INFINITESIMAL DEFORMATIONS OF PARABOLIC CONNECTIONS AND PARABOLIC OPERS

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Abstract. We compute the infinitesimal deformations of quadruples of the form

$$(X,\, S,\, E_*,\, D),$$

where $(X,\, S)$ is a compact Riemann surface with n marked points, E_* is a parabolic vector bundle on X with parabolic structure over S, and D is a parabolic connection on E_*. Using it we compute the infinitesimal deformations of $(X,\, S,\, D)$, where D is a parabolic $\mathrm{SL}(r, \mathbb{C})$–oper on $(X,\, S)$. It is shown that the monodromy map, from the moduli space of triples $(X,\, S,\, D)$, where D is a parabolic $\mathrm{SL}(r, \mathbb{C})$–oper on $(X,\, S)$, to the $\mathrm{SL}(r, \mathbb{C})$–character variety of $X \setminus S$, is an immersion.

Contents

1. Introduction

Opers were introduced by Beilinson and Drinfeld [BD1], [BD2]. They have turned out to be very important in diverse topics, for example in geometric Langlands correspondence, nonabelian Hodge theory, some branches of mathematics physics, differential equations et cetera; see [BF], [DFK+], [FT], [FG1], [FG2], [CS], [Fr1], [Fr2], [KSZ], [MR], [BSY] and references therein.

Parabolic vector bundles were introduced by Mehta and Seshadri in [MS]. In [BDP], $\mathrm{SL}(r, \mathbb{C})$–opers in the set-up of parabolic vector bundles were introduced. The aim here is to further investigate the $\mathrm{SL}(r, \mathbb{C})$–opers in the context of parabolic vector bundles.

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Being inspired by the interesting work [Sa], we study the infinitesimal deformations, and the monodromy map, of the parabolic $\text{SL}(r,\mathbb{C})$–opers. Computation of the infinitesimal deformations of parabolic opers entails computation of the infinitesimal deformations of quadruples of the form $(X,\, S,\, E_*,\, D)$, where $(X,\, S)$ is a compact Riemann surface with n marked points, E_* is a parabolic vector bundle on X with parabolic structure over S, and D is a parabolic connection on E_*.

We introduce the parabolic analog of the Atiyah bundle and the Atiyah exact sequence. Given a parabolic vector bundle E_* on $(X,\, S)$, its Atiyah bundle $\text{At}(E_*)$ fits in the short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \text{End}^P(E_*) \longrightarrow \text{At}(E_*) \stackrel{\sigma}{\longrightarrow} TX\otimes \mathcal{O}_X(-S) \longrightarrow 0\,, \tag{1.1}$$

where $\text{End}^P(E_*)$ denotes the sheaf of quasiparabolic flag preserving endomorphisms of E_* (see (2.2) for the quasiparabolic flags); the sequence in (1.1) is the Atiyah exact sequence in the set-up of parabolic bundles (see Definition 2.3). We show the following:

(1) *A holomorphic splitting of* (1.1) *produces a logarithmic connection on the holomorphic vector bundle E underlying E_* such that the residues preserve the quasiparabolic flags of E_*, and*
(2) *a parabolic connection on E_* is a holomorphic splitting of* (1.1) *such that the eigenvalues of the residues are given by the parabolic weights (see* (2.3) *for the parabolic weights).*

(See Lemma 2.4 and Corollary 2.5.)

We prove the following (see Lemma 3.1):

The infinitesimal deformations of the triple $(X,\, S,\, E_*)$ are parametrized by $H^1(X,\, \text{At}(E_*))$.

Now let D be a connection on the parabolic vector bundle E_* over $(X,\, S)$. We assume that the local monodromy of D around each point of S is semisimple (meaning diagonalizable). Let

$$\mathcal{D}_0\,:\, \text{End}^P(E_*) \longrightarrow \text{End}^n(E_*)\otimes K_X$$

be the corresponding logarithmic connection on $\text{End}^P(E_*)$, where $\text{End}^n(E_*)\,\subset\, \text{End}^P(E_*)$ is the sheaf of endomorphisms nilpotent with respect to the quasiparabolic flags of E_*. We show that this operator $\mathcal{D}_0$ extends to a holomorphic differential operator

$$\mathcal{D}\,:\, \text{At}(E_*) \longrightarrow \text{End}^n(E_*)\otimes K_X\otimes \mathcal{O}_X(S)$$

(see (2.24)). Let $\mathcal{B}_\bullet$ denote the following two-term complex of sheaves on X

$$\mathcal{B}_\bullet\ :\ \mathcal{B}_0\,=\,\text{At}(E_*) \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{B}_1\,=\,\text{End}^n(E_*)\otimes K_X\otimes \mathcal{O}_X(S)\,,$$

where $\mathcal{B}_i$ is at the i-th position.

We prove the following (see Lemma 3.4):

The infinitesimal deformations of the quadruple $(X,\, S,\, E_*,\, D)$ are parametrized by the first hypercohomology $\mathbb{H}^1(\mathcal{B}_\bullet)$.

Now assume that D is a parabolic $\mathrm{SL}(r,\mathbb{C})$–oper (the definition of a parabolic $\mathrm{SL}(r,\mathbb{C})$–oper is recalled in Section 4). Since D is a parabolic $\mathrm{SL}(r,\mathbb{C})$–oper, the local monodromy of D around each point of S is semisimple (see Lemma 4.2). In (4.7) and (4.9) we construct the holomorphic vector bundles $\mathrm{At}_X(r)$ and $\mathrm{ad}_1^n\left(\mathrm{Sym}^{r-1}(\mathcal{E}_*)\right)$ respectively on X. Using D we construct a differential operator

$$\mathcal{D}_B\,:\,\mathrm{At}_X(r)\,\longrightarrow\,\mathrm{ad}_1^n\left(\mathrm{Sym}^{r-1}(\mathcal{E}_*)\right)\otimes K_X\otimes\mathcal{O}_X(S)$$

(see (4.13)). Let $\mathcal{C}_\bullet$ denote the following two-term complex of sheaves on X

$$\mathcal{C}_\bullet\,:\,\mathcal{C}_0\,=\,\mathrm{At}_X(r)\,\xrightarrow{\mathcal{D}_B}\,\mathcal{C}_1\,=\,\mathrm{ad}_1^n\left(\mathrm{Sym}^{r-1}(\mathcal{E}_*)\right)\otimes K_X\otimes\mathcal{O}_X(S)\,,$$

where $\mathcal{C}_i$ is at the i-th position.

We prove the following (see Theorem 4.5):

The space of all infinitesimal deformation of the triple $(X,\,S,\,D)$, where D is a parabolic $\mathrm{SL}(r,\mathbb{C})$–oper on $(X,\,S)$, is given by the hypercohomology $\mathbb{H}^1(\mathcal{C}_\bullet)$.

A reformulation of the above result is proved in Corollary 4.6.

In Theorem 5.1 we prove the following:

The monodromy map from the moduli space of triples $(X,\,S,\,D)$, where D is a parabolic $\mathrm{SL}(r,\mathbb{C})$–oper on $(X,\,S)$, to the $\mathrm{SL}(r,\mathbb{C})$–character variety for $X\setminus S$ is an immersion.

The isomonodromy condition defines a holomorphic foliation on the moduli space of quadruples $(X,\,S,\,E_*,\,D)$, where D is a connection on the parabolic vector bundle E_* over $(X,\,S)$. The proof of Theorem 5.1 involves computing this foliation. This computation is carried out in Lemma 3.5.

In the appendix we give an alternative definition of a parabolic $\mathrm{SL}(r,\mathbb{C})$–oper in terms of $\mathbb{R}$-filtered sheaves as introduced and studied by Maruyama and Yokogawa. This definition is conceptually closer to the definition of an ordinary $\mathrm{SL}(r,\mathbb{C})$–oper and clarifies the one given in [BDP].

2. Holomorphic connections and the Atiyah bundle

2.1. **Atiyah bundle for parabolic bundles.** Let X be a compact connected Riemann surface. Fix a finite subset of n distinct points

$$S\,:=\,\{x_1,\,\cdots,\,x_n\}\,\subset\,X. \tag{2.1}$$

The reduced effective divisor $x_1+\ldots+x_n$ on X will also be denoted by S.

A quasiparabolic structure on a holomorphic vector bundle E on X is a filtration of subspaces of the fiber E_{x_i} of E over x_i

$$E_{x_i}\,=\,E_{i,1}\,\supset\,E_{i,2}\,\supset\,\cdots\,\supset\,E_{i,l_i}\,\supset\,E_{i,l_i+1}\,=\,0 \tag{2.2}$$

for every $1 \leq i \leq n$. A parabolic structure on E is a quasiparabolic structure as above together with a string of real numbers

$$0 \leq \alpha_{i,1} < \alpha_{i,2} < \cdots < \alpha_{i,l_i} < 1 \tag{2.3}$$

for every $1 \leq i \leq n$. The above number $\alpha_{i,j}$ is called the parabolic weight of the subspace $E_{i,j}$ in (2.2). The divisor S is known as the parabolic divisor. (See [MS], [MY], [Bh].)

A parabolic vector bundle is a holomorphic vector bundle E with a parabolic structure $(\{E_{i,j}\}, \{\alpha_{i,j}\})$. For notational convenience, $(E, (\{E_{i,j}\}, \{\alpha_{i,j}\}))$ will be denoted by E_*.

Assumption 2.1. Throughout we will work with the assumption that all the parabolic weights $\alpha_{i,j}$ are rational numbers. We assume that each x_i has at least one nonzero parabolic weight. We also assume that $3(\text{genus}(X) - 1) + n > 0$.

For any $1 \leq i \leq n$ and $1 \leq j \leq l_i + 1$, let $\mathcal{E}_{i,j} \longrightarrow X$ be the holomorphic vector bundle defined by the following short exact sequence of coherent analytic sheaves on X:

$$0 \longrightarrow \mathcal{E}_{i,j} \longrightarrow E \longrightarrow E_{x_i}/E_{i,j} \longrightarrow 0 \tag{2.4}$$

(see (2.2)). Since $E_{x_i}/E_{i,j}$ is supported on x_i, the subsheaf $\mathcal{E}_{i,j}$ of E actually coincides with E over the open subset $X \setminus \{x_i\} \subset X$.

The space of all holomorphic sections, over an open subset $U \subset X$, of a holomorphic vector bundle $V \longrightarrow X$ is denoted by $\Gamma(U, V)$.

Let

$$\text{End}^P(E_*) \subset \text{End}(E) \tag{2.5}$$

be the coherent analytic subsheaf such that for any open subset $U \subset X$, the subspace

$$\Gamma\left(U, \text{End}^P(E_*)\right) \subset \Gamma(U, \text{End}(E))$$

consists of all $\mathcal{O}_U$–linear homomorphisms $s : E\big|_U \longrightarrow E\big|_U$ satisfying the condition that

$$s(\mathcal{E}_{i,j}\big|_U) \subset \mathcal{E}_{i,j}\big|_U$$

for all $x_i \in U$ and all $1 \leq j \leq l_i$, where $\mathcal{E}_{i,j}$ is defined in (2.4).

Let

$$\text{End}^n(E_*) \subset \text{End}^P(E_*) \tag{2.6}$$

be the coherent analytic subsheaf consists of all $\mathcal{O}_U$–linear homomorphisms $s : E\big|_U \longrightarrow E\big|_U$ satisfying the condition that for any open subset $U \subset X$,

$$s(\mathcal{E}_{i,j}\big|_U) \subset \mathcal{E}_{i,j+1}\big|_U$$

for all $x_i \in U$ and all $1 \leq j \leq l_i$.

Remark 2.2. It is customary to define $\text{End}^P(E_*)$ as the subsheaf of $\text{End}(E)$ that preserves the subspace $E_{i,j} \subset E_{x_i}$ for all $x_i \in U$ and all $1 \leq j \leq l_i$. Similarly, $\text{End}^n(E_*)$ is defined to be the subsheaf of $\text{End}^P(E_*)$ that takes any $E_{i,j}$ to $E_{i,j+1}$. While the definitions in (2.5) and (2.6) are equivalent to these, we will see that the definitions in (2.5) and (2.6) are more useful for our purpose.

Using the pairing $(\text{End}(E)\otimes\mathcal{O}(S))^{\otimes 2} \longrightarrow \mathcal{O}(2S)$ defined by trace

$$A\otimes B \longmapsto \text{trace}(AB)\,,$$

we have

$$\text{End}^P(E_*)^* \,=\, \text{End}^n(E_*)\otimes\mathcal{O}(S)\,.$$

For any integer $k \geq 0$, let $\text{Diff}^k(E,\,E)$ be the holomorphic vector bundle on X given by the sheaf of all holomorphic differential operators, of order at most k, from E to itself. We have the following short exact sequence of holomorphic vector bundles on X

$$0 \longrightarrow \text{Diff}^0(E,\,E) \,=\, \text{End}_{\mathcal{O}_X}(E) \longrightarrow \text{Diff}^1(E,\,E) \tag{2.7}$$
$$\xrightarrow{\sigma_0} \text{End}_{\mathcal{O}_X}(E)\otimes TX \,=\, E\otimes E^*\otimes TX \longrightarrow 0\,,$$

where TX is the holomorphic tangent bundle of X and σ_0 is the symbol map on the first order differential operators. Let

$$\text{Diff}^1_P(E,\,E) \,\subset\, \text{Diff}^1(E,\,E) \tag{2.8}$$

be the coherent analytic subsheaf consists of all differential operators $D_U \,:\, E\big|_U \longrightarrow E\big|_U$, where $U\,\subset\,X$ is any open subset, satisfying the condition that for any $s\,\in\,\Gamma(U,\,\mathcal{E}_{i,j})$,

$$D_U(s)\,\in\,\Gamma(U,\,\mathcal{E}_{i,j})$$

for all $x_i\,\in\,U$ and all $1\,\leq\,j\,\leq\,l_i$, where $\mathcal{E}_{i,j}$ is constructed in (2.4).

Comparing the definitions given in (2.8) and (2.5) we conclude that

$$\text{Diff}^1_P(E,\,E)\bigcap\text{End}_{\mathcal{O}_X}(E) \,=\, \text{End}^P(E_*)\,; \tag{2.9}$$

the above intersection takes place inside $\text{Diff}^1(E,\,E)$ (see (2.7) and (2.8)). Consider the subsheaf

$$TX\otimes\mathcal{O}_X(-S) \,\subset\, TX \,=\, \text{Id}_E\otimes TX \,\subset\, \text{End}(E)\otimes TX\,, \tag{2.10}$$

where S is the divisor in (2.1). Define

$$\text{At}(E_*) \,:=\, \text{Diff}^1_P(E,\,E)\bigcap\sigma_0^{-1}(TX\otimes\mathcal{O}_X(-S)) \,\subset\, \text{Diff}^1(E,\,E)\,, \tag{2.11}$$

where σ_0 is the projection in (2.7) and $\text{Diff}^1_P(E,\,E)\,\subset\,\text{Diff}^1(E,\,E)$ is the subsheaf in (2.8). Now from (2.7) and (2.9) we get the following short exact sequence of holomorphic vector bundles on X

$$0 \longrightarrow \text{End}^P(E_*) \longrightarrow \text{At}(E_*) \xrightarrow{\sigma} TX\otimes\mathcal{O}_X(-S) \longrightarrow 0\,, \tag{2.12}$$

where σ is the restriction of σ_0 to $\text{At}(E_*)\,\subset\,\text{Diff}^1(E,\,E)$. It is straightforward to check that the homomorphism σ in (2.12) is surjective; indeed, this follows immediately from the fact that $TX\otimes\mathcal{O}_X(-S)\,\subset\,\sigma_0(\text{Diff}^1_P(E,\,E))$.

Definition 2.3. The vector bundle $\text{At}(E_*)$ in (2.11) will be called the *Atiyah bundle* for the parabolic bundle E_*, and the sequence in (2.12) will be called the *Atiyah exact sequence* for the parabolic bundle E_*.

When S is the zero divisor (meaning $n\,=\,0$ in (2.1)), then $\text{At}(E_*)$ is the usual Atiyah bundle $\text{At}(E)$ for E, and (2.12) is the usual Atiyah exact sequence for E. (See [At].)

2.2. **Holomorphic connections on a parabolic bundle.** The holomorphic cotangent bundle of X will be denoted by K_X.

Let V be a holomorphic vector bundle on X. A *logarithmic connection* on V singular over S is a holomorphic differential operator of order one

$$D\,:\,V\,\longrightarrow\,V\otimes K_X\otimes\mathcal{O}_X(S)$$

satisfying the Leibniz identity, which says that

$$D(fs)\,=\,fD(s)+s\otimes df \tag{2.13}$$

for any locally defined holomorphic function f on X and any locally defined holomorphic section s of V.

We note that any logarithmic connection on V is flat because $\Omega^{2,0}_X\,=\,0$.

Take a point $x_i\,\in\,S$. The fiber of $K_X\otimes\mathcal{O}_X(S)$ over x_i is identified with $\mathbb{C}$ by the Poincaré adjunction formula [GH, p. 146]. To explain this isomorphism

$$(K_X\otimes\mathcal{O}_X(S))_{x_i}\,\xrightarrow{\,\sim\,}\,\mathbb{C}\,, \tag{2.14}$$

let z be a holomorphic coordinate function on X defined on an analytic open neighborhood of x_i such that $z(x_i)\,=\,0$. Then we have the isomorphism $\mathbb{C}\,\longrightarrow\,(K_X\otimes\mathcal{O}_X(S))_{x_i}$ that sends any $c\,\in\,\mathbb{C}$ to $c\cdot\frac{dz}{z}(x_i)\,\in\,(K_X\otimes\mathcal{O}_X(S))_{x_i}$. It is straightforward to check that this map $\mathbb{C}\,\longrightarrow\,(K_X\otimes\mathcal{O}_X(S))_{x_i}$ is actually independent of the choice of the above holomorphic coordinate function z.

