The Frobenius map, rank 2 vector bundles and Kummer's quartic surface in characteristic 2 and 3

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December 19, 2007

Abstract

Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$. Let $\mathrm{M}_X(r)$ be the moduli space of semi-stable rank r vector bundles with fixed trivial determinant. The relative Frobenius map $F : X \rightarrow X_1$ induces by pull-back a rational map $V : \mathrm{M}_{X_1}(r) \rightarrow \mathrm{M}_X(r)$. We determine the equations of V in the following two cases (1) $(g, r, p) = (2, 2, 2)$ and X nonordinary with Hasse-Witt invariant equal to 1 (see math.AG/0005044 for the case X ordinary), and (2) $(g, r, p) = (2, 2, 3)$. We also show the existence of base points of V, i.e., semi-stable bundles E such that F^*E is not semi-stable, for any triple (g, r, p).

2000 Mathematics Subject Classification. Primary 14H60, 14D20, Secondary 14H40

1 Introduction

Let X be a smooth projective curve of genus 2 defined over an algebraically closed field k of characteristic $p > 0$. The moduli space M_X of semi-stable rank 2 vector bundles with fixed trivial determinant is isomorphic to the linear system $|2\Theta| \cong \mathbb{P}^3$ over $\mathrm{Pic}^1(X)$ and the k-linear relative Frobenius map $F : X \rightarrow X_1$ induces by pull-back a rational map (the Verschiebung)

$$\begin{array}{ccc} \mathrm{M}_{X_1} & \xrightarrow{V} & \mathrm{M}_X \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D} \\ |2\Theta_1| & \xrightarrow{\tilde{V}} & |2\Theta| \end{array} \tag{1.1}$$

The vertical maps D are isomorphisms and the Verschiebung $V : E \mapsto F^*E$ coincides via D with a rational map $\tilde{V}$ given by polynomial equations of degree p (Proposition 7.2). The Kummer surfaces Kum_X and Kum_{X_1} are canonically contained in the linear systems $|2\Theta|$ and $|2\Theta_1|$ and coincide with the semi-stable boundary of the moduli spaces M_X and M_{X_1}. Moreover $\tilde{V}$ maps Kum_{X_1} onto Kum_X.

Our interest in diagram (1.1) comes from questions related to the action of the Frobenius map on vector bundles like e.g. surjectivity of V, density of Frobenius-stable bundles, loci of Frobenius-destabilized bundles (see [LP]). These questions are largely open when the rank of the bundles, the genus of the curve or the characteristic of the field are arbitrary. In [LP] we made use of the exceptional isomorphism $D : \mathrm{M}_X \rightarrow |2\Theta|$ in the genus 2, rank 2 case and determined the equations of $\tilde{V}$ when X is an ordinary curve and $p = 2$, which allowed us to answer the above mentioned questions. In this paper we obtain the equations of $\tilde{V}$ in two more cases:

(1) $p = 2$ and X nonordinary with Hasse-Witt invariant equal to 1,

(2) $p = 3$ and any X.

In case (1) we consider a family $\mathcal{X}$ of genus 2 curves parametrized by a discrete valuation ring with ordinary generic fibre $\mathcal{X}_\eta$ and special fibre isomorphic to X. We obtain the equations of $\tilde{V}$ for X (Theorem 5.1) by specializing the quadrics P_{ij} defining the Verschiebung $V_\eta : \mathrm{M}_{\mathcal{X}_{1\eta}} \longrightarrow \mathrm{M}_{\mathcal{X}_\eta}$ associated to $\mathcal{X}_\eta$ (section 5). In order to determine the limit of the P_{ij}'s we use the explicit formulae (Proposition 3.1) of the coefficients of the P_{ij}'s, which coincide with the coefficients of Kummer's quartic surface $\mathrm{Kum}_{\mathcal{X}_\eta}$, in terms of the coefficients of an affine equation of the ordinary curve $\mathcal{X}_\eta$. As in the ordinary case we easily deduce from the equations of $\tilde{V}$ a full description of the Verschiebung V (Proposition 5.4).

In case (2) we show that the cubic equations of $\tilde{V}$ are given by the polar equations of a Kummer surface $S \subset |2\Theta_1|$ (Theorem 6.1). Moreover S is isomorphic to Kum_X and the 16 nodes of S correspond to the 16 base points of $\tilde{V}$. We deduce that V is surjective and of degree 11 (Corollary 6.4).

In the appendix we show that over *any* smooth curve X of genus $g \geq 2$ defined over an algebraically closed field of characteristic $p > 0$ and for any integer $r \geq 2$, there exist Frobenius-destabilized bundles of rank r, i.e., semi-stable bundles E such that F^*E is not semi-stable (Theorem 7.4). This improves Gieseker's theorem [G], which asserts existence of Frobenius-destabilized bundles over a general curve.

We thank M. Raynaud for helpful discussions.

2 Preliminaries on genus 2 curves in characteristic 2

We consider a smooth *ordinary* curve X of genus 2 defined over a field K of characteristic 2. We denote by $\overline{K}$ the algebraic closure of K, by $k(X)$ the function field of X, by JX the Jacobian of X and by ω_X the canonical line bundle of X. In this section we recall some results on equations and moduli of the curve X.

2.1 Weierstrass points

After taking a finite extension of K and applying an automorphism of $\mathbb{P}^1_K$ we can assume that the three Weierstrass points of X are $0, 1$ and ∞. We consider the birational Abel-Jacobi map

$$AJ : \ \mathrm{Sym}^2(X) \longrightarrow JX, \qquad P_1 + P_2 \longmapsto \mathcal{O}_X(P_1 + P_2) \otimes \omega_X^{-1}. \tag{2.1}$$

We observe that the three nonzero elements $[0], [1], [\infty]$ of the group scheme of 2-torsion points $JX[2]$ are K-rational and are given by

$$[0] := AJ(1 + \infty), \qquad [1] := AJ(0 + \infty), \qquad [\infty] := AJ(0 + 1). \tag{2.2}$$

For later use we mention that the sheaf of locally exact differential forms (see [R] section 4.1) equals

$$B = \mathcal{O}_X(0 + 1 + \infty) \otimes \omega_X^{-1}.$$

We recall that B is a theta-characteristic of X, i.e., $B^{\otimes 2} \cong \omega_X$.

2.2 Level 2 structures

A level 2 structure is an isomorphism $\psi : JX[2] \xrightarrow{\sim} \mathbb{F}_2^2$. Note that two level 2 structures differ by an automorphism of $\mathbb{F}_2^2$, i.e., an element of $\mathrm{GL}(2, \mathbb{F}_2) = \mathfrak{S}_3$, where $\mathfrak{S}_3$ is the symmetric group. It is well-known that a level 2 structure ψ is equivalent to an ordering of the Weierstrass points of X. We refer e.g. to [DO] page 141 for the characteristic zero case, which can easily be adapted to the characteristic two case. Because of the choices involved in the degeneration of X to a nonordinary curve (section 4.3), we consider the ordering $1, \infty, 0$ of the Weierstrass points. With the notation of (2.2) the corresponding level 2 structure ψ is given by

$$\psi([0]) = (1, 0), \qquad \psi([1]) = (0, 1), \qquad \psi([\infty]) = (1, 1). \tag{2.3}$$

A level 2 structure ψ allows us to construct ([LP] section 2) the theta basis $\{X_g\}_{g \in \mathbb{F}_2^2}$ of the space $H^0(JX, 2\Theta)$. We denote the four sections by X_B, X_0, X_1, X_∞ and introduce the rational functions $Z_\bullet \in k(JX)$ defined by

$$Z_0 = \frac{X_0}{X_B}, \qquad Z_1 = \frac{X_1}{X_B}, \qquad Z_\infty = \frac{X_\infty}{X_B}. \tag{2.4}$$

We recall that X_B is the theta function whose associated nonreduced zero divisor equals $2\Theta_B$, with

$$\mathrm{supp}(\Theta_B) := \{L \in JX \mid h^0(X, L \otimes B) \geq 1\}.$$

2.3 Birational models

Let X be an ordinary smooth curve of genus 2 defined over $\overline{K}$ and ψ a level 2 structure. It follows from [L] page 28 and [B] Proposition 1.4 that the pair (X, ψ) is *uniquely* represented by an affine equation of the form

$$y^2 + x(x+1)y = x(x+1)(ax^3 + (a+b)x^2 + cx + c), \tag{2.5}$$

with $a, b, c \in \overline{K}$. Moreover if X is defined over K, the coefficients a, b, c lie in a finite extension of K. The next lemma is an immediate consequence of [B] Proposition 1.5.

2.1 Lemma. *The curve X defined by the equation* (2.5) *is smooth if and only if $abc \neq 0$.*

Let $\widetilde{\mathcal{M}}_3$ denote the moduli space parametrizing pairs (X, ψ) of smooth ordinary genus 2 curves X defined over $\overline{K}$ equipped with a level 2 structure. It follows from the previous remark that $\widetilde{\mathcal{M}}_3$ is an affine variety,

$$\widetilde{\mathcal{M}}_3 = \mathrm{Spec}\ \overline{K}[a, b, c, \frac{1}{abc}].$$

Fixing the curve X, the symmetric group $\mathfrak{S}_3$ acts naturally on the level 2 structures ψ. It can be shown that this $\mathfrak{S}_3$-action on $\widetilde{\mathcal{M}}_3$ coincides with the permutation action of $\mathfrak{S}_3$ on the coefficients a, b, c.

