# On Pryms, rank 2 bundles and nonabelian theta functions

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## 1 Introduction

Let $\mathcal{M}_0$ (resp. $\mathcal{M}_p$) denote the moduli space parametrizing semistable rank 2 bundles with determinant equal to $\mathcal{O}_C$ (resp. $\mathcal{O}_C(p)$) over a smooth, projective curve $C$ of genus $g \geq 2$; $p$ is a fixed point of $C$. The Picard group of both moduli spaces is isomorphic to $\mathbb{Z}$ and we denote by $\mathcal{L}$ (resp. $\mathcal{L}_p$) their ample generators. Then the Verlinde formula gives the dimension of the vector spaces $H^0(\mathcal{M}_0, \mathcal{L}^k)$ and $H^0(\mathcal{M}_p, \mathcal{L}_p^k)$ which consist of what is called generalized or nonabelian theta functions of level $k$.

Several authors have studied the geometry of the moduli space $\mathcal{M}_0$ in connection with the Jacobian $J$ and the Prym variety $P_x$ of an unramified double cover of the curve $C$ associated to a nonzero 2-torsion point $x$. The Kummers of all these abelian varieties can be mapped naturally to $\mathcal{M}_0$ and the intersection points of two distinct Kummers give the Schottky-Jung and Donagi relations between their theta-nulls [vG-P1]. As a consequence of these identities, van Geemen and Previato [vG-P

$$m_4 : S^4 H^0(\mathcal{M}_0, \mathcal{L}) \longrightarrow H^0(\mathcal{M}_0, \mathcal{L}^4)$$

is surjective.

In analogy with $\mathcal{M}_0$, the moduli space $\mathcal{M}_p$ also contains the Prym varieties $P_x$ and (a blown-up of) the Jacobian $\hat{J}$. We observe that the varieties $\hat{J}$ and $P_x$ intersect and that two orthogonal Pryms $P_x$ and $P_y$, although they don't intersect, verify a geometric property, which lead to new relations among theta-constants (see section 4). Finally, we can adapt the method of [vG-P1,2] to prove the main theorem:

**Theorem 1.1** *For a generic curve, the multiplication map*

$$m_2 : S^2 H^0(\mathcal{M}_p, \mathcal{L}_p) \longrightarrow H^0(\mathcal{M}_p, \mathcal{L}_p^2)$$

*is surjective*

As was shown in [O-P], surjectivity of $m_2$ implies that the natural homomorphism

$$\bigoplus_{x \in J[2]} H^0_-(P_x, \mathcal{O}(3\Xi_x))^\vee \longrightarrow H^0(\mathcal{M}_p, \mathcal{L}_p^2)$$

is an isomorphism. Similarly, surjectivity of $m_4$ implies an analogous isomorphism of $H^0(\mathcal{M}_0, \mathcal{L}^4)$ with theta spaces. Recently, Ramanan [R] obtained a different (and more general) proof of both isomorphisms.

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# 2 Theta functions and the geometry of $\mathcal{M}_p$

## 2.1 Theta characteristics and the Heisenberg group

Let $A$ be a principally polarized abelian variety of dimension $g$ and let $\Theta$ be a symmetric divisor defining the principal polarization. The line bundle $L := \mathcal{O}_A(2\Theta)$ does not depend on the choice of $\Theta$. We will denote by $V$ the vector space of global sections $H^0(A, L)$ and by $A[2]$ the group of points of order 2. The line bundle $L$ defines a skew-symmetric bilinear pairing $\langle\ ,\ \rangle : A[2] \times A[2] \to \{\pm 1\}$.

A *theta characteristic* [B2] of a ppav $A$ is a function $\kappa : A[2] \to \{\pm 1\}$ such that

$$\kappa(x+y) = \kappa(x)\kappa(y)\langle x, y\rangle$$

The group $A[2]$ acts on the set $\vartheta(A)$ of theta characteristics of $A$ by:

$$(x \cdot \kappa)(y) = \langle x, y\rangle \kappa(y)$$

A theta characteristic $\kappa$ is *even* ($\varepsilon(\kappa) = 1, \kappa \in \vartheta^+(A)$) if it takes $2^{g-1}(2^g+1)$ times the value $+1$ and $2^{g-1}(2^g-1)$ times the value $-1$. Otherwise $\kappa$ is *odd* ($\varepsilon(\kappa) = -1, \kappa \in \vartheta^-(A)$). We have the following formula:

$$\varepsilon(x \cdot \kappa) = \kappa(x)\varepsilon(\kappa) \tag{1}$$

Suppose that $A$ is the Jacobian $J$ of a smooth projective curve $C$ of genus $g$. A theta characteristic of $C$ is a line bundle $\kappa$ such that $\kappa^{\otimes 2} \cong K_C$. One can associate to $\kappa$ a theta characteristic of $J$ by

$$\kappa(x) = (-1)^{h^0(\kappa \otimes x) + h^0(\kappa)}$$

This correspondence gives a bijection between theta characteristics of $J$ and those of $C$. Furthermore, $x \cdot \kappa$ corresponds to the line bundle $\kappa \otimes x$ and $\varepsilon(\kappa) = (-1)^{h^0(\kappa)}$.

We fix a *theta structure* for $L$, that is an isomorphism between the group $\mathcal{G}(L) := \{(x, \varphi) \mid \varphi : t_x^* L \xrightarrow{\sim} L\}$, where $t_x$ denotes translation by $x$, and the *Heisenberg group* [M1] defined as a set by

$$Heis(g) = \mathbb{C}^* \times K(g) \times \widehat{K(g)}$$

where $K(g) = (\mathbb{Z}/2)^g$, $\widehat{K(g)} = \mathrm{Hom}((\mathbb{Z}/2)^g, \mathbb{C}^*)$ and multiplication is defined by

$$(s, a, \alpha)(t, b, \beta) = (st\beta(a), a+b, \alpha\beta)$$

Via the theta structure $Heis(g)$ acts on the vector space $V$ and, by Mumford's theta theory, $V$ is an irreducible representation of $Heis(g)$. There exists a unique (up to scalar) basis $\{X_b\}$, with $b \in K(g)$, for $V$ such that

$$(s, a, \alpha)X_b = s\alpha(b + a)X_{b+a} \tag{2}$$

The theta structure allows us to define for $(c, \gamma) \in (\mathbb{Z}/2)^g \times \mathrm{Hom}((\mathbb{Z}/2)^g, \mathbb{C}^*)$ a theta characteristic $\kappa = \kappa \begin{bmatrix} c \\ \gamma \end{bmatrix}$ by the formula

$$\kappa(a, \alpha) = \gamma(a)\alpha(a + c)$$

One has the formula $\varepsilon(\kappa) = \gamma(c)$.

The Abelian variety $A$ can be identified to the quotient of the tangent space $T_0(A)$ by a lattice $\Gamma$. We choose a symplectic basis $(\gamma_1, \ldots, \gamma_{2g})$ of $\Gamma$ with respect to the non-degenerate skew-symmetric bilinear pairing given by the principal polarization on $A$. The first $g$ vectors form a basis of $T_0(A)$ and there exists a matrix $\Omega$ of the Siegel upper half-space such that $\Gamma = \Gamma_\Omega = \mathbb{Z}^g \oplus \Omega\mathbb{Z}^g$. The space $V$ can be identified to the space of $\Gamma$-quasi-periodic second order theta functions. The symplectic basis $(\gamma_1, \ldots, \gamma_{2g})$ defines an isomorphism between $K(g) \times \widehat{K(g)}$ and $A[2]$, which extends to a theta structure for $L$.

For $(c, \gamma) \in K(g) \times \widehat{K(g)}$ we can define a theta function

$$\theta \begin{bmatrix} c \\ \gamma \end{bmatrix} (z, \Omega) = \sum_m \exp\pi i({}^t(m + \tfrac{1}{2}\tilde{c})\Omega(m + \tfrac{1}{2}\tilde{c}) + 2{}^t(m + \tfrac{1}{2}\tilde{c})(z + \tfrac{1}{2}\tilde{\gamma}))$$

where we sum over $m \in \mathbb{Z}^g$, and $\tilde{c}$ and $\tilde{\gamma}$ are representatives in $\mathbb{Z}^g$ of $c$ and $\gamma$ (via the canonical isomorphism of $K(g)$ with $\widehat{K(g)}$). This theta function will also be denoted by $\theta_\kappa(z, \Omega)$. A basis, verifying relation (2), is given by the functions

$$X_b = \theta \begin{bmatrix} b \\ 0 \end{bmatrix} (2z, 2\Omega) \qquad \forall b \in K(g)$$

We can associate to a theta characteristic $\kappa = \kappa \begin{bmatrix} c \\ \gamma \end{bmatrix}$ a character $\chi_\kappa$ of order 2 of the Heisenberg group $Heis(g)$ defined by $\chi_\kappa(s, a, \alpha) = s^2\gamma(a)\alpha(c)$, and an element

$$\xi_\kappa = \sum_{b \in K(g)} \gamma(b) X_b \otimes X_{b+c} \in V \otimes V$$

$\xi_\kappa$ is an eigenvector under the action of $Heis(g)$ associated to the character $\chi_\kappa$ and the vectors $\{\xi_\kappa\}$ for $\kappa \in \vartheta^+(A)$ (resp. $\kappa \in \vartheta^-(A)$) form a basis of $S^2V$ (resp. $\Lambda^2V$). We shall identify $S^2V$ (resp. $\Lambda^2V$) with the invariant (resp. anti-invariant) subspace of $V \otimes V$ under the involution $\varphi \otimes \psi \mapsto \psi \otimes \varphi$. In the sequel, we will use the *addition formula*

$$\theta_\kappa(z + u, \Omega)\theta_\kappa(z - u, \Omega) = \sum_{b \in K(g)} \gamma(b)\theta \begin{bmatrix} b \\ 0 \end{bmatrix} (2z, 2\Omega)\theta \begin{bmatrix} b + c \\ 0 \end{bmatrix} (2u, 2\Omega) \tag{3}$$

Finally, recall that the theta functions $\theta_\kappa(2z, \Omega)$ of order 4 are eigenvectors under the natural action of $Heis(g)$ on $H^0(A, L^{\otimes 2})$ associated to the character $\chi_\kappa$ and that they form a basis of $H^0(A, L^{\otimes 2})$ for $\kappa \in \vartheta(A)$.