Let $D_V\,:\,V\,\longrightarrow\,V\otimes K_X\otimes\mathcal{O}_X(S)$ be a logarithmic connection on V. Using the Leibniz identity in (2.13) it is straightforward to deduce that the composition of homomorphisms

$$V\,\xrightarrow{\,D_V\,}\,V\otimes K_X\otimes\mathcal{O}_X(S)\,\longrightarrow\,(V\otimes K_X\otimes\mathcal{O}_X(S))_{x_i}\,\xrightarrow{\,\sim\,}\,V_{x_i} \tag{2.15}$$

is $\mathcal{O}_X$–linear; the above isomorphism $(V\otimes K_X\otimes\mathcal{O}_X(S))_{x_i}\,\xrightarrow{\,\sim\,}\,V_{x_i}$ is given by the isomorphism in (2.14). Therefore, the composition of homomorphisms in (2.15) produces a $\mathbb{C}$–linear homomorphism

$$\text{Res}(D_V,\,x_i)\,:\,V_{x_i}\,\longrightarrow\,V_{x_i}\,, \tag{2.16}$$

which is called the *residue* of D_V at x_i; see [De]. If $\lambda_1,\,\cdots,\,\lambda_r$ are the generalized eigenvalues of $\text{Res}(D_V,\,x_i)$ with multiplicity, where $r\,=\,\text{rank}(V)$, then the generalized eigenvalues of the local monodromy of D around x_i are

$$\exp(-2\pi\sqrt{-1}\lambda_1),\,\exp(-2\pi\sqrt{-1}\lambda_1),\,\cdots,\,\exp(-2\pi\sqrt{-1}\lambda_r \tag{2.17}$$

[De].

We now recall another description of the logarithmic connections on V. Consider the short exact sequence in (2.7)

$$0\,\longrightarrow\,\text{End}(V)\,\longrightarrow\,\text{Diff}^1(V,\,V)\,\xrightarrow{\,\widehat{\sigma}_V\,}\,TX\otimes\text{End}(V)\,\longrightarrow\,0 \tag{2.18}$$

for V. Define

$$\text{At}(V,S)\,:=\,\widehat{\sigma}_V^{-1}(TX\otimes\mathcal{O}_X(-S))\,\subset\,\text{Diff}^1(V,\,V)\,,$$

where $TX \otimes \mathcal{O}_X(-S) \subset TX \otimes \text{End}(V)$ is the subbundle defined as in (2.10) for V. So from (2.18) we have the short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \text{End}(V) \longrightarrow \text{At}(V, S) \xrightarrow{\sigma_V} TX \otimes \mathcal{O}_X(-S) \longrightarrow 0\,, \tag{2.19}$$

where σ_V is the restriction of the projection $\widehat{\sigma}_V$ in (2.18) to the subsheaf $\text{At}(V, S) \subset \text{Diff}^1(V,\, V)$. A logarithmic connection on V is a holomorphic splitting of the short exact sequence in (2.19); in other words, a logarithmic connection on V is a holomorphic homomorphism of vector bundles

$$h\,:\, TX \otimes \mathcal{O}_X(-S) \longrightarrow \text{At}(V, S)$$

such that $\sigma_V \circ h \,=\, \text{Id}_{TX\otimes\mathcal{O}_X(-S)}$, where σ_V is the homomorphism in (2.19).

Take a parabolic vector bundle $E_* \,=\, (E,\, (\{E_{i,j}\},\, \{\alpha_{i,j}\}))$; see (2.2), (2.3).

A *connection* on E_* is a logarithmic connection D on E, singular over S, such that

(1) $\text{Res}(D, x_i)(E_{i,j}) \,\subset\, E_{i,j}$ for all $1 \,\leq\, j \,\leq\, l_i$, $1 \,\leq\, i \,\leq\, n$ (see (2.2)), and
(2) the endomorphism of $E_{i,j}/E_{i,j+1}$ induced by $\text{Res}(D, x_i)$ coincides with multiplication by the parabolic weight $\alpha_{i,j}$ for all $1 \,\leq\, j \,\leq\, l_i$, $1 \,\leq\, i \,\leq\, n$.

(See [BL, Section 2.2].) A necessary and sufficient condition for E_* to admit a connection is given in [BL]. The condition in question says that E_* admits a connection if and only if the parabolic degree of every direct summand of E_* is zero [BL, p. 594, Theorem 1.1].

A *holomorphic splitting* of the Atiyah exact sequence in (2.12) (see Definition 2.3) is a holomorphic homomorphism of vector bundles

$$\mathbf{h}\,:\, TX \otimes \mathcal{O}_X(-S) \longrightarrow \text{At}(E_*)$$

such that

$$\sigma \circ \mathbf{h} \,=\, \text{Id}_{TX\otimes\mathcal{O}_X(-S)}\,, \tag{2.20}$$

where σ is the projection in (2.12).

Lemma 2.4. *Giving a holomorphic splitting* $\mathbf{h}$ *of the Atiyah exact sequence for* E_* *(see* (2.20)*) is equivalent to giving a logarithmic connection* D *on* E *satisfying the following condition: for every* $x_i \,\in\, S$,

$$\text{Res}(D,\, x_i)(E_{i,j}) \,\subset\, E_{i,j}$$

for all $1 \,\leq\, j \,\leq\, l_i$ *(see* (2.2)*), where* $\text{Res}(D,\, x_i)(E_{i,j})$ *is constructed in* (2.16).

Proof. Let $\mathbf{h}$ be a holomorphic splitting of the Atiyah exact sequence in (2.12). In other words, $\mathbf{h}\,:\, TX \otimes \mathcal{O}_X(-S) \longrightarrow \text{At}(E_*)$ is a holomorphic homomorphism such that (2.20) holds. Recall from (2.11) that $\text{At}(E_*) \subset \text{Diff}^1_P(E,\, E)$. The composition of homomorphisms

$$TX \otimes \mathcal{O}_X(-S) \xrightarrow{\mathbf{h}} \text{At}(E_*) \hookrightarrow \text{Diff}^1_P(E,\, E)$$

will be denoted by $\widetilde{\mathbf{h}}$. Let $\widetilde{\mathbf{h}}'$ denote the composition of homomorphisms

$$TX \otimes \mathcal{O}_X(-S) \xrightarrow{\widetilde{\mathbf{h}}} \text{Diff}^1_P(E,\, E) \xhookrightarrow{\iota_0} \text{Diff}^1(E,\, E) \tag{2.21}$$

(see (2.8) for the inclusion map ι_0). Since (2.20) holds, we conclude that

$$\sigma_0 \circ \widetilde{\mathbf{h}}' \,=\, \text{Id}_{TX\otimes\mathcal{O}_X(-S)}\,,$$

where σ_0 is the projection in (2.7). Therefore, $\widetilde{\mathbf{h}}'$ is a logarithmic connection on E; this logarithmic connection on E will be denoted by D. Since

$$\widetilde{\mathbf{h}}' = \iota_0 \circ \widetilde{\mathbf{h}}$$

(see (2.21)), the residues of the above defined logarithmic connection D satisfy the following condition: for every $x_i \in S$,

$$\mathrm{Res}(D,\, x_i)(E_{i,j}) \subset E_{i,j}$$

for all $1 \leq j \leq l_i$.

To prove the converse, let D be a logarithmic connection on E such that

$$\mathrm{Res}(D,\, x_i)(E_{i,j}) \subset E_{i,j} \tag{2.22}$$

for all $x_i \in S$ and all $1 \leq j \leq l_i$. So D gives a holomorphic homomorphism

$$\mathbf{h} : TX \otimes \mathcal{O}_X(-S) \longrightarrow \mathrm{At}(E, S)$$

(see (2.19) for $\mathrm{At}(E, S)$) such that the composition of homomorphisms

$$TX \otimes \mathcal{O}_X(-S) \xrightarrow{\mathbf{h}} \mathrm{At}(E, S) \longrightarrow TX \otimes \mathcal{O}_X(-S)$$

coincides with the identity map of $TX \otimes \mathcal{O}_X(-S)$; see (2.19) for the above projection $\mathrm{At}(E, S) \longrightarrow TX \otimes \mathcal{O}_X(-S)$. The given condition in (2.22) implies that

$$\mathbf{h}(TX \otimes \mathcal{O}_X(-S)) \subset \mathrm{Diff}^1_P(E,\, E) \bigcap \mathrm{At}(E, S) = \mathrm{At}(E_*)\,;$$

note that from (2.11) we have

$$\mathrm{At}(E_*) := \mathrm{Diff}^1_P(E,\, E) \bigcap \sigma_0^{-1}(TX \otimes \mathcal{O}_X(-S)) \subset \sigma_0^{-1}(TX \otimes \mathcal{O}_X(-S)) = \mathrm{At}(E, S)\,.$$

Consequently, the homomorphism $\mathbf{h}$ gives a holomorphic splitting, as in (2.20), of the Atiyah exact sequence for E_*. □

Corollary 2.5. *Giving a connection on the parabolic bundle E_* is equivalent to giving a holomorphic splitting* $\mathbf{h}$ *of the Atiyah exact sequence for E_* (see* (2.12)*) such that the logarithmic connection D on E associated to* $\mathbf{h}$ *(see Lemma 2.4) satisfies the following condition: for every $x_i \in S$, the residue* $\mathrm{Res}(D,\, x_i)$ *induces the endomorphism* $\alpha_{i,j} \cdot \mathrm{Id}_{E_{i,j}/E_{i,j+1}}$ *of the quotient space $E_{i,j}/E_{i,j+1}$ for all* $1 \leq j \leq l_i$.

Proof. Note that from Lemma 2.4 we know that

$$\mathrm{Res}(D,\, x_i)(E_{i,j}) \subset E_{i,j}$$

for all $x_i \in S$ and all $1 \leq j \leq l_i$. Therefore, $\mathrm{Res}(D,\, x_i)$ induces an endomorphism of the quotient space $E_{i,j}/E_{i,j+1}$.

The result follows from Lemma 2.4 and the definition of a connection on E_*. □

2.3. **A homomorphism associated to a connection.** Let $E_* = (E, (\{E_{i,j}\}, \{\alpha_{i,j}\}))$ be a parabolic vector bundle on X with parabolic divisor S. Let D be a connection on E_*. Using D we will construct a first order holomorphic differential operator

$$\mathcal{D}_0 : \text{End}^P(E_*) \longrightarrow \text{End}^n(E_*) \otimes K_X \otimes \mathcal{O}_X(S) \tag{2.23}$$

(see (2.5) and (2.6) for $\text{End}^P(E_*)$ and $\text{End}^n(E_*)$ respectively).

To construct $\mathcal{D}_0$, take any holomorphic section $\Phi \in \Gamma(U, \text{End}(E))$, where $U \subset X$ is any open subset. Let

$$A_U : \Gamma(U, E) \longrightarrow \Gamma(U, E \otimes K_X \otimes \mathcal{O}_X(S))$$

be the homomorphism defined by

$$s \longmapsto D(\Phi(s)) - (\Phi \otimes \text{Id}_{K_X \otimes \mathcal{O}_X(S)})(D(s)) .$$

This A_U is evidently $\mathcal{O}_U$–linear. Hence we have

$$A_U \in \Gamma(U, \text{End}(E) \otimes K_X \otimes \mathcal{O}_X(S)) .$$

From the properties of D it follows that

$$A_U \in \Gamma(U, \text{End}^n(E_*) \otimes K_X \otimes \mathcal{O}_X(S)) \subset \Gamma(U, \text{End}(E) \otimes K_X \otimes \mathcal{O}_X(S))$$

if $\Phi \in \Gamma(U, \text{End}^P(E_*))$. The homomorphism $\mathcal{D}_0$ in (2.23) is defined by $\Phi \longmapsto A_U$.

Recall from (2.12) that $\text{End}^P(E_*)$ is a holomorphic subbundle of $\text{At}(E_*)$. We will now extend $\mathcal{D}_0$ in (2.23) to a first order holomorphic differential operator

$$\mathcal{D} : \text{At}(E_*) \longrightarrow \text{End}^n(E_*) \otimes K_X \otimes \mathcal{O}_X(S). \tag{2.24}$$

To construct $\mathcal{D}$, take holomorphic sections

$$\Phi \in \Gamma(U, \text{At}(E_*)) \quad \text{and} \quad v \in \Gamma(U, TX \otimes \mathcal{O}_X(-S)),$$

where $U \subset X$ is any open subset. Denote

$$w := \sigma(\Phi) \in \Gamma(U, TX \otimes \mathcal{O}_X(-S)), \tag{2.25}$$

where σ is the projection in (2.12). Take any $s \in \Gamma(U, E)$. So

$$\Phi(s) \in \Gamma(U, E)$$

(recall from (2.11) that $\text{At}(E_*) \subset \text{Diff}^1(E, E)$), and hence

$$D(\Phi(s)) \in \Gamma(U, E \otimes K_X \otimes \mathcal{O}_X(S)) .$$

Therefore, we have

$$\langle D(\Phi(s)), v \rangle \in \Gamma(U, E) , \tag{2.26}$$

where $\langle -, - \rangle$ is the natural duality pairing

$$(K_X \otimes \mathcal{O}_X(S)) \otimes (TX \otimes \mathcal{O}_X(-S)) \longrightarrow \mathcal{O}_X . \tag{2.27}$$

We have $\langle D(s), v \rangle \in \Gamma(U, E)$, so

$$\Phi(\langle D(s), v \rangle) \in \Gamma(U, E) , \tag{2.28}$$

where $\langle -, - \rangle$ is the pairing in (2.27). Consider the Lie bracket of vector fields

$$[v, w] \in \Gamma(U, TX \otimes \mathcal{O}_X(-S)) ,$$

where w is defined in (2.25). We have

$$\langle D(s),\, [v,\, w]\rangle \,\in\, \Gamma(U,\, E)\,. \tag{2.29}$$

Let $B_U\,:\,\Gamma(U,\, E)\,\longrightarrow\,\Gamma(U,\, E)$ be the homomorphism defined by

$$s\,\longmapsto\,\langle D(\Phi(s)),\, v\rangle - \Phi(\langle D(s),\, v\rangle) - \langle D(s),\, [v,\, w]\rangle$$

(see (2.26), (2.28) and (2.29)). This homomorphism B_U is evidently $\mathcal{O}_U$–linear, and hence we have

$$B_U\,\in\,\Gamma(U,\, \mathrm{End}(E))\,. \tag{2.30}$$

It is now straightforward to check that

$$B_U\,\in\,\Gamma(U,\, \mathrm{End}^n(E_*))\,\subset\,\Gamma(U,\, \mathrm{End}(E))\,.$$

The homomorphism $\mathcal{D}$ in (2.24) is uniquely defined by the following property: For any open subset $U\,\subset\, X$ and sections

$$\Phi\,\in\,\Gamma(U,\, \mathrm{At}(E_*)) \quad\text{and}\quad v\,\in\,\Gamma(U,\, TX\otimes\mathcal{O}_X(-S)),$$

the equality

$$\langle\mathcal{D}(\Phi),\, v\rangle\,=\,B_U$$

holds, where B_U is constructed in (2.30) from Φ and v using D, and $\langle -,\, -\rangle$ is the pairing in (2.27).

From the constructions of $\mathcal{D}$ and $\mathcal{D}_0$ (see (2.23)) it follows immediately that the restriction of $\mathcal{D}$ to $\mathrm{End}^P(E_*)\,\subset\,\mathrm{At}(E_*)$ actually coincides with $\mathcal{D}_0$.

3. Infinitesimal deformations and isomonodromy

3.1. **Infinitesimal deformations.** Let E_* be a parabolic vector bundle over X with parabolic divisor S. Then the infinitesimal deformations of E_*, keeping the pair $(X,\, S)$ fixed, are parametrized by $H^1(X,\, \mathrm{End}^P(E_*))$, where $\mathrm{End}^P(E_*)$ is defined in (2.5) [MS]. We recall that the infinitesimal deformations of the pair $(X,\, S)$ are parametrized by $H^1(X,\, TX\otimes\mathcal{O}_X(-S))$.

Lemma 3.1. *The infinitesimal deformations of the triple*

$$(X,\, S,\, E_*)$$

are parametrized by $H^1(X,\, \mathrm{At}(E_*))$*, where* $\mathrm{At}(E_*)$ *is defined in* (2.11)*.*

Proof. The proof is very similar to that of [Ch2, p. 1413, Proposition 4.3] (see also [Ch1]).

The lemma actually follows from [Ch2, p. 1413, Proposition 4.3] once we invoke the correspondence between the parabolic bundles and the orbifold bundles. This is explained below.

For any $x_i\,\in\, S$, let N_i be the smallest positive integer such that for all $1\,\leq\, j\,\leq\, l_i$,

$$\alpha_{i,j}\,=\,\frac{m_{i,j}}{N_i}\,,$$

where $m_{i,j}$ are nonnegative integers; see (2.3) and Assumption 2.1. There is a ramified Galois covering

$$\varphi\,:\,Y\,\longrightarrow\, X \tag{3.1}$$

satisfying the following two conditions:

- φ is unramified over the complement $X \setminus S$, and
- for every $x_i \in S$ and one (hence every) point $y \in \varphi^{-1}(x_i)$, the order of ramification of φ at y is N_i.

