We are now in a position to prove Theorem 1.3. First we show the inclusion $\Gamma_{000} \subset \Gamma_{00}^{(2)}$. We fix $f \in \Gamma_{000}$ and three points $a_2, a_3, a_4 \in \mathcal{C}$. We consider a_1 as a canonical coordinate and derive two (resp. three) times with respect to a_1 and take the value at the point $a_1 = a_4$. This way, we obtain two equations among vectors in $\mathbb{C}^{2^g}/\mathbb{T}_0$

$$D_{a_4}^2\theta_2(w(a_2-a_3)) + D_{a_4}\theta_2(w(a_2-a_3))\left[4\partial\log\frac{q_{42}}{q_{43}}\right] = 0$$

$$D_{a_4}^2\theta_2(w(a_2-a_3))\,[\partial\log q_{42}q_{43}] + D_{a_4}\theta_2(w(a_2-a_3))\left[\partial^2\log\frac{q_{42}}{q_{43}} + 4\partial\log q_{42}q_{43}\cdot\partial\log\frac{q_{42}}{q_{43}}\right] = 0$$

where ∂ means derivative with respect to the first variable of the prime form q. Hence we get, for all $a_2, a_3, a_4 \in \mathcal{C}$, a system of two linear equations involving the vectors $D_{a_4}^2\theta_2(w(a_2-a_3))$ and $D_{a_4}\theta_2(w(a_2-a_3))$ whose determinant

$$\partial^2\log\frac{q_{42}}{q_{43}}$$

is non-zero on an open subset of $\mathcal{C}^3$. Hence the two vectors $D_{a_4}^2\theta_2(w(a_2-a_3))$ and $D_{a_4}\theta_2(w(a_2-a_3))$ are zero on an open subset of $\mathcal{C}^3$, so they are identically zero. This implies that $f \in \Gamma_{00}^{(2)}$.

The inclusion $\Gamma_{11} \subset \Gamma_{000}$ and the second statement of Theorem 1.3 can easily be deduced from the commutativity of the diagram

$$\begin{array}{ccc} \Gamma_{00}^{(2)} & \xrightarrow{\phi_2^*} & \mathrm{Sym}^2 I(2) \\ \downarrow^{\alpha} & \swarrow & \\ I(4) & & \end{array} \tag{6.2}$$

where α is the map which associates to a 2θ-function its quartic tangent cone at the origin (i.e. the degree 4 term of the Taylor expansion at the origin), ϕ_2^* is as in (5.22) and the diagonal arrow is the multiplication map m. By definition we have

$$\ker\alpha = \Gamma_{000} \qquad \ker\phi_2^* = \Gamma_{11}$$

To check the commutativity of (6.2), since by Theorem 1.1 (2) $\langle\mathrm{Sing}\Theta\rangle = \mathbb{P}\Gamma_{00}^{(2)}$, it is enough to check that $\alpha(D(E)) = m(\phi_2^*(D(E)))$ for the bundle $E = \xi \oplus K\xi^{-1}$, with $\xi \in \mathrm{Sing}\,\Theta$ and $h^0(\xi) = 2$, i.e. that $D(E) = \Theta_\xi + \Theta_{K\xi^{-1}}$. This follows from Prop. 5.12 and the description of Q_E (see proof of Theorem 5.14).

6.2. Remark. We don't know how to find a general formula for $\dim\Gamma_{000}$. In the examples that follow, we give $\dim\Gamma_{000}$ for $g \le 7$.

6.2 Examples

6.2.1 Curves of genus less than 6

For any non-hyperelliptic curve of genus $g \le 5$, we have

$$\dim\Gamma_{000} = 0.$$

This follows from a straight-forward dimension count using Thms. 1.1, 1.2 and 1.3 (For a trigonal genus 5 curve, we also use Prop. 7.2).