## 2.2 The space of sections $H^0(\mathcal{M}_p, \mathcal{L}_p)$

We recall the main results of [B2]. Let $l \in J$, such that $l^{\otimes 2} \not\cong \mathcal{O}_C$. We can define a stable rank 2 bundle $F_l \in \mathcal{M}_p$ which fits in an exact sequence

$$0 \longrightarrow l \oplus l^{-1} \longrightarrow F_l \longrightarrow \mathbb{C}_p \longrightarrow 0$$

The map $l \mapsto F_l$ can be extended to a morphism $j_p : \hat{J} \mapsto \mathcal{M}_p$ where $\hat{J}$ denotes the blow-up at the points of order 2 of $J$. Let $F \in \mathcal{M}_p$. For any surjective morphism $u : F \to \mathbb{C}_p$, the vector bundle $\ker(u)$ is semistable. Thus we obtain a morphism from the projective line $\mathbb{P}F_p$, which parametrize, up to homothety, nonzero morphisms $F \to \mathbb{C}_p$, to the moduli space $\mathcal{M}_0$. In [B1], Beauville gives an identification of $H^0(\mathcal{M}_0, \mathcal{L})$ with $V := H^0(J, \mathcal{O}(2\Theta))$. Therefore we obtain a morphism $\varphi_{\mathcal{L}} : \mathcal{M}_0 \to \mathbb{P}V$. The image of the composite $\mathbb{P}F_p \to \mathcal{M}_0 \to \mathbb{P}V$ is a line [B2], i.e. a point in $\mathbb{P}\Lambda^2 V$ by the Plücker embedding. This defines a map $\varphi_p : \mathcal{M}_p \to \mathbb{P}\Lambda^2 V$ and the composite $\varphi_p \circ j_p : \hat{J} \to \mathbb{P}\Lambda^2 V$ is the Gauss map associated to the vector field $D$, tangent to $C$ at $p$, where $C$ is embedded in $J$ by the map $q \mapsto \mathcal{O}_C(q-p)$. The following diagram is commutative:

$$\begin{array}{ccc} \Lambda^2 V & \xrightarrow{\varphi_p^*} & H^0(\mathcal{M}_p, \mathcal{L}_p) \\ & \searrow^{w_D} & \downarrow j_p^* \\ & & H^0_-(J, \mathcal{O}(4\Theta)) \end{array}$$

where $w_D$ is the Wahl map associated to the vector field $D$ and the subscript $-$ means odd theta functions. Choose a period matrix $\tau$ for the Jacobian $J$. In this paper we will assume that

$$(*) \begin{cases} \forall \kappa \in \vartheta^-(J)\ ,\ h^0(C, \kappa(-p)) = 0 \iff D\theta_\kappa(0,\tau) \neq 0 \\ \forall \kappa \in \vartheta^+(J)\ ,\ h^0(C, \kappa) = 0 \iff \theta_\kappa(0,\tau) \neq 0 \end{cases}$$

These conditions are verified for a generic curve. Under these assumptions all the linear maps of the diagram above are isomorphisms and there exist a nonzero constant $c$ such that $\forall \kappa \in \vartheta^-(J)$

$$w_D(\xi_\kappa) = c \cdot D\theta_\kappa(0,\tau) \cdot \theta_\kappa(2z,\tau)$$

# 3 Prym varieties

We can associate to any nonzero 2-torsion point $x \in J[2]$ an unramified double cover $\pi_x : C_x \to C$. We shall denote by $\sigma$ the involution of $C_x$ given by sheet-interchange over $C$. The norm map $\mathrm{Nm}_x : \mathrm{Jac}(C_x) \to J$ maps $D \mapsto \pi_{x*}D$ for any divisor $D$ on $C_x$. The kernel $\ker(\mathrm{Nm}_x)$ has two connected components. We will denote by $P_x$, the *Prym variety*, the component containing 0 and by $P_x^-$ the other component. $P_x$ has a natural principal polarization $\Xi_x$. Let's recall the following facts ([M2] and [vG-P1]):

We have $\ker(\pi_x^*) = \langle x \rangle$ and $\pi_x^*$ induces a symplectic isomorphism

$$\pi_x^* : x^\perp / \langle x \rangle \xrightarrow{\sim} P_x[2]$$

where $x^\perp = \{y \in J[2] \mid \langle x, y \rangle = 1\}$. Any $y \in J[2]$ with $\langle x, y \rangle = -1$ gives by tensor product with $\pi_x^*(y)$ an isomorphism $P_x \xrightarrow{\sim} P_x^-$. Fix $z_x \in J$ with $z_x^{\otimes 2} \cong x$, we get a map:

$$\begin{array}{rcl} \psi_x : \ker(\mathrm{Nm}_x) = P_x \cup P_x^- & \longrightarrow & \mathcal{M}_0 \\ M & \longmapsto & \pi_{x*}(M) \otimes z_x \end{array}$$

The image of $\psi_x$ does not depend on the choice of $z_x$ (although $\psi_x$ does!) and $\psi_x$ is equivariant for the actions of $J[2]$ on $\ker(\mathrm{Nm}_x)$ and $\mathcal{M}_0$. The group $J[2]$ acts by tensorization on $\mathcal{M}_0$ and this action induces a projective representation of $J[2]$ in $\mathbb{P}H^0(\mathcal{M}_0, \mathcal{L})$, which can be identified with $\mathbb{P}V$. The composite map $\Phi_x = \varphi_{\mathcal{L}} \circ \psi_x : P_x \cup P_x^- \to \mathbb{P}V$ is also $J[2]$-equivariant. The (projective) linear map corresponding to $x$ has two eigenspaces $\mathbb{P}V_x$ and $\mathbb{P}V_x^-$ in $\mathbb{P}V$ and there is one component of $\ker(\mathrm{Nm}_x)$ in each eigenspace. By proposition 1 [vG-P1] the (restricted) map $\phi_x : P_x \to \mathbb{P}V_x$ is given by the linear series $\mid 2\Xi_x \mid$ and by an identification $\mathbb{P}V_x \cong \mathbb{P}H^0(P_x, \mathcal{O}(2\Xi_x))$.

We can summarize these facts in the following commutative diagram:

$$\begin{array}{ccc} P_x \cup P_x^- & \xrightarrow{\phi_x} & \mathbb{P}V_x \cup \mathbb{P}V_x^- \\ \downarrow \psi_x & & \cap \\ \mathcal{M}_0 & \xrightarrow{\varphi_{\mathcal{L}}} & \mathbb{P}V \end{array}$$

Now we will describe an "odd degree" version of this diagram. Fix a point $p_x \in C_x$ with $\pi_x(p_x) = p$ and consider the map:

$$\begin{array}{rcl} \psi'_x : P_x \cup P_x^- & \longrightarrow & \mathcal{M}_p \\ M & \longmapsto & \pi_{x*}(M(p_x)) \otimes z_x \end{array}$$

This map is $J[2]$-equivariant and the image of $P_x \cup P_x^-$ in $\mathcal{M}_p$ does not depend on the choice of the point $p_x$. Moreover, the images of both components are the same. Thus we will restrict $\psi'_x$ to the zero component $P_x$.

**Lemma 3.1**

$$(\psi'_x \circ t_{\alpha_x^{-1}})^* \mathcal{L}_p \cong \mathcal{O}_{P_x}(4\Xi_x)$$

*where $\alpha_x \in P_x$ is any point satisfying $\alpha_x^{\otimes 4} \cong \mathcal{O}_{C_x}(2p_x - 2\sigma p_x)$, and $t_{\alpha_x^{-1}}$ denotes translation by $\alpha_x^{-1}$.*

*Proof:* $\mathcal{L}_p = \mathcal{O}(D_\kappa)$ for all $\kappa \in \vartheta^-(J)$, where $D_\kappa$ is the divisor on $\mathcal{M}_p$ with support $\{F \in \mathcal{M}_p \mid h^0(C, \mathcal{E}nd_0(F) \otimes \kappa) > 0\}$. By lemma 1.7 [B2], $(\psi'_x \circ t_{\alpha_x^{-1}})^*(D_\kappa) = [2]^*\Xi_N$. Here [2] denotes the duplication map on $P_x$ and $\Xi_N$ the theta divisor on $P_x$ with support $\{M \in P_x \mid h^0(C_x, M \otimes N) > 0\}$. The line bundle $N$ has degree $2g-2$ and can be computed (lemma 1.5 [B2]) $N = \mathcal{O}_{C_x}(p_x - \sigma p_x) \otimes \alpha_x^{\otimes -2} \otimes \pi_x^*\kappa$. A basic fact for Prym varieties is that $\mathcal{O}_{C_x}(p_x - \sigma p_x) \in P_x^-$. Hence, there exists $x' \in J[2]$ with $\langle x, x' \rangle = -1$ such that $\pi_x^*(x') = \alpha_x^{\otimes 2} \otimes \mathcal{O}_{C_x}(\sigma p_x - p_x)$ and we have $N = \pi_x^*(\kappa \otimes x')$. Therefore $N^{\otimes 2} = \pi_x^*(K_C) = K_{C_x}$. □

The decomposition into eigenspaces $V = V_x \oplus V_x^-$ induces a decomposition:

$$\Lambda^2 V = \Lambda^2 V_x \oplus \Lambda^2 V_x^- \oplus W_x$$

where $W_x$ is the quotient of $V_x \otimes V_x^- \oplus V_x^- \otimes V_x$ by the subspace generated by the vectors $v \otimes v^- + v^- \otimes v$, for $v \in V_x$ and $v^- \in V_x^-$. The projective space $\mathbb{P}W_x$ is linearly spanned by the Plücker image of lines in $\mathbb{P}V$ joining $\mathbb{P}V_x$ to $\mathbb{P}V_x^-$.