Such a covering φ exists; see [Na, p. 26, Proposition 1.2.12] and Assumption 2.1.

Let $\Gamma_\varphi = \text{Gal}(\varphi) \subset \text{Aut}(Y)$ be the Galois group for the ramified covering φ, so $X = Y/\Gamma_\varphi$. An equivariant vector bundle over Y is a holomorphic vector bundle $V \longrightarrow Y$ equipped with a lift of the action of Γ_φ. In other words,

- Γ_φ acts holomorphically on the total space of V, and
- the action of any $g \in \Gamma_\varphi$ on V is a holomorphic automorphism of the vector bundle V over the automorphism g of Y. In particular, the projection map $V \longrightarrow Y$ is Γ_φ–equivariant.

There is a natural equivalence of categories between the parabolic vector bundles on X whose parabolic weights at each x_i are integral multiples of $\frac{1}{N_i}$ and the equivariant vector bundles on Y [Bi1], [Bo1], [Bo2].

Let F_* be a parabolic vector bundle on X whose parabolic weights at each x_i are integral multiples of $\frac{1}{N_i}$. The holomorphic vector bundle underlying F_* will be denoted by F. Let V be the equivariant vector bundle on Y corresponding to F_*. The action Γ_φ on V produces a homomorphism from Γ_φ to the group $\text{Aut}(\varphi_* V)$ of holomorphic automorphisms of the direct image $\varphi_* V$, over the identity map of X. Then we have

$$F = (\varphi_* V)^{\Gamma_\varphi} \subset \varphi_* V ,$$

where $(\varphi_* V)^{\Gamma_\varphi}$ denotes the invariant part for the action of Γ_φ on $\varphi_* V$.

Take a holomorphic family of compact Riemann surfaces equipped with n ordered marked points. Assume that this family is parametrized by T, and that there is a point $t_0 \in T$ such that the fiber over t_0 is the given pair (X, S). Then the construction of the ramified Galois covering of X, done in [Na, p. 26, Proposition 1.2.12], extends to produce a family of ramified Galois coverings of all Riemann surfaces over an open neighborhood of t_0 in T.

Let V be the equivariant bundle on Y corresponding to the parabolic vector bundle E_* in the lemma. Then $\text{End}^P(E_*)$ is the holomorphic vector bundle underlying the parabolic bundle that corresponds to the equivariant vector bundle $\text{End}(V)$ on Y. The action of Γ_φ on Y produces an action of Γ_φ on TY, and $TX \otimes \mathcal{O}_X(-S)$ is the holomorphic line bundle underlying the corresponding parabolic line bundle on X. The actions of Γ_φ on V and Y together produce an action of Γ_φ on $\text{At}(V)$. The Atiyah bundle $\text{At}(E_*)$ is the holomorphic vector bundle underlying the parabolic bundle corresponding to the equivariant vector bundle $\text{At}(V)$ on Y. This implies that

$$H^1(X, \text{At}(E_*)) = H^1(Y, \text{At}(V))^{\Gamma_\varphi} . \tag{3.2}$$

In view of(3.2), the lemma follows immediately from [Ch2, p. 1413, Proposition 4.3]. □

Let

$$0 \,=\, H^0(X,\, TX\otimes \mathcal{O}_X(-S)) \,\longrightarrow\, H^1\left(X,\, \text{End}^P(E_*)\right) \,\xrightarrow{p_1}\, H^1(X,\, \text{At}(E_*)) \tag{3.3}$$
$$\xrightarrow{p_2}\, H^1(X,\, TX\otimes \mathcal{O}_X(-S)) \,\longrightarrow\, H^2\left(X,\, \text{End}^P(E_*)\right) \,=\, 0$$

be the long exact sequence of cohomologies associated to the short exact sequence in (2.12); we have $H^0(X,\, TX\otimes \mathcal{O}_X(-S)) \,=\, 0$ by Assumption 2.1. The projection p_2 in (3.3) is the forgetful map that sends an infinitesimal deformation of the triple $(X,\, S,\, E_*)$ to the infinitesimal deformation of the pair $(X,\, S)$ obtained from it by simply forgetting E_*. The injective homomorphism p_1 in (3.3) sends an infinitesimal deformation of E_* to the infinitesimal deformation of $(X,\, S,\, E_*)$ obtained from it by keeping the pair $(X,\, S)$ fixed.

Lemma 3.2. *Assume that the parabolic bundle E_* has a connection D such that the local monodromy of D around each point of S is semisimple (meaning diagonalizable). Then the local monodromy of every connection on E_* around each point of S is also semisimple.*

Proof. Take a ramified Galois covering

$$\varphi\,:\, Y\,\longrightarrow\, X,$$

as in (3.1), satisfying the following two conditions:

- φ is unramified over the complement $X\setminus S$, and
- for every $x_i\,\in\, S$ and one (hence every) point $y\,\in\, \varphi^{-1}(x_i)$, the order of ramification of φ at y is N_i.

Let V be the equivariant vector bundle on Y corresponding to E_*. Then D corresponds to an equivariant holomorphic connection on V. The space of all connections on E_* is an affine space for the vector space $H^0(X,\, \text{End}^n(E_*)\otimes K_X\otimes\mathcal{O}_X(S))$, where $\text{End}^n(E_*)$ is defined in (2.6). On the other hand the space of all equivariant holomorphic connections on V is an affine space for the vector space $H^0(Y,\, \text{End}(V)\otimes K_Y)^\Gamma$, where Γ is the Galois group for the covering map φ.

Since we have

$$H^0(X,\, \text{End}^n(E_*)\otimes K_X\otimes\mathcal{O}_X(S)) \,=\, H^0(Y,\, \text{End}(V)\otimes K_Y)^\Gamma,$$

we conclude that every connection D' on E_* is given by an equivariant connection on V. This implies that the order of the local monodromy of D' around any point $x_i\,\in\, S$ is a sub-multiple of N_i. This implies that the local monodromy of D' around every point of S is semisimple. □

Let D be a connection on the parabolic bundle E_*. We assume that the local monodromy of D around each point of S is semisimple. As mentioned above, the space of all connections on E_* is an affine space for the vector space $H^0(X,\, \text{End}^n(E_*)\otimes K_X\otimes\mathcal{O}_X(S))$, where $\text{End}^n(E_*)$ is defined in (2.6). This implies that the infinitesimal deformations of the connection D, keeping $(X,\, S,\, E_*)$ fixed, are parametrized by $H^0(X,\, \text{End}^n(E_*)\otimes K_X\otimes\mathcal{O}_X(S))$.

Let $\mathcal{A}_\bullet$ be the following two-term complex of sheaves on X

$$\mathcal{A}_\bullet\ :\ \mathcal{A}_0\,=\,\text{End}^P(E_*)\,\xrightarrow{\mathcal{D}_0}\,\mathcal{A}_1\,=\,\text{End}^n(E_*)\otimes K_X\otimes\mathcal{O}_X(S)\,, \tag{3.4}$$

where $\mathcal{D}_0$ is the differential operator in (2.23), and $\mathcal{A}_i$ is at the i-th position.

The following lemma is a standard fact.

Lemma 3.3. *The infinitesimal deformations of the pair* $(E_*,\, D)$*, keeping* $(X,\, S)$ *fixed, are parametrized by the first hypercohomology* $\mathbb{H}^1(\mathcal{A}_\bullet)$*, where* $\mathcal{A}_\bullet$ *is the complex in* (3.4)*.*

The complex $\mathcal{A}_\bullet$ in (3.4) fits in the following short exact sequence of complexes of sheaves on X

$$\begin{array}{ccccc}
 & 0 & & 0 \\
 & \downarrow & & \downarrow \\
 & 0 & \longrightarrow & \mathrm{End}^n(E_*)\otimes K_X\otimes \mathcal{O}_X(S) \\
 & \downarrow & & \downarrow \mathrm{id} \\
\mathcal{A}_\bullet \ : & \mathrm{End}^P(E_*) & \stackrel{\mathcal{D}_0}{\longrightarrow} & \mathrm{End}^n(E_*)\otimes K_X\otimes \mathcal{O}_X(S) \\
 & \downarrow \mathrm{id} & & \downarrow \\
 & \mathrm{End}^P(E_*) & \longrightarrow & 0 \\
 & \downarrow & & \downarrow \\
 & 0 & & 0
\end{array}$$

Let

$$\longrightarrow H^0(X,\, \mathrm{End}^n(E_*)\otimes K_X\otimes \mathcal{O}_X(S)) \stackrel{\alpha_1}{\longrightarrow} \mathbb{H}^1(\mathcal{A}_\bullet) \stackrel{\alpha_2}{\longrightarrow} H^1\left(X,\, \mathrm{End}^P(E_*)\right) \longrightarrow \tag{3.5}$$

be the long exact sequence of hypercohomologies associated to the above short exact sequence of complexes. The homomorphism α_2 in (3.5) sends an infinitesimal deformation of the pair $(E_*,\, D)$ to the infinitesimal deformation of E_* obtained from it by simply forgetting the connection. The homomorphism α_1 in (3.5) sends an infinitesimal deformation of the connection D to the infinitesimal deformation of the pair $(E_*,\, D)$ obtained from it by keeping the parabolic bundle E_* fixed.

Let $\mathcal{B}_\bullet$ denote the following two-term complex of sheaves on X

$$\mathcal{B}_\bullet \ : \ \mathcal{B}_0 \,=\, \mathrm{At}(E_*) \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{B}_1 \,=\, \mathrm{End}^n(E_*)\otimes K_X\otimes \mathcal{O}_X(S)\,, \tag{3.6}$$

where $\mathcal{D}$ is the homomorphism in (2.24), and $\mathcal{B}_i$ is at the i-th position.

Lemma 3.4. *The infinitesimal deformations of the quadruple*

$$(X,\, S,\, E_*,\, D)$$

are parametrized by the first hypercohomology $\mathbb{H}^1(\mathcal{B}_\bullet)$*, where* $\mathcal{B}_\bullet$ *is the complex in* (3.6)*.*

Proof. The proof is very similar to the proof of [Ch2, p. 1415, Proposition 4.4] (see also [Ch1]). In fact, it can also be deduced from [Ch2, p. 1415, Proposition 4.4], as done in the proof of Lemma 3.1. This is elaborated below.

Take the ramified Galois covering $(Y,\, \varphi)$ in (3.1). As in the proof of Lemma 3.1, V denotes the equivariant vector bundle on Y corresponding to the parabolic vector bundle E_*. The connection D on E_* corresponds to a Γ_φ–invariant holomorphic connection on V, where Γ_φ

is, as before, the Galois group for φ; this Γ_φ–invariant holomorphic connection on V will be denoted by D'. Let

$$\mathcal{B}'_\bullet \,:\, \mathcal{B}'_0 \,:=\, \text{At}(V) \xrightarrow{\mathcal{D}'} \mathcal{B}_1 \,:=\, \text{End}(V)\otimes K_Y \tag{3.7}$$

be the complex in [Ch2, p. 1415, Proposition 4.4] for D'; it is the same complex as in (3.6) when there is no parabolic structure (meaning the parabolic structure is trivial). We note that the differential operator $\mathcal{D}'$ in (3.7) is Γ_φ–equivariant, because the connection D' on V is Γ_φ–invariant. The holomorphic vector bundle $\text{End}^n(E_*)\otimes K_X\otimes \mathcal{O}_X(S)$ coincides with the holomorphic vector bundle underlying the parabolic bundle that corresponds to the Γ_φ–equivariant bundle $\text{End}(V)\otimes K_Y$ on Y. It was noted in the proof of Lemma 3.1 that $\text{At}(E_*)$ is the holomorphic vector bundle underlying the parabolic bundle corresponding to the equivariant bundle $\text{At}(V)$ on Y. Moreover, the operator $\mathcal{D}$ in (3.6) coincides with the one given by $\mathcal{D}'$ on the Γ_φ–invariant part of the direct image. These together imply that

$$\mathbb{H}^1(\mathcal{B}_\bullet) \,=\, \mathbb{H}^1(\mathcal{B}'_\bullet)^{\Gamma_\varphi}\,, \tag{3.8}$$

where $\mathbb{H}^1(\mathcal{B}'_\bullet)^{\Gamma_\varphi} \subset \mathbb{H}^1(\mathcal{B}'_\bullet)$ is the invariant part for the action of Γ_φ.

In view of (3.8), the lemma follows from [Ch2, p. 1415, Proposition 4.4]. □

The complex $\mathcal{B}_\bullet$ in (3.6) fits in the following short exact sequence of complexes of sheaves on X

$$\begin{array}{cccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
\mathcal{A}_\bullet & : & \text{End}^P(E_*) & \xrightarrow{\mathcal{D}_0} & \text{End}^n(E_*)\otimes K_X\otimes\mathcal{O}_X(S) \\
 & & \downarrow & & \downarrow \text{id} \\
\mathcal{B}_\bullet & : & \text{At}(E_*) & \xrightarrow{\mathcal{D}} & \text{End}^n(E_*)\otimes K_X\otimes\mathcal{O}_X(S) \\
 & & \downarrow & & \downarrow \\
 & & TX\otimes\mathcal{O}_X(-S) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

where the vertical exact sequence in the left is the one in (2.12); see (3.4) and (3.6) for $\mathcal{A}_\bullet$ and $\mathcal{B}_\bullet$ respectively. Let

$$\longrightarrow \mathbb{H}^1(\mathcal{A}_\bullet) \xrightarrow{\beta_1} \mathbb{H}^1(\mathcal{B}_\bullet) \xrightarrow{\beta_2} H^1(X,\, TX\otimes\mathcal{O}_X(-S)) \longrightarrow \tag{3.9}$$

be the long exact sequence of hypercohomologies associated to the above short exact sequence of complexes. The homomorphism β_2 in (3.9) sends an infinitesimal deformation of the quadruple $(X,\, S,\, E_*,\, D)$ to the infinitesimal deformation of $(X,\, S)$ obtained from it by simply forgetting the pair $(E_*,\, D)$. The homomorphism β_1 in (3.9) sends an infinitesimal deformation of the pair $(E_*,\, D)$ to the infinitesimal deformation of the quadruple $(X,\, S,\, E_*,\, D)$ obtained from it by keeping the pair $(X,\, S)$ fixed.

3.2. **Character variety and the monodromy map.** Fix an integer $r \geq 1$. Fix a base point $b_0 \in X \setminus S$, and denote by $\widetilde{\text{Hom}}(\pi_1(X \setminus S, b_0), \text{GL}(r, \mathbb{C}))$ the space of all homomorphisms from $\pi_1(X \setminus S, b_0)$ to $\text{GL}(r, \mathbb{C})$. Given any $\rho \in \widetilde{\text{Hom}}(\pi_1(X \setminus S, b_0), \text{GL}(r, \mathbb{C}))$, we consider $\mathbb{C}^r$ as a $\pi_1(X \setminus S, b_0)$–module by combining ρ with the standard r–dimensional representation of $\text{GL}(r, \mathbb{C})$. We recall that ρ is called *completely reducible* if the $\pi_1(X \setminus S, b_0)$–module $\mathbb{C}^r$ corresponding to ρ is a direct sum of irreducible $\pi_1(X \setminus S, b_0)$–modules. Let

$$\text{Hom}(\pi_1(X \setminus S, b_0), \text{GL}(r, \mathbb{C})) \subset \widetilde{\text{Hom}}(\pi_1(X \setminus S, b_0), \text{GL}(r, \mathbb{C}))$$

be the space of all completely reducible representations. The adjoint action of $\text{GL}(r, \mathbb{C})$ on itself produces an action of $\text{GL}(r, \mathbb{C})$ on $\text{Hom}(\pi_1(X \setminus S, b_0), \text{GL}(r, \mathbb{C}))$. The quotient space

$$\mathcal{R}_X(r) = \text{Hom}(\pi_1(X \setminus S, b_0), \text{GL}(r, \mathbb{C}))/\text{GL}(r, \mathbb{C}) \tag{3.10}$$

is a normal quasiprojective variety defined over $\mathbb{C}$. (See [Si], [LS], and references therein, for $\mathcal{R}_X(r)$.)

For another base point $b'_0 \in X \setminus S$, the two groups $\pi_1(X \setminus S, b_0)$ and $\pi_1(X \setminus S, b'_0)$ are identified up to inner automorphisms, and hence $\mathcal{R}_X(r)$ does not depend on the choice of the base point b_0. The complex structure of X does not play any role in the construction of $\mathcal{R}_X(r)$; the space $\mathcal{R}_X(r)$ depends only on the topological surface underlying $X \setminus S$.

A connection D on a parabolic vector bundle E_* is called *completely reducible* if it is a direct sum of irreducible logarithmic connections.

Fix the dimensions of the subspaces $E_{i,j}$ and fix the parabolic weights $\alpha_{i,j}$; see (2.2), (2.3). Let $\mathcal{M}_X(r)$ denote the moduli space of pairs (E_*, D), where E_* is a parabolic vector bundle of rank r on X having the given parabolic type and D is a completely reducible connection on E_*; see [In], [IIS1], [IIS2], [Iw], [BBP] and references therein for the moduli space.

Since any logarithmic connection on a holomorphic vector bundle on X is flat, we can associate a monodromy representation to any logarithmic connection. Consequently, we have a holomorphic map

$$\mathbb{M}_X : \mathcal{M}_X(r) \longrightarrow \mathcal{R}_X(r), \tag{3.11}$$

where $\mathcal{R}_X(r)$ is constructed in (3.10), that sends any connection to its monodromy.