6.2.2 Curves of genus 6

Consider first a genus 6 curve which is not trigonal or a smooth plane quintic. In order to determine the vector space $\ker m$ of Thm 1.3, we consider the rational map induced by the linear system $|I(2)|$ on the canonical space $|K|^* = \mathbb{P}^5$

$$\pi : |K|^* \longrightarrow |I(2)|^* = \mathbb{P}^5.$$

Note that π is defined away from the canonical curve. We claim that π is dominant, which we see as follows. By the Enriques-Petri theorem the canonical curve is the set-theoretic intersection of all quadrics in $I(2)$. The four general elements of $I(2)$ cut out C_{can} and some other curves, and a general fifth element of $I(2)$ does not contain any of the other curves. Thus the general fiber of π does not contain any curve, so π is dominant.

Therefore since the image of π is not contained in a quadric, we have the equality $\Gamma_{000} = \Gamma_{11}$. By Thm. 8.1 (1) [OPP] and Thm. 5.1 (1) [M2], there exists a unique stable bundle $E_{max} \in \mathcal{SU}_C(2, K)$ with maximal number of sections $h^0(E_{max}) = 5$. Since $C_2 - C_2 \subset D(E_{max})$, we have $D(E_{max}) \in \mathbb{P}\Gamma_{11}$. Moreover by a dimension count using Thms. 1.1 and 1.2, we have $\dim \Gamma_{11} = 1$, from which we deduce that

$$\mathbb{P}\Gamma_{000} = \mathbb{P}\Gamma_{11} = D(E_{max}).$$

As a complement to the examples of Brill-Noether loci of $\mathcal{SU}_C(2, K)$ provided in [OPP] we add a geometric description of the divisor $D(E_{max}) \subset JC$. Let $L = g^1_4$ be a tetragonal series on C and φ_L be the associated surjective Abel-Jacobi map

$$\begin{array}{rcl} \varphi_L : C_6 & \longrightarrow & JC \\ (p_1, \ldots, p_6) & \longmapsto & K^{-1}L(p_1 + \ldots + p_6) \end{array}$$

Let π_L be the map induced by the base point free linear series $|KL^{-1}|$

$$\pi_L : C \longrightarrow |KL^{-1}|^* = \mathbb{P}^2.$$

If C is bi-elliptic, the image is a smooth plane cubic. Otherwise (general case) π_L maps C birationally to a nodal plane sextic.

6.3. Proposition. *Let $\mathcal{S} \subset C_6$ be the divisor consisting of sextuples $(p_1, \ldots, p_6)$ such that the points $\pi_L(p_i)$ lie on a conic in $|KL^{-1}|^* = \mathbb{P}^2$. Then*

$$\varphi_L(\mathcal{S}) = D(E_{max})$$

Proof. By [M2] Prop. 3.1 (see also [OPP] example 3.4), we can write E_{max} as an extension

$$0 \longrightarrow L \longrightarrow E_{max} \longrightarrow KL^{-1} \longrightarrow 0 \tag{6.3}$$

By definition, we have $\lambda \in D(E_{max}) \iff h^0(E_{max} \otimes \lambda) > 0$. First, we see that if $h^0(L\lambda) > 0$ or $h^0(L\lambda^{-1}) > 0$, then the exact sequence (6.3) implies that $\lambda \in D(E_{max})$ (note that $D(E_{max})$ is symmetric). Therefore we can assume that $h^0(L\lambda) = h^0(L\lambda^{-1}) = 0$ or equivalently that $h^0(KL^{-1}\lambda) = h^0(KL^{-1}\lambda^{-1}) = 1$. Writing the long exact sequence associated to (6.3), we see that $H^0(E_{max} \otimes \lambda) = \ker(\delta : H^0(KL^{-1}\lambda) \longrightarrow H^1(L\lambda))$. Therefore $h^0(E_{max} \otimes \lambda) > 0$ if and only if the image of the multiplication map

$$H^0(KL^{-1}\lambda) \otimes H^0(KL^{-1}\lambda^{-1}) \longrightarrow H^0(K^2L^{-2})$$