**Lemma 3.2** *The image of the composite $h_x := \varphi_p \circ \psi'_x : P_x \to \mathcal{M}_p \to \mathbb{P}\Lambda^2 V$ is contained in the subspace $\mathbb{P}W_x$.*

*Proof:* Fix a theta structure for $\mathcal{O}(2\Theta)$ on $J$. The point $x \in J[2]$ is represented by $x = \begin{bmatrix} a \\ \alpha \end{bmatrix}$ and $(1, a, \alpha) \cdot (1, a, \alpha) = (\alpha(a), 0, 1)$. The eigenvalues of $(1, a, \alpha)$ corresponding to the eigenspaces $V_x$ and $V_x^-$ are of opposite sign, hence $(1, a, \alpha)$ acts on $\Lambda^2 V_x$ and $\Lambda^2 V_x^-$ as $\alpha(a) Id$ and on $W_x$ as $-\alpha(a) Id$. Consider $\kappa \in \vartheta^-(J)$. The relation

$$(1, a, \alpha)\xi_\kappa = \chi_\kappa(1, a, \alpha)\xi_\kappa = \alpha(a)\kappa(x)\xi_\kappa$$

implies that

$$\begin{array}{rcll} \xi_\kappa \in W_x & \iff & \kappa(x) = -1 & \\ & \iff & \varepsilon(x \cdot \kappa) = 1 & \text{relation}(1) \\ & \iff & h^0(C, \kappa \otimes x) = 0 & \text{assumption}(*) \end{array}$$

Now we can conclude using lemma 1.5 a) and b) of [B2]. □

Thus we obtain a commutative diagram:

$$\begin{array}{ccc} P_x \cup P_x^- & \xrightarrow{h_x} & \mathbb{P}W_x \\ \downarrow{\scriptstyle \psi'_x} & & \cap \\ \mathcal{M}_p & \xrightarrow{\varphi_p} & \mathbb{P}\Lambda^2 V \end{array}$$

**Lemma 3.3** *$\forall M \in P_x$, the line $h_x(M)$ in $\mathbb{P}V$ passes through the points $\phi_x(M \otimes \alpha_x^{-1}) \in \mathbb{P}V_x$ and $\phi_x(M \otimes \alpha_x^{-1}(p_x - \sigma p_x)) \in \mathbb{P}V_x^-$.*

*Proof:* By definition of the line $\varphi_p(\psi'_x(M))$, it suffices to prove that $\psi_x(M\otimes\alpha_x^{-1})$ and $\psi_x(M\otimes\alpha_x^{-1}(p_x-\sigma p_x))$ are subbundles of $\psi'_x(M)$. Consider the exact sequence on $C_x$

$$0\longrightarrow \mathcal{O}_{C_x}\longrightarrow \mathcal{O}_{C_x}(p_x)\longrightarrow \mathbb{C}_{p_x}\longrightarrow 0$$

Tensorizing with $M\otimes\alpha_x^{-1}\otimes\pi_x^*z_x$, taking direct image by $\pi_x$ and using the projection formula gives

$$0\longrightarrow \pi_{x*}(M\otimes\alpha_x^{-1})\longrightarrow \psi'_x(M)\longrightarrow \mathbb{C}_p\longrightarrow 0$$

The other point is obtained in the same way from the exact sequence

$$0\longrightarrow \mathcal{O}_{C_x}\longrightarrow \mathcal{O}_{C_x}(\sigma p_x)\longrightarrow \mathbb{C}_{\sigma p_x}\longrightarrow 0$$

after tensorization with $M\otimes\alpha_x^{-1}(p_x-\sigma p_x)\otimes\pi_x^*z_x$ □

Using lemma 3.2 and 3.3, we can decompose the composite $h_x: P_x\to \mathbb{P}W_x$

$$P_x\longrightarrow \mathbb{P}V_x\times\mathbb{P}V_x^-\xrightarrow{\mu}\mathbb{P}W_x$$

where the first arrow maps $M\mapsto(\phi_x(M\otimes\alpha_x^{-1}),\phi_x(M\otimes\alpha_x^{-1}(p_x-\sigma p_x))$ and $\mu$ maps the pair of points $(\nu,\eta)$ to the line in $\mathbb{P}V$ passing through $\nu$ and $\eta$. Recall that the choice of $\alpha_x$ determines a 2-torsion point $x'\in J[2]$ with $\langle x,x'\rangle=-1$ such that $\pi_x^*(x')=\alpha_x^{\otimes 2}\otimes\mathcal{O}_{C_x}(\sigma p_x-p_x)$ .

Fix a theta structure on $J$, such that $x=\begin{bmatrix}0\,0\\0\,1\end{bmatrix}$ and $x'=\begin{bmatrix}0\,1\\0\,0\end{bmatrix}$. We can choose e.g. a symplectic basis $(\gamma_1,\dots,\gamma_{2g})$ such that $x=\frac{1}{2}\gamma_g+\Gamma_\tau\in\mathbb{C}^g/\Gamma_\tau\cong J$ and $x'=\frac{1}{2}\gamma_{2g}+\Gamma_\tau$, where $\Gamma_\tau$ is the lattice associated to the period matrix $\tau$ of $J$. By the canonical isomorphism $\pi_x^*: x^\perp/\langle x\rangle\xrightarrow{\sim}P_x[2]$, we can identify $P_x[2]$ with the subgroup of $K(g)\times\widehat{K(g)}$ given by $\{\begin{bmatrix}d\,0\\\delta\,0\end{bmatrix};(d,\delta)\in K(g-1)\times\widehat{K(g-1)}\}$. We have

$$\begin{array}{cccc}\xi_\kappa\in W_x & \Longleftrightarrow & \kappa=\kappa\begin{bmatrix}d\,1\\\delta\,0\end{bmatrix} & \text{with } \delta(d)=-1\\ & & \text{or } \kappa=\kappa\begin{bmatrix}d\,1\\\delta\,1\end{bmatrix} & \text{with } \delta(d)=1\end{array}$$

Consider $\kappa\in\vartheta^-(J)$, with $\varepsilon(x\cdot\kappa)=1$ (i.e. $\xi_\kappa\in W_x$). Then $(x'\cdot\kappa)$ induces a theta characteristic on $x^\perp/\langle x\rangle$, because

$$\begin{aligned}\forall y\in x^\perp\ (x'\cdot\kappa)(y+x) &= (x'\cdot\kappa)(y)\,(x'\cdot\kappa)(x)\langle y,x\rangle\\ &= (x'\cdot\kappa)(y)\,\kappa(x)\langle x,x'\rangle\langle y,x\rangle\\ &= (x'\cdot\kappa)(y)\end{aligned}$$

This theta characteristic of $P_x$ will be denoted by $\bar\kappa=\bar\kappa\begin{bmatrix}d\\\delta\end{bmatrix}$. This correspondence gives a (noncanonical) bijection

$$\begin{array}{ccc} \vartheta_x^-(J) & \xrightarrow{\sim} & \vartheta(P_x) \\ \kappa & \longmapsto & \bar{\kappa} \end{array}$$

where $\vartheta_x^-(J) = \{\kappa \in \vartheta^-(J) \mid \varepsilon(x \cdot \kappa) = 1\}$. A basis of $V_x$ (resp. $V_x^-$) is given by $\{X_{(b\,0)}\}$ (resp. $\{X_{(b\,1)}\}$) for $b \in K(g-1)$.

**Lemma 3.4** *Choose a period matrix $\omega_x$ for the Prym $P_x$. Then the composite $h_x$ induces by pull-back (up to homothety) the linear map*

$$\begin{array}{rcl} h_x^* : W_x & \longrightarrow & H^0(P_x, \mathcal{O}(4\Xi_x)) \\ \xi_\kappa & \longmapsto & \theta_{\bar{\kappa}}(-2\tilde{\alpha}_x, \omega_x)\theta_{\bar{\kappa}}(2z, \omega_x) \end{array}$$

*Proof:* An easy computation shows that the pull-back $\mu^* : W_x \to V_x \otimes V_x^-$ maps $\xi_\kappa \mapsto 2\sum_b \delta(b) X_{(b\,0)} \otimes X_{(b+d\,1)}$, with $b$ running over $K(g-1)$. The 2-torsion point $x' = \begin{bmatrix} 0\,1 \\ 0\,0 \end{bmatrix}$ gives a linear isomorphism, up to homothety, $V_x^- \xrightarrow{\sim} V_x$ with $X_{(b\,1)} \mapsto X_{(b\,0)}$. Under the natural identifications of $V_x$ with $H^0(P_x, \mathcal{O}(2\Xi_x))$, and $H^0(P_x, \mathcal{O}(4\Xi_x))$ with the space of $\Gamma_{\omega_x}$-quasi-periodic theta functions of order 4, we obtain a map (up to homothety)

$$\begin{aligned} \xi_\kappa \quad \mapsto \quad & \sum_b \delta(b)\theta \begin{bmatrix} b \\ 0 \end{bmatrix} (2z - 2\tilde{\alpha}_x, 2\omega_x)\theta \begin{bmatrix} b+d \\ 0 \end{bmatrix} (2z + 2\tilde{\alpha}_x, 2\omega_x) \\ & = \theta_{\bar{\kappa}}(-2\tilde{\alpha}_x, \omega_x)\theta_{\bar{\kappa}}(2z, \omega_x) \end{aligned}$$

where the last equality is obtained by the addition formula (3) and $\tilde{\alpha}_x$ is a representative in $\mathbb{C}^g$ of $\alpha_x \in \mathbb{C}^g/\Gamma_{\omega_x} \cong P_x$. □

# 4 Schottky-Jung and Donagi relations

We shall now describe the intersection points of (the Kummers of) the Jacobian and the Pryms mapped to $\mathcal{M}_0$ and $\mathcal{M}_p$. This method was used [vG-P1] to give a vector bundle theoretic proof of the Schottky-Jung and Donagi relations. We denote by $\varphi_L : J \to \mathbb{P}V$ the map given by the linear series $\mid 2\Theta \mid$. We consider two orthogonal nonzero 2-torsion points $x, y \in J[2]$, $\langle x, y \rangle = 1$, which define 2-torsion points of $P_x$ and $P_y$, namely $\bar{x} := x + \langle y \rangle \in y^\perp/\langle y \rangle \cong P_y[2]$ and $\bar{y} := y + \langle x \rangle = x^\perp/\langle x \rangle \cong P_x[2]$.

**Proposition 4.1** *(a) the Schottky-Jung relations: $\exists z_x \in J$ with $z_x^{\otimes 2} \cong x$ such that*

$$\varphi_L(z_x) = \phi_x(0)$$

*(b) the Donagi relations: $\exists u_y \in P_y$ with $u_y^{\otimes 2} \cong \bar{x}$ and $\exists u_x \in P_x$ with $u_x^{\otimes 2} \cong \bar{y}$ such that*

$$\phi_x(u_x) = \phi_y(u_y)$$

The following proposition is an "odd degree" version of the classical relations.

**Proposition 4.2** *With the same notation as above, we have*

*(a)* $\varphi_p \circ j_p(z_x) = h_x(\alpha_x)$

$$\begin{array}{ccc} \hat{J} & \xrightarrow{\varphi_p \circ j_p} & I\!P\Lambda^2 V \\ & & \cup \\ P_x & \xrightarrow{h_x} & I\!PW_x \end{array}$$

*(b)* $p_{x,y}[h_x(u_x \otimes \alpha_x)] = p_{x,y}[h_y(u_y \otimes \alpha_y)]$

$$\begin{array}{ccccc} P_x & \xrightarrow{h_x} & I\!PW_x & \overset{p_{x,y}}{\searrow} & \\ & & & & I\!PW_{x,y} \\ P_y & \xrightarrow{h_y} & I\!PW_y & \underset{p_{x,y}}{\nearrow} & \end{array}$$

*where* $p_{x,y}$ *is the projection* $I\!P\Lambda^2 V \to I\!PW_{x,y} := I\!PW_x \cap W_y$.