2.4 Normal form

We introduce the rational function $Y \in k(X)$ defined by $Y = \frac{y}{x(x+1)}$. Then (2.5) becomes

$$Y^2 + Y = R(x), \qquad \text{with } R(x) = \frac{ax^3 + (a+b)x^2 + cx + c}{x(x+1)}. \tag{2.6}$$

We also observe that, given a polynomial $S \in \overline{K}[x]$, the involution $i_S : \mathbb{A}^2_{\overline{K}} \to \mathbb{A}^2_{\overline{K}}$, $(x, y) \mapsto (x, y + x(x+1)S)$ transforms the equation (2.6) of X into $Y^2 + Y = R + S^2 + S$.

2.5 Kummer's quartic equation

For a pair (X, ψ) it has been shown in [LP] Proposition 4.1 that there exist constants $\lambda_0, \lambda_1, \lambda_\infty \in \overline{K}$ such that the following equality holds in $k(JX)$,

$$\lambda_0^2(Z_0^2 + Z_1^2 Z_\infty^2) + \lambda_1^2(Z_1^2 + Z_0^2 Z_\infty^2) + \lambda_\infty^2(Z_\infty^2 + Z_0^2 Z_1^2) + \lambda_0\lambda_1\lambda_\infty Z_0 Z_1 Z_\infty = 0. \tag{2.7}$$

The constants $\lambda_0, \lambda_1, \lambda_\infty$ are related via ψ (2.3) to the $\{\lambda_g\}_{g\in\mathbb{F}_2^2}$ used in [LP] Proposition 4.1 as follows: $\lambda_0 = \frac{\lambda_{10}}{\lambda_{00}}$, $\lambda_1 = \frac{\lambda_{01}}{\lambda_{00}}$, $\lambda_\infty = \frac{\lambda_{11}}{\lambda_{00}}$.

3 The coefficients $\lambda_\bullet$ of Kummer's quartic surface Kum_X for an ordinary curve X

In this section we determine the coefficients $\lambda_\bullet$ of the Kummer surface Kum_X for an ordinary curve X. This result will be used in the proof of Theorem 5.1 (section 5).

3.1 Proposition. *Given a curve X with a level 2 structure ψ represented by an affine equation (2.5) with $a, b, c \in K$. Then the coefficients of the equation (2.7) of its Kummer surface Kum_X are*

$$\lambda_0^2 = \frac{1}{ab}, \qquad \lambda_1^2 = \frac{1}{ac}, \qquad \lambda_\infty^2 = \frac{1}{bc}.$$

Let $\{x_g\}$ be the dual basis of the theta basis $\{X_g\}_{g\in\mathbb{F}_2^2}$ of $|2\Theta| = \mathbb{P}^3$. Then the homogeneous equation of Kum_X is

$$c(x_{00}^2 x_{10}^2 + x_{01}^2 x_{11}^2) + b(x_{00}^2 x_{01}^2 + x_{10}^2 x_{11}^2) + a(x_{00}^2 x_{11}^2 + x_{01}^2 x_{10}^2) + x_{00}x_{01}x_{10}x_{11} = 0.$$

The idea of the proof is to consider the pull-back of the rational functions $Z_\bullet$ (2.4) by the Abel-Jacobi map (2.1) to the symmetric product $\mathrm{Sym}^2(X)$ and to do the computations in the function field $k(S^2X) \hookrightarrow k(X \times X)$. Since X is given by the equation (2.5), we have natural coordinates x_1, y_1 and x_2, y_2 on $X \times X$. For notational convenience we introduce $Y_i = \frac{y_i}{x_i(x_i+1)}$, for $i = 1, 2$.
We will use the following two lemmas.

3.2 Lemma. *Suppose that there exist polynomials $A, B \in K[x_1, x_2]$ which satisfy*

$$(Y_1 + Y_2)A(x_1, x_2) = B(x_1, x_2). \tag{3.1}$$

Then $A = B = 0$.

Proof. Squaring relation (3.1) and using (2.6) leads to the equation

$$(Y_1 + Y_2)A^2 + (R(x_1) + R(x_2))A^2 + B^2 = 0.$$

Applying again (3.1), the first term transforms into AB. Clearing denominators, we arrive at a polynomial equation, which only holds if $A = B = 0$ (e.g. by taking the total degree of A and B). $\square$

3.3 Lemma. *The pull-back by the Abel-Jacobi map AJ of the rational function $Z_\infty \in k(JX)$ equals*

$$AJ^*(Z_\infty) = \alpha_\infty \frac{P(x_1, x_2)}{(Y_1 + Y_2)^2}, \qquad \textit{with } P(x_1, x_2) = \frac{(x_1 + x_2)^2}{x_1 x_2 (x_1 + 1)(x_2 + 1)}$$

Similarly we have

$$AJ^*(Z_0) = \alpha_0 \frac{P(x_1,x_2)}{(Y_1+Y_2)^2} x_1 x_2, \qquad AJ^*(Z_1) = \alpha_1 \frac{P(x_1,x_2)}{(Y_1+Y_2)^2}(x_1+1)(x_2+1),$$

for some nonzero constants $\alpha_0, \alpha_1, \alpha_\infty \in K$.

Proof. The first equality follows immediately from Theorem 2 [AG] and the other two from Proposition 5 [AG]. ☐

Proof of Proposition 3.1. We write $Q = \frac{P(x_1,x_2)}{(Y_1+Y_2)^2}$. Using Lemma 3.3, the pull-back to $\mathrm{Sym}^2(X)$ of equation (2.7) equals

$$\lambda_0^2\left[\alpha_0^2 x_1^2 x_2^2 Q^2 + \alpha_1^2\alpha_\infty^2 (x_1+1)^2(x_2+1)^2 Q^4\right] + \lambda_1^2\left[\alpha_1^2(x_1+1)^2(x_2+1)^2Q^2 + \alpha_0^2\alpha_\infty^2 x_1^2x_2^2Q^4\right] + \lambda_\infty^2\left[\alpha_\infty^2 Q^2 + \alpha_0^2\alpha_1^2x_1^2x_2^2(x_1+1)^2(x_2+1)^2Q^4\right] + \lambda_0\lambda_1\lambda_\infty\left[\alpha_0\alpha_1\alpha_\infty x_1x_2(x_1+1)(x_2+1)Q^3\right] = 0.$$

Dividing by Q^2 and multiplying by $(Y_1+Y_2)^4$, we obtain

$$(Y_1+Y_2)^4\left[\lambda_0^2\alpha_0^2x_1^2x_2^2 + \lambda_1^2\alpha_1^2(x_1+1)^2(x_2+1)^2 + \lambda_\infty^2\alpha_\infty^2\right] + (Y_1+Y_2)^2\left[\lambda_0\lambda_1\lambda_\infty\alpha_0\alpha_1\alpha_\infty P x_1x_2(x_1+1)(x_2+1)\right] + S = 0, \tag{3.2}$$

where S is the sum of the remaining terms (not containing Y_1, Y_2). After taking the square root (note that the entire expression is a square of a polynomial in the x_i's and Y_i's), applying (2.6) and Lemma 3.2, we obtain that the coefficients of $(Y_1+Y_2)^2$ and $(Y_1+Y_2)^4$ are the same, i.e.,

$$\lambda_0^2\alpha_0^2x_1^2x_2^2 + \lambda_1^2\alpha_1^2(x_1+1)^2(x_2+1)^2 + \lambda_\infty^2\alpha_\infty^2 = \lambda_0\lambda_1\lambda_\infty\alpha_0\alpha_1\alpha_\infty P x_1x_2(x_1+1)(x_2+1).$$

An easy computation shows that this equality holds only if

$$\lambda_0\alpha_0 = \lambda_1\alpha_1 = \lambda_\infty\alpha_\infty = 1.$$

Now we replace the α's and the sum S by their expressions in the square root of the equation (3.2)

$$(Y_1+Y_2)^2(x_1+x_2) + (Y_1+Y_2)(x_1+x_2) + P\left[\frac{\lambda_0}{\lambda_1\lambda_\infty}(x_1+1)(x_2+1) + \frac{\lambda_1}{\lambda_0\lambda_\infty}x_1x_2 + \frac{\lambda_\infty}{\lambda_0\lambda_1}x_1x_2(x_1+1)(x_2+1)\right] = 0.$$

We introduce the constants $\mu_0, \mu_1, \mu_\infty \in K$ defined by $\mu_\infty = \frac{\lambda_\infty}{\lambda_0\lambda_1}, \mu_0 = \frac{\lambda_0}{\lambda_1\lambda_\infty}, \mu_1 = \frac{\lambda_1}{\lambda_0\lambda_\infty}$. The previous equality becomes after replacing P by its expression and dividing by (x_1+x_2)

$$\left[Y_1^2 + Y_1 + \frac{\mu_0}{x_1} + \frac{\mu_1}{x_1+1} + \mu_\infty x_1\right] + \left[Y_2^2 + Y_2 + \frac{\mu_0}{x_2} + \frac{\mu_1}{x_2+1} + \mu_\infty x_2\right] = 0.$$