is contained in the hyperplane $\mathrm{Sym}^2 H^0(KL^{-1}) \subset H^0(K^2L^{-2})$ which defines the extension class of E_{max} in $|K^2L^{-2}|^*$ (see [OPP] example 3.4). Since $h^0(KL^{-1}\lambda) = 1$, there exists a unique sextuple $(p_1, \ldots, p_6) \in C_6$ such that $\lambda = \varphi_L((p_i))$. Then we deduce from the commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\varphi_{K^2L^{-2}}} & |K^2L^{-2}|^* = \mathbb{P}^6 \\ \downarrow \pi_L & & \downarrow pr \\ |KL^{-1}|^* = \mathbb{P}^2 & \xrightarrow{Ver} & \mathbb{P}\mathrm{Sym}^2 H^0(KL^{-1})^* = \mathbb{P}^5 \end{array}$$

that $h^0(E_{max} \otimes \lambda) > 0$ if and only if the 6 points $\varphi_{K^2L^{-2}}(p_i)$ lie on a hyperplane in $\mathbb{P}^6$ which is the inverse image under the projection map pr of a hyperplane in $\mathbb{P}^5$. But this last condition says that the 6 points $\pi_L(p_i)$ lie on a conic in $|KL^{-1}|^*$. Finally, we notice that if $h^0(L\lambda^{-1}) > 0$ ($\iff h^0(\mathcal{O}(\sum p_i)) \geq 2$) then there exists a divisor $D = \sum q_i$ in the linear system $|\sum p_i|$ such that the $\pi_L(q_i)$ lie on a conic (e.g. we can show that $\mathcal{S}$ is an ample divisor on C_6). $\square$

Consider now the case of a *smooth plane quintic*. By Thm. 8.1 (3) [OPP] there exists a unique S-equivalence class ϵ_{max} such that if $h^0(E) \geq 5$, then $[E] = \epsilon_{max}$. In particular, $\epsilon_{max} = [g^2_5 \oplus g^2_5]$. Moreover, using an explicit basis of quadrics of $I(2)$, we can show that the map π is birational (for more details see [OPP] Thm. 5.5), hence $\ker m = \{0\}$. As before, we deduce from Thms. 1.1 and 1.2 that

$$\mathbb{P}\Gamma_{000} = \mathbb{P}\Gamma_{11} = D(\epsilon_{max}) = 2\Theta_{g^2_5}.$$

Consider now the case of a *trigonal curve*. By Thm. 8.1 (2) [OPP] there exists a projective line $\mathbb{P}^1_{bun}$ of stable bundles E with $h^0(E) = 5$, hence $\mathbb{P}^1_{bun} \subset \mathbb{P}\Gamma_{11}$. Using Prop. 7.2, we compute that $\dim \Gamma_{11} = 2$ and (using section 7) that $\dim \ker m = 1$. Hence we have

$$\mathbb{P}^1_{bun} = \mathbb{P}\Gamma_{11} \qquad \dim \Gamma_{000} = 3$$

6.2.3 Curves of genus 7

For a non-tetragonal genus 7 curve, we have $\dim \Gamma_{000}/\Gamma_{11} = 1$ (by [M1] thm 4.2). So we obtain $\dim \Gamma_{11} = 9$ and $\dim \Gamma_{000} = 10$.

7 Trigonal curves

Let C be a trigonal curve with $g \geq 5$ and g^1_3 its unique trigonal series. By remark 5.11, we obtain as in (5.22) a linear map $\phi_2^* : \Gamma_{00}^{(2)} \longrightarrow \mathrm{Sym}^2 I(2)$. The aim of this section is to compute the rank of ϕ_2^* (cf. section 5.3.2).

First we need to quote some results about quadrics containing a rational normal scroll from [AH], which we state here for the case of the degree $g-2$ surface $S \subset |K|^*$ ruled by the pencil of trisecants to the canonical curve. For a trigonal curve, the space $I(2)$ and the space of quadrics $I_S(2)$ containing the surface S are isomorphic. Let $V = H^0(C, K - g^1_3)$. We choose two sections $s_0, s_1 \in H^0(g^1_3)$ and consider the isomorphism β_0 (resp. β_1) induced by multiplication by the section s_0 (resp. s_1)

$$\beta_0 : V \xrightarrow{\sim} V_0 \subset H^0(K) \qquad \beta_1 : V \xrightarrow{\sim} V_1 \subset H^0(K)$$