*Remark:* One can show, using proposition 2.4 [R], that the Pryms $P_x$ and $P_y$ don't intersect in $\mathcal{M}_p$. However, two Pryms associated to nonorthogonal 2-torsion points intersect in $\mathcal{M}_p$. This intersection property, although natural, is not used in the sequel of this paper.

*Proof:* Let $z_x$, with $z_x^{\otimes 2} \cong x$, define the map $\psi_x : P_x \to \mathcal{M}_0$. Then

$$\psi_x(0) = \pi_{x*}(\mathcal{O}_{C_x}) \otimes z_x = (\mathcal{O}_C \oplus x) \otimes z_x = z_x \oplus z_x^{-1}$$

Taking the image by $\varphi_{\mathcal{L}}$ gives the classical Schottky-Jung relation (prop. 4.1(a)). The two rank 2 bundles $\psi'_x(\alpha_x) = \pi_{x*}(\mathcal{O}_{C_x}(p_x)) \otimes z_x$ and $F_{z_x} = j_p(z_x)$ (see section 2) fit into the exact sequence

$$0 \longrightarrow z_x \oplus z_x^{-1} \longrightarrow \psi'_x(\alpha_x) \longrightarrow \mathbb{C}_p \longrightarrow 0$$

As $\psi'_x(\alpha_x)$ is a stable bundle, it is nonsplit and hence isomorphic to $j_p(z_x)$. Now apply $\varphi_p$ to get the first relation.

The main step of the proof given by van Geemen of the Donagi relations is the existence of a stable rank 2 bundle

$$E := \psi_x(u_x) = \psi_y(u_y) \in \mathcal{M}_0$$

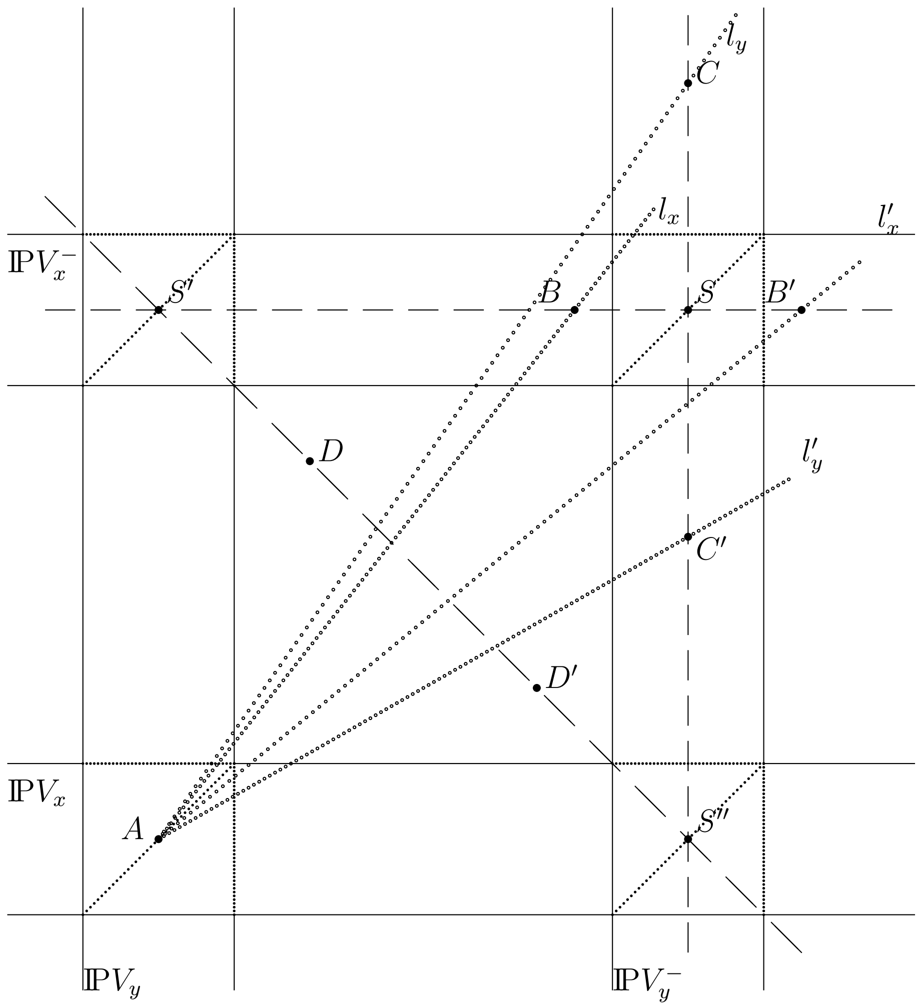
Consider the projective line $\mathbb{P} := \mathbb{P}\mathrm{Ext}^1(\mathbb{C}_p, E)$, parametrizing isomorphism classes of extensions of $\mathbb{C}_p$ by $E$, and the classifying morphism $\mathbb{P} \to \mathcal{M}_p$. The proof of the following lemma is similar to the proof of lemma 3.4 of [B2].

**Lemma 4.3** *The image of the composite* $I\!\!P \to \mathcal{M}_p \xrightarrow{\varphi_p} I\!\!P\Lambda^2 V$ *is a conic, hence contained in a plane.*

Consider the four lines in $\mathbb{P}V$ (here we identify bundles in $\mathcal{M}_p$ with their image by $\varphi_p$ in $\mathbb{P}\Lambda^2 V$)

$$
\begin{aligned}
l_x &:= \pi_{x*}(u_x(p_x)) \otimes z_x &= h_x(u_x \otimes \alpha_x) \\
l'_x &:= \pi_{x*}(u_x(\sigma p_x)) \otimes z_x & \\
l_y &:= \pi_{y*}(u_y(p_y)) \otimes z_y &= h_y(u_y \otimes \alpha_y) \\
l'_y &:= \pi_{y*}(u_y(\sigma p_y)) \otimes z_y &
\end{aligned}
$$

with $l_x, l'_x \in \mathbb{P}W_x$ and $l_y, l'_y \in \mathbb{P}W_y$. They pass through the point $A = \phi_x(u_x) = \phi_y(u_y) \in \mathbb{P}V_x \cap V_y$.




By lemma 3.3 the line $l_x$ passes through the point

$$B := \phi_x(u_x \otimes \alpha_x^{\otimes 2}) \otimes x' = \phi_x(u_x(p_x - \sigma p_x) \in \mathbb{P}V_x^-$$

Similarly we define the points (see picture)

$$\begin{aligned} B' &:= \phi_x(u_x(\sigma p_x - p_x)) \in \mathbb{P}V_x^- \\ C &:= \phi_y(u_y(p_y - \sigma p_y)) \in \mathbb{P}V_y^- \\ C' &:= \phi_y(u_y(\sigma p_y - p_y)) \in \mathbb{P}V_y^- \end{aligned}$$

Recall that the eigenspaces $\mathbb{P}V_x$ and $\mathbb{P}V_x^-$ are stable under the action of $y$, since $\langle x, y\rangle = 1$. We have $\forall M \in \ker(\mathrm{Nm}_x)$, $\phi_x(M) = \phi_x(M^{-1})$, which implies that

$$B' = \phi_x(u_x^{-1}(p_x - \sigma p_x)) = \phi_x(u_x(p_x - \sigma p_x)) \otimes \bar{y}) = y.B$$

where $y.B$ denotes the image of $B$ by the projective linear action of $y$ on $\mathbb{P}V_x^-$. Similarly $C' = x.C$.

Consider the pencil $\wp_x$ of lines generated by the lines $l_x$ and $l'_x$. Each line of $\wp_x$ passes through the point $A$ and intersects the eigenspace $\mathbb{P}V_x^-$ in a point of the line $(BB')$. Similarly we consider the pencil $\wp_y$ generated by $l_y$ and $l'_y$ (see picture). The pencils $\wp_x$ and $\wp_y$ determine a line in $\mathbb{P}\Lambda^2 V$ passing through the points $l_x$ and $l'_x$ (for $\wp_x$) and through $l_y$ and $l'_y$ (for $\wp_y$). Now the four points $l_x, l'_x, l_y, l'_y \in \mathbb{P}\Lambda^2 V$ lie on a conic (lemma 4.3) and therefore are contained in a plane. Hence the lines $(l_x l'_x)$ and $(l_y l'_y)$ intersect in $\mathbb{P}\Lambda^2 V$, i.e. the pencils $\wp_x$ and $\wp_y$ have a common line $l$. This line $l$ intersects $\mathbb{P}V_x^-$ in a point on $(BB')$ and $\mathbb{P}V_y^-$ in a point on $(CC')$. Thus $l$ intersects the subspace $\mathbb{P}V_x^- \cap V_y^-$ in a point $S = (CC') \cap (BB')$.

Consider the direct sum:

$$W_y = W_{x,y} \oplus W_y \cap (\Lambda^2 V_x \oplus \Lambda^2 V_x^-)$$

where $W_{x,y} = W_x \cap W_y$. We will show that $p_{x,y}(l_y) = l$ (up to a scalar) where $p_{x,y} : W_x \to W_{x,y}$ is the projection given by the direct sum. Combining this result with its analogue $p_{x,y}(l_x) = l$ allows us to conclude.

Choose a theta structure for $L = \mathcal{O}(2\Theta)$ on $J$. The point $x \in J[2]$ is represented by $x = (1, a, \alpha)$. As in the proof of lemma 3.2, $x$ acts on $W_{x,y}$ as $-\alpha(a) Id$ and on $W_y \cap (\Lambda^2 V_x \oplus \Lambda^2 V_x^-)$ as $\alpha(a) Id$. Choose a representative in $W_y$, which we also denote by $l_y$, of the line $l_y = (AC) \in \mathbb{P}W_y$. As $C' = x.C$ and $A = x.A$, we have $x.l_y = (AC') = l'_y$ and $x.l'_y = l_y$. Thus we obtain the direct sum decomposition:

$$l_y = \frac{1}{2}(l_y - \alpha(a) l'_y) + \frac{1}{2}(l_y + \alpha(a) l'_y)$$

Now the line $\frac{1}{2}(l_y - \alpha(a) l'_y) \in \mathbb{P}W_x$ belongs to the pencil $\wp_y$ and intersects $\mathbb{P}V_x^-$. Hence $l = \frac{1}{2}(l_y - \alpha(a) l'_y) = p_{x,y}(l_y)$ (up to a scalar). □

We choose a symplectic basis $(\gamma_1, \ldots, \gamma_{2g})$ for $J$ such that $x = \frac{1}{2}\gamma_g + \Gamma_\tau$, $x' = \frac{1}{2}\gamma_{2g} + \Gamma_\tau$, $y = \frac{1}{2}\gamma_{g-1} + \Gamma_\tau$ and $y' = \frac{1}{2}\gamma_{2g-1} + \Gamma_\tau$, with $\tau$ the period matrix of $J$. In the corresponding theta structure these points are represented by

$$x = \begin{bmatrix} 0\,0\,0 \\ 0\,0\,1 \end{bmatrix} \quad x' = \begin{bmatrix} 0\,0\,1 \\ 0\,0\,0 \end{bmatrix} \quad y = \begin{bmatrix} 0\,0\,0 \\ 0\,1\,0 \end{bmatrix} \quad y' = \begin{bmatrix} 0\,1\,0 \\ 0\,0\,0 \end{bmatrix} \tag{4}$$

where the zeros in the first column belong to $K(g-2)$. We also choose period matrices $\omega_x$ and $\omega_y$ for the Pryms $P_x$ and $P_y$. We are now able to express the preceding relations in terms of theta-constants.