Let

$$\varpi : X_T \longrightarrow T \tag{3.12}$$

be a holomorphic family of compact connected Riemann surfaces parametrized by a simply connected complex manifold T. For any $t \in T$, the fiber $\varpi^{-1}(t)$ will be denoted by X_t. For $1 \leq i \leq n$, let

$$\phi_i : T \longrightarrow X_T \tag{3.13}$$

be a holomorphic section such that $\phi_i(T) \bigcap \phi_j(T) = \emptyset$ for all $i \neq j$. Fix a base point $t_0 \in T$. Denote X_{t_0} by X, and also denote $s_i(t_0)$ by x_i for every $1 \leq i \leq n$. As before, denote the subset $\{x_1, \cdots, x_n\} \subset X$ by S. For any $t \in T$, the subset $\{\phi_1(t), \cdots, \phi_n(t)\} \subset X_t := \varpi^{-1}(t)$ will be denoted by S_t.

Since the parameter space T is simply connected, $\pi_1(X \setminus S, b_0)$ and $\pi_1(X_t \setminus S_t, b_t)$ are identified up to inner automorphisms. This implies that the character variety $\mathcal{R}_{X_t}(r)$ constructed as in (3.10) is canonically identified with $\mathcal{R}_X(r)$.

Let

$$\mathcal{M}^T(r) \longrightarrow T \tag{3.14}$$

be the relative moduli space of parabolic bundles with connection for the family X_T in (3.12). In view of the above observation that $\mathcal{R}_{X_t}(r)$ is identified with $\mathcal{R}_X(r)$, the monodromy maps $\mathbb{M}_X$ in (3.11) for points of T actually fit together to produce a holomorphic map

$$\mathbb{M} : \mathcal{M}^T(r) \longrightarrow \mathcal{R}_X(r)\,. \tag{3.15}$$

Let E^T_* be a holomorphic family of parabolic bundles on X_T, and let D^T be a relative connection E^T_*. In other words, the pair $(E^T_*,\, D^T)$ corresponds to a holomorphic section

$$\Psi : T \longrightarrow \mathcal{M}^T(r) \tag{3.16}$$

of the holomorphic family of moduli spaces in (3.14).

A holomorphic family of the above type $(E^T_*,\, D^T)$, of parabolic bundles with connection, is called isomonodromic if the composition $\mathbb{M}\circ\Psi$ is a constant map, where $\mathbb{M}$ is the monodromy map in (3.15) and Ψ is the map in (3.16). This condition of being isomonodromy defines a holomorphic foliation on $\mathcal{M}^T(r)$ which is transversal to the holomorphic foliation given by the projection $\mathcal{M}^T(r) \longrightarrow T$ in (3.14). In other words, the direct sum of these two distributions coincides with the full tangent bundle $T\mathcal{M}^T(r) \longrightarrow \mathcal{M}^T(r)$, on the smooth locus of $\mathcal{M}^T(r)$.

Let D be a connection on a parabolic vector bundle E_* over X with parabolic structure over S. We assume that the local monodromy of D around each point of S is semisimple. The foliation given by isomonodromy produces a homomorphism from the space of infinitesimal deformations of the pair $(X,\, S)$ to the space of infinitesimal deformations of the quadruple $(X,\, S,\, E_*,\, D)$. In view of Lemma 3.4, this homomorphism, from the space of infinitesimal deformations of $(X,\, S)$ to the space of infinitesimal deformations of $(X,\, S,\, E_*,\, D)$, is given by a homomorphism

$$\gamma : H^1(X,\, TX\otimes \mathcal{O}_X(-S)) \longrightarrow \mathbb{H}^1(\mathcal{B}_\bullet)\,, \tag{3.17}$$

where $\mathcal{B}_\bullet$ is the complex in (3.6).

From Corollary 2.5 we know that the connection D produces a homomorphism

$$\mathbf{h} : TX\otimes \mathcal{O}_X(-S) \longrightarrow \mathrm{At}(E_*)$$

such that $\sigma\circ\mathbf{h} = \mathrm{Id}_{TX\otimes\mathcal{O}_X(-S)}$, where σ is the projection in (2.12). This homomorphism $\mathbf{h}$ produces a homomorphism of complexes of sheaves on X

$$\begin{array}{ccccc} & TX\otimes \mathcal{O}_X(-S) & \longrightarrow & 0 \\ & \big\downarrow \mathbf{h} & & \big\downarrow \\ \mathcal{B}_\bullet\ : & \mathrm{At}(E_*) & \stackrel{\mathcal{D}}{\longrightarrow} & \mathrm{End}^n(E_*)\otimes K_X\otimes\mathcal{O}_X(S) \end{array} \tag{3.18}$$

We note that $\mathcal{D}\circ\mathbf{h} = 0$ because the connection D is flat. The homomorphism of complexes in (3.18) produces a homomorphism of hypercohomologies

$$\delta : H^1(X,\, TX\otimes \mathcal{O}_X(-S)) \longrightarrow \mathbb{H}^1(\mathcal{B}_\bullet)\,. \tag{3.19}$$

Lemma 3.5. *The homomorphism* γ *constructed in* (3.17) *coincides with the homomorphism* δ *constructed in* (3.19).

Proof. The proof of Lemma 3.5 is similar to the proof of [Ch2, p. 1417, Proposition 5.1] (see also [Ch1]). In fact, as done in the proofs of Lemma 3.1 and Lemma 3.4, the lemma can be deduced from [Ch2, p. 1417, Proposition 5.1] using the correspondence between the parabolic bundles and the orbifold bundles. We omit the details. □

4. Infinitesimal deformations of parabolic opers

We recall from [BDP] the definition of parabolic $\text{SL}(r, \mathbb{C})$-opers.

Consider the subset $S \subset X$ in (2.1); assume that the integer $n = \#S$ is even. Fix a holomorphic line bundle $\mathbb{L}$ on X of degree $-\frac{n}{2}$ such that $\mathbb{L}^{\otimes 2}$ is isomorphic to $\mathcal{O}_X(-S)$; also fix a holomorphic isomorphism

$$\varphi_0 : \mathbb{L}^{\otimes 2} \longrightarrow \mathcal{O}_X(-S)$$

of line bundles. Fix a holomorphic line bundle $K_X^{1/2}$ on X of degree $\text{genus}(X) - 1$ such that $\left(K_X^{1/2}\right)^{\otimes 2}$ is isomorphic to K_X, in other words, $K_X^{1/2}$ is a theta characteristic on X; fix a holomorphic isomorphism

$$I_X : \left(K_X^{1/2}\right)^{\otimes 2} \longrightarrow K_X$$

of line bundles. Since

$$H^1(X, \text{Hom}(K_X^{-1/2} \otimes \mathbb{L}, K_X^{1/2} \otimes \mathbb{L})) = H^1(X, K_X) = H^0(X, \mathcal{O}_X)^* = \mathbb{C}$$

(the isomorphism $H^1(X, K_X) = H^0(X, \mathcal{O}_X)^*$ is given by Serre duality), we have a nontrivial extension

$$0 \longrightarrow K_X^{1/2} \otimes \mathbb{L} \longrightarrow \mathcal{E} \xrightarrow{q} K_X^{-1/2} \otimes \mathbb{L} \longrightarrow 0 \tag{4.1}$$

corresponding to $1 \in H^1(X, \text{Hom}(K_X^{-1/2} \otimes \mathbb{L}, K_X^{1/2} \otimes \mathbb{L})) = \mathbb{C}$.

We will put a parabolic structure on the rank two vector bundle $\mathcal{E}$ in (4.1). Fix a function

$$\mathbf{c} : S \longrightarrow \{t \in \mathbb{Z} \mid t \geq 2\}; \tag{4.2}$$

the integer $\mathbf{c}(x_i)$ will also be denoted by c_i.

Equip the vector bundle $\mathcal{E}$ in (4.1) with the following parabolic structure over S: For any $x_i \in S$, the quasiparabolic filtration of $\mathcal{E}_{x_i}$ is

$$(K_X^{1/2} \otimes \mathbb{L})_{x_i} \subset \mathcal{E}_{x_i},$$

where $K_X^{1/2} \otimes \mathbb{L}$ is the line subbundle in (4.1). The parabolic weight of $(K_X^{1/2} \otimes \mathbb{L})_{x_i}$ is $\frac{2c_i - 1}{2c_i}$ and the parabolic weight of $\mathcal{E}_{x_i}$ is $\frac{1}{2c_i}$ (see (4.2)). The resulting parabolic vector bundle of rank two on X will be denoted by $\mathcal{E}_*$. The parabolic degree of $\mathcal{E}_*$ is zero. We note that the parabolic exterior product $\bigwedge^2 \mathcal{E}_*$ of $\mathcal{E}_*$ is the trivial holomorphic line bundle on X equipped with the trivial holomorphic structure; see [BDP, Section 5], [Bi2] for the parabolic exterior product and the parabolic symmetric product.

Remark 4.1. It can be shown that the above rank two parabolic bundle $\mathcal{E}_*$ is indecomposable. To prove this assume that $\mathcal{E}_* = L_* \oplus L'_*$, where L_* and L'_* are parabolic line bundles. Suppose that $\text{par} - \deg(L_*) \geq \text{par} - \deg(L'_*)$. Consider the composition of homomorphism

$$L_* \hookrightarrow \mathcal{E}_* \xrightarrow{q} (K_X^{-1/2} \otimes \mathbb{L})_*,$$

where q is the projection in (4.1) and $(K_X^{-1/2}\otimes\mathbb{L})_*$ is the quotient line bundle $K_X^{-1/2}\otimes\mathbb{L}$ in (4.1) equipped with the parabolic structure induced by $\mathcal{E}_*$. This composition of homomorphisms vanishes identically, because $\text{par}-\deg(L_*) > \text{par}-\deg((K_X^{-1/2}\otimes\mathbb{L})_*)$. This implies that the short exact sequence in (4.1) splits. But the short exact sequence in (4.1) does not split. In view of this contradiction we conclude that the parabolic bundle $\mathcal{E}_*$ is indecomposable.

We note that a connection on a parabolic vector bundle V_* induces connections on $\bigwedge^i V_*$ and $\text{Sym}^i(V_*)$ for every i.

Take any integer $r\geq 2$. Let $\text{Sym}^{r-1}(\mathcal{E}_*)$ be the parabolic symmetric product of $\mathcal{E}_*$. We have $\text{rank}(\text{Sym}^{r-1}(\mathcal{E}_*)) = r$, and the parabolic exterior product $\bigwedge^r \text{Sym}^{r-1}(\mathcal{E}_*)$ is the trivial holomorphic line bundle on X equipped with the trivial parabolic structure.

A *parabolic* $\text{SL}(r,\mathbb{C})$*–oper* on X is a connection D on the parabolic vector bundle $\text{Sym}^{r-1}(\mathcal{E}_*)$ such that the connection on $\bigwedge^r \text{Sym}^{r-1}(\mathcal{E}_*)$ induced by D coincides with the connection on $\mathcal{O}_X$ given by the de Rham differential d; see [BDP, p. 511, Definition 5.2].

Lemma 4.2. *For any parabolic* $\text{SL}(r,\mathbb{C})$*–oper connection D, the local monodromy of D around any point of S is semisimple.*

Proof. First note that the parabolic $\mathcal{E}_*$ admits a connection, because $\mathcal{E}_*$ is indecomposable (this was shown in Remark 4.1) and its parabolic degree is zero [BL, p. 594, Theorem 1.1]. Since the two parabolic weights at every $x_i \in S$ do not differ by an integer, we conclude that for any connection on the parabolic $\mathcal{E}_*$ the local monodromy around any point of S is semisimple (see (2.17)). Therefore, the connection on $\text{Sym}^{r-1}(\mathcal{E}_*)$ induced by a connection on $\mathcal{E}_*$ also has the property that the local monodromy around every point of S is semisimple. From this it follows that every connection on $\text{Sym}^{r-1}(\mathcal{E}_*)$ has the property that the local monodromy around every point of S is semisimple; see Lemma 3.2. □

The subbundle $K_X^{1/2}\otimes\mathbb{L}\subset\mathcal{E}$ in (4.1) equipped with the induced parabolic structure will be denoted by F_*. So the holomorphic line bundle underlying F_* is $K_X^{1/2}\otimes\mathbb{L}$, and the parabolic weight of F_* and any $x_i\in S$ is $\frac{2c_i-1}{2c_i}$. The parabolic subbundle

$$F_* \stackrel{\iota_F}{\hookrightarrow} \mathcal{E}_* \tag{4.3}$$

produces a filtration of parabolic subbundles of $\text{Sym}^{r-1}(\mathcal{E}_*)$ in the following way. For any $0\leq j\leq r-1$, consider the parabolic vector bundle $(F_*)^{\otimes(r-1-j)}\otimes\text{Sym}^j(\mathcal{E}_*)$; by definition, $(F_*)^{\otimes 0}$ and $\text{Sym}^0(\mathcal{E}_*)$ coincide with the trivial holomorphic line bundle with the trivial parabolic structure (see [Bi2], [BDP] for the tensor product of parabolic vector bundles). So the rank of $(F_*)^{\otimes(r-1-j)}\otimes\text{Sym}^j(\mathcal{E}_*)$ is $j+1$. This parabolic vector bundle $(F_*)^{\otimes(r-1-j)}\otimes\text{Sym}^j(\mathcal{E}_*)$ will be denoted by $\mathcal{F}_*^{(j+1)}$ (since $j+1$ is its rank). We note that

$$\mathcal{F}_*^{(1)}\subset\mathcal{F}_*^{(2)}\subset\cdots\subset\mathcal{F}_*^{(r-1)}\subset\mathcal{F}_*^{(r)} = \text{Sym}^{r-1}(\mathcal{E}_*) \tag{4.4}$$

is a filtration of parabolic subbundles of $\text{Sym}^{r-1}(\mathcal{E}_*)$. For any $1\leq i\leq r-1$, the inclusion map

$$\mathcal{F}_*^{(i)}\hookrightarrow\mathcal{F}_*^{(i+1)}$$

in (4.4) is constructed, in a straightforward way, using the inclusion map $F_* \hookrightarrow \mathcal{E}_*$ in (4.3) together with the natural projection $\mathcal{E}_* \otimes \text{Sym}^j(\mathcal{E}_*) \longrightarrow \text{Sym}^{j+1}(\mathcal{E}_*)$. More, precisely, we have

$$(\iota_F)^{\otimes(r-i)} \,:\, (F_*)^{\otimes(r-i)} \hookrightarrow (F_*)^{\otimes(r-i-1)} \otimes \mathcal{E}_*,$$

where ι_F is the inclusion map in (4.3). This implies that

$$\mathcal{F}^{(i)}_* \,=\, (F_*)^{\otimes(r-i)} \otimes \text{Sym}^{i-1}(\mathcal{E}_*) \hookrightarrow (F_*)^{\otimes(r-i-1)} \otimes \mathcal{E}_* \otimes \text{Sym}^{i-1}(\mathcal{E}_*)$$
$$\longrightarrow (F_*)^{\otimes(r-i-1)} \otimes \text{Sym}^{i}(\mathcal{E}_*) \,=\, \mathcal{F}^{(i+1)}_*\,.$$

The above composition of homomorphisms produces the inclusion map $\mathcal{F}^{(i)}_* \subset \mathcal{F}^{(i+1)}_*$ in (4.4).

For any $1 \leq j \leq r$, the holomorphic vector bundle of rank j underlying the parabolic vector bundle $\mathcal{F}^{(j)}_*$ will be denoted by $\mathcal{F}^{(j)}$. So the filtration in (4.4) produce a filtration of holomorphic subbundles

$$\mathcal{F}^{(1)} \subset \mathcal{F}^{(2)} \subset \cdots \subset \mathcal{F}^{(r-1)} \subset \mathcal{F}^{(r)} \tag{4.5}$$

of $\mathcal{F}^{(r)}$. Note that $\mathcal{F}^{(r)}$ is the holomorphic vector bundle underlying the parabolic vector bundle $\text{Sym}^{r-1}(\mathcal{E}_*)$.

Remark 4.3. It should be clarified that although the holomorphic vector bundle underlying the parabolic bundle $\mathcal{E}_*$, namely the holomorphic vector bundle $\mathcal{E}$, does not depend on the function $\mathbf{c}$ in (4.2), the vector bundle $\mathcal{F}^{(r)}$ depends on $\mathbf{c}$ in general. It should also be mentioned that the fact that the parabolic exterior product $\bigwedge^r \text{Sym}^{r-1}(\mathcal{E}_*)$ is the trivial holomorphic line bundle equipped with the trivial parabolic structure does not imply that $\bigwedge^r \mathcal{F}^{(r)}$ is the trivial holomorphic line bundle. In fact we have degree $\left(\bigwedge^r \mathcal{F}^{(r)}\right) \,<\, 0$. Note that degree$(\mathcal{E}) \,=\, -n$.

Remark 4.4. It is a straightforward computation to check that

$$\bigwedge^r \mathcal{F}^{(r)} \,=\, \mathcal{O}_X(-\sum_{i=1}^n d_i x_i)\,,$$

where

$$d_i \,=\, \sum_{k=0}^{r-1}\left(\frac{2k(c_i-1)+r-1}{2c_i} - \left\lfloor \frac{2k(c_i-1)+r-1}{2c_i}\right\rfloor\right)\,;$$

the integral part of $b \,\in\, \mathbb{Q}$ is denoted $\lfloor b \rfloor$, so $\lfloor b \rfloor \,\in\, \mathbb{Z}$ and $0 \,\leq\, b - \lfloor b \rfloor \,<\, 1$. It is straightforward to check that each d_i is an integer.