where $V_i = H^0(C, K - D_i)$ and D_i is the zero divisor of the section s_i. We then define a linear map

$$\beta : \Lambda^2 V \longrightarrow \mathrm{Sym}^2 H^0(K)$$

by setting

$$\beta(v \wedge w) = \beta_0(v) \otimes \beta_1(w) - \beta_0(w) \otimes \beta_1(v) \tag{7.1}$$

which is a quadric of rank less than or equal to 4 containing S. One checks that $\beta(v \wedge w)$ does not depend on the choice of the sections s_0, s_1. Then Prop. 2.14 [AH] says that β induces an isomorphism

$$\beta : \Lambda^2 V \xrightarrow{\sim} I_S(2) \tag{7.2}$$

We also define a rational map $\delta : \mathcal{W}(4) \longrightarrow Gr(2,V)$ as follows. Consider a semi-stable bundle $E \in \mathcal{W}(4)$ (see (5.23)). Then, by [M2], Prop. 3.1, $h^0(E(-g^1_3)) \geq 1$; it can be shown that $h^0(E(-g^1_3)) = 1$ for a general bundle $E \in \mathcal{W}(4)$, i.e. E can be uniquely written as an extension

$$0 \longrightarrow g^1_3 \longrightarrow E \stackrel{\pi}{\longrightarrow} K - g^1_3 \longrightarrow 0 \tag{7.3}$$

and we define $\delta(E) = \text{im } (H^0(E) \stackrel{H^0(\pi)}{\longrightarrow} H^0(K - g^1_3) = V)$. Then we can prove the following

7.1. Lemma. *The map $Q : \mathcal{W}(4) \longrightarrow |I(2)|$ defined in section 5.2 factorizes as follows*

$$\mathcal{W}(4) \stackrel{\delta}{\longrightarrow} Gr(2,V) \stackrel{Pl}{\longrightarrow} \mathbb{P}(\Lambda^2 V) \stackrel{\beta}{\longrightarrow} |I_S(2)| = |I(2)|$$

where Pl is the Plücker embedding of the Grassmannian.

Proof. Consider $E \in \mathcal{W}(4)$ with $h^0(E(-g^1_3)) = 1$ and identify $H^0(g^1_3)$ with a 2-dimensional subspace of $H^0(E)$. We choose a basis $\{s_0, s_1\}$ of $H^0(g^1_3)$. Let $R = \beta \circ Pl \circ \delta(E) \in |I(2)|$ be the associated quadric. In order to show that $R = Q_E$ it is enough to show that their associated polar forms, which we view as global sections of the line bundle $\mathcal{M}$ over C_2 (see (5.20)), coincide. Hence it is enough to show the implication

$$\forall p, q \in C,\ p \neq q,\ h^0(g^1_3(-p-q)) = 0 \qquad \tilde{Q}_E(p,q) = 0 \Longrightarrow \tilde{R}(p,q) = 0$$

But by Lemma 5.5, the assumption $\tilde{Q}_E(p,q) = 0$ means that there exists a section $a \in H^0(E)$ vanishing at p and q. Since $h^0(g^1_3(-p-q)) = 0$, we have $a \notin H^0(g^1_3)$ and we can find a section $b \in H^0(E)$ such that $\{s_0, s_1, a, b\}$ is a basis of $H^0(E)$. Then $H^0(\pi)$ induces a linear isomorphism $\mathbb{C}a \oplus \mathbb{C}b \stackrel{\sim}{\longrightarrow} \delta(E) \subset V$. Let $u, v \in \delta(E)$ be the images of a, b under this isomorphism. Then we see that

$$\beta_0(u) = s_0 \wedge a \in H^0(K) \qquad \beta_0(v) = s_0 \wedge b \in H^0(K).$$