**Proposition 4.4 ("even version")** *There exist nonzero constants $c_1, c_2$ such that*

*(a) the Schottky-Jung relations :* $\forall b \in K(g-1)$

$$\theta \begin{bmatrix} b\,0 \\ 0\,1 \end{bmatrix} (0, 2\tau) = c_1 \theta \begin{bmatrix} b \\ 0 \end{bmatrix} (0, 2\omega_x)$$

*(b) the Donagi relations :* $\forall d \in K(g-2)$

$$\theta \begin{bmatrix} d\,0 \\ 0\,1 \end{bmatrix} (0, 2\omega_x) = c_2 \theta \begin{bmatrix} d\,0 \\ 0\,1 \end{bmatrix} (0, 2\omega_y)$$

*Proof:* We refer to proposition 5 of [vG] for (i) and to lemma 1 of [vG-P1] for (ii).

**Proposition 4.5 ("odd version")** *There exist nonzero constants $d_1, \ldots, d_4$ such that*

*(a)* $\forall \kappa \in \vartheta_x^-(J) \cong \vartheta(P_x)$

$$\varepsilon(\bar{\kappa})\theta_{\bar{\kappa}}^2(2\tilde{\alpha}_x, \omega_x) = d_1 \theta_{x \cdot \kappa}(0, \tau) D\theta_\kappa(0, \tau)$$

*(a')* $\forall b \in K(g-1)$

$$D\theta \begin{bmatrix} b\,1 \\ 0\,1 \end{bmatrix} (0, 2\tau) = d_2 \theta \begin{bmatrix} b \\ 0 \end{bmatrix} (4\tilde{\alpha}_x, 2\omega_x)$$

*(b)* $\forall d \in K(g-2)$

$$\theta \begin{bmatrix} d\,1 \\ 0\,1 \end{bmatrix} (4\tilde{\alpha}_x, 2\omega_x) = d_3 \theta \begin{bmatrix} d\,1 \\ 0\,1 \end{bmatrix} (4\tilde{\alpha}_y, 2\omega_y)$$

*(b')* $\forall \kappa \in \vartheta_x^-(J) \cap \vartheta_y^-(J)$

$$\theta_{\bar{\kappa}}(2\tilde{\alpha}_x, \omega_x)\theta_{\bar{y} \cdot \bar{\kappa}}(2\tilde{\alpha}_x, \omega_x) = d_4 \theta_{\bar{\kappa}}(2\tilde{\alpha}_y, \omega_y)\theta_{\bar{x} \cdot \bar{\kappa}}(2\tilde{\alpha}_y, \omega_y)$$

*Proof:*

(a) We express the coordinates of the point $\varphi_p \circ j_p(z_x) = h_x(\alpha_x)$ in the basis $\{\xi_\kappa\}$ of $W_x$ with $\kappa \in \vartheta_x^-(J)$. The map $\varphi_p \circ j_p$ induces by pull-back the Wahl map $w_D$ and $w_D(\xi_\kappa) = cD\theta_\kappa(0, \tau)\theta_\kappa(2z, \tau)$. Therefore the coordinates of $\varphi_p \circ j_p(z_x)$ are (up to scalar)

$$D\theta_\kappa(0, \tau)\theta_{x \cdot \kappa}(0, \tau)$$

since $\theta_\kappa(\frac{1}{2}\gamma_g, \tau) = \theta_{x \cdot \kappa}(0, \tau)$. By lemma 3.4 the coordinates of $h_x(\alpha_x)$ are (up to scalar)

$$\theta_{\bar{\kappa}}(-2\tilde{\alpha}_x, \omega_x)\theta_{\bar{\kappa}}(2\tilde{\alpha}_x, \omega_x) = \varepsilon(\bar{\kappa})\theta_{\bar{\kappa}}^2(2\tilde{\alpha}_x, \omega_x)$$

(a') The line $t$ in $\mathbb{P}V$, corresponding to the intersection point in $\mathbb{P}\Lambda^2 V$ of $\hat{J}$ and $P_x$, has two descriptions: first, $t$ passes through $\varphi_L(z_x) = \phi_x(0) \in \mathbb{P}V_x$ and its direction is given by a tangent vector $v$ (to $J$), whose coordinates are (up to a scalar)

$$[\underbrace{\ldots D\theta \begin{bmatrix} b\,0 \\ 0\,1 \end{bmatrix} (0, 2\tau) \ldots}_{V_x}, \underbrace{\ldots D\theta \begin{bmatrix} b\,1 \\ 0\,1 \end{bmatrix} (0, 2\tau) \ldots}_{V_x^-}]$$

Since $\theta \begin{bmatrix} b\,0 \\ 0\,1 \end{bmatrix} (z, 2\tau)$ is an even theta function, $D\theta \begin{bmatrix} b\,0 \\ 0\,1 \end{bmatrix} (z, 2\tau)$ is odd and thus vanishes at the origin. Hence the tangent vector $v$ is contained in $\mathbb{P}V_x^-$.

Secondly, by lemma 3.3 $t$ passes through the point $\phi_x(p_x - \sigma p_x) \in \mathbb{P}V_x^-$ and since $t$ intersects $\mathbb{P}V_x^-$ in a unique point, the coordinates of this point can be expressed in two different ways.

*Remark:* One should notice that the statements (a) and (a') are equivalent, namely by applying the differential operator $D$ to the addition formula (3) -treat one variable as a constant- evaluating at the origin and using the "even" Schottky-Jung relations.

(b) A basis of $V_y$ (resp. $V_y^-$) is given by $\{X_{(d\,0\,\varepsilon)}\}$ (resp. $\{X_{(d\,1\,\varepsilon)}\}$) for $d \in K(g-2)$ and $\varepsilon \in \mathbb{Z}/2$. A basis of the intersection $V_x^- \cap V_y^-$ is given by $\{X_{(d\,1\,1)}\}$ for $d \in K(g-2)$. We can decompose $V_x^- = V_x^- \cap V_y \oplus V_x^- \cap V_y^-$. The point $D = (BB') \cap (CC') \in \mathbb{P}V_x^- \cap V_y^-$ (see proof of proposition 4.2 (b)) is the image of $B$ by the projection $\mathbb{P}V_x^- \to \mathbb{P}V_x^- \cap V_y^-$. Therefore the coordinates of $S$ are (up to a scalar)

$$\theta \begin{bmatrix} d\,1 \\ 0\,0 \end{bmatrix} (2\tilde{u}_x + 4\tilde{\alpha}_x, 2\omega_x) = \theta \begin{bmatrix} d\,1 \\ 0\,1 \end{bmatrix} (4\tilde{\alpha}_x, 2\omega_x)$$

where $\tilde{u}_x$ is a representative of $u_x \in \mathbb{C}^{g-1}/\Gamma_{\omega_x} \cong P_x$ and $2\tilde{u}_x = \bar{y} + \Gamma_{\omega_x}$ with $\bar{y} = \begin{bmatrix} 0\,0 \\ 0\,1 \end{bmatrix}$. We obtain the relation (b) doing the same reasoning for $V_y^-$ and the point $C$.

(b') We can deduce relation (b') from (b) and the Donagi relations using the addition formula (3) with $z = 2\tilde{\alpha}_x + \tilde{u}_x$ and $u = \tilde{u}_x$. Another method is to express the coordinates of the line $l = (AS) = p_{x,y}[h_x(u_x \otimes \alpha_x)] = p_{x,y}[h_y(u_y \otimes \alpha_y)]$ in the basis $\{\xi_\kappa\}$ of $W_{x,y}$. We immediately get the result from lemma 3.4 . □

*Remark:* For a generic curve $C$, the conditions (*) are verified. Hence by relation (a) $\theta_{\bar{\kappa}}(2\tilde{\alpha}_x, \omega_x) \neq 0$, $\forall \bar{\kappa} \in \vartheta(P_x)$, i.e. $h^* : W_x \to H^0(P_x, \mathcal{O}(4\Xi_x))$ is an isomorphism.

**Proposition 4.6 ("symmetry")** *There exists a nonzero constant $e$ such that $\forall d \in K(g-2)$*

$$\theta \begin{bmatrix} d\,1 \\ 0\,1 \end{bmatrix} (4\tilde{\alpha}_x, 2\omega_x) = e\theta \begin{bmatrix} d\,0 \\ 0\,1 \end{bmatrix} (4\tilde{\alpha}_x, 2\omega_x)$$

*Proof:* We need more information about the configuration of lines described in the proof of prop. 4.2 (b). First, the intersection point $A$ (see picture) also lies on the Kummer of the Prym $P_{x+y}$ (see [D]). Analogously, we can consider the lines $l_{x+y}$ and $l'_{x+y}$, which intersect the eigenspace $\mathbb{P}V_{x+y}^{-}$ in two points $D$ and $D'$. Therefore we obtain six points $B, B', C, C', D, D'$ lying on a conic and, by doing the same reasoning as above, we get intersection points

$$(BB') \cap (DD') = S' \in \mathbb{P}V_x^{-} \cap V_y \quad \text{and} \quad (CC') \cap (DD') = S'' \in \mathbb{P}V_x \cap V_y^{-}$$

Consider the subgroup $G$ of $K(g) \times \widehat{K(g)}$ generated by the elements $x$ and $y$. Since $G$ is totally isotropic, there exists a level subgroup $\tilde{G} \subset Heis(g)$ over $G$. We can choose $\tilde{x}, \tilde{y} \in \tilde{G}$ such that they act as $-Id$ on $V_x^{-}$ and $V_y^{-}$. We have a decomposition

$$V_{x+y}^{-} = V_x^{-} \cap V_y \oplus V_x \cap V_y^{-} \tag{5}$$

Consider the involution $\tau \in \mathrm{Aut}(K(g) \times \widehat{K(g)})$ interchanging the points $x$ and $y$ and leaving the subgroup $K(g-2) \times \widehat{K(g-2)}$ invariant. Then $\tau$ lifts to an involution $\tilde{\tau}$ of $Heis(g)$, which interchanges $\tilde{x}$ and $\tilde{y}$. By prop. 3 [M1], $Heis(g)$ has a unique irreducible representation $\rho : Heis(g) \to GL(V)$ on which $\mathbb{C}^*$ acts by its natural character. Hence, there exists a linear involution $i$ of $V$ such that

$$\rho \circ \tilde{\tau}(\alpha) = i \circ \rho(\alpha) \circ i \quad \forall \alpha \in Heis(g)$$

In particular: $i(V_x) = V_y$, $i(V_x^{-}) = V_y^{-}$ and $i$ acts on the eigenspace $V_{x+y}^{-}$ interchanging the two factors of the decomposition (5). The points $D$ and $D'$ are invariant under $i$, therefore the line $(DD')$ is preserved under $i$ and we can conclude that $i(S') = S''$. Now we express the coordinates of $S'$ and $S''$ in there natural bases and we obtain the relation for the theta-constants on the Prym $P_{x+y}$. The same method gives the symmetric relations for $P_x$ and $P_y$. □

# 5 Quadrics in $\mathbb{P}\Lambda^2 V$

We can associate to any $x \in J[2]$ a character of $J[2]$, also denoted by $x$, defined by $y \mapsto \langle x, y \rangle$. This correspondence is one-to-one and allows us to identify $J[2]$ with its character group.