Consider the Atiyah bundle

$$\text{At}\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right) \,\subset\, \text{Diff}^1\left(\mathcal{F}^{(r)},\, \mathcal{F}^{(r)}\right)$$

constructed as in (2.11) for the parabolic bundle $E_* \,=\, \text{Sym}^{r-1}(\mathcal{E}_*)$, where $\mathcal{F}^{(r)}$ is the vector bundle in (4.5) and $\text{Sym}^{r-1}(\mathcal{E}_*)$ is the parabolic vector bundle in (4.4). Let

$$\text{At}'_X(r) \,\subset\, \text{At}\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right) \,\subset\, \text{Diff}^1\left(\mathcal{F}^{(r)},\, \mathcal{F}^{(r)}\right) \tag{4.6}$$

be the holomorphic subbundle of $\text{At}\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)$ constructed as follows: The space of all holomorphic sections of $\text{At}'_X(r)$ over any open subset $U \subset X$ is the space of all first order differential operations

$$D_U : \Gamma\left(U, \text{Sym}^{r-1}(\mathcal{E}_*)\right) \longrightarrow \Gamma\left(U, \text{Sym}^{r-1}(\mathcal{E}_*)\right)$$

such that

- $D_U \in \Gamma\left(U, \text{At}\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)\right)$, and
- $D_U\left(\mathcal{F}^{(j)}\right) \subset \mathcal{F}^{(j)}$ for all $1 \leq j \leq r$, where $\mathcal{F}^{(j)}$ is the subbundle in (4.5).

Consider the subbundle

$$\mathcal{O}_X \subset \text{Hom}\left(\mathcal{F}^{(r)}, \mathcal{F}^{(r)}\right) = \text{Diff}^0\left(\mathcal{F}^{(r)}, \mathcal{F}^{(r)}\right) \subset \text{Diff}^1\left(\mathcal{F}^{(r)}, \mathcal{F}^{(r)}\right)$$

given by pointwise multiplication. We note that

$$\mathcal{O}_X \subset \text{At}'_X(r),$$

where $\text{At}'_X(r)$ is the subsheaf in (4.6). Let

$$\text{At}_X(r) := \text{At}'_X(r)/\mathcal{O}_X \tag{4.7}$$

be the quotient bundle.

Construct the holomorphic vector bundle $\text{End}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)$ by substituting, in (2.6), the parabolic vector bundle $\text{Sym}^{r-1}(\mathcal{E}_*)$ in place of E_*. Let

$$\text{ad}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right) \subset \text{End}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)$$

be the holomorphic subbundle of co-rank one given by the intersection of $\text{End}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)$ with the sheaf of endomorphisms of $\mathcal{F}^{(r)}$ of trace zero. So using the natural inclusion map $\mathcal{O}_X(-S) \hookrightarrow \text{End}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)$ defined by pointwise multiplication, we have

$$\text{End}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right) = \text{ad}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right) \oplus \mathcal{O}_X(-S). \tag{4.8}$$

Let

$$\text{End}^n_1\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right) \subset \text{End}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)$$

be the subbundle defined by imposing the condition that the subbundle $\mathcal{F}^{(i)}$ in (4.5) is mapped to $\mathcal{F}^{(i+1)}$ for all $1 \leq i \leq r-1$. In other words, a locally defined holomorphic section s of the vector bundle $\text{End}^n(\text{Sym}^{r-1}(\mathcal{E}_*))$ is a locally defined section of $\text{End}^n_1\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)$ if and only if $s\left(\mathcal{F}^{(i)}\right) \subset \mathcal{F}^{(i+1)}$ for every $1 \leq i \leq r-1$. Define the intersection

$$\text{ad}^n_1\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right) := \text{ad}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right) \bigcap \text{End}^n_1\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right); \tag{4.9}$$

this intersection is taking place inside $\text{End}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)$. From (4.8) we have

$$\text{End}^n_1\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right) = \text{ad}^n_1\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right) \oplus \mathcal{O}_X(-S). \tag{4.10}$$

Now let D be a connection on the parabolic vector bundle $\text{Sym}^{r-1}(\mathcal{E}_*)$ defining a parabolic $\text{SL}(r, \mathbb{C})$–oper on X. In other words, the connection on $\bigwedge^r \text{Sym}^{r-1}(\mathcal{E}_*)$ induced by D coincides with the connection on $\mathcal{O}_X$ given by the de Rham differential d. Consider the first order holomorphic differential operator

$$\mathcal{D} : \text{At}\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right) \longrightarrow \text{End}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right) \otimes K_X \otimes \mathcal{O}_X(S) \tag{4.11}$$

constructed as in (2.24) from D; substitute $(\text{Sym}^{r-1}(\mathcal{E}_*),\, D)$ in place of $(E_*,\, D)$ in (2.24). The restriction of the differential operator $\mathcal{D}$ to the subbundle $\text{At}'_X(r)$ constructed in (4.6) has its image contained in

$$\text{End}^n_1\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)\otimes K_X\otimes \mathcal{O}_X(S)\ \subset\ \text{End}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)\otimes K_X\otimes \mathcal{O}_X(S)$$

(see (4.10)), in other words,

$$\mathcal{D}(\text{At}'_X(r))\ \subset\ \text{End}^n_1\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)\otimes K_X\otimes \mathcal{O}_X(S)\,. \tag{4.12}$$

Furthermore, the differential operator $\mathcal{D}$ takes the subbundle $\mathcal{O}_X\ \subset\ \text{At}'_X(r)$ (see (4.7)) to

$$K_X\ =\ \mathcal{O}_X(-S)\otimes K_X\otimes \mathcal{O}_X(S)\ \subset\ \text{End}^n_1\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)\otimes K_X\otimes \mathcal{O}_X(S)\,;$$

see (4.10) for the subbundle $\mathcal{O}_X(-S)\ \subset\ \text{End}^n_1\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)$. In fact, the restriction of $\mathcal{D}$ to $\mathcal{O}_X\ \subset\ \text{At}'_X(r)$ coincides with the de Rham differential d. Consequently, using (4.12) and the decomposition in (4.10), the differential operator $\mathcal{D}\big\vert_{\text{At}'_X(r)}$ in (4.11) produces a differential operator

$$\mathcal{D}_B\ :\ \text{At}_X(r)\ \longrightarrow\ \text{ad}^n_1\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)\otimes K_X\otimes \mathcal{O}_X(S)\,. \tag{4.13}$$

Let $\mathcal{C}_\bullet$ denote the following two-term complex of sheaves on X

$$\mathcal{C}_\bullet\ :\ \mathcal{C}_0\ =\ \text{At}_X(r)\ \xrightarrow{\mathcal{D}_B}\ \mathcal{C}_1\ =\ \text{ad}^n_1\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)\otimes K_X\otimes \mathcal{O}_X(S)\,, \tag{4.14}$$

where $\mathcal{D}_B$ is the homomorphism in (4.13), and $\mathcal{C}_i$ is at the i-th position.

Theorem 4.5. *The space of all infinitesimal deformation of the triple* $(X,\, S,\, D)$*, where* D *is a parabolic* $\text{SL}(r,\mathbb{C})$*–oper on* X*, is given by the hypercohomology*

$$\mathbb{H}^1(\mathcal{C}_\bullet),$$

where $\mathcal{C}_\bullet$ *is the complex in* (4.14).

Proof. As in (3.1), take a ramified Galois covering

$$\varphi\ :\ Y\ \longrightarrow\ X \tag{4.15}$$

satisfying the following two conditions:

- φ is unramified over the complement $X\setminus S$, and
- for every $x_i\ \in\ S$ and each point $y\ \in\ \varphi^{-1}(x_i)$, the order of ramification of ρ at y is $c_i\ =\ \mathbf{c}(x_i)$ (see (4.2)).

Such a covering φ exists by [Na, p. 26, Proposition 1.2.12]. As before, denote by Γ_φ the Galois group $\text{Gal}(\varphi)$ of the map φ. From [BDP, p. 514, Theorem 6.3] we know that the parabolic $\text{SL}(r,\mathbb{C})$–oper D on X corresponds to a Γ–invariant $\text{PSL}(r,\mathbb{C})$–oper $\mathbb{D}$ on Y. We note that there is a natural bijection between the $\text{PSL}(r,\mathbb{C})$–opers on Y and each connected component of the $\text{SL}(r,\mathbb{C})$–opers on Y. We will recall a description of this $\text{PSL}(r,\mathbb{C})$–oper $\mathbb{D}$ on Y corresponding to D.

We first note that the parabolic vector bundle $\text{Sym}^{r-1}(\mathcal{E}_*)$ defines a holomorphic parabolic principal $\text{SL}(r,\mathbb{C})$–bundle on X, because $\bigwedge^r \text{Sym}^{r-1}(\mathcal{E}_*)$ is the trivial parabolic line bundle;

see [BBN1], [BBN2], [BBP] for the parabolic analog of principal bundles. Using the quotient map $\mathrm{SL}(r, \mathbb{C}) \longrightarrow \mathrm{PSL}(r, \mathbb{C})$, a parabolic principal $\mathrm{SL}(r, \mathbb{C})$–bundle produces parabolic principal $\mathrm{PSL}(r, \mathbb{C})$–bundle. Let

$$\widetilde{\mathbb{P}}_* \longrightarrow X \tag{4.16}$$

denote the parabolic principal $\mathrm{PSL}(r, \mathbb{C})$–bundle given by the parabolic principal $\mathrm{SL}(r, \mathbb{C})$–bundle on X defined by $\mathrm{Sym}^{r-1}(\mathcal{E}_*)$.

Fix a Borel subgroup

$$B \subset \mathrm{PSL}(r, \mathbb{C}) . \tag{4.17}$$

The filtration of parabolic vector bundles $\{\mathcal{F}_*^{(j)}\}_{j=1}^{r-1}$ in (4.4) produces a reduction of structure group of the parabolic principal $\mathrm{PSL}(r, \mathbb{C})$–bundle $\widetilde{\mathbb{P}}_*$ in (4.16) to the subgroup B of $\mathrm{PSL}(r, \mathbb{C})$ in (4.17). Let

$$\widetilde{\mathbb{P}}(B)_* \subset \widetilde{\mathbb{P}}_* \tag{4.18}$$

be this reduction of structure group to B.

The parabolic principal $\mathrm{PSL}(r, \mathbb{C})$–bundle $\widetilde{\mathbb{P}}_*$ in (4.16) corresponds to an equivariant holomorphic principal $\mathrm{PSL}(r, \mathbb{C})$–bundle on Y [BBN2], [BBN1]; let

$$\mathbb{P}(r) \longrightarrow Y \tag{4.19}$$

be the equivariant holomorphic principal $\mathrm{PSL}(r, \mathbb{C})$–bundle corresponds to $\widetilde{\mathbb{P}}_*$. We note that $\mathbb{P}(r)$ is the unique holomorphic principal $\mathrm{PSL}(r, \mathbb{C})$–bundle on Y underlying the $\mathrm{PSL}(r, \mathbb{C})$–opers on Y. We will briefly recall a description of $\mathbb{P}(r)$.

Take a theta characteristic $K_Y^{1/2}$ on Y. Let

$$0 \longrightarrow K_Y^{1/2} \longrightarrow W \longrightarrow \left(K_Y^{1/2}\right)^* \longrightarrow 0 \tag{4.20}$$

be the nontrivial extension corresponding to

$$1 \in \mathbb{C} = H^0(Y, \mathcal{O}_Y)^* = H^1(Y, K_Y) = H^1(Y, \mathrm{Hom}((K_Y^{1/2})^*, K_Y^{1/2})) .$$

The holomorphic principal $\mathrm{PSL}(r, \mathbb{C})$–bundle $\mathbb{P}(r)$ in (4.19) coincides with the holomorphic principal $\mathrm{PSL}(r, \mathbb{C})$–bundle defined by the projective bundle $\mathbb{P}\left(\mathrm{Sym}^{r-1}(W)\right)$. (By $\mathbb{P}(V)$ we denote the projective bundle defined by the spaces of lines in the fibers of V.) While the vector bundle $\mathrm{Sym}^{r-1}(W)$ depends on the choice of the theta characteristic $K_Y^{1/2}$ when r is even (the vector bundle $\mathrm{Sym}^{r-1}(W)$ does not depend on the choice of $K_Y^{1/2}$ when r is odd), the projective bundle $\mathbb{P}\left(\mathrm{Sym}^{r-1}(W)\right)$ is actually independent of the choice of the theta characteristic $K_Y^{1/2}$. Indeed, if we replace $K_Y^{1/2}$ by $K_Y^{1/2} \otimes \xi$, where ξ is a holomorphic line bundle on Y of order two, then W gets replaced by $W \otimes \xi$, and hence $\mathrm{Sym}^{r-1}(W)$ gets replaced by $\mathrm{Sym}^{r-1}(W) \otimes \xi^{\otimes(r-1)}$.

Since any holomorphic automorphism of Y takes a theta characteristic on Y to a (possibly different) theta characteristic on Y, from the above property of $\mathbb{P}\left(\mathrm{Sym}^{r-1}(W)\right)$ it follows immediately that $\mathbb{P}(r)$ is an equivariant holomorphic principal $\mathrm{PSL}(r, \mathbb{C})$–bundle on Y. In other words, the action of Γ_φ on Y lifts to an holomorphic action of Γ_φ on $\mathbb{P}(r)$ that commutes with the action of $\mathrm{PSL}(r, \mathbb{C})$ on the principal bundle $\mathbb{P}(r)$.

In particular, $\mathbb{P}(W)$ is an equivariant vector bundle. The action of Γ_φ on $\mathbb{P}(W)$ preserves the holomorphic section of the projective bundle $\mathbb{P}(W)$ given by the line subbundle $K_Y^{1/2}$ in (4.20).

The connection D on the parabolic vector bundle $\text{Sym}^{r-1}(\mathcal{E}_*)$ produces a connection on the parabolic principal $\text{PSL}(r,\mathbb{C})$–bundle $\widetilde{\mathbb{P}}_*$ on X given by $\mathbb{P}\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)$. This connection on $\widetilde{\mathbb{P}}_*$ in turn produces a Γ_φ–invariant holomorphic connection on the principal $\text{PSL}(r,\mathbb{C})$–bundle $\mathbb{P}(r)$ on Y. This Γ_φ–invariant holomorphic connection on $\mathbb{P}(r)$ will be denoted by

$$D^Y. \tag{4.21}$$

Any holomorphic connection on $\mathbb{P}(r)$ is a $\text{PSL}(r,\mathbb{C})$–oper on Y [BD1], [BD2], [Fr1]. In particular, D^Y in (4.21) is a $\text{PSL}(r,\mathbb{C})$–oper on Y. Given an oper, a complex of sheaves was constructed by Sanders in [Sa]. We will briefly recall his construction of complex of sheaves for the $\text{PSL}(r,\mathbb{C})$–oper D^Y in (4.21).

Let

$$\mathcal{W}_1 \subset \mathcal{W}_2 \subset \cdots \subset \mathcal{W}_{r-1} \subset \mathcal{W}_r = \text{Sym}^{r-1}(W)$$

be the filtration of holomorphic subbundles, where $\mathcal{W}_j = (K_Y^{1/2})^{\otimes(r-j)} \otimes \text{Sym}^{j-1}(W)$ (see (4.20)); in particular $\text{rank}(\mathcal{W}_j) = j$. Let

$$\mathbb{P}(\mathcal{W}_1) \subset \mathbb{P}(\mathcal{W}_2) \subset \cdots \subset \mathbb{P}(\mathcal{W}_{r-1}) \subset \mathbb{P}(\mathcal{W}_r) = \mathbb{P}(\text{Sym}^{r-1}(W)) \tag{4.22}$$

be the corresponding filtration of projective bundles; recall that $\mathbb{P}(V)$ denotes the projective bundle defined by the spaces of lines in the fibers of V. The filtration in (4.22) produces a holomorphic reduction of structure group

$$\mathbb{P}(r)_B \subset \mathbb{P}(r) \tag{4.23}$$

to the Borel subgroup B in (4.17), where $\mathbb{P}(r)$ is the holomorphic principal $\text{PSL}(r,\mathbb{C})$–bundle in (4.19).

Recall that $\mathbb{P}(W)$ is an equivariant vector bundle, and the action of Γ_φ on $\mathbb{P}(W)$ preserves the holomorphic section of $\mathbb{P}(W)$ given by the line subbundle $K_Y^{1/2}$ in (4.20). Therefore, from the construction of the filtration in (4.22) it follows immediately that the action of Γ_φ on $\mathbb{P}(\text{Sym}^{r-1}(W))$ preserves each projective subbundle $\mathbb{P}(\mathcal{W}_j)$. Consequently, the action of Γ_φ on $\mathbb{P}(r)$ preserves the reduction of the structure group $\mathbb{P}(r)_B$ in (4.23). The principal B–bundle $\mathbb{P}(r)_B$ in (4.23) in fact corresponds to the reduction $\widetilde{\mathbb{P}}(B)_*$ in (4.18).

The holomorphic reduction $\mathbb{P}(r)_B$ in (4.23) coincides with the holomorphic reduction of structure group of $\mathbb{P}(r)$ to the subgroup $B \subset \text{PSL}(r,\mathbb{C})$ that appears in the definition of a $\text{PSL}(r,\mathbb{C})$–oper on Y. Let

$$\text{At}(\mathbb{P}(r)_B) \longrightarrow Y$$

be the Atiyah bundle for the holomorphic principal B–bundle $\mathbb{P}(r)_B$ (see [At]). Let

$$\text{At}(\mathbb{P}(r)) \longrightarrow Y$$

be the Atiyah bundle for the principal $\text{PSL}(r,\mathbb{C})$–bundle $\mathbb{P}(r)$ in (4.19). We have

$$\text{At}(\mathbb{P}(r)_B) \subset \text{At}(\mathbb{P}(r)) \tag{4.24}$$

because of the reduction of structure group in (4.23).