The same holds for β_1 and s_1. By (7.1) we have

$$\begin{aligned} 2\tilde{R}(p,q) = (s_0 \wedge a)(p) \cdot (s_1 \wedge b)(q) + (s_0 \wedge a)(q) \cdot (s_1 \wedge b)(p) \\ - (s_0 \wedge b)(p) \cdot (s_1 \wedge a)(q) - (s_0 \wedge b)(q) \cdot (s_1 \wedge a)(p) \end{aligned}$$

and this expression, which does not depend on the choice of the basis $\{s_0, s_1, a, b\}$, is obviously zero if $a \in H^0(E(-p-q))$. □

We consider now the commutative diagram (5.23). Lemma 7.1 implies that the inverse image under the Veronese map of a hyperplane in $\mathrm{Sym}^2 I(2)$ containing im ϕ_2^* is a quadric in $|I(2)|$ containing the Grassmannian $Gr(2,V)$. Moreover any such quadric comes from an element in the annihilator $(\text{im } \phi_2^*)^\perp$. Hence we obtain an isomorphism

$$I_{Gr(2,V)}(2) \cong (\text{im } \phi_2^*)^\perp. \tag{7.4}$$

But the degree 2 part $I_{Gr(2,V)}(2)$ of the ideal of the Grassmannian $Gr(2,V)$ is isomorphic to the vector space $\Lambda^4 V$ generated by the Plücker equations (see e.g. [M2]). Hence we have shown

7.2. Proposition. *The corank of ϕ_2^* is $\binom{g-2}{4}$.*

8 Concluding remarks

1. It is natural to ask whether the three main theorems can be generalized to analogous subseries. For this purpose, we introduce the following subspaces of $\mathbb{P}\Gamma_{00}$

$$\begin{aligned}
\mathbb{P}\Gamma_{[n]\cdot 0} &= \{D \mid \mathrm{mult}_0(D) \geq 2n\} && \text{for } n \geq 2 \\
\mathbb{P}\Gamma_{dd} &= \{D \mid C_{d+1} - C_{d+1} \subset D\} && \text{for } d \geq 0 \\
\mathbb{P}\Gamma_{dd}^{(2)} &= \{D \mid \mathrm{mult}_{C_{d+1}-C_{d+1}}(D) \geq 2\} && \text{for } d \geq 0
\end{aligned}$$

where $C_{d+1} - C_{d+1}$ is the image of the difference map ($d \geq 0$)

$$\begin{aligned}
\phi_{d+1} : C_{d+1} \times C_{d+1} &\longrightarrow JC \\
(D, D') &\longmapsto \mathcal{O}(D - D')
\end{aligned}$$

The following inclusions are obvious

$$\Gamma_{(d+1)(d+1)} \subset \Gamma_{dd}^{(2)} \subset \Gamma_{dd} \tag{8.1}$$

and one might expect that the following holds (see Thm. 1.3)

$$\Gamma_{(d+1)(d+1)} \subset \Gamma_{[d+3]\cdot 0} \subset \Gamma_{dd}^{(2)} \tag{8.2}$$

$$\Gamma_{[d+3]\cdot 0}/\Gamma_{(d+1)(d+1)} \cong \ker \mathrm{Sym}^2 I^{(d+1)}(d+2) \longrightarrow I(2d+4) \tag{8.3}$$

where $I^{(d+1)}(d+2)$ is the space of degree $d+2$ polynomials vanishing at order $d+1$ along C_{can}. Some previous work towards (8.2) has been done in [Gu3]. Statement (8.3) follows from (8.2).

2. An important ingredient of the proof of Thm. 1.1 (resp. Thm. 1.2) is the use of rank 2 vector bundles with 3 (resp. 4) sections. The constructions involved may be viewed as examples of a general construction. Following Mukai [M2], we associate to any $E \in \mathcal{W}(n)$ a commutative diagram (as in (5.13): replace V by $H^0(E)$)

$$\begin{array}{ccc}
C & \xrightarrow{\gamma} & Gr(2, H^0(E)^*) \\
\downarrow{\scriptstyle \varphi_K} & & \downarrow \\
|K|^* & \xrightarrow{\lambda^*} & \mathbb{P}(\Lambda^2 H^0(E)^*) = \mathbb{P}^{\binom{n}{2}-1}
\end{array} \tag{8.4}$$

The definitions of the morphisms γ and λ^* are as in (5.14) and (5.15).