The Heisenberg group $Heis(g)$ acts on $S^2(\Lambda^2 V)$ and its center $\mathbb{C}^*$ acts by $t \mapsto t^4$. This action factors over the abelian group $J[2]$. Consider the decomposition into character spaces

$$S^2(\Lambda^2 V) = \bigoplus_{x \in J[2]} S^2(\Lambda^2 V)_x$$

**Lemma 5.1** *a) The elements $\{\xi_\kappa \otimes \xi_\kappa\}$ for $\kappa \in \vartheta^{-}(J)$ form a basis of $S^2(\Lambda^2 V)_0$. In particular,* $\dim\, S^2(\Lambda^2 V)_0 = 2^{g-1}(2^g - 1)$.

*b) For a nonzero $x \in J[2]$, the distinct elements $\{\xi_\kappa \otimes \xi_{x\cdot\kappa}\}$ for $\kappa \in \vartheta^{-}(J)$ with $x \cdot \kappa \in \vartheta^{-}(J)$ form a basis of $S^2(\Lambda^2 V)_x$. In particular,* $\dim\, S^2(\Lambda^2 V)_x = 2^{g-2}(2^{g-1} - 1)$.

*Proof:* Choose a theta structure for $\mathcal{O}(2\Theta)$ on $J$. A point $x \in J[2]$ is represented by $x = (1, a, \alpha)$. Then for any theta characteristic $\kappa = \kappa\begin{bmatrix} c \\ \gamma \end{bmatrix}$ and $x \cdot \kappa = \kappa\begin{bmatrix} c+a \\ \alpha\gamma \end{bmatrix}$

$$\begin{aligned}(1, b, \beta)\xi_\kappa &= \gamma(b)\beta(c)\xi_\kappa \\ (1, b, \beta)\xi_{x\cdot\kappa} &= (\alpha\gamma)(b)\beta(c+a)\xi_{x\cdot\kappa}\end{aligned}$$

Hence $(1, b, \beta)\xi_\kappa \otimes \xi_{x\cdot\kappa} = \alpha(b)\beta(a)\xi_\kappa \otimes \xi_{x\cdot\kappa}$ i.e. $\xi_\kappa \otimes \xi_{x\cdot\kappa} \in S^2(\Lambda^2 V)_x$. Now the elements $\{\xi_\kappa \otimes \xi_\kappa\}$ and $\{\xi_\kappa \otimes \xi_{x\cdot\kappa}\}$ for all $x \neq 0$ generate the space $S^2(\Lambda^2 V)$, since $\{\xi_\kappa\}$ is a basis of $\Lambda^2 V$ with $\kappa \in \vartheta^-(J)$. Identifying $\xi_\kappa \otimes \xi_{x\cdot\kappa}$ and $\xi_{x\cdot\kappa} \otimes \xi_\kappa$, the number of all these elements is equal to dim $S^2(\Lambda^2 V)$. □

## 5.1 Invariant quadrics

Fix a theta structure for $\mathcal{O}(2\Theta)$ on $J$. We define a linear map for any nonzero $x = \begin{bmatrix} a \\ \alpha \end{bmatrix}$

$$\begin{aligned}M_x : S^2(\Lambda^2 V)_0 &\longrightarrow (S^2 V \otimes \Lambda^2 V)_x \\ \xi_\kappa \otimes \xi_\kappa &\longmapsto \gamma(a)\xi_{x\cdot\kappa} \otimes \xi_\kappa \text{ if } \kappa \in \vartheta_x^-(J) \\ & \qquad 0 \qquad\qquad \text{otherwise}\end{aligned}$$

for all theta characteristics $\kappa = \kappa\begin{bmatrix} c \\ \gamma \end{bmatrix}$. The space $(S^2 V \otimes \Lambda^2 V)_x$ denotes the subspace of $S^2 V \otimes \Lambda^2 V$ corresponding to the character $x$.

*Remark:* We defined the map $M_x$ using a theta structure on $J$. However, in the proposition 5.2, we will use the fact that $M_x$ admits an intrinsic definition (up to homothety): consider for $x \in J[2]$ the linear automorphism $U(x)$ of $V$, unique up to homothety, given by the projective representation of $J[2]$ on $\mathbb{P}V$. Let $v_\kappa \otimes v_\kappa \in V \otimes V$ be an eigenvector (unique up to scalar) w.r.t. the character $\chi_\kappa$ for the action of $\mathcal{G}(L)$ on $V \otimes V$ (see section 2). Recall that $v_\kappa$ and $\xi_\kappa$ are proportional via a theta structure on $J$. We consider the automorphism $\widehat{U(x)} = U(x) \otimes Id$ of $V \otimes V$ and define $M_x : v_\kappa \otimes v_\kappa \mapsto v_\kappa \otimes \widehat{U(x)} v_\kappa$. This defines a linear map, up to homothety, since the definition does not depend on the choice of the eigenvectors $v_\kappa$. Let us check that the two definitions are equivalent. With the same notation as above:

$$\begin{aligned}(U(x) \otimes Id)\xi_\kappa &= \sum_b \gamma(b)\alpha(b+a)\psi_{b+a} \otimes \psi_{b+c} \\ &= \sum_b \gamma(b+a)\alpha(b)\psi_b \otimes \psi_{b+a+c} \\ &= \gamma(a)\xi_{x\cdot\kappa}\end{aligned}$$

Consider the product morphism $\hat{J} \to \mathbb{P}V \times \mathbb{P}\Lambda^2 V$. The elements of $(S^2 V \otimes \Lambda^2 V)_x$ are polynomials on $\mathbb{P}V \times \mathbb{P}\Lambda^2 V$ of degree 2 on the first factor and linear on the second one.

$$\begin{array}{ccccc} \hat{J} & & & & \\ & \searrow^{j_p} & & & \\ & & \mathcal{M}_p & \xrightarrow{\varphi_p} & \mathbb{P}\Lambda^2 V \\ & \nearrow_{\psi'_x} & & & \\ P_x & & & & \end{array}$$

**Proposition 5.2** *An invariant quadric* $F \in S^2(\Lambda^2 V)_0$ *vanishes on the Prym* $P_x$ *if and only if* $M_x(F)$ *vanishes on* $\hat{J}$.

*Proof:* Since $M_x$ is defined intrinsically, we can choose a theta structure such that the conditions preceding proposition 4.4 hold, i.e. $x = \begin{bmatrix} 0\,0 \\ 0\,1 \end{bmatrix}$ where the zeros of the first column belong to $K(g-1)$. In particular, $a = 0$. By lemma 5.1 a generic polynomial $F \in S^2(\Lambda^2 V)_0$ and its image $M_x(F)$ can be written (with $a_\kappa \in \mathbb{C}$):

$$\begin{aligned} F &= \sum_{\kappa \in \vartheta^-(J)} a_\kappa \xi_\kappa \otimes \xi_\kappa \\ M_x(F) &= \sum_{\kappa \in \vartheta_x^-(J)} a_\kappa \xi_{x\cdot\kappa} \otimes \xi_\kappa \end{aligned}$$

First, we rephrase the condition that $F$ vanishes on $P_x$: if $\varepsilon(x\cdot\kappa) = -1$, the polynomial $\xi_\kappa \otimes \xi_\kappa \in S^2(\Lambda^2 V)_0$ vanishes on $P_x$ (lemma 3.2); if $\kappa \in \vartheta_x^-(J)$, the polynomial $\xi_\kappa \otimes \xi_\kappa$ restricts to the theta function $\theta^2_{\bar\kappa}(-2\tilde\alpha_x, \omega_x)\theta^2_{\bar\kappa}(2z, \omega_x)$. By the addition formula (3) we have:

$$\theta^2_{\bar\kappa}(2z, \omega_x) = \sum_b \delta(b)\theta \begin{bmatrix} b \\ 0 \end{bmatrix} (4z, 2\omega_x)\theta \begin{bmatrix} b+d \\ 0 \end{bmatrix} (0, 2\omega_x)$$

with $\bar\kappa = \bar\kappa \begin{bmatrix} d \\ \delta \end{bmatrix}$ and $b$ running over $K(g-1)$. Note that the family of theta functions of order 8 $\{\theta \begin{bmatrix} b \\ 0 \end{bmatrix} (4z, 2\omega_x)\}$ is linearly independent. Therefore $F$ vanishes on $P_x$

$$\begin{aligned} &\iff \quad \sum_\kappa a_\kappa \sum_b \delta(b)\theta^2_{\bar\kappa}(2\tilde\alpha_x, \omega_x)\theta \begin{bmatrix} b+d \\ 0 \end{bmatrix} (0, 2\omega_x)\theta \begin{bmatrix} b \\ 0 \end{bmatrix} (4z, 2\omega_x) = 0 \\ &\iff \quad \forall b \in K(g-1) \qquad \sum_{\kappa \in \vartheta_x^-(J)} a_\kappa \delta(b)\theta^2_{\bar\kappa}(2\tilde\alpha_x, \omega_x)\theta \begin{bmatrix} b+d \\ 0 \end{bmatrix} (0, 2\omega_x) = 0 \qquad (6) \end{aligned}$$

Now we rephrase the condition that $M_x(F)$ vanishes on $\hat{J}$: the polynomial $\xi_{x\cdot\kappa} \otimes \xi_\kappa \in (S^2 V \otimes \Lambda^2 V)_x$ restricts to the theta function (see section 2)

$$\theta_{x\cdot\kappa}(0, \tau)\theta_{x\cdot\kappa}(2z, \tau)D\theta_\kappa(0, \tau)\theta_\kappa(2z, \tau)$$