This equation holds in $k(X \times X)$ and since the variables (x_1, Y_1) and (x_2, Y_2) are separated, each of the two terms equals zero. So we can drop the indices and we obtain an equation

$$Y^2 + Y = \mu_\infty x + \frac{\mu_0}{x} + \frac{\mu_1}{x+1} = \frac{\mu_\infty x^3 + \mu_\infty x^2 + (\mu_0+\mu_1)x + \mu_0}{x(x+1)}, \tag{3.3}$$

which has to be equivalent (after applying an automorphism of $\mathbb{A}^2_{\overline{K}}$) to the normal form (2.6) of the equation of X. The automorphism is given by i_S (see section 2.4) with $S(x) = s \in \overline{K}$ satisfying $s^2 + s = \mu_1$. Hence (3.3) is equivalent to $Y^2 + Y = R(x)$ with $R(x) = \frac{\mu_\infty x^3 + (\mu_\infty+\mu_1)x^2 + \mu_0 x + \mu_0}{x(x+1)}$. Hence by uniqueness of the normal form, we obtain $a = \mu_\infty, b = \mu_1, c = \mu_0$ and therefore also the relations claimed in the proposition. ☐

4 Degeneration of an ordinary genus 2 curve

The aim of this section is to describe (see formulae (4.10)) the degeneration of the theta basis of $H^0(JC, 2\Theta)$ when the ordinary curve C degenerates to a nonordinary one. This result will be central in the proof of Theorem 5.1(section 5).

For that purpose we denote by X/k a smooth curve with Hasse-Witt invariant equal to 1 and we introduce a family $\mathcal{X}$ over $R = k[[t]]$ such that the special fibre $\mathcal{X}_0$ is isomorphic to X and the generic fibre $\mathcal{X}_\eta$ is an ordinary genus 2 curve. Here η (resp. 0) is the generic (resp. closed) point of Spec(R). Let $\mathcal{J}\mathcal{X}$ be its associated Jacobian scheme over Spec(R).

4.1 2-divisible groups

Let $\mathcal{J}\mathcal{X}(2)$ be the 2-divisible group (see e.g. [D]) of $\mathcal{J}\mathcal{X}$, which is finite and flat over Spec(R). We consider the canonical exact sequence

$$0 \longrightarrow \mathcal{J}\mathcal{X}(2)^0 \longrightarrow \mathcal{J}\mathcal{X}(2) \longrightarrow \mathcal{J}\mathcal{X}(2)^{et} \longrightarrow 0, \tag{4.1}$$

where $\mathcal{J}\mathcal{X}(2)^0$ (resp. $\mathcal{J}\mathcal{X}(2)^{et}$) is a connected (resp. étale) 2-divisible group. Taking again the connected component of the Cartier dual $\left(\mathcal{J}\mathcal{X}(2)^0\right)^D$ of $\mathcal{J}\mathcal{X}(2)^0$, we obtain the two inclusions

$$\mathcal{J}\mathcal{X}(2)^{00} \subset \left(\mathcal{J}\mathcal{X}(2)^0\right)^D, \qquad \mathcal{J}\mathcal{X}(2)^0 \subset \mathcal{J}\mathcal{X}(2),$$

with quotients given by the 2-divisible groups

$$\mathcal{J}\mathcal{X}(2)/\mathcal{J}\mathcal{X}(2)^0 = \mathcal{J}\mathcal{X}(2)^{et} \cong \mathbb{Q}_2/\mathbb{Z}_2, \qquad \left(\mathcal{J}\mathcal{X}(2)^0\right)^D/\mathcal{J}\mathcal{X}(2)^{00} \cong \mathbb{G}_m(2).$$

The 2-divisible group $\mathcal{J}\mathcal{X}(2)^{00}$ is self-dual, of dimension 1 and of height 2. Because of the uniqueness of 2-divisible groups over k with these properties (see e.g. [D] Examples page 93), the special fibre $\mathcal{J}\mathcal{X}(2)^{00}_0$ $(= \mathcal{J}\mathcal{X}(2)^{00} \otimes_R k)$ is isomorphic to the 2-divisible group associated to the supersingular elliptic curve E^{ss}/k. We recall that there exists a unique (up to isomorphism) supersingular curve E^{ss}, which is defined by $j = 0$. Therefore by a theorem of Serre-Tate [K], there exists an elliptic curve $\mathcal{E}_\mathcal{X}$ over Spec(R) such that $(\mathcal{E}_\mathcal{X})_0 \cong E^{ss}$ and the associated 2-divisible group $\mathcal{E}_\mathcal{X}(2)$ is isomorphic to $\mathcal{J}\mathcal{X}(2)^{00}$ over Spec(R).

4.2 Degeneration of elliptic curves

In this section we compute the linear action of the 2-torsion point of $\mathcal{E}$ on the space of second order theta functions $H^0(\mathcal{E}, 2\Theta)$ for a family of elliptic curves $\mathcal{E}/\text{Spec}(R)$ with supersingular special fibre $\mathcal{E}_0 \cong E^{ss}$ and ordinary generic fibre $\mathcal{E}_\eta$.

4.2.1 Addition on an ordinary elliptic curve

Let E be an ordinary elliptic curve defined over a field K of characteristic 2 by the homogeneous equation

$$Y^2Z + a_1XYZ = X^3 + a_2X^2Z + a_4XZ^2 \tag{4.2}$$

with $a_1, a_2, a_4 \in K$ and $a_1 \neq 0$. We take as origin the inflection point ∞ with projective coordinates $(0 : 1 : 0)$. The projection with center ∞ gives a $2:1$ morphism $E \stackrel{\pi}{\longrightarrow} \mathbb{P}^1_K$, with X, Z projective coordinates on $\mathbb{P}^1_K$. The Abel-Jacobi map $E \to JE$, $e \mapsto \mathcal{O}_E(e - \infty)$ identifies E with JE. Under this identification the theta divisor Θ_B, associated to the canonical theta-characteristic

$B = \mathcal{O}_E(P - \infty) \in JE[2]$, becomes the 2-torsion point P with projective coordinates $(0:0:1)$. Moreover, using this identification, we have $\mathcal{O}_{JE}(2\Theta) = \mathcal{O}_E(2P) = \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$.

The point $B \in JE[2]$ induces a linear involution, denoted by g, on the space

$$W = H^0(JE, \mathcal{O}(2\Theta)) = H^0(E, \pi^*\mathcal{O}_{\mathbb{P}^1}(1)), \tag{4.3}$$

such that for all nonzero $s \in W$ we have $T_B \mathrm{div}(s) = \mathrm{div}(g.s)$. Here T_B denotes translation in JE by the point B. The space W has two distinguished bases: first the coordinate functions $\{X, Z\}$ and secondly the theta basis $\{X_0, X_1\}$ (see [LP] section 2). Since the canonical section $X_0 \in W$ (associated to the divisor Θ_B) is proportional to X, there exists a nonzero $a \in K$ such that

$$g.X = aZ, \qquad g.Z = a^{-1}X.$$

In order to determine a in terms of the coefficients $a_i \in K$, we choose one of the two points on E with projective coordinates of the form $(1 : Y : 1)$ and call it A. By construction we have $A \in \mathrm{div}(X + Z)$ and, after applying T_B, we obtain $A + P \in T_B\mathrm{div}(X + Z)$. Since

$$T_B \mathrm{div}(X + Z) = \mathrm{div}(g.X + g.Z) = \mathrm{div}(aZ + a^{-1}X),$$

we deduce that

$$\left(\frac{X}{Z}\right)(A + P) = a^2.$$

Now the addition formula for elliptic curves (see e.g. [S] page 59) implies that $\left(\frac{X}{Z}\right)(A + P) = a_4$. Hence $a = \sqrt{a_4}$ and the theta basis of W is given by

$$X_0 = X, \qquad X_1 = g.X_0 = \sqrt{a_4}Z. \tag{4.4}$$

4.2.2 An example

We consider the family of elliptic curves $\mathcal{E}$ over $\mathrm{Spec}(R)$ defined by the homogeneous equation

$$V^2Z + t^4UVZ + VZ^2 = U^3 + UZ^2 \tag{4.5}$$

with origin $\infty = (0:1:0)$. The generic fiber $\mathcal{E}_\eta$ is an ordinary elliptic curve over $\mathrm{Spec}(K)$, with $K = k((t))$, and the special fibre is supersingular,i.e., $\mathcal{E}_0 \cong E^{ss}$. The 2-torsion point P_η of $\mathcal{E}_\eta$ has projective coordinates

$$P_\eta = t^6(u_0 : v_0 : 1),$$

with $u_0 = t^{-4}$, $v_0 = t^{-2} + t^{-6}$, which specializes to $\infty_0 \in \mathcal{E}_0$.