If n is even, $n = 2d+4$ for $d \geq 0$, the Plücker space $\Lambda^2 H^0(E)^*$ carries canonically a symmetric multilinear form

$$\tilde{\mathrm{Pf}}\,(\omega_1, \ldots, \omega_{d+2}) = \omega_1 \wedge \ldots \wedge \omega_{d+2} \in \Lambda^{2d+4} H^0(E)^* \cong \mathbb{C}$$

which defines a degree $d+2$ polynomial $\mathrm{Pf} \in \mathrm{Sym}^{d+2}(\Lambda^2 H^0(E)^*)$ vanishing to order $d+1$ along the Grassmannian $Gr(2, H^0(E)^*)$. Notice that Pf is the Pfaffian if we represent $\omega \in \Lambda^2 H^0(E)^*$ as an $n \times n$ skew-symmetric matrix. Therefore we can define for any $E \in \mathcal{W}(2d+4)$ a polynomial $Q_E = (\lambda^*)^{-1}(\mathrm{Pf}) \in |I^{(d+1)}(d+2)|$. A straighforward generalization of Prop. 5.12 leads to

8.1. Proposition. *For all $d \geq 0$, we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{W}(2d+4) & \xrightarrow{D} & \mathbb{P}\Gamma_{dd}^{(2)} \\ \downarrow{\scriptstyle Q} & & \downarrow{\scriptstyle \phi_{d+2}^*} \\ |I^{d+1}(d+2)| & \xrightarrow{Ver} & \mathbb{P}\mathrm{Sym}^2 I^{d+1}(d+2) \end{array} \tag{8.5}$$

We expect that the linear maps ϕ_{d+2}^* are surjective (at least for a general curve),i.e. that one has isomorphisms (generalizing Theorem 1.2)

$$\Gamma_{dd}^{(2)}/\Gamma_{(d+1)(d+1)} \xrightarrow{\sim} \mathrm{Sym}^2 I^{d+1}(d+2).$$

Some evidence for this is given by considering the case of cubics singular along the canonical curve ($d = 1$). One computes, using results proved by J. Wahl [Wa], dim $I^2(3) = \binom{g-5}{3} + \binom{g-7}{2}$ for a general curve C.

- for $g = 8$ and C without g_7^2, the space $I^2(3)$ is one-dimensional, generated by the cubic $Q_{E_{max}}$, where $E_{max} \in \mathcal{W}(6)$ is the unique stable bundle with maximal number of sections.
- for $g = 9$ and C general, the image of the morphism $\mathcal{W}(6) \longrightarrow |I^2(3)| = \mathbb{P}^4$ is a quartic threefold with 21 singular points (i.e. the 42 g_8^1's modulo involution) [M3].

In both cases we immediately see that ϕ_3^* is surjective. For $d > 1$, it seems that very little is known about the spaces $|I^{d+1}(d+2)|$.

If n is odd and $n > 3$, we do not have a natural equation like the Pfaffian Pf . If $n = 3$, the composite $\lambda^* \circ \varphi_K : C \longrightarrow \mathbb{P}^2 = \mathbb{P}(\Lambda^2 H^0(E)^*)$ is the morphism described in remark 2.5. Already for the next step in the filtration (8.1), i.e. the quotient $\Gamma_{11}/\Gamma_{11}^{(2)}$, finding an isomorphic vector space attached to the canonical curve seems rather complicated.

3. Finally, we note that the bundles E_W, which were introduced in section 2.3, can be used to work out a vector-bundle theoretical proof of Welters' theorem [We], namely the statement that the base locus of Γ_{00} equals the surface $C - C$ for $g \geq 5$.

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Christian Pauly
Laboratoire J.-A. Dieudonné
Université de Nice Sophia Antipolis
Parc Valrose
F-06108 Nice Cedex 02
France
E-mail: pauly@math.unice.fr

Emma Previato
Department of Mathematics
Boston University
Boston, MA 02215-2411
U.S.A.
E-mail: ep@math.bu.edu