Again by the addition formula (3), we have:

$$\theta_\kappa(2z, \tau)\theta_{x\cdot\kappa}(2z, \tau) = \delta(d)\sum_b \delta(b)\theta \begin{bmatrix} b\,1 \\ 0\,1 \end{bmatrix} (4z, 2\tau)\theta \begin{bmatrix} b+d\;0 \\ 0\quad\;1 \end{bmatrix} (0, 2\tau)$$

Note that the family of theta functions of order 8 $\{\theta\begin{bmatrix} b\,1 \\ 0\,1 \end{bmatrix}(4z, 2\tau)\}$ is linearly independent. Therefore $M_x(F)$ vanishes on $\hat{J}$ if and only if

$$\sum_{\kappa,b} a_\kappa \delta(d)\delta(b)\theta_{x\cdot\kappa}(0,\tau)D\theta_\kappa(0,\tau)\theta\begin{bmatrix} b+d & 0 \\ 0 & 1 \end{bmatrix}(0,2\tau)\theta\begin{bmatrix} b\,1 \\ 0\,1 \end{bmatrix}(4z,2\tau) = 0$$

$$\forall b \in K(g-1) \ \sum_{\kappa} \varepsilon(\bar{\kappa}) a_\kappa \delta(b)\theta_{x\cdot\kappa}(0,\tau)D\theta_\kappa(0,\tau)\theta\begin{bmatrix} b+d & 0 \\ 0 & 1 \end{bmatrix}(0,2\tau) = 0 \qquad (7)$$

The Schottky-Jung relations of proposition 4.4 and 4.5 implie that the set of equations (6) and (7) are proportional, hence the two vanishing conditions are equivalent. □

Let us introduce the linear map

$$\begin{aligned} N : S^2(\Lambda^2 V)_0 &\longrightarrow [V \otimes \Lambda^2 V]_0 \\ \xi_\kappa \otimes \xi_\kappa &\longmapsto X_c \otimes \xi_\kappa \end{aligned}$$

where $\kappa = \kappa\begin{bmatrix} c \\ \gamma \end{bmatrix}$ and $[V \otimes \Lambda^2 V]_0$ is the subspace of $V \otimes \Lambda^2 V$ invariant under the maximal level subgroup $\widehat{K(g)} \subset Heis(g)$. By abuse of notation we also call $\widehat{K(g)}$ its lift in $Heis(g)$. It is easy to check that $N$ is an isomorphism. Consider the multiplication map of theta functions

$$m : V \otimes \Lambda^2 V \xrightarrow{\sim} H^0(J, \mathcal{O}(2\Theta)) \otimes H^0_-(J, \mathcal{O}(4\Theta)) \longrightarrow H^0_-(J, \mathcal{O}(6\Theta))$$

where the first arrow is the Wahl map, which is an isomorphism under the assumption (*). The composite $m$ is $Heis(g)$-equivariant. By Mumford's theta theory [M1], $H^0_-(J, \mathcal{O}(6\Theta))$ is a direct sum of irreducible representations of dimension $2^g$ ( the center $\mathbb{C}^*$ of $Heis(g)$ acts by $t \mapsto t^3$). A result of Kempf [K] asserts that $m$ is surjective. Take invariants under the level subgroup $\widehat{K(g)}$, which gives a surjective map:

$$[m]_0 : [V \otimes \Lambda^2 V]_0 \longrightarrow [H^0_-(J, \mathcal{O}(6\Theta))]_0$$

Now we can state the main theorem of this subsection

**Theorem 5.3** *An invariant quadric $F \in S^2(\Lambda^2 V)_0$ vanishes on all Pryms if and only if $N(F) \in \ker[m]_0$*

*Proof:* Let $G_0 = N(F)$ and $\forall b \in K(g)$ $G_b = (1, b, 0).G_0 \in V \otimes \Lambda^2 V$. Consider a point $x = \begin{bmatrix} a \\ \alpha \end{bmatrix} \in J[2]$. Let us prove the following relation, for $x \neq 0$:

$$M_x(F) = \sum_{b \in K(g)} \alpha(b+a) X_{b+a} \otimes G_b \qquad (8)$$

The right-hand side is considered as an element of $S^2 V \otimes \Lambda^2 V$ (after the projection $V \otimes V \otimes \Lambda^2 V \to S^2 V \otimes \Lambda^2 V$). By linearity, it suffices to prove this relation for $F = \xi_\kappa \otimes \xi_\kappa$. If $\kappa \in \vartheta^-_x(J)$, $M_x(F) = \gamma(a)\xi_{x\cdot\kappa} \otimes \xi_\kappa$ by definition. We compute

$$\begin{aligned} G_b &= (1,b,1)X_c \otimes \xi_\kappa \\ &= X_{b+c} \otimes \chi_\kappa(1,b,1)\xi_\kappa \\ &= \gamma(b)X_{b+c} \otimes \xi_\kappa \end{aligned}$$

Hence, the right-hand side is equal to

$$\begin{aligned} &\left(\sum_b \alpha(b+a)\gamma(b)X_{b+a} \otimes X_{b+c}\right) \otimes \xi_\kappa \\ &= \gamma(a)\left(\sum_b \alpha(b)\gamma(b)X_b \otimes X_{b+a+c}\right) \otimes \xi_\kappa \\ &= \gamma(a)\xi_{x\cdot\kappa} \otimes \xi_\kappa \end{aligned}$$

If $\varepsilon(x\cdot\kappa) = -1$, $M_x(F) = 0$ and the right-hand side $\gamma(a)\xi_{x\cdot\kappa}\otimes\xi_\kappa$ vanishes by projection to $S^2V \otimes \Lambda^2 V$, which proves relation (8).

Assume $N(F) = G_0 \in \ker[m]_0$. Since $m$ is $Heis(g)$-equivariant, $G_b \in \ker m$, $\forall b \in K(g)$. Hence, by relation (8), $M_x(F)$ vanishes on $\hat{J}$ for all nonzero $x \in J[2]$. By proposition 5.2 we deduce that $F$ vanishes on $P_x$.

Conversely, assume that $F$ vanishes on all Pryms. Then, by proposition 5.2, $M_x(F)$ vanishes on $\hat{J}$ for all nonzero $x \in J[2]$. Fix a nonzero $a \in K(g)$. In particular, by relation (8), $\forall \alpha \in \widehat{K(g)}$ $\sum_b \alpha(b+a)X_{b+a}\otimes G_b$ vanishes on $\hat{J}$. Taking suitable linear combinations of these polynomials, we deduce that $\forall b \in K(g)$, $X_{b+a} \otimes G_b$ vanishes on $\hat{J}$. Since $X_{b+a}$ does not vanish on $J$, we obtain that $\forall b \in K(g)$, $G_b \in \ker m$. □

## 5.2 Noninvariant quadrics

In this subsection we fix a nonzero $x \in J[2]$ and we shall prove a similar result concerning the vanishing on all Pryms of a polynomial $F \in S^2(\Lambda^2 V)_x$.

**Proposition 5.4** *Consider $y \in J[2]$ with $\langle x, y\rangle = 1$ and $x \neq y$. Then a quadric $F \in S^2(\Lambda^2 V)_x$ vanishes on $P_y$ if and only if $F$ vanishes on $P_{y+x}$.*

*Proof:* We can choose a theta structure such that $x + y = \begin{bmatrix} 0\,0\,0 \\ 0\,0\,1 \end{bmatrix}$ and $y = \begin{bmatrix} 0\,0\,0 \\ 0\,1\,0 \end{bmatrix}$. By lemma 5.1 a generic polynomial $F \in S^2(\Lambda^2 V)_x$ can be written $F = \sum a_\kappa \xi_\kappa \otimes \xi_{x\cdot\kappa}$ where we sum over $\kappa \in \vartheta^-(J)$ with $\varepsilon(x\cdot\kappa) = -1$. For any such $\kappa$, we have $1 = \kappa(x) = \kappa(x+y)\kappa(y)\langle x+y, y\rangle = \kappa(x)\kappa(y)$, which is equivalent to $\varepsilon(x+y\cdot\kappa) = \varepsilon(y\cdot\kappa)$.

If $\varepsilon(y\cdot\kappa) = -1$, the polynomial $\xi_\kappa \otimes \xi_{x\cdot\kappa}$ vanishes on $P_y$ (lemma 3.2) ; if $\varepsilon(y\cdot\kappa) = 1$, the polynomial $\xi_\kappa \otimes \xi_{x+y\cdot\kappa}$ restricts to the theta function

$$\theta_{\bar\kappa}(-2\tilde\alpha_y, \omega_y)\theta_{\bar\kappa}(2z, \omega_y)\theta_{\bar x\bar\kappa}(-2\tilde\alpha_y, \omega_y)\theta_{\bar x\bar\kappa}(2z, \omega_y)$$

By the addition formula, we have

$$\theta_{\bar\kappa}(2z,\omega_y)\theta_{\bar x\bar\kappa}(2z,\omega_y)=\epsilon(e)\sum_d \epsilon(d)\theta\begin{bmatrix} d\,1\\ 0\,1\end{bmatrix}(4z,2\omega_y)\theta\begin{bmatrix} d+e & 0\\ 0 & 1\end{bmatrix}(0,2\omega_y)$$

with $\bar\kappa=\bar\kappa\left[\begin{smallmatrix} e\,1\\ \epsilon\,0\end{smallmatrix}\right]$ and $\bar x\bar\kappa=\bar\kappa\left[\begin{smallmatrix} e\,1\\ \epsilon\,1\end{smallmatrix}\right]$. We have $\varepsilon(\bar\kappa)=\epsilon(e)$ and $\varepsilon(\bar\kappa)\cdot\varepsilon(\bar x\bar\kappa)=-1$. The family $\{\theta\left[\begin{smallmatrix} d\,1\\ 0\,1\end{smallmatrix}\right](4z,2\omega_x)\}$ is linearly independent. Hence $F$ vanishes on $P_y$ if and only if $\forall d\in K(g-2)$

$$\sum_\kappa \varepsilon(\bar\kappa)a_\kappa\epsilon(d)\theta_{\bar\kappa}(2\tilde\alpha_y,\omega_y)\theta_{\bar x\bar\kappa}(2\tilde\alpha_y,\omega_y)\theta\begin{bmatrix} d+e & 0\\ 0 & 1\end{bmatrix}(0,2\omega_y)=0 \tag{9}$$

where we sum over $\kappa\in\vartheta_y^-(J)\cap\vartheta_{x+y}^-(J)$. The vanishing condition for the Prym $P_{x+y}$ is obtained from equation (9) by replacing $\bar x$ by $\bar y$ and the subscript $y$ by $x+y$. The Donagi relations and proposition 4.5 (b') -written for the orthogonal pair (y,x+y)- implie that the two conditions are equivalent. □