The R-module $H^0(\mathcal{E}, 2\Theta)$ is free, of rank 2, and an R-basis is given by $\{U, Z\}$. In order to compute the linear action g of the 2-torsion point $P \in \mathcal{E}$ on $H^0(\mathcal{E}, 2\Theta)$, we consider the generic fibre $\mathcal{E}_\eta$. The change of variables $X = U + u_0Z$ and $Y = V + v_0Z$ transforms equation (4.5) into (4.2), with $a_1 = t^4$, $a_2 = u_0$, and $a_4 = 1 + u_0^2 + t^4v_0 = 1 + t^{-8} + t^2 + t^{-2}$. With the notation of section 4.2.1 we find $a = t^{-4}(1 + t^3 + t^4 + t^5) = \sqrt{a_4}$. Therefore the action of g on $H^0(\mathcal{E}, 2\Theta)$ is given by the formulae

$$g.U = \frac{1}{1 + t^3 + t^4 + t^5}U + \frac{t^2 + t^4 + t^6}{1 + t^3 + t^4 + t^5}Z,$$
$$g.Z = \frac{t^4}{1 + t^3 + t^4 + t^5}U + \frac{1}{1 + t^3 + t^4 + t^5}Z,$$

and, using (4.4), the theta functions X_0 and X_1 can be expressed in the R-basis $\{U, Z\}$ as follows

$$\begin{aligned} X_0 &= t^4 U + Z, \\ X_1 &= g.X_0 = (1 + t^3 + t^4 + t^5)Z. \end{aligned}$$

Note that we multiplied both expressions by t^4, which will not affect the final calculations of section 5 as we will work on the projectivization of the R-module $\mathcal{W}$. We also note that the two sections $X_0 \otimes_R k$ and $X_1 \otimes_R k$ coincide at the special fibre $H^0(\mathcal{E}_0, 2\Theta_{|\mathcal{E}_0}) \cong H^0(\mathcal{E}, 2\Theta) \otimes_R k$.

4.3 Specializing an ordinary curve

Let X/k be a smooth genus 2 curve with Hasse-Witt invariant equal to 1. Following e.g. [L] X is birational to an affine curve given by an equation of the form

$$y^2 + xy = \lambda x^5 + \mu x^3 + x,$$

with $\lambda, \mu \in k$ and $\lambda \neq 0$. The projection $(x, y) \mapsto x$ defines a separable double cover $X \to \mathbb{P}^1_k$ ramified at 0 and ∞. Let $\mathbb{P}^1_R$ be the projective line over $R = k[[s]]$ with affine coordinate x. We introduce the family $\mathcal{X} \to \mathbb{P}^1_R$ defined by the projective closure of the affine curve with equation

$$y^2 + (sx^2 + x)y = \lambda x^5 + \mu x^3 + x.$$

The special fibre $\mathcal{X}_0/k$ equals X and the generic fibre $\mathcal{X}_\eta/K$ of the family $\mathcal{X}$ is a smooth ordinary curve of genus 2, which is birational to the curve (defined over a finite extension of K) given by the standard equation (2.5) with coefficients

$$a = \lambda/s^3, \ b = \alpha^2 + \alpha, \ c = s, \qquad \text{and} \qquad \alpha^2 = \lambda/s^3 + \mu/s + s. \tag{4.6}$$

Let $\mathcal{J}\mathcal{X}$ be the associated Jacobian scheme and $\mathcal{J}\mathcal{X}[2]/R$ be the group scheme of 2-torsion points. Then we have the following isomorphisms

$$\mathcal{J}\mathcal{X}[2]_\eta \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mu_2^2/K, \qquad \mathcal{J}\mathcal{X}[2]_0 \cong JX[2] \cong (\mathbb{Z}/2\mathbb{Z}) \times \mu_2 \times G_{l,l}/k,$$

where $G_{l,l}$ is the unique self-dual local-local group scheme of dimension 1 and length 4. Note that $G_{l,l}$ is isomorphic to the group scheme of 2-torsion points $E^{ss}[2]$ (see section 4.2.2). The étale parts of both fibres can be described in terms of Weierstrass points as follows.

The 3 Weierstrass points of $\mathcal{X}_\eta \to \mathbb{P}^1_K$ are 0_η, ∞_η, and 1_η with affine coordinate $1/s$, which specialize to 0, ∞ and ∞ respectively. We obtain by (2.2) the three nonzero elements of $\mathcal{J}\mathcal{X}[2]^{et}_\eta$, which we denote by $[0]_\eta$, $[1]_\eta$ and $[\infty]_\eta$. At the special fibre the nonzero 2-torsion point in $JX[2]^{et} \cong \mathbb{Z}/2\mathbb{Z}$ equals $AJ(0 + \infty)$, which we denote by $[1]_0$. We see that $[1]_\eta$ and $[\infty]_\eta$ specialize to $[1]_0 \in JX[2]^{et}$, and $[0]_\eta$ specializes to 0.

4.4 Decomposing Heisenberg groups

We are interested in the linear action of the Heisenberg group scheme $\mathcal{H}/R$, which is a central extension (see [M] page 221)

$$0 \longrightarrow \mu_2 \longrightarrow \mathcal{H} \longrightarrow \mathcal{J}\mathcal{X}[2] \longrightarrow 0, \tag{4.7}$$

on the free R-module $\mathcal{W} = H^0(\mathcal{J}\mathcal{X}[2], 2\Theta)$ of rank 4. We choose the splitting over R of the connected-étale exact sequence

$$0 \longrightarrow \mathcal{J}\mathcal{X}[2]^0 \longrightarrow \mathcal{J}\mathcal{X}[2] \longrightarrow \mathcal{J}\mathcal{X}[2]^{et} \longrightarrow 0$$

determined by the nonzero 2-torsion point $[1] = AJ(0+\infty) \in \mathcal{JX}[2]$. Note that $\mathcal{JX}[2]^{et} \cong \mathbb{Z}/2\mathbb{Z}$ and that $[\infty] \in \mathcal{JX}[2]$ determines a different splitting. Passing to the Cartier dual we obtain a decomposition over R,

$$\mathcal{JX}[2] = \mathbb{Z}/2\mathbb{Z} \times \mu_2 \times \mathcal{JX}[2]^{00}.$$

Pulling-back the central extension (4.7) by the canonical inclusions of $\mathbb{Z}/2\mathbb{Z} \times \mu_2$ and $\mathcal{JX}[2]^{00}$ into $\mathcal{JX}[2]$ we obtain the two Heisenberg groups $\mathcal{H}^{et}$ and $\mathcal{H}^0$

$$0 \longrightarrow \mu_2 \longrightarrow \mathcal{H}^{et} \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mu_2 \longrightarrow 0, \qquad 0 \longrightarrow \mu_2 \longrightarrow \mathcal{H}^0 \longrightarrow \mathcal{JX}[2]^{00} \longrightarrow 0.$$

It is clear that the Heisenberg group scheme $\mathcal{H}$ (4.7) is isomorphic to the quotient $\mathcal{H}^0 \times \mathcal{H}^{et}/\mu_2$, where μ_2 acts diagonally on $\mathcal{H}^0 \times \mathcal{H}^{et}$. Let $\mathcal{W}^{et}$ and $\mathcal{W}^0$ be the sub-R-modules of $\mathcal{W}$ fixed by the subgroups $\mathcal{H}^0$ and $\mathcal{H}^{et}$ of $\mathcal{H}$. By general theory of Heisenberg groups, $\mathcal{W}$ is the unique irreducible $\mathcal{H}$-module of weight 1, which implies an $\mathcal{H}$-isomorphism

$$\mathcal{W} \cong \mathcal{W}^0 \otimes \mathcal{W}^{et}.$$

Moreover $\mathcal{W}^0$ (resp. $\mathcal{W}^{et}$) is the unique irreducible $\mathcal{H}^0$ (resp. $\mathcal{H}^{et}$)-module of weight 1.

Let H/k be the Heisenberg group scheme associated to $\mathbb{Z}/2\mathbb{Z} \times \mu_2$ ($\cong E[2]$ for any ordinary elliptic curve E/k) and let W be the unique irreducible H-module of weight 1. Note that W is isomorphic (as H-module) to the space (4.3). It is clear that we have the following isomorphisms

$$\mathcal{H}^{et} \cong H \otimes_k R, \qquad \mathcal{W}^{et} \cong W \otimes_k R.$$

We will denote by $\{Z_0, Z_1\}$ the "constant" R-basis of $\mathcal{W}^{et}$ induced by the theta basis $\{X_0, X_1\}$ of W,i.e., $Z_i := X_i \otimes_k 1$.

The structure of the $\mathcal{H}^0$-module $\mathcal{W}^0$ is determined by analyzing the group scheme $\mathcal{JX}[2]^{00}$. In section 4.1 we considered the 2-divisible group $\mathcal{JX}(2)^{00}$ and we showed the existence of an elliptic curve $\mathcal{E}_{\mathcal{X}}/R$ such that $\mathcal{E}_{\mathcal{X}}(2) \cong \mathcal{JX}(2)^{00}$. In particular $\mathcal{E}_{\mathcal{X}}[2] \cong \mathcal{JX}[2]^{00}$. We observe that the j-invariants of the elliptic curves $\mathcal{E}_{\mathcal{X}}/R$ and $\mathcal{E}/R$ (section 4.2.2) lie in the maximal ideal of R, because $(\mathcal{E}_{\mathcal{X}})_0 \cong \mathcal{E}_0 \cong E^{ss}$. Therefore there exists a relation of the form

$$s^n = ut^m \tag{4.8}$$

between the two uniformizing parameters s in (4.6) and t in (4.5), with u invertible in $k[[t]]$ and $n, m \in \mathbb{N}^*$, and we can assume, after passing to the ramified cover given by (4.8), that the $\mathcal{H}^0$-module $\mathcal{W}^0$ equals $H^0(\mathcal{E}, 2\Theta)$.