Let $\text{ad}(\mathbb{P}(r)) \longrightarrow Y$ be the adjoint bundle of the holomorphic principal $\text{PSL}(r,\mathbb{C})$–bundle $\mathbb{P}(r)$. We recall that $\text{ad}(\mathbb{P}(r))$ is the holomorphic vector bundle associated to the principal $\text{PSL}(r,\mathbb{C})$–bundle $\mathbb{P}(r)$ for the adjoint action of $\text{PSL}(r,\mathbb{C})$ on its Lie algebra. We will describe $\text{ad}(\mathbb{P}(r))$ explicitly. Let

$$\gamma\,:\,\mathbb{P}(\text{Sym}^{r-1}(W))\,\longrightarrow\, Y$$

be the natural projection. Let

$$T_\gamma\,\subset\, T\mathbb{P}(\text{Sym}^{r-1}(W))$$

be the relative holomorphic tangent bundle for the projection γ, meaning T_γ is the kernel of the differential $d\gamma\,:\, T\mathbb{P}(\text{Sym}^{r-1}(W))\,\longrightarrow\,\gamma^*TY$ of γ. Then we have

$$\text{ad}(\mathbb{P}(r))\,=\,\gamma_* T_\gamma\,. \tag{4.25}$$

Given any holomorphic connection on the principal $\text{PSL}(r,\mathbb{C})$–bundle $\mathbb{P}(r)$, there is a holomorphic differential operator of order one from $\text{At}(\mathbb{P}(r))$ to $\text{ad}(\mathbb{P}(r))\otimes K_Y$ [Ch2, p. 1415, (1)], [Ch2, p. 1415, Proposition 4.4]. Consider the differential operator

$$\text{At}(\mathbb{P}(r))\,\longrightarrow\,\text{ad}(\mathbb{P}(r))\otimes K_Y$$

corresponding to the connection D^Y in (4.21). Let

$$\widetilde{D}^Y\,:\,\text{At}(\mathbb{P}(r)_B)\,\longrightarrow\,\text{ad}(\mathbb{P}(r))\otimes K_Y \tag{4.26}$$

be its restriction to the subbundle $\text{At}(\mathbb{P}(r)_B)$ (see (4.24)).

Let $\text{ad}(\mathbb{P}(r)_B)\,\longrightarrow\, Y$ be the adjoint bundle for the holomorphic principal B–bundle $\mathbb{P}(r)_B$. Note that we have

$$\text{ad}(\mathbb{P}(r)_B)\,\subset\,\text{ad}(\mathbb{P}(r))$$

because of the reduction of structure group in (4.23). Recall the description of $\text{ad}(\mathbb{P}(r))$ in (4.25). For any $y\,\in\, Y$, the subspace

$$\text{ad}(\mathbb{P}(r)_B)_y\,\subset\,\text{ad}(\mathbb{P}(r))_y$$

consists of all holomorphic vector fields v on $\mathbb{P}(\text{Sym}^{r-1}(W))_y$ satisfying the following condition: for any $1\,\leq\, j\,\leq\, r-1$, and any

$$z\,\in\,\mathbb{P}(\mathcal{W}_j)_y\,\subset\,\mathbb{P}(\text{Sym}^{r-1}(W))_y$$

(see (4.22)),

$$v(z)\,\in\, T_z\mathbb{P}(\mathcal{W}_j)_y\,,$$

note that $T_z\mathbb{P}(\mathcal{W}_j)_y\,\subset\, T_z\mathbb{P}(\text{Sym}^{r-1}(W))_y$ because $\mathbb{P}(\mathcal{W}_j)_y\,\subset\,\mathbb{P}(\text{Sym}^{r-1}(W))_y$. Let

$$R_n(\text{ad}(\mathbb{P}(r)_B))\,\subset\,\text{ad}(\mathbb{P}(r)_B)$$

be the subbundle given by the nilpotent radical bundle; so for any $y\,\in\, Y$, the subspace $R_n(\text{ad}(\mathbb{P}(r)_B))_y\,\subset\,\text{ad}(\mathbb{P}(r)_B)_y$ is the nilpotent radical. So, for any $y\,\in\, Y$, we have

$$R_n(\text{ad}(\mathbb{P}(r)_B))_y\,=\,[\text{ad}(\mathbb{P}(r)_B)_y,\,\text{ad}(\mathbb{P}(r)_B)_y]\,.$$

Let

$$[R_n(\text{ad}(\mathbb{P}(r)_B),\,R_n(\text{ad}(\mathbb{P}(r)_B)]\,\subset\,R_n(\text{ad}(\mathbb{P}(r)_B)$$

be the commutator. For any $y\,\in\, Y$, we have

$$[R_n(\text{ad}(\mathbb{P}(r)_B),\,R_n(\text{ad}(\mathbb{P}(r)_B)]_y\,=\,[R_n(\text{ad}(\mathbb{P}(r)_B)_y,\,R_n(\text{ad}(\mathbb{P}(r)_B)_y]\,.$$

Let

$$\mathrm{ad}_1(\mathbb{P}(r)_B) \,:=\, [R_n(\mathrm{ad}(\mathbb{P}(r)_B),\, R_n(\mathrm{ad}(\mathbb{P}(r)_B)]^{\perp} \,\subset\, \mathrm{ad}(\mathbb{P}(r)) \tag{4.27}$$

be the annihilator of $[R_n(\mathrm{ad}(\mathbb{P}(r)_B)),\, R_n(\mathrm{ad}(\mathbb{P}(r)_B))]$ for the fiberwise Killing form on the adjoint bundle $\mathrm{ad}(\mathbb{P}(r))$. The vector bundle $\mathrm{ad}_1(\mathbb{P}(r)_B)$ has the following description in terms of the isomorphism in (4.25). For any $y \in Y$, a holomorphic vector field v on $\mathbb{P}(\mathrm{Sym}^{r-1}(W))_y$ lies in the fiber $\mathrm{ad}_1(\mathbb{P}(r)_B)_y$ if and only if the following condition holds: for any $1 \leq j \leq r-1$ and any $z \in \mathbb{P}(\mathcal{W}_j)_y$,

$$v(z) \,\in\, T_z\mathbb{P}(\mathcal{W}_{j+1})_y\,.$$

Since the connection D^Y in (4.21) is a $\mathrm{PSL}(r,\mathbb{C})$–oper on Y, the image of the differential operator $\widetilde{D}^Y$ in (4.26) is contained in the subbundle

$$\mathrm{ad}_1(\mathbb{P}(r)_B)\otimes K_Y \,\subset\, \mathrm{ad}(\mathbb{P}(r))\otimes K_Y$$

defined in (4.27). Therefore, $\widetilde{D}^Y$ defines a differential operator

$$D^Y_1 \,:\, \mathrm{At}(\mathbb{P}(r)_B) \,\longrightarrow\, \mathrm{ad}_1(\mathbb{P}(r)_B)\otimes K_Y\,; \tag{4.28}$$

see [Sa, (5.8)]. Let $\mathcal{H}_\bullet$ be the following two-term complex of sheaves on Y

$$\mathcal{H}_\bullet \,:\, \mathcal{H}_0 \,=\, \mathrm{At}(\mathbb{P}(r)_B) \,\stackrel{D^Y_1}{\longrightarrow}\, \mathcal{H}_1 \,=\, \mathrm{ad}_1(\mathbb{P}(r)_B)\otimes K_Y\,, \tag{4.29}$$

where D^Y_1 is the homomorphism in (4.28), and $\mathcal{H}_i$ is at the i-th position.

The space of all infinitesimal deformations of the $\mathrm{PSL}(r,\mathbb{C})$–oper $(Y,\, D^Y)$ is given by the hypercohomology $\mathbb{H}^1(\mathcal{H}_\bullet)$, where $\mathcal{H}_\bullet$ is the complex constructed in (4.29) [Sa, Theorem 5.9].

We noted earlier that the action of Γ_φ on $\mathbb{P}(r)$ preserves the reduction of structure group $\mathbb{P}(r)_B$ in (4.23). Since $\mathbb{P}(r)_B$ is an equivariant bundle, we conclude that that both $\mathrm{At}(\mathbb{P}(r)_B)$ and $\mathrm{ad}(\mathbb{P}(r)_B)$ are equivariant vector bundles. Hence $R_n(\mathrm{ad}(\mathbb{P}(r)_B))$ in (4.27) is an equivariant subbundle of $\mathrm{ad}(\mathbb{P}(r)_B)$, which in turn implies that $\mathrm{ad}_1(\mathbb{P}(r)_B)$ in (4.27) is an equivariant subbundle of $\mathrm{ad}(\mathbb{P}(r))$; the fiberwise Killing form on $\mathrm{ad}(\mathbb{P}(r))$ is evidently Γ_φ–invariant. The operator D^Y_1 in (4.28) is Γ_φ–equivariant, because the connection D^Y in (4.21) is invariant under the action of Γ_φ on $\mathbb{P}(r)$. Consequently, the complex $\mathcal{H}_\bullet$ in (4.29) is Γ_φ–equivariant. Therefore, the group Γ_φ acts on the hypercohomology $\mathbb{H}^1(\mathcal{H}_\bullet)$. Let

$$\mathbb{H}^1(\mathcal{H}_\bullet)^{\Gamma_\varphi} \,\subset\, \mathbb{H}^1(\mathcal{H}_\bullet) \tag{4.30}$$

be the invariant part for the action of Γ_φ.

We have

$$(\varphi_*\mathrm{At}(\mathbb{P}(r)_B))^{\Gamma_\varphi} \,=\, \mathrm{At}_X(r)$$

and

$$(\varphi_*(\mathrm{ad}_1(\mathbb{P}(r)_B)\otimes K_Y))^{\Gamma_\varphi} \,=\, \mathrm{ad}^n_1\left(\mathrm{Sym}^{r-1}(\mathcal{E}_*)\right)\otimes K_X\otimes \mathcal{O}_X(S)$$

(see (4.14) for $\mathrm{At}_X(r)$ and $\mathrm{ad}^n_1(\mathrm{Sym}^{r-1}(\mathcal{E}_*))\otimes K_X\otimes\mathcal{O}_X(S)$). Moreover, the differential operator D^Y_1 in (4.28) gives the differential operator $\mathcal{D}_B$ in (4.14). Therefore, we conclude that

$$\mathbb{H}^1(\mathcal{C}_\bullet) \,=\, \mathbb{H}^1(\mathcal{H}_\bullet)^{\Gamma_\varphi}\,,$$

where $\mathcal{C}_\bullet$ is the complex in (4.14) (see (4.30)). Now the theorem follows from the earlier mentioned result of [Sa] that the infinitesimal deformations of the $\mathrm{PSL}(r,\mathbb{C})$–oper $(Y,\, D^Y)$ are parametrized by $\mathbb{H}^1(\mathcal{H}_\bullet)$. $\square$

Consider the short exact sequence of holomorphic vector bundles on X

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathrm{At}'_X(r) \longrightarrow \mathrm{At}_X(r) \longrightarrow 0 \tag{4.31}$$

in (4.7). We will show that it splits holomorphically.

Take any

$$\delta \in \Gamma\left(U,\, \mathrm{At}\left(\mathrm{Sym}^{r-1}(\mathcal{E}_*)\right)\right) \subset \Gamma\left(U,\, \mathrm{Diff}^1\left(\mathcal{F}^{(r)},\, \mathcal{F}^{(r)}\right)\right)$$

(see (4.6)), where $U \subset X$ is an open subset. Then δ produces a holomorphic differential operator

$$\widetilde{\delta} \in \Gamma\left(U,\, \mathrm{Diff}^1\left(\bigwedge\nolimits^r \mathcal{F}^{(r)},\, \bigwedge\nolimits^r \mathcal{F}^{(r)}\right)\right) \tag{4.32}$$

which is constructed as follows: Take any

$$s = s_1 \wedge \cdots \wedge s_r \in \Gamma\left(U,\, \bigwedge\nolimits^r \mathcal{F}^{(r)}\right) \tag{4.33}$$

where $s_i \in \Gamma\left(U,\, \mathcal{F}^{(r)}\right)$ for all $1 \leq i \leq r$. Now define

$$\widetilde{\delta}(s) := \sum_{j=1}^{r} s_1 \wedge \cdots \wedge s_{j-1} \wedge \delta(s_j) \wedge s_{j+1} \wedge \cdots \wedge s_r \in \Gamma\left(U,\, \bigwedge\nolimits^r \mathcal{F}^{(r)}\right).$$

It is straightforward to check that $\widetilde{\delta}(s)$ is indeed independent of the choice of the decomposition of of the section s in (4.33).

Let

$$\eta : \mathrm{At}'_X(r) \longrightarrow TX \otimes \mathcal{O}_X(-S) \tag{4.34}$$

be the restriction of the natural projection $\mathrm{At}\left(\mathrm{Sym}^{r-1}(\mathcal{E}_*)\right) \longrightarrow TX \otimes \mathcal{O}_X(-S)$ (see (2.12)) to the subbundle $\mathrm{At}'_X(r) \subset \mathrm{At}\left(\mathrm{Sym}^{r-1}(\mathcal{E}_*)\right)$ in (4.6).

We recall from Remark 4.4 that $\bigwedge^r \mathcal{F}^{(r)} = \mathcal{O}_X(-\sum_{i=1}^n d_i x_i)$. Therefore, the de Rham differential d on $\mathcal{O}_X$ produces a logarithmic connection on $\mathcal{O}_X(-\sum_{i=1}^n d_i x_i) = \bigwedge^r \mathcal{F}^{(r)}$. Let

$$\widetilde{d} : \bigwedge\nolimits^r \mathcal{F}^{(r)} \longrightarrow \bigwedge\nolimits^r \mathcal{F}^{(r)} \otimes K_X \otimes \mathcal{O}_X(\sum_{i=1}^n x_i) \tag{4.35}$$

be the logarithmic connection on $\bigwedge^r \mathcal{F}^{(r)}$ given by the de Rham differential.

Let

$$\mathrm{At}^0_X(r) \subset \mathrm{At}'_X(r) \tag{4.36}$$

be the holomorphic subbundle whose holomorphic sections over any open subset $U \subset X$ consist of all

$$\delta \in \Gamma(U,\, \mathrm{At}'_X(r))$$

satisfying the following condition:

$$\widetilde{\delta}(s) = \langle \widetilde{d}(s),\, \eta(\delta)\rangle$$

for all $s \in \Gamma\left(U,\, \bigwedge^r \mathcal{F}^{(r)}\right)$, where $\widetilde{\delta}$ is constructed in (4.32) from δ and η is the homomorphism in (4.34), while and $\langle -,\, -\rangle$ is the duality pairing in (2.27) and $\widetilde{d}$ is constructed in (4.35).

The composition of homomorphisms

$$\mathrm{At}^0_X(r) \hookrightarrow \mathrm{At}'_X(r) \longrightarrow \mathrm{At}_X(r)$$

(see (4.36) and (4.31) for these homomorphisms) is evidently an isomorphism. Therefore, the holomorphic subbundle $\mathrm{At}^0_X(r)$ in (4.36) produces a holomorphic splitting of the short exact sequence in (4.31).

Let D be a connection on $\mathrm{Sym}^{r-1}(\mathcal{E}_*)$ defining a parabolic $\mathrm{SL}(r,\mathbb{C})$–oper on X. Consider the differential operator $\mathcal{D}$ in (4.11) constructed from the $\mathrm{SL}(r,\mathbb{C})$–oper D. Clearly, we have

$$\mathcal{D}\left(\mathrm{At}^0_X(r)\right) \subset \mathrm{ad}^n_1\left(\mathrm{Sym}^{r-1}(\mathcal{E}_*)\right) \otimes K_X \otimes \mathcal{O}_X(S)\,,$$

where $\mathrm{ad}^n_1\left(\mathrm{Sym}^{r-1}(\mathcal{E}_*)\right)$ and $\mathrm{At}^0_X(r)$ are constructed in (4.9) and (4.36) respectively; recall from (4.12) that

$$\mathcal{D}\left(\mathrm{At}'_X(r)\right) \subset \mathrm{End}^n_1\left(\mathrm{Sym}^{r-1}(\mathcal{E}_*)\right) \otimes K_X \otimes \mathcal{O}_X(S)$$

and $\mathrm{End}^n_1\left(\mathrm{Sym}^{r-1}(\mathcal{E}_*)\right)$ decomposes as in (4.10).

Consequently, the complex of sheaves $\mathcal{C}_\bullet$ in (4.14) is equivalent to the following complex $\mathcal{C}'_\bullet$ of sheaves on X

$$\mathcal{C}'_\bullet \,:\, \mathcal{C}'_0 \,=\, \mathrm{At}^0_X(r) \xrightarrow{\mathcal{D}} \mathcal{C}'_1 \,=\, \mathrm{ad}^n_1\left(\mathrm{Sym}^{r-1}(\mathcal{E}_*)\right) \otimes K_X \otimes \mathcal{O}_X(S)\,. \tag{4.37}$$

Hence Theorem 4.5 gives the following:

Corollary 4.6. *The space of all infinitesimal deformation of the triple* $(X,\, S,\, D)$*, where* D *is a parabolic* $\mathrm{SL}(r,\mathbb{C})$*–oper on* X*, is given by the hypercohomology*

$$\mathbb{H}^1(\mathcal{C}'_\bullet),$$

where $\mathcal{C}'_\bullet$ *is the complex in* (4.37).