From now on, we fix a theta structure for $\mathcal{O}(2\Theta)$ such that $x=\left[\begin{smallmatrix} 0\,0\\ 0\,1\end{smallmatrix}\right]$ and $x'=\left[\begin{smallmatrix} 0\,1\\ 0\,0\end{smallmatrix}\right]$. We define a linear map $M_y^{(x)}$ for any nonzero $y=\left[\begin{smallmatrix} a\,0\\ \alpha\,0\end{smallmatrix}\right]$

$$\begin{array}{rcll} M_y^{(x)}: S^2(\Lambda^2 V)_x & \longrightarrow & (W_x^+\otimes W_x^-)_{x+y} & \\ \xi_\kappa\otimes\xi_{x\cdot\kappa} & \longmapsto & \delta(a)\xi_{(x+y)\cdot\kappa'}\otimes\xi_{\kappa'} & \text{if } \kappa\in\vartheta_y^-(J)\\ & & 0 & \text{otherwise}\end{array}$$

here $W^\pm$ is the subspace of $W_x$ isomorphic (via $h_x^*$) to $H^0_\pm(P_x,\mathcal{O}(4\Xi_x))$. A basis of $S^2(\Lambda^2V)_x$ is given by the tensors $\xi_\kappa\otimes\xi_{x\cdot\kappa}$ for $\kappa=\kappa\left[\begin{smallmatrix} d\,0\\ \delta\,0\end{smallmatrix}\right]$ such that $\delta(d)=-1$. We note $\kappa'=x'\cdot\kappa$. The linear map $M_y^{(x)}$ also admits an intrinsic definition (up to homothety) -see remark in subsection 5.1.

**Proposition 5.5** *A quadric $F\in S^2(\Lambda^2V)_x$ vanishes on $P_y$ if and only if $M_y^{(x)}(F)$ vanishes on $P_x$*

*Proof:* Choose a theta structure on $J$ such that conditions (4) hold. The theta characteristics $\kappa=\kappa\left[\begin{smallmatrix} d\,0\\ \delta\,0\end{smallmatrix}\right]\in\vartheta_y^-(J)$ are of the form

$$\kappa=\kappa\begin{bmatrix} e\,1\,0\\ \epsilon\,0\,0\end{bmatrix}\quad \epsilon(e)=-1 \quad\text{or}\quad \kappa=\kappa\begin{bmatrix} e\,1\,0\\ \epsilon\,1\,0\end{bmatrix}\quad \epsilon(e)=1 \tag{10}$$

They restrict to the theta characteristics $\bar\kappa=\bar\kappa\left[\begin{smallmatrix} e\,0\\ \epsilon\,0\end{smallmatrix}\right]$ on $P_y$. A generic polynomial $F=\sum a_\kappa\xi_\kappa\otimes\xi_{x\cdot\kappa}\in S^2(\Lambda^2V)_x$ restricts to the theta function on $P_y$ (lemma 3.4)

$$\sum_\kappa a_\kappa\theta_{\bar\kappa}(-2\tilde\alpha_y,\omega_y)\theta_{\bar\kappa}(2z,\omega_y)\theta_{\bar x\bar\kappa}(-2\tilde\alpha_y,\omega_y)\theta_{\bar x\bar\kappa}(2z,\omega_y)$$
$$=\sum_\kappa\sum_d a_\kappa\epsilon(d)\theta_{\bar\kappa}(2\tilde\alpha_y,\omega_y)\theta_{\bar x\bar\kappa}(2\tilde\alpha_y,\omega_y)\theta\begin{bmatrix} d\,0\\ 0\,1\end{bmatrix}(4z,2\omega_y)\theta\begin{bmatrix} d+e & 0\\ 0 & 1\end{bmatrix}(0,2\omega_y) \tag{11}$$

where we sum over $\kappa$ of the form (10) and $d \in K(g-2)$. The last equation is obtained using the addition formula (3). The polynomial $M_y^{(x)}(F) = \sum a_\kappa \xi_{(x+y)\cdot\kappa'} \otimes \xi_{\kappa'}$ restricts to the theta function on $P_x$

$$\sum_\kappa a_\kappa \theta_{\bar{\kappa}'}(-2\tilde{\alpha}_x, \omega_x)\theta_{\bar{\kappa}'}(2z, \omega_x)\theta_{\overline{y\kappa}'}(-2\tilde{\alpha}_x, \omega_x)\theta_{\overline{y\kappa}'}(2z, \omega_x)$$

$$= \sum_\kappa \sum_d a_\kappa \epsilon(d)\theta_{\bar{\kappa}'}(2\tilde{\alpha}_x, \omega_x)\theta_{\overline{y\kappa}'}(2\tilde{\alpha}_x, \omega_x)\theta\begin{bmatrix} d\,1 \\ 0\,1 \end{bmatrix}(4z, 2\omega_x)\theta\begin{bmatrix} d+e\ 0 \\ 0 \quad 1 \end{bmatrix}(0, 2\omega_x) \qquad (12)$$

with the same summation as in (11). We apply the addition formula again: $\forall(e,\epsilon) \in K(g-2) \times \widehat{K(g-2)}$

$$\begin{aligned} \theta_{\bar{\kappa}}(2\tilde{\alpha}_y, \omega_y)\theta_{\bar{x}\bar{\kappa}}(2\tilde{\alpha}_y, \omega_y) &= \sum_d \epsilon(d) \underbrace{\theta\begin{bmatrix} d\,0 \\ 0\,1 \end{bmatrix}(4\tilde{\alpha}_y, 2\omega_y)}_{\text{prop.4.5(b) and prop.4.6}} \underbrace{\theta\begin{bmatrix} d+e\ 0 \\ 0 \quad 1 \end{bmatrix}(0, 2\omega_y)}_{\text{prop.4.4(a)}} \\ &= (cste) \sum_d \epsilon(d) \overbrace{\theta\begin{bmatrix} d\,1 \\ 0\,1 \end{bmatrix}(4\tilde{\alpha}_x, 2\omega_x)} \overbrace{\theta\begin{bmatrix} d+e\ 0 \\ 0 \quad 1 \end{bmatrix}(0, 2\omega_x)} \\ &= (cste) \ \theta_{\bar{\kappa}'}(2\tilde{\alpha}_x, \omega_x)\theta_{\overline{y\kappa}'}(2\tilde{\alpha}_x, \omega_x) \end{aligned}$$

Thus the two vanishing conditions (11) and (12) are proportional, hence equivalent. □

We introduce the linear isomorphism

$$\begin{aligned} N^{(x)} : S^2(\Lambda^2 V)_x &\longrightarrow [V_x^- \otimes W_x^-]_0 \\ \xi_\kappa \otimes \xi_{x\cdot\kappa} &\longmapsto X_{(d\,1)} \otimes \xi_{\kappa'} \end{aligned}$$

As above, $[V_x^- \otimes W_x^-]_0$ denotes the invariant subspace of $V_x^- \otimes W_x^-$ under the level subgroup $\widehat{K(g-1)}$ of $Heis(g-1)$ acting on $V_x^- \otimes W_x^-$. Doing the same reasoning as in subsection 5.1, we get a surjective map

$$[m^{(x)}]_0 : [V_x^- \otimes W_x^-]_0 \longrightarrow [H_-^0(P_x, 6\Xi_x)]_0$$

The following theorem is the analogue of theorem 5.3

**Theorem 5.6** *A quadric $F \in S^2(\Lambda^2 V)_x$ vanishes on all Pryms if and only if $N^{(x)}(F) \in \ker[m^{(x)}]_0$.*

*Proof:* First, by prop. 5.4, it is sufficient to look at the vanishing of $F$ on all the Pryms $P_y$ with $y = \begin{bmatrix} a\,0 \\ \alpha\,0 \end{bmatrix}$. The rest follows, as in the proof of theorem 5.3, from relation (8) which we can write

$$M_y^{(x)}(F) = \sum_{b \in K(g)} \hat{\alpha}(b+\hat{a}) X_{b+\hat{a}} \otimes G_b$$

where $G_b = (1, b, 0)N^{(x)}(F) \in V \otimes W_x$ and $\hat{\alpha} = (\alpha\ 1)$, $\hat{a} = (a\ 0)$. □

# 6 Proof of the main theorem

Now we are able to prove theorem 1.1: consider a curve $C$ verifying condition (*). The following diagram is commutative

$$\begin{array}{ccc} \bigoplus_{x\in J[2]} S^2(\Lambda^2 V)_x & \xrightarrow{\oplus(m_2)_x} & H^0(\mathcal{M}_p, \mathcal{L}_p^2) \\ & \searrow^{\oplus n_x} & \downarrow \text{res} \\ & & \bigoplus_{\substack{x\in J[2] \\ x\neq 0}} H^0(P_x, 8\Xi_x) \end{array}$$

The vertical arrow is the restriction map to all the Pryms lying in $\mathcal{M}_p$. We identify $H^0(\mathcal{M}_p, \mathcal{L}_p)$ with $\Lambda^2 V$ via $\varphi_p^*$. The horizontal arrow is the multiplication map $m_2 = \oplus(m_2)_x$, which is decomposed under the action of $Heis(g)$. We deduce from theorem 5.3 and 5.6 that

$$\begin{aligned} \text{dim ker } n_0 &= 2^{g-1}(2^g-1) - \frac{3^g-1}{2} \\ \text{dim ker } n_x &= 2^{g-2}(2^{g-1}-1) - \frac{3^{g-1}-1}{2} \quad \text{for } x \neq 0 \end{aligned}$$

Commutativity of the diagram above implies that dim ker $(m_2)_x \leq$ dim ker $n_x$. Hence:

$$\begin{aligned} \dim H^0(\mathcal{M}_p, \mathcal{L}_p^2) &\geq \sum_{x\in J[2]} \dim S^2(\Lambda^2 V)_x - \text{dim ker } (m_2)_x \\ &\geq \sum_{x\in J[2]} \dim S^2(\Lambda^2 V)_x - \text{dim ker } n_x \\ &= \frac{3^g-1}{2} + (2^{2g}-1)\frac{3^{g-1}-1}{2} \end{aligned}$$

Now the Verlinde formula (see e.g. [vG-P2]) tells us that we have equality, hence equality at each level and in particular $m_2$ is surjective. □

**Corollary 6.1** *A quadric on $\mathbb{P}\Lambda^2 V$ vanishes on $\mathcal{M}_p$ if and only if it vanishes on all Pryms.*

This corollary is the analogue of theorem 4.2(ii) [vG-P2]:

*A quartic on $\mathbb{P}V$ vanishes on $\mathcal{M}_0$ if and only if it vanishes on the Kummers of the Jacobian and of all the Pryms.*

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