In order to have a consistent notation we denote the R-basis $\{U, Z\}$ of $\mathcal{W}^0 = H^0(\mathcal{E}, 2\Theta)$ by $\{Z_0, Z_1\}$ and recall from section 4.2.2 the transition formulae

$$X_0 = t^4 Z_0 + Z_1, \qquad X_1 = (1 + t^3 + t^4 + t^5) Z_1. \tag{4.9}$$

Let $\{x_i\}$ and $\{z_i\}$ denote the dual K-bases of $\{X_i\}$ and $\{Z_i\}$ in both spaces $\mathcal{W}^0$ and $\mathcal{W}^{et}$. Then the 4 tensors $z_{ij} := z_i \otimes z_j \in \mathcal{W}^*$ form an R-basis and the dual theta functions $x_{ij} := x_i \otimes x_j$ can be expressed as follows (after chasing denominators)

$$\begin{aligned} x_{00} &= (1 + t^3 + t^4 + t^5) z_{00}, \qquad & x_{10} &= z_{00} + t^4 z_{10}, \\ x_{01} &= (1 + t^3 + t^4 + t^5) z_{01}, \qquad & x_{11} &= z_{01} + t^4 z_{11}. \end{aligned} \tag{4.10}$$

Note that the coordinate x_{10} specializes to x_{00} and x_{11} specializes to x_{01}. Via the level 2 structure (2.3) this parallels the specialization of the 2-torsion points $[0]_\eta$, $[1]_\eta$, and $[\infty]_\eta$ (section 4.3).

5 Equations of $\tilde{V}$ for nonordinary X in characteristic 2

It can be shown as in [LP] section 5 that the identification $\mathrm{M}_X \longrightarrow |2\Theta|$ extends to the relative case $\mathcal{X} \longrightarrow \mathrm{Spec}(R)$,i.e., we have an isomorphism $\mathrm{M}_{\mathcal{X}} \longrightarrow \mathbb{P}(\mathcal{W})$ over $\mathrm{Spec}(R)$. Therefore the relative Frobenius morphism $\mathcal{X} \longrightarrow \mathcal{X}_1$ (over $\mathrm{Spec}(R)$) induces by pull-back a rational map

$$\begin{array}{ccc} \mathbb{P}(\mathcal{W}_1) & \stackrel{\tilde{\mathcal{V}}}{\longrightarrow} & \mathbb{P}(\mathcal{W}) \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(R) & \end{array}$$

with $\mathcal{W}_1 = H^0(\mathcal{J}\mathcal{X}_1, 2\Theta_1)$. We recall that the map $\tilde{\mathcal{V}}$ is given by a linear system of 4 quadrics. Over the generic point $\eta \in \mathrm{Spec}(R)$ the equations of the map $\tilde{\mathcal{V}}_\eta : \mathbb{P}(\mathcal{W}_1)_\eta \longrightarrow \mathbb{P}(\mathcal{W})_\eta$ are of the form ([LP] Proposition 3.1 (3))

$$\tilde{\mathcal{V}} \ : \ x = (x_{ij}) \longmapsto (\lambda_{00}P_{00}(x) : \lambda_{01}P_{01}(x) : \lambda_{10}P_{10}(x) : \lambda_{11}P_{11}(x)) \tag{5.1}$$

with

$$\begin{array}{ll} P_{00}(x) = x_{00}^2 + x_{01}^2 + x_{10}^2 + x_{11}^2, & P_{10}(x) = x_{00}x_{10} + x_{01}x_{11}, \\ P_{01}(x) = x_{00}x_{01} + x_{10}x_{11}, & P_{11}(x) = x_{00}x_{11} + x_{10}x_{01}. \end{array}$$

Here we use the K-basis of theta coordinates x_{ij} on $\mathbb{P}(\mathcal{W}_1)_\eta$ and $\mathbb{P}(\mathcal{W})_\eta$. Moreover Proposition 3.1 relates the coefficients λ_{ij} (defined up to a scalar) to the coefficients $a, b, c \in K$ (4.6),

$$(\lambda_{00} : \lambda_{01} : \lambda_{10} : \lambda_{11}) = (\sqrt{abc} : \sqrt{b} : \sqrt{c} : \sqrt{a}).$$

Since the theta coordinates x_{ij} are no longer independent after specialization, we express the equations (5.1) of $\tilde{\mathcal{V}}$ in the R-basis $\{z_{ij}\}$ using the transition formulae (4.10). A straightforward computation shows that the map

$$\tilde{\mathcal{V}} \ : \ z = (z_{ij}) \longmapsto (R_{00}(z) : R_{01}(z) : R_{10}(z) : R_{11}(z)) \tag{5.2}$$

is given by the quadrics

$$\begin{array}{rcl} R_{00}(z) & = & \frac{\sqrt{abc}}{1+t^3+t^4+t^5} \left[(t^{12} + t^{16} + t^{20})(z_{00}^2 + z_{01}^2) + t^{16}(z_{10}^2 + z_{11}^2) \right], \\ R_{01}(z) & = & \frac{\sqrt{b}}{1+t^3+t^4+t^5} \left[(t^{12} + t^{16} + t^{20}) z_{00}z_{01} + t^8(z_{00}z_{11} + z_{10}z_{01}) + t^{16} z_{10}z_{11} \right], \\ R_{10}(z) & = & \frac{1}{t^4} \left[R_{00} + \sqrt{c}(1 + t^6 + t^8 + t^{10}) \left(z_{00}^2 + z_{01}^2 + t^8(z_{00}z_{10} + z_{11}z_{01}) \right) \right], \\ R_{11}(z) & = & \frac{1}{t^4} \left[R_{01} + \sqrt{a}(1 + t^6 + t^8 + t^{10}) t^8 (z_{00}z_{11} + z_{01}z_{10}) \right]. \end{array} \tag{5.3}$$

Note that we square the coefficients in (4.10) when considering coordinates on $\mathbb{P}(\mathcal{W}_1)$. At the special fibre the map (5.2) specializes to $\tilde{\mathcal{V}}_0$ obtained by putting $t = 0$ after having divided the R_{ij}'s by t^α, where α is the lowest valuation appearing in the expressions (5.3). The map $\tilde{\mathcal{V}}_0$ coincides with the Verschiebung $\tilde{V}$ of the curve X, because both maps extend to rational maps over R and coincide over K. Since the image of $\tilde{V}$ is nondegenerate (it contains the Kummer surface $\mathrm{Kum}_X \subset |2\Theta|$), the lowest valuations for each of the 4 quadrics R_{ij} coincide (otherwise the image of $\tilde{\mathcal{V}}_0$ is contained in a hyperplane).

We work out the specialization of the quadrics as follows. We write $\nu = \frac{m}{n}$ and replace s by vt^ν, with $v \in k[[t]]$ invertible, in the expression of the coefficients a, b, c of (4.6). Note that the (rational) valuations of $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are $-\frac{3}{2}\nu, -\frac{3}{2}\nu, \frac{1}{2}\nu$ respectively. First we observe that the valuations of R_{01} and R_{00} equal $8 - \frac{3}{2}\nu$ and $12 - \frac{5}{2}\nu$ respectively. Since they coincide, we obtain $\nu = 4$,i.e.,

$$R_{00} = t^2(z_{00}^2 + z_{01}^2) + \text{ h.o.t.}, \qquad R_{01} = t^2(z_{00}z_{11} + z_{10}z_{01}) + \text{ h.o.t.},$$

up to some multiplicative nonzero constants. Next we see that the expansions of R_{10} and R_{11} are given by

$$R_{10} = t^2(z_{00}^2 + z_{01}^2 + z_{10}^2 + z_{11}^2) + \text{ h.o.t.}, \qquad R_{11} = t^2(z_{00}z_{01}) + \text{ h.o.t.},$$

up to some multiplicative nonzero constants and some multiple of R_{00} and R_{01} respectively. Thus we have shown

5.1 Theorem. *Let X be a smooth curve with Hasse-Witt invariant equal to 1. There exist coordinates $\{z_{ij}\}$ on $|2\Theta_1|$ and $\{y_{ij}\}$ on $|2\Theta|$ such that the equations of $\tilde{V}$ are given by*

$$|2\Theta_1| \xrightarrow{\tilde{V}} |2\Theta|, \qquad z = (z_{ij}) \longmapsto y = (y_{ij}) = (\lambda_{00}Q_{00}(z) : \lambda_{01}Q_{01}(z) : \lambda_{10}Q_{10}(z) : \lambda_{11}Q_{11}(z))$$

with

$$\begin{aligned} Q_{00}(z) &= z_{00}^2 + z_{01}^2, & Q_{10}(z) &= z_{00}^2 + z_{01}^2 + z_{10}^2 + z_{11}^2, \\ Q_{01}(z) &= z_{00}z_{11} + z_{10}z_{01}, & Q_{11}(z) &= z_{00}z_{01}, \end{aligned}$$

and the λ_{ij}'s are nonzero constants depending on the curve X.

5.2 Remark. We note that the equations of $\tilde{V}$ given in Theorem 5.1 are written in two sets of coordinates on $|2\Theta|$ and $|2\Theta_1|$ which do not necessarily correspond under the k-semi-linear isomorphism $JX_1 \to JX$.