5. Monodromy of parabolic opers

Consider a family of n–pointed Riemann surfaces

$$\{(X_T \xrightarrow{\varpi} T),\, (\phi_1,\, \cdots,\, \phi_n)\} \tag{5.1}$$

as in (3.12) and (3.13). Assume that this family is locally universal. The relative canonical line bundle on X_T for the projection ϖ will be denoted by K_ϖ.

Fix a holomorphic line bundle $\mathbf{L}$ on X_T such that $\mathbf{L}\otimes\mathbf{L}$ is holomorphically isomorphic to $K_\varpi \otimes \mathcal{O}_{X_T}(-\sum_{i=1}^n \phi_i(T))$; such a line bundle $\mathbf{L}$ exists locally with respect to T. Fix a holomorphic isomorphism between $\mathbf{L}\otimes\mathbf{L}$ and $K_\varpi \otimes \mathcal{O}_{X_T}(-\sum_{i=1}^n \phi_i(T))$. Fix a function

$$\mathbf{c} \,:\, \{1,\, \cdots,\, n\} \longrightarrow \{t \,\in\, \mathbb{Z} \mid t \,\geq\, 2\}$$

as in (4.2).

Fix the parabolic structure to be that of an $\mathrm{SL}(r,\mathbb{C})$–oper. As in (3.14),

$$\beta \,:\, \mathcal{M}^T(r) \longrightarrow T \tag{5.2}$$

is the corresponding relative moduli space of parabolic bundles with connection. Let

$$\mathcal{O}^T(r) \hookrightarrow \mathcal{M}^T(r) \tag{5.3}$$

be the locus of parabolic $\mathrm{SL}(r,\mathbb{C})$–opers.

Consider the monodromy map

$$\mathbb{M} \,:\, \mathcal{M}^T(r) \,\longrightarrow\, \mathcal{R}_X(r)$$

in (3.15). Let

$$\mathbb{M}^0 \,:\, \mathcal{O}^T(r) \,\longrightarrow\, \mathcal{R}_X(r) \tag{5.4}$$

be the restriction of $\mathbb{M}$ to the subspace $\mathcal{O}^T(r)$ in (5.3).

We will prove that the holomorphic map $\mathbb{M}^0$ in (5.4) is an immersion. Our proof is modeled on the proof of Sanders that $\mathbb{M}^0$ is an immersion under the assumption that $n = 0$ (parabolic points are absent); see [Sa, Theorem 6.3].

Take a point $t_0 \in T$. Denote the Riemann surface $\varpi^{-1}(t_0) = X_{t_0}$ by X. Denote the divisor $\sum_{i=1}^n \phi_i(t_0)$ on X by S. We have

$$T_{t_0}T \,=\, H^1(X,\, TX\otimes\mathcal{O}_X(-S));$$

recall that $(\varpi,\, \{\phi_i\}_{i=1}^n)$ is a locally complete family. Take any parabolic bundle with connection

$$(E_*,\, D) \,\in\, \beta^{-1}(t_0)$$

on X, where β is the projection in (5.2). From Lemma 3.4 it follows that

$$T_{(E_*,D)}\mathcal{M}^T(r) \,=\, \mathbb{H}^1(\mathcal{B}_\bullet), \tag{5.5}$$

where $\mathcal{B}_\bullet$ is the complex in (3.6). Let

$$T_\beta \,\subset\, T\mathcal{M}^T(r)$$

be the relative tangent bundle for the projection β in (5.2). From Lemma 3.3 it follows that $(T_\beta)_{(E,D)} = \mathbb{H}^1(\mathcal{A}_\bullet)$, where $\mathcal{A}_\bullet$ is the complex in (3.4). Consider the homomorphism

$$\delta \,:\, T_{t_0}T \,=\, H^1(X,\, TX\otimes\mathcal{O}_X(-S)) \,\longrightarrow\, T\mathcal{M}^T(r) \,=\, \mathbb{H}^1(\mathcal{B}_\bullet)$$

in (3.19). Recall that the two holomorphic foliations on $\mathcal{M}^T(r)$, one given by the isomonodromy condition and the other given by the projection β in (5.2), are transversal. Therefore, from Lemma 3.5 and (5.5) we know that

$$\mathbb{H}^1(\mathcal{B}_\bullet) \,=\, (T_\beta)_{(E_*,D)}\oplus\delta(T_{t_0}T) \,=\, \mathbb{H}^1(\mathcal{A}_\bullet)\oplus\delta(H^1(X,\, TX\otimes\mathcal{O}_X(-S))). \tag{5.6}$$

Let

$$\Phi \,:\, \mathbb{H}^1(\mathcal{B}_\bullet) \,\longrightarrow\, \mathbb{H}^1(\mathcal{A}_\bullet) \tag{5.7}$$

be the projection corresponding to the decomposition in (5.6). We will now describe this homomorphism Φ explicitly.

The connection D on E_* gives a holomorphic decomposition

$$\mathrm{At}(E_*) \,=\, \mathrm{End}^P(E_*)\oplus(TX\otimes\mathcal{O}_X(-S)) \tag{5.8}$$

(see Lemma 2.4). As in (3.18), $\mathbf{h} \,:\, TX\otimes\mathcal{O}_X(-S) \,\longrightarrow\, \mathrm{At}(E_*)$ is the homomorphism given by the decomposition in (5.8). Consider $\mathcal{D}$ constructed in (2.24). Recall that

$$\mathcal{D}\circ\mathbf{h} \,=\, 0$$

in (3.18). Consequently, the decomposition in (5.8) produces a homomorphism $\mathcal{P}$ of complexes

$$\begin{array}{ccccc} \mathcal{B}_\bullet & : & \text{At}(E_*) & \xrightarrow{\mathcal{D}} & \text{End}^n(E_*) \otimes K_X \otimes \mathcal{O}_X(S) \\ \big\downarrow \mathcal{P} & & \big\downarrow p & & \big\| \\ \mathcal{A}_\bullet & : & \text{End}^P(E_*) & \xrightarrow{\mathcal{D}_0} & \text{End}^n(E_*) \otimes K_X \otimes \mathcal{O}_X(S) \end{array} \tag{5.9}$$

where p is the projection given by the decomposition in (5.8). Let

$$\mathcal{P}_* \,:\, \mathbb{H}^1(\mathcal{B}_\bullet) \longrightarrow \mathbb{H}^1(\mathcal{A}_\bullet) \tag{5.10}$$

be the homomorphism of hypercohomologies corresponding to the homomorphism of complexes $\mathcal{P}$ in (5.9). The homomorphism $\mathcal{P}_*$ in (5.10) evidently coincides with the projection Φ in (5.7).

Now set E_* to be the rank r parabolic vector bundle $\text{Sym}^{r-1}(\mathcal{E}_*)$ in (4.4), and let D be a connection on $\text{Sym}^{r-1}(\mathcal{E}_*)$ such that

$$\big(\text{Sym}^{r-1}(\mathcal{E}_*),\, D\big) \,\in\, \mathcal{O}^T(r) \bigcap \beta^{-1}(t_0)$$

(see (5.3) and (5.2)). From Corollary 4.6 we know that

$$T_{(\text{Sym}^{r-1}(\mathcal{E}_*),D)}\mathcal{O}^T(r) \,=\, \mathbb{H}^1(\mathcal{C}'_\bullet)\,,$$

where $\mathcal{C}'_\bullet$ is the complex in (4.37). We have the following two homomorphisms P and Q of complexes

$$\begin{array}{ccccc} \mathcal{C}'_\bullet & : & \mathcal{C}'_0 \,=\, \text{At}^0_X(r) & \xrightarrow{\mathcal{D}} & \mathcal{C}'_1 \,=\, \text{ad}^n_1\big(\text{Sym}^{r-1}(\mathcal{E}_*)\big) \otimes K_X \otimes \mathcal{O}_X(S) \\ \big\downarrow Q & & \big\downarrow Q_0 & & \big\downarrow Q_1 \\ \widetilde{\mathcal{B}}_\bullet & : & \widetilde{\mathcal{B}}_0 \,=\, \text{At}\big(\text{Sym}^{r-1}(\mathcal{E}_*)\big) & \xrightarrow{\mathcal{D}} & \widetilde{\mathcal{B}}_1 \,=\, \text{End}^n\big(\text{Sym}^{r-1}(\mathcal{E}_*)\big) \otimes K_X \otimes \mathcal{O}_X(S) \\ \big\downarrow P & & \big\downarrow P_0 & & \big\downarrow P_1 \\ \widetilde{\mathcal{A}}_\bullet & : & \widetilde{\mathcal{A}}_0 \,=\, \text{End}^P\big(\text{Sym}^{r-1}(\mathcal{E}_*)\big) & \xrightarrow{\mathcal{D}_0} & \widetilde{\mathcal{A}}_1 \,=\, \text{End}^n\big(\text{Sym}^{r-1}(\mathcal{E}_*)\big) \otimes K_X \otimes \mathcal{O}_X(S) \end{array} \tag{5.11}$$

where

- $\widetilde{\mathcal{B}}_\bullet$ is the complex in (3.6), with $\big(\text{Sym}^{r-1}(\mathcal{E}_*),\, D\big)$ substituted in place in $(E_*,\, D)$,
- $\widetilde{\mathcal{A}}_\bullet$ is the complex in (3.4), with $\big(\text{Sym}^{r-1}(\mathcal{E}_*),\, D\big)$ substituted in place in $(E_*,\, D)$,
- $\mathcal{C}'_\bullet$ is the complex in (4.37),
- for $i \,=\, 0,\, 1$, the homomorphism $\mathcal{C}'_i \,\longrightarrow\, \widetilde{\mathcal{B}}_i$ in (5.11) is the natural inclusion map, and
- the homomorphism P is the homomorphism $\mathcal{P}$ in (5.9) with $\big(\text{Sym}^{r-1}(\mathcal{E}_*),\, D\big)$ substituted in place in $(E_*,\, D)$. So P_1 is the identity map.

Let

$$(P \circ Q)_* \,:\, \mathbb{H}^1(\mathcal{C}'_\bullet) \longrightarrow \mathbb{H}^1(\widetilde{\mathcal{A}}_\bullet) \tag{5.12}$$

be the homomorphism of hypercohomologies induced by the homomorphism $P \circ Q$ in (5.11).

Since the homomorphism $\mathcal{P}_*$ in (5.10) coincides with the projection Φ in (5.7), to prove that the map $\mathbb{M}^0$ in (5.4) is an immersion, it suffices to show that the homomorphism $(P \circ Q)_*$ in (5.12) is injective.

Let

$$q\,:\,\text{End}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)\otimes K_X\otimes\mathcal{O}_X(S)\,\longrightarrow$$
$$\left(\text{End}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)\otimes K_X\otimes\mathcal{O}_X(S)\right)/P_1Q_1\left(\text{ad}_1^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)\otimes K_X\otimes\mathcal{O}_X(S)\right)$$
$$=\,\left(\text{End}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)/\text{ad}_1^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)\right)\otimes K_X\otimes\mathcal{O}_X(S)$$

be the quotient map. From the commutativity of (5.11) it follows immediately that the composition of maps

$$q\circ\mathcal{D}_0\circ(P_0\circ Q_0)\,:\,\text{At}_X^0(r)\,\longrightarrow\,\left(\text{End}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)/\text{ad}_1^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)\right)\otimes K_X\otimes\mathcal{O}_X(S)$$

vanishes on the subbundle $\text{At}_X^0(r)\bigcap\text{End}^P\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)\,\subset\,\text{At}_X^0(r)$ (see (2.12) for the subbundle $\text{End}^P\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)$ of $\text{At}(\text{Sym}^{r-1}(\mathcal{E}_*)))$. Therefore, $q\circ\mathcal{D}_0\circ(P_0\circ Q_0)$ produces a homomorphism

$$\mathcal{S}\,:\,TX\otimes\mathcal{O}_X(-S)\,\longrightarrow\,\left(\text{End}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)/\text{ad}_1^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)\right)\otimes K_X\otimes\mathcal{O}_X(S)\,.$$

This homomorphism $\mathcal{S}$ coincides with the second fundamental form of the reduction of structure group $\widetilde{\mathbb{P}}(B)_*$ in (4.18) for the connection on $\widetilde{\mathbb{P}}_*$ given by D. We know that this second fundamental form is everywhere nonzero, because D is a parabolic $\text{SL}(r,\mathbb{C})$–oper. Consequently, $\mathcal{S}$ is everywhere nonzero. This implies that the homomorphism $P_0\circ Q_0$ in (5.11) is injective.

Since $P_0\circ Q_0$ in (5.11) is injective, it follows that the kernel of the homomorphism $(P\circ Q)_*$ in (5.12) is the quotient of a subspace of $H^0\left(X,\,\text{End}^P\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)/P_0Q_0\left(\text{At}_X^0(r)\right)\right)$. More precisely, let

$$\mathbf{V}\,\subset\,H^0\left(X,\,\text{End}^P\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)/P_0Q_0(\text{At}_X^0(r))\right)$$

be the subspace consisting of all sections s such that $\mathcal{D}_0(s)\,=\,0$; note that from the commutativity of the diagram in (5.11) it follows that $\mathcal{D}_0$ produces a homomorphism

$$H^0\left(X,\,\text{End}^P\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)/P_0Q_0(\text{At}_X^0(r))\right)\,\longrightarrow$$
$$H^0\left(X,\,\left(\text{End}^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)/\text{ad}_1^n\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)\right)\otimes K_X\otimes\mathcal{O}_X(S)\right)\,.$$

Let

$$\mathbf{W}\,\subset\,\mathbf{V}$$

be the subspace consisting of all sections s such that there is a section

$$\widetilde{s}\,\in\,H^0\left(X,\,\text{End}^P\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)\right)$$

satisfying the following two conditions:

- $\mathcal{D}_0(\widetilde{s})\,=\,0$, where $\mathcal{D}_0$ is the homomorphism in (5.11), and
- $\widetilde{s}$ projects to s under the natural map

$$H^0\left(X,\,\text{End}^P\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)\right)\,\longrightarrow\,H^0\left(X,\,\text{End}^P\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)/P_0Q_0(\text{At}_X^0(r))\right)\,.$$

Then we have

$$\text{kernel}((P\circ Q)_*)\,=\,\mathbf{V}/\mathbf{W}\,,\tag{5.13}$$

where $(P\circ Q)_*$ is the homomorphism in (5.12).

Now

$$\text{End}^P\left(\text{Sym}^{r-1}(\mathcal{E}_*)\right)/P_0Q_0(\text{At}_X^0(r))\,=\,\mathcal{O}_X\oplus\mathcal{W}\,,$$

where $\mathcal{W}$ admits a filtration of holomorphic subbundles such that every successive quotient is of the form $(TX \otimes \mathcal{O}_X(-S))^{\otimes m}$, $m \geq 1$. From Assumption 2.1 it follows that

$$H^0(X, (TX \otimes \mathcal{O}_X(-S))^{\otimes m}) = 0$$

for all $m \geq 1$. Hence

$$H^0\left(X, \operatorname{End}^P\left(\operatorname{Sym}^{r-1}(\mathcal{E}_*)\right)/P_0Q_0(\operatorname{At}^0_X(r))\right) = H^0(X, \mathcal{O}_X).$$

But $\mathcal{O}_X \subset \operatorname{End}^P\left(\operatorname{Sym}^{r-1}(\mathcal{E}_*)\right)$, and $\mathcal{D}_0(H^0(X, \mathcal{O}_X)) = 0$. Consequently, we have

$$H^0(X, \mathcal{O}_X) \subset \mathbf{W},$$

where $\mathbf{W}$ is the subspace in (5.13). Therefore, from (5.13) it follows that

$$\operatorname{kernel}((P \circ Q)_*) = 0. \tag{5.14}$$

In other words, the homomorphism $(P \circ Q)_*$ is injective.

As noted above, the map $\mathbb{M}^0$ in (5.4) is an immersion if the homomorphism $(P \circ Q)_*$ in (5.12) is injective. Therefore, we have proved the following:

Theorem 5.1. *The map* $\mathbb{M}^0$ *in* (5.4) *is an immersion.*

Appendix A. Parabolic opers

The purpose of this appendix is to recall some results by K. Yokogawa [Yo] on Hom-sheaves, tensor products and extension classes of parabolic bundles and to give an alternative definition of a parabolic $\mathrm{SL}(r)$-oper [BDP] which is conceptually closer to the definition of an ordinary $\mathrm{SL}(r)$-oper.

A.1. **Correspondence: flags and $\mathbb{R}$-filtered sheaves.** We first recall the correspondence between a parabolic vector bundle as defined in section 2.1 and an $\mathbb{R}$-filtered sheaf $\{E_t\}_{t\in\mathbb{R}}$ as introduced and studied in [MY], [Yo], [BY]. Using the notation of section 2.1 we define for $t \in [0,1]$ the vector bundle E_t by the following equalities

$$\begin{aligned} E_t^i &= E && \text{for } 0 \leq t \leq \alpha_{i,1} \\ E_t^i &= \ker(E \to E_{x_i}/E_{i,j}) && \text{for } \alpha_{i,j-1} < t \leq \alpha_{i,j} \\ E_t^i &= E(-x_i) && \text{for } \alpha_{i,l_i} < t \leq 1 \\ E_t &= \bigcap_{i=1}^n E_t^i \end{aligned}$$

We extend to $\mathbb{R}$ by the formula $E_{t+1} = E_t(-S)$. We also denote this $\mathbb{R}$-filtered sheaf by E_*. Note that $E_t \subset E_{t'}$ for any $t \geq t'$.