5.3 Remark. In case X is a nonordinary curve with Hasse-Witt invariant equal to 0, i.e., X is supersingular, we observe that the 2-divisible group $JX(2) = JX(2)^{00}$ (see section 4.1) is self-dual, of dimension 2 and height 4. There exists a finite number of isomorphism classes of such 2-divisible groups over k (see [D] page 93). Moreover one can show that $JX(2)$ cannot be isomorphic to the product $E^{ss}(2) \times E^{ss}(2)$.

As in [LP] section 6 we can easily deduce from Theorem 5.1 a full description of the Verschiebung $\tilde{V}$. Since the computations are analogous to those of [LP] Proposition 6.1, we leave them to the reader.

5.4 Proposition. *Let X be a smooth genus 2 curve with Hasse-Witt invariant equal to 1.*

1. *There exists a unique stable bundle $E_{BAD} \in \mathrm{M}_{X_1}$, which is destabilized by the Frobenius map, i.e., F^*E_{BAD} is not semi-stable. We have $E_{BAD} = F_*B^{-1}$ and its projective coordinates are $(0:0:1:1)$.*

2. *Let H_1 be the hyperplane in $|2\Theta_1|$ defined by $z_{00} + z_{01} = 0$. The map $\tilde{V}$ contracts H_1 to the conic $\mathrm{Kum}_X \cap H$, where H is the hyperplane in $|2\Theta|$ defined by $y_{00} = 0$.*

3. *The fiber of $\tilde{V}$ over a point $[E] \in \mathrm{M}_X$ is*

   - *a nondegenerate $\mathbb{Z}/2\mathbb{Z}$-orbit of a point $[E_1] \in \mathrm{M}_{X_1}$, if $[E] \notin H$*
   - *empty, if $[E] \in H \setminus (H \cap \mathrm{Kum}_X)$*
   - *a projective line passing through E_{BAD}, if $[E] \in H \cap \mathrm{Kum}_X$*

   In particular, $\tilde{V}$ is dominant and nonsurjective. The separable degree of $\tilde{V}$ is 2.

6 Equations of $\tilde{V}$ in characteristic 3

Let X be a smooth curve of genus 2 defined over a field of characteristic 3. The main result of this section is

6.1 Theorem. *1. There exists an embedding* $\alpha : \mathrm{Kum}_X \hookrightarrow |2\Theta_1|$ *such that the equality of hypersurfaces in* $|2\Theta_1|$

$$\tilde{V}^{-1}(\mathrm{Kum}_X) = \mathrm{Kum}_{X_1} \cup \alpha(\mathrm{Kum}_X)$$

holds set-theoretically.

2. The cubic equations of $\tilde{V}$ *are given by the 4 partials of the quartic equation of the Kummer surface* $\alpha(\mathrm{Kum}_X) \subset |2\Theta_1|$. *In other words,* $\tilde{V}$ *is the polar map of the surface* $\alpha(\mathrm{Kum}_X)$.

Proof. Given $L \in JX$ and a section $\varphi \in H^0(X, \omega \otimes L^2)$, we can consider, using adjunction and relative duality for the map F, the homomorphism

$$F_*L \xrightarrow{\varphi} L \otimes \omega,$$

where we also write L for the pull-back ι^*L under the k-semi-linear isomorphism $\iota : X_1 \to X$. The map φ is surjective as is seen as follows: suppose φ vanishes at $x \in X$, then, again by adjunction and relative duality, we obtain a nonzero map $F^*(L^{-1}\omega^{-1}(x)) = L^{-3}\omega^{-3}(3x) \to L^{-1}\omega^{-2}$, which is impossible for degree reasons.

We define $E_L := \ker(\varphi)$. Hence there is an exact sequence of vector bundles over X_1

$$0 \longrightarrow E_L \longrightarrow F_*L \xrightarrow{\varphi} L \otimes \omega \longrightarrow 0. \qquad (6.1)$$

It is staightforward to show that E_L has determinant $\mathcal{O}_{X_1}$ (since $\det F_*L = \det(F_*\mathcal{O}_X) \otimes L = \det B \otimes L = \omega L$), that E_L is semi-stable, that $E_L = E_{L^{-1}}$ (if $L^{-1} \neq L$), and that F^*E_L is S-equivalent to $L \oplus L^{-1}$ (by adjunction the inclusion $E_L \to F_*L$ induces a nonzero map $F^*E_L \to L$). Here B denotes the sheaf of locally exact differential forms of X_1 (see [R] section 4.1).

We observe that for L such that $L^2 \neq \mathcal{O}$, $\dim H^0(\omega L^2) = 1$, hence φ is uniquely defined (up to a scalar). For L such that $L^2 = \mathcal{O}$, we obtain a projective line $\mathbb{P}^1_L$ of rank 2 vector bundles $E_{L,\varphi}$ with $\varphi \in |\omega|$. The variety of pairs (L, φ) is isomorphic to the blowing-up $\mathrm{Bl}_2(JX)$ of JX at the 16 2-torsion points $L \in JX[2]$. Hence we obtain a morphism

$$e : \mathrm{Bl}_2(JX) \longrightarrow \mathrm{M}_{X_1} \cong |2\Theta_1|, \qquad (L, \varphi) \longmapsto E_L.$$

Now we will determine the image of the morphism e. We consider the set of semi-stable bundles

$$\tilde{V}^{-1}(\mathrm{Kum}_X) = \{E \in \mathrm{M}_{X_1} \mid F^*E \text{ strictly semi-stable}\}.$$

First we will show that $\tilde{V}^{-1}(\mathrm{Kum}_X)$ has two irreducible components: the Kummer surface Kum_{X_1} and the image $e(\mathrm{Bl}_2(JX))$. To do that we observe that for any stable $E \in \tilde{V}^{-1}(\mathrm{Kum}_X)$, there exists a nonzero map $F^*E \longrightarrow L$ for some $L \in JX$. By adjunction we obtain a nonzero map $E \longrightarrow F_*L$, which is shown to be injective (same method as above), hence E is of the form E_L.

By Proposition 7.2 the map $\tilde{V}$ is given by a linear system $|\mathcal{L}|$ of 4 cubics on $|2\Theta_1|$. The key fact underlying Theorem 6.1 is a striking relationship between cubics and quartics on $|2\Theta_1|$ ([vG] Proposition 2): the 4 cubics in $|\mathcal{L}|$ are the 4 partial derivatives of a Heisenberg invariant quartic, whose zero divisor we denote by $Q \subset |2\Theta_1|$. By [R] Remarque 4.1.2 (2) the 16 points $B \otimes \kappa^{-1}$ (a

$JX_1[2]$-orbit) are contained in the base locus of $|\mathcal{L}|$, so that the quartic Q is singular at the 16 points $B \otimes \kappa^{-1}$. Hence Q is a Kummer surface with polar map $\tilde{V}$. We will show that

$$Q \cong \mathrm{Kum}_X \qquad \text{and} \qquad Q = e(\mathrm{Bl}_2(JX)).$$

The 16 nodes of the quartic Q form a so-called 16_6-configuration (see e.g. [GD]): there are 16 hyperplanes containing each 6 nodes, which moreover lie on a conic. In order to determine the image $\tilde{V}(Q)$, which will be isomorphic to Q — since Q is self-dual —, it will be enough to determine the images of these 16 conics. Indeed the 16 conics will be contracted by $\tilde{V}$ to the 16 nodes of $\tilde{V}(Q)$.

We now give an additional description of these 16 conics through the 16 6-tuples of nodes.

6.2 Lemma. *For any* $L \in JX[2]$

1. *the image* $e(\mathbb{P}^1_L)$ *in* M_{X_1} *is a conic,*
2. *for a Weierstrass point* $w \in X$*, we have* $e(\varphi) \cong B \otimes L^{-1}(-w)$*, with* $\mathrm{Div}(\varphi) = 2w$.

Proof. (1) Without loss of generality we can assume $L = \mathcal{O}$. Then we have a family $\{E_\varphi\}$ of semi-stable rank 2 vector bundles parametrized by $\mathbb{P}^1 = |\omega|$ and defined (pointwise) by the exact sequence

$$0 \longrightarrow E_\varphi \longrightarrow F_*\mathcal{O} \xrightarrow{\varphi} \omega \longrightarrow 0. \tag{6.2}$$

Let us define the family $\{E_\varphi\}$: we denote by p and q the projections of $X \times \mathbb{P}^1$ on the factors X and $\mathbb{P}^1$. Then we have a "universal" section over $X \times \mathbb{P}^1$

$$q^*\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow p^*\omega.$$

Now we tensorize with $p^*\omega^{-3} \otimes q^*\mathcal{O}(1)$ and we obtain

$$(Id \times F)^* p^*\omega^{-1} = p^*\omega^{-3} \longrightarrow p^*\omega^{-2} \otimes q^*\mathcal{O}(1),$$

which, by adjunction, transforms into a map

$$p^*\omega^{-1} \longrightarrow (Id \times F)_* \big(p^*\omega^{-2} \otimes q^*\mathcal{O}(1)\big) = p^*F_*\omega^{-2} \otimes q^*\mathcal{O}(1),$$

Taking the dual and using relative duality $F_*\omega^{-2} = (F_*\mathcal{O})^\vee$ we define the family $\mathcal{E}$ over $X \times \mathbb{P}^1$ as the kernel (again we check surjectivity as above)

$$0 \longrightarrow \mathcal{E} \longrightarrow p^*F_*\mathcal{O} \otimes q^*\mathcal{O}(-1) \longrightarrow p^*\omega \longrightarrow 0. \tag{6.3}$$

By construction we have $\mathcal{E}_{|X \times \{\varphi\}} \cong E_\varphi$. We consider a hyperplane in M_{X_1} of the form

$$H = \{E \in \mathrm{M}_{X_1} \mid \dim H^0(X_1, E \otimes M) > 0\},$$

with $M \in \mathrm{Pic}^1(X_1)$. We recall that H is defined as a determinant divisor and by functoriality of determinant divisors, the pull-back divisor $e^*(H)$ (we also denote by e the restricted morphism $e : \mathbb{P}^1 \longrightarrow \mathrm{M}_{X_1}$) equals the determinant divisor of the vector bundle map

$$q_*(p^*(F_*\mathcal{O} \otimes M) \otimes q^*\mathcal{O}(-1)) = H^0(M^3) \otimes \mathcal{O}(-1) \longrightarrow q_*(p^*\omega M) = H^0(\omega M),$$

obtained by tensorizing (6.3) with p^*M and taking direct image under q. Note that the higher direct images $R^1 q_*$ are zero. Since $h^0(M^3) = h^0(\omega M) = 2$, we obtain $\deg e^*(H) = 2$.

(2) We tensorize the exact sequence (6.1) with L, using $F_*L \otimes L = F_*(L \otimes F^*L) = F_*\mathcal{O}_X$ (projection formula)

$$\begin{array}{ccccccccc} & & & & \mathcal{O}_{X_1} & & & & \\ & & & & \downarrow & \searrow^{\varphi} & & & \\ 0 & \longrightarrow & E_{L,\varphi} \otimes L & \longrightarrow & F_*\mathcal{O}_X & \longrightarrow & \omega_{X_1} & \longrightarrow & 0 \\ & & & \searrow^{\hat{\varphi}} & \downarrow & & & & \\ & & & & B & & & & \end{array}$$

The vertical arrows form an exact sequence (see [R] section 4.1). The upper diagonal map defined as composite map $\mathcal{O}_{X_1} \to F_*\mathcal{O}_X \to \omega_{X_1}$ equals φ, which implies that the lower diagonal map $\hat{\varphi} : E_{L,\varphi} \otimes L \to B$ vanishes at w. Hence, using stability of B and $\det B = \omega$, we obtain $E_{L,\varphi} \otimes L \cong B(-w)$. □

It immediately follows from this lemma that the 16 conics of the 16_6-configuration are the conics $e(\mathbb{P}^1_L)$, with $L \in JX[2]$. It is clear by construction that the conic $e(\mathbb{P}^1_L)$ is contracted by $\tilde{V}$ to the point $[L \oplus L^{-1}] \in \mathrm{M}_X$. Hence the image $\tilde{V}(Q)$ is a Kummer surface singular along the 16 points $[L \oplus L^{-1}]$, with $L \in JX[2]$. Since a Kummer surface is uniquely defined by its 16 nodes (Lemma 2.19 [GD]), we obtain $\tilde{V}(Q) = \mathrm{Kum}_X$ and by self-duality of Q, we conclude that $Q \cong \mathrm{Kum}_X$.

Finally we also have obtained that $Q \subset \tilde{V}^{-1}(\mathrm{Kum}_X)$. Obviously $Q \neq \mathrm{Kum}_{X_1}$, so $Q = e(\mathrm{Bl}_2(JX))$, since both surfaces are irreducible. This completes the proof of Theorem 6.1. □

6.3 Remark. 1. The rational map $e : \mathrm{Kum}_X \longrightarrow \mathrm{M}_{X_1}$ (defined away from the 16 nodes of Kum_X) is the birational inverse of $\tilde{V}$.

2. One has the following scheme-theoretical equality (among divisors in $|2\Theta_1|$)

$$\tilde{V}^{-1}(\mathrm{Kum}_X) = \mathrm{Kum}_{X_1} + 2\alpha(\mathrm{Kum}_X).$$

6.4 Corollary. *1. The map $\tilde{V}$ has exactly 16 base points, which correspond bijectively to the*

- 16 *nodes of the surface* $\alpha(\mathrm{Kum}_X) \subset |2\Theta_1|$
- 16 *stable rank* 2 *vector bundles* $B \otimes \kappa^{-1} \in \mathrm{M}_{X_1} \cong |2\Theta_1|$, *where B is the bundle of locally exact differentials and κ a theta-characteristic of X_1.*

2. The map $\tilde{V}$ is surjective, separable and of degree 11.

Proof. We only have to show part 2, since part 1 is clear from the proof of Theorem 6.1. We recall that the rational map $\tilde{V}$, which is defined away from the 16 points $B \otimes \kappa^{-1}$, coincides with the polar map of the Kummer surface Kum_X. It is well-known that $\tilde{V}$ can be resolved into a morphism $\mathcal{V} : \mathrm{Bl}(|2\Theta_1|) \longrightarrow |2\Theta|$ by blowing-up these 16 points in $|2\Theta_1|$. We denote by E_κ the exceptional divisor over $B \otimes \kappa^{-1}$ and by $H_\kappa \subset |2\Theta|$ the hyperplane $\mathcal{V}(E_\kappa)$. Note that the H_κ are the 16 tropes of $\mathrm{Kum}_X \subset |2\Theta|$ and that $\mathcal{V}_{|E_\kappa}$ is a linear isomorphism. It is clear that the image of $\tilde{V}$ contains the complement of the 16 hyperplanes H_κ.

Let us check that the H_κ are also contained in the image of $\tilde{V}$: a simple computation shows that the cubic $C_\kappa := \tilde{V}^{-1}(H_\kappa) \subset |2\Theta_1|$ is singular at the point $B \otimes \kappa^{-1}$ and that the restriction of $\tilde{V}$ to the cubic C_κ concides with the (birational) projection with center $B \otimes \kappa^{-1}$. Moreover the projectivized tangent cone at $B \otimes \kappa^{-1}$ to C_κ is the conic $Q_\kappa \subset H_\kappa$ through the 6 nodes (recall that $2Q_\kappa = H_\kappa \cap \mathrm{Kum}_X$). Therefore any point in $H_\kappa \setminus Q_\kappa$ lies in the image of $\tilde{V}$. To finish the argument we observe that $Q_\kappa \subset \mathrm{Kum}_X$ and that $\tilde{V} : \mathrm{Kum}_{X_1} \longrightarrow \mathrm{Kum}_X$ is surjective. □

6.5 Remark. 1. We recall ([LP] Remark 6.2) that surjectivity only holds for S-equivalence classes (not isomorphism classes!). In fact, there always exist semi-stable bundles E which do not descend by Frobenius.

2. The number of base points and the degree of $\tilde{V}$ was also obtained in [O] by computing the number of connections (on certain unstable bundles) with zero p-curvature.

3. It would be interesting to have an explicit description of the 11 vector bundles in a general fiber $\tilde{V}^{-1}(E)$ of the polar map $\tilde{V}$.

7 Appendix: base points of V

In this section we consider a smooth curve X of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$. We denote by $\mathrm{M}_X(r)$ (resp. $\mathrm{M}_{X_1}(r)$) the moduli space of semi-stable rank r vector bundles over X (resp. X_1) with fixed trivial determinant and by $\mathcal{L}$ (resp. $\mathcal{L}_1$) the determinant line bundle over $\mathrm{M}_X(r)$ (resp. $\mathrm{M}_{X_1}(r)$). The relative Frobenius map $F : X \to X_1$ induces by pull-back a rational map

$$V : \mathrm{M}_{X_1}(r) \longrightarrow \mathrm{M}_X(r),$$

called the Verschiebung. Let $\mathcal{I}$ be the indeterminacy locus of V, i.e., the closed subscheme of $\mathrm{M}_{X_1}(r)$ consisting of semi-stable bundles E such that F^*E is not semi-stable. Let $U = \mathrm{M}_{X_1}(r) \setminus \mathcal{I}$ be the open subset where V is a morphism.

7.1 General facts

7.1 Proposition. *We have an isomorphism* $V^*(\mathcal{L}) \cong \left(\mathcal{L}_1^{\otimes p}\right)_{|U}$.

Proof. Let $\mathcal{M}_X(r)$ and $\mathcal{M}_{X_1}(r)$ be the moduli stacks of rank r vector bundles over X and X_1 and let $\mathcal{E}$ and $\mathcal{E}_1$ be the universal bundles with trivialized determinant on $X \times \mathcal{M}_X(r)$ and $X_1 \times \mathcal{M}_{X_1}(r)$. It is well-known that the inverses of the determinant of cohomology, which we denote by $\det Rp_*\mathcal{E}$ and $\det Rp_{1*}\mathcal{E}_1$ descend (after restriction to the semi-stable loci) to the line bundles $\mathcal{L}$ and $\mathcal{L}_1$ on the moduli spaces $\mathrm{M}_X(r)$ and $\mathrm{M}_{X_1}(r)$. Now, since $\det Rp_*$ commutes with base change, we have an isomorphism over the moduli stack $\mathcal{M}_{X_1}(r)$

$$V^*(\det Rp_*\mathcal{E}) \cong \det Rp_* \left((F \times \mathrm{Id})^*\mathcal{E}_1\right).$$

Moreover we have a commutative diagram

$$\begin{array}{ccc} X \times \mathcal{M}_{X_1}(r) & \xrightarrow{F \times \mathrm{Id}} & X_1 \times \mathcal{M}_{X_1}(r) \\ & \searrow^{p} \quad \swarrow^{p_1} & \\ & \mathcal{M}_{X_1}(r) & \end{array}$$

where p and p_1 denote the projections on the second factor. Since $F \times \mathrm{Id}$ is an affine morphism, we have $R^1(F \times \mathrm{Id})_* = 0$. Hence

$$\det Rp_* \left((F \times \mathrm{Id})^*\mathcal{E}_1\right) \cong \det Rp_{1*} \left((F \times \mathrm{Id})_*(F \times \mathrm{Id})^*\mathcal{E}_1\right) \cong \det Rp_{1*}(\mathcal{E}_1 \boxtimes F_*\mathcal{O}_X).$$

The last equality follows from the projection formula. Using a filtration by line bundles of the rank p bundle $F_*\mathcal{O}_X$ and by showing that $\det Rp_{1*}(\mathcal{E}_1 \boxtimes \mathcal{O}_{X_1}(D)) = \det Rp_{1*}(\mathcal{E}_1)$ for an effective divisor D — here we use the fact that $\det \mathcal{E}_1$ is trivialized — we show that $\det Rp_{1*}(\mathcal{E}_1 \boxtimes F_*\mathcal{O}_X) \cong (\det Rp_{1*}\mathcal{E}_1)^{\otimes p}$. We obtain the isomorphism of the lemma by descent on U. $\square$

7.2 Proposition. *If $g = 2$ and $r = 2$, then $\dim \mathcal{I} = 0$ and the rational map $\tilde{V}$ is given by polynomials of degree p.*

Proof. The fact that $\dim \mathcal{I} = 0$ is proved in Theorem 3.2 [JX]. This implies that $V^*(\mathcal{L})$ extends uniquely to $\mathcal{L}_1^{\otimes p}$ over M_{X_1} and the lemma follows from the isomorphism $\mathcal{L}_1 \cong \mathcal{O}_{\mathbb{P}^3}(1)$. □

7.3 Remark. For general g, r, p we do not know an estimate of the dimension of $\mathcal{I}$.

7.2 Existence of base points

7.4 Theorem. *The indeterminacy locus $\mathcal{I}$ is nonempty.*

Proof. First it will be enough to show nonemptiness of $\mathcal{I}$ in the case $r = 2$, since taking direct sums with the trivial bundle implies nonemptiness for arbitrary r. Secondly it suffices to show nonemptiness of $\mathcal{I}$ after a field extension k'/k, with k' algebraically closed.

Let $\overline{\mathcal{M}_g}$ be the coarse moduli space of stable genus g curves defined over k, which is an irreducible projective variety [DM]. Let η be the generic point of $\overline{\mathcal{M}_g}$. The choice of a geometric point $\overline{\eta}$ over η defines a smooth curve $\mathcal{X}_{\overline{\eta}}$ over $\overline{k(\eta)}$, the algebraic closure of the function field $k(\eta)$ of $\overline{\mathcal{M}_g}$. The curve $\mathcal{X}_{\overline{\eta}}$ is defined over a finite extension K of $k(\eta)$ and we denote by $\mathcal{X}_K$ some model of $\mathcal{X}_{\overline{\eta}}$, i.e., $\mathcal{X}_K \times_K \overline{k(\eta)} \cong \mathcal{X}_{\overline{\eta}}$.

The curve X/k defines a k-rational point x of $\overline{\mathcal{M}_g}$, which lies in the closure of η. The local ring A_x at the generic point of the exceptional divisor of the blowing-up of $\overline{\mathcal{M}_g}$ at the point x is a discrete valuation ring with fraction field $k(\eta)$ and residue field containing k. By the stable reduction theorem (Corollary 2.7 [DM]) there exists a finite extension L of K, and therefore also of $k(\eta)$, such that $\mathcal{X}_L$ is the generic fibre of a stable curve $\mathcal{X}$ over the integral closure A of A_x in L. Note that A is a discrete valuation ring with fraction field L and with residue field, denoted by $k(s)$, containing k. Moreover the diagram

$$\begin{array}{ccc} \mathrm{Spec}(A) & & \\ \downarrow & \searrow^{\mathcal{X}} & \\ \mathrm{Spec}(A_x) & \longrightarrow & \overline{\mathcal{M}_g} \end{array}$$

commutes when restricted to $\mathrm{Spec}(L) \hookrightarrow \mathrm{Spec}(A)$ and therefore commutes because $\overline{\mathcal{M}_g}$ is separated. It follows that the special point $s \in \mathrm{Spec}(A)$ maps to x, i.e., there exists an isomorphism $X \times_k \overline{k(s)} \cong \mathcal{X}_s \times_{k(s)} \overline{k(s)}$.

In summary, we have constructed a stable curve $\mathcal{X}$ over a discrete valuation ring A with generic fibre $\mathcal{X}_L$ and geometric special fibre isomorphic to $X \times_k \overline{k(s)}$. The fraction field of A is L and its residue field $k(s)$.

We now choose a tree of $\mathbb{P}^1_k$'s, denoted by X', defining a closed point x' in the boundary of $\overline{\mathcal{M}_g}$. Repeating the above contruction with x' instead of x, we obtain a stable curve $\mathcal{X}'$ over a discrete valuation ring A' with generic fibre $\mathcal{X}_{L'}$ and geometric special fibre isomorpic to $X' \times_k \overline{k(s')}$. The fraction field of A' is L', a finite extension of $k(\eta)$, and its residue field is $k(s')$. Moreover the isomorphism $X' \times_k \overline{k(s')} \cong \mathcal{X}_{s'} \times_{k(s')} \overline{k(s')}$ is defined over a finite extension of $k(s')$.

We choose a finite extension of $k(\eta)$ containing both L and L', which we call again L, and take the integral closures in L of A and A', which we call again A and A'. Thus we have constructed two stable curves $\mathcal{X}$ and $\mathcal{X}'$ over A and A' such that $\mathcal{X}_L \cong \mathcal{X}'_L$ and which specialize to X and X' respectively.

Let $\hat{L}$ be the fraction field of the completion $\hat{A}'$ of A'. By construction the curve $\mathcal{X}_{\hat{L}} \cong \mathcal{X}'_{\hat{L}}$ is a Mumford-Tate curve and, by the main result of [G], there exists a stable rank 2 vector bundle $\hat{\mathcal{E}}$ over $\mathcal{X}_{\hat{L}}$ such that $F^*\hat{\mathcal{E}}$ is not semi-stable.

7.5 Lemma. *There exists a finite extension L_1 of L contained in the field $\hat{L}$ and a stable bundle $\mathcal{E}_1$ over $\mathcal{X}_{L_1}$ such that $\mathcal{E}_1 \otimes_{L_1} \hat{L} \cong \hat{\mathcal{E}}$ and $F^*\mathcal{E}_1$ is not semi-stable.*

Proof. Let $\hat{\pi} : F^*\hat{\mathcal{E}} \to \hat{\mathcal{L}}$ be a maximal destabilizing quotient of $\hat{\mathcal{E}}$. There exist a models $\mathcal{E}_{k(S)}, \mathcal{L}_{k(S)}$ and $\pi_{k(S)}$ of $\hat{\pi}$ over $\mathcal{X}_{k(S)}$, where $k(S)$ is an extension of finite type of L. The field $k(S)$ is the function field for some algebraic variety S over L. Shrinking S if necessary, one can assume that $\pi_{k(S)}$ comes from

$$\pi_S : F^*\mathcal{E}_S \to \mathcal{L}_S,$$

where $\mathcal{E}_S$ is a family of stable bundles over $\mathcal{X}_L$ parametrized by S (stability is an open condition). We now choose a closed point $s \in S$ and pull-back the family $\mathcal{E}_S$ under the inclusion $s \hookrightarrow S$. We thus obtain a stable bundle $\mathcal{E}_{L_1}$ over $\mathcal{X}_{L_1}$, where L_1 is the residue field at the point s, which is a finite extension of L. □

Again we take the integral closures A_1 and A'_1 of the discrete valuation rings A and A' in L_1. By the previous lemma we have a stable bundle $\mathcal{E}_1$ and a destabilizing quotient $\mathcal{L}_1$ over $\mathcal{X}_{L_1} = \mathcal{X}'_{L_1}$

$$\pi_{L_1} : F^*\mathcal{E}_1 \to \mathcal{L}_1.$$

After possibly taking a finite extension of L_1, we can assume [La] that $\mathcal{E}_1$ and $\mathcal{L}_1$ have models over $\mathcal{X} \to \mathrm{Spec}(A)$ with $(\mathcal{E}_1)_{\overline{s}}$ semi-stable over $X \times_k \overline{k(s)}$. By semi-continuity, we have

$$\mathrm{Hom}(F^*\mathcal{E}_{1\overline{s}}, \mathcal{L}_{\overline{s}}) \neq 0,$$

which shows that $F^*\mathcal{E}_{1\overline{s}}$ is not semi-stable.

□

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