We recall that the family E_* is left-continuous, meaning that for any $t \in \mathbb{R}$

$$\lim_{\substack{s\to t\\ s<t}} E_s = E_t.$$

The family E_* is not right-continuous and we will denote for any $t \in \mathbb{R}$

$$E_{t+} := \lim_{\substack{s\to t\\ s>t}} E_s.$$

Then E_{t+} is a subsheaf of E_t and the quotient E_t/E_{t+} is a torsion-sheaf supported at the parabolic divisor S.

A.2. **Special structure and shifts.** Every vector bundle E can be considered as a parabolic vector bundle E_* with the special structure, defined either by the properties $l_i = 1, \alpha_{i,1} = 0$ for $1 \leq i \leq n$, or equivalently by the equalities

$$E_t := E \qquad \text{for } t \in]-1, 0].$$

Given a parabolic vector bundle E_* and an n-tuple $\underline{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$ we define the shift $E[\underline{\beta}]_*$ by the equalities

$$E[\underline{\beta}]_t = \bigcap_{i=1}^n E^i_{t+\beta_i} \qquad \text{for } t \in \mathbb{R}.$$

Also, in order to simplify notation, we define for $\beta \in \mathbb{R}$ the shift of $E[\beta]_*$ as $E[\beta]_* = E[\underline{\beta}]_*$ with $\beta_i = \beta$ for every i.

A.3. **Tensor products, Hom-sheaves and duals.** Given two parabolic vector bundles E_* and E'_* we define their parabolic tensor product (see e.g. [Yo] section 3) by the formula

$$(E_* \otimes E'_*)_t := \sum_{s \in \mathbb{R}} E_s \otimes E'_{t-s} \subset E_0 \otimes E'_0(*S).$$

Here $E_0 \otimes E'_0(*S)$ denotes the (non-coherent) sheaf of rational sections of $E_0 \otimes E'_0$ admitting arbitrary poles at the parabolic divisor S. We note that it is enough to consider the sum for s running over a subinterval of $\mathbb{R}$ of length 1 (because of the invariance of the tensor product $E_s \otimes E'_{t-s}$ under the shift $s \mapsto s+1$) and that only a finite number of subsheaves $E_s \otimes E'_{t-s}$ occur.

Before defining the parabolic Hom-sheaf of two parabolic vector bundles E_* and E'_*, we define the sheaf $Hom(E_*, E'_*)$ as the subsheaf of $Hom(E_0,\, E'_0)$ consisting of parabolic homomorphisms, i.e., homomorphisms $f \,:\, E_0 \longrightarrow E'_0$ satisfying

$$f(E_t) \,\subset\, E'_t$$

for any $t \,\in\, [0,\, 1[$ and thus for any $t \,\in\, \mathbb{R}$. Note that if E_* and E'_* are vector bundles, then $Hom(E_*,\, E'_*)$ is also a vector bundle. Now, we define the parabolic Hom-sheaf $Hom(E_*,\, E'_*)_*$ by the formula

$$Hom(E_*,\, E'_*)_t \,:=\, Hom(E_*,\, E'[t]_*)$$

for any $t \,\in\, \mathbb{R}$.

The parabolic dual $E_*^\vee$ of a parabolic vector bundle E_* is by definition

$$E_*^\vee \,:=\, Hom(E_*,\, \mathcal{O}_*)_*,$$

where $\mathcal{O}_*$ denotes the trivial bundle with the special structure.

The above definitions of parabolic tensor products, Hom-sheaves and duals extend the standard operations on vector bundles, when considering a vector bundle as a parabolic vector bundle with its special structure. Also, the following relations are easy to check :

$$E[\underline{\beta}]_* \otimes E'[\underline{\beta}']_* = E \otimes E'[\underline{\beta} + \underline{\beta}']_*$$
$$E[\underline{\beta}]_*^\vee = E^\vee[-\underline{\beta}]_*$$
$$E_*^\vee \otimes E'_* = Hom(E_*, E'_*)_*$$

A.4. **Cohomology of a parabolic bundle.** Given a parabolic vector bundle E_* over the curve X we define the cohomology of E_* as the cohomology of the vector bundle E_0

$$H^i(X, E_*) = H^i(X, E_0).$$

A.5. **Parabolic subbundles and parabolic degree.** We say that E'_* is a parabolic subbundle of E_* if there is an injective parabolic homomorphism

$$E'_* \hookrightarrow E_*$$

with torsion-free cokernel, or equivalently, for any $t \in \mathbb{R}$, the subsheaf E'_t is a subbundle of E_t.

The parabolic degree of a parabolic bundle E_* is defined as

$$\mathrm{pardeg}(E_*) = \int_0^1 \deg(E_t)dt + n\mathrm{rk}(E_*),$$

where n is the number of parabolic points. We have the following formulae :

$$\mathrm{pardeg}(E_* \otimes E'_*) = \mathrm{rk}(E'_*)\mathrm{pardeg}(E_*) + \mathrm{rk}(E_*)\mathrm{pardeg}(E'_*),$$

$$\mathrm{pardeg}(E[\underline{\beta}]_*) = \mathrm{pardeg}(E_*) - \mathrm{rk}(E_*)\sum_{i=1}^{n} \beta_i.$$

A.6. **Canonical injections and quasi-isomorphisms.** Given a parabolic line bundle L_* and a vector $\underline{\gamma} \in \mathbb{R}^n$ with $\gamma_i \geq 0$ for all i, we have a canonical parabolic injection

$$\iota : L_* \longrightarrow L[-\underline{\gamma}]_*$$

induced by the natural inclusions $L_t \subset L_{t-\gamma_i}$.

Definition A.1. We say that a parabolic homomorphism between two parabolic line bundles

$$\varphi : L_* \longrightarrow M_*$$

is a quasi-isomorphism, if there exists a vector $\underline{\gamma} \in \mathbb{R}^n$ with $0 \leq \gamma_i < 1$ and a parabolic isomorphism $M_* \cong L[-\underline{\gamma}]_*$ such that via this isomorphism φ identifies with the canonical injection ι. In that case we say that φ is a quasi-isomorphism of weight $\underline{\gamma} \in \mathbb{R}^n$.

A.7. **Extensions of parabolic bundles.** Given two parabolic vector bundles E_* and E'_* we say that the parabolic vector bundle F_* is an extension of E_* by E'_* if there exists a short exact sequence of parabolic homomorphisms

$$0 \longrightarrow E'_* \longrightarrow F_* \longrightarrow E_* \longrightarrow 0.$$

By [Yo] Lemma 1.4 and Lemma 3.6 the isomorphism classes of extensions of E_* by E'_* are in one-to-one correspondence with the cohomology space $\mathrm{Ext}^1(E_*, E'_*) = H^1(X, Hom(E_*, E'_*))$.

A.8. **Connections on parabolic bundles.** Given a parabolic vector bundle E_* with parabolic divisor $S \subset X$ we define a connection ∇_* on E_* as a $\mathbb{C}$-linear homomorphism between the parabolic bundles E_* and $E_* \otimes K[-1]_*$

$$\nabla_* : E_* \longrightarrow E_* \otimes K[-1]_* \tag{A.1}$$

such that for every $t \in \mathbb{R}$ the map $\nabla_t : E_t \to (EK[-1])_t = E_tK(S)$ is a logarithmic connection with poles at the parabolic divisor S and for any $t \leq t'$ we have a commutative diagram

$$\begin{array}{ccc} E_{t'} & \longrightarrow & E_{t'}K(S) \\ \big\downarrow & & \big\downarrow \\ E_t & \longrightarrow & E_tK(S) \end{array}$$

where the vertical maps are the natural inclusions.

On the trivial parabolic bundle $\mathcal{O}_*$ there is a natural connection given by the de Rham differentiation and which is denoted by d_*

$$d_* : \mathcal{O}_* \to K[-1]_*$$

and, fixing an integer n, is given for $t \in]n-1, n]$ by differentiation of regular functions having zeros or poles of order n at S

$$d_t : \mathcal{O}_t = \mathcal{O}(-nS) \to K[-1]_t = K(-(n-1)S).$$

We now describe the properties of ∇_* in terms of the parabolic structure given by the flags at the parabolic divisor.

Lemma A.2. *Consider a connection ∇_* on E_* as defined in (A.1). Then the logarithmic connection ∇_0 on E_0 obtained by putting $t = 0$ satisfies*

$$\mathrm{Res}(\nabla_0, x_i)(E_{i,j}) \subset E_{i,j} \tag{A.2}$$

for any $i = 1, \ldots, n$ and any $j = 1, \ldots, l_i$. Conversely, any logarithmic connection ∇_0 on E_0 satisfying (A.2) gives rise to a connection ∇_* on E_*.

The proof of this lemma is standard and therefore left to the reader.

A.9. **Parabolic connections on parabolic bundles.** Consider a connection ∇_* on a parabolic bundle E_* as defined in (A.1). Then for any $t \in \mathbb{R}$ we can consider the residue at $x_i \in S$ of the logarithmic connection ∇_t

$$\mathrm{Res}(\nabla_t, x_i) \in \mathrm{End}_{\mathbb{C}}((E_t)_{x_i}).$$

Since ∇_t preserves all subsheaves $E_{t'}$ for $t' \geq t$ we obtain by passing to the quotient $(E_t/E_{t+})_{x_i}$ a linear map

$$\overline{\mathrm{Res}}(\nabla_t, x_i) \in \mathrm{End}_{\mathbb{C}}((E_t/E_{t+})_{x_i}).$$

and therefore an endomorphism, which we simply denote by $\overline{\mathrm{Res}}(\nabla_t)$, of the torsion sheaf E_t/E_{t+}.

With this notation, we can now define a parabolic connection on a parabolic bundle.

Definition A.3. Let E_* be a parabolic bundle. We say that a connection ∇_* on E_* is a *parabolic* connection if for any $t \in \mathbb{R}$

$$\overline{\mathrm{Res}}(\nabla_t) = t\mathrm{Id}.$$

We leave it as an exercise to the reader to check that this definition is equivalent via the correspondence of section A.1 to the definition given in section 2.2, i.e. for any parabolic point $x_i \in S$ the residue of the logarithmic connection ∇_0 acts as $\alpha_{i,j}\mathrm{Id}$ on the quotient space $E_{i,j}/E_{i,j+1}$.

We also mention the following useful facts, whose proofs are standard.

Let (E_*, ∇_*) and (E'_*, ∇'_*) be parabolic vector bundles equipped with parabolic connections. Then

- the parabolic tensor product $E_* \otimes E'_*$ is naturally equipped with the tensor product connection $(\nabla \otimes \nabla')_*$, which is also parabolic.

- the parabolic symmetric power $\mathrm{Sym}^m E_*$ is naturally equipped with the symmetric power connection $(\mathrm{Sym}^m \nabla)_*$, which is also parabolic.
- the de Rham differentiation d_* on the trivial parabolic bundle $\mathcal{O}_*$ is a parabolic connection.

Remark A.4. Note that, when considering parabolic bundles from the "flag"-point of view, the relation between the parabolic weights of E_* and those of its symmetric powers $\mathrm{Sym}^m E_*$ is quite complicated (as one needs to take fractional parts and reorder them in order to obtain an increasing sequence of parabolic weights in the interval $[0, 1[$). Therefore the "$\mathbb{R}$-filtered sheaf"-point of view is more adapted when considering symmetric powers, as we will need to do in the sequel.

With this notation we can reformulate the following existence theorem.

Theorem A.5 ([BL])**.** *The parabolic bundle E_* admits a parabolic connection ∇_* if and only if any direct summand of E_* has parabolic degree equal to 0.*

A.10. **Parabolic** $\mathrm{SL}(2)$**-opers.** We now define the parabolic analogue of the Gunning bundle. Given the parabolic weights $\alpha_{i,1}, \alpha_{i,2}$ for $1 \leq i \leq n$ satisfying the inequalities (2.3) and the additional assumption $\alpha_{i,1} + \alpha_{i,2} = 1$ for all i, we introduce the real numbers

$$\beta_i = \alpha_{2,i} - \alpha_{1,i} \in]0, 1[$$

and we define

$$\underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{R}^n.$$

Let K be the canonical bundle of the curve X. Then we define the canonical parabolic bundle by

$$K_*^{par} = K[-\underline{\beta}]_*,$$

where we equip K with the special structure. We also define a parabolic theta-characteristic θ_*^{par} as a parabolic line bundle satisfying

$$\theta_*^{par} \otimes \theta_*^{par} = K_*^{par}.$$

One can easily check that the parabolic line bundle θ_*^{par} (resp. $(\theta_*^{par})^{-1}$) corresponds via the above correspondence to a line bundle M (resp. Q) satisfying $M^2 = K(-S)$ (resp. $Q^2 = K^{-1}(-S)$) with parabolic weight $\alpha_{i,2}$ (resp. $\alpha_{i,1}$) at the parabolic point x_i for $1 \leq i \leq n$.

We define the parabolic Gunning bundle $\mathcal{G}_*^{par}$ as the unique non-split parabolic extension

$$0 \longrightarrow \theta_*^{par} \longrightarrow \mathcal{G}_*^{par} \longrightarrow (\theta_*^{par})^{-1} \longrightarrow 0. \tag{A.3}$$

We note that the space of parabolic extensions of $(\theta_*^{par})^{-1}$ by θ_*^{par} is one-dimensional, since

$$\begin{aligned} \dim \mathrm{Ext}^1((\theta_*^{par})^{-1}, \theta_*^{par}) &= \dim H^1(\theta_*^{par} \otimes \theta_*^{par}) \\ &= \dim H^1(K[-\underline{\beta}]_*) = \dim H^1(K) = 1. \end{aligned}$$

Clearly, the parabolic bundle $\mathcal{G}_*^{par}$ has trivial parabolic determinant since

$$\det \mathcal{G}_*^{par} = \theta_*^{par} \otimes (\theta_*^{par})^{-1} = \mathcal{O}_*.$$

It is easy to check that the exact sequence (A.3) equals the exact sequence (3.4) in [BDP] defining the underlying parabolic bundle of a parabolic SL(2)-oper.

Furthermore, any parabolic connection ∇_* on $\mathcal{G}_*^{par}$ induces a second fundamental form, which is a $\mathcal{O}_X$-linear parabolic homomorphism

$$\psi : \theta_*^{par} \longrightarrow (\theta_*^{par})^{-1} \otimes K[-1]_* \cong \theta_*^{par}[-1+\underline{\beta}].$$

The same argument as in the non-parabolic case shows that $\psi \neq 0$, since $\psi = 0$ would imply the existence of a parabolic connection on the parabolic bundle θ_*^{par} having parabolic degree $g-1+\frac{1}{2}(\sum_{i=1}^n \beta_i) > 0$, which contradicts Theorem A.5. Thus ψ is a quasi-isomorphism of weight $1-\underline{\beta}$.

A.11. **Parabolic** $\mathrm{SL}(r)$**-opers.** With the above introduced notation we give a new definition of a parabolic $\mathrm{SL}(r)$-oper.

Definition A.6. A parabolic $\mathrm{SL}(r)$-oper is a triple $(E_*, E_{\bullet *}, \nabla_*)$ consisting of a rank-r parabolic vector bundle E_*, a filtration $E_{\bullet *}$ of E_* by parabolic subbundles

$$0 = E_{0*} \subset E_{1*} \subset E_{2*} \subset \ldots \subset E_{r-1*} \subset E_{r*} = E_*$$

with $\mathrm{rk}(E_{i*}) = i$ and a parabolic connection ∇_* on E_* satisfying the following conditions

- $\det(E_*, \nabla_*) = (\mathcal{O}_*, d_*)$
- $\nabla_*(E_{i*}) \subset E_{i+1*} \otimes K[-1]_*$ for any $i = 1, \ldots, r-1$
- There exists a vector $\underline{\beta} \in \mathbb{R}^n$ with $0 < \beta_i < 1$ such that for any $i = 1, \ldots, r-1$ the parabolic homomorphisms induced by ∇_* between parabolic line bundles
$$(E_{i*}/E_{i-1*}) \longrightarrow (E_{i+1*}/E_{i*}) \otimes K[-1]_*$$
are quasi-isomorphisms of weight $1-\underline{\beta}$ (see Definition (A.1)).

Remark A.7. If $r = 2$ one can easily show that any parabolic $\mathrm{SL}(2)$-oper is of the form $(\mathcal{G}_*^{par}, \mathcal{G}_{\bullet *}^{par}, \nabla_*)$, where $\mathcal{G}_*^{par}$ is the parabolic Gunning bundle introduced in section A.10, $\mathcal{G}_{\bullet *}^{par}$ is given by the exact sequence (A.3) and ∇_* is any parabolic connection satisfying $\det \nabla_* = d_*$.

We now show that, similar to the non-parabolic case, the underlying parabolic bundle of a parabolic $\mathrm{SL}(r)$-oper is a parabolic symmetric power of the parabolic Gunning bundle.

Theorem A.8. *Let $(E_*, E_{\bullet *}, \nabla_*)$ be a parabolic $\mathrm{SL}(r)$-oper associated to the vector $\underline{\beta} \in \mathbb{R}^n$. Then, up to tensor product with an r-torsion parabolic line bundle, we have an isomorphism between parabolic bundles*

$$E_* \cong \mathrm{Sym}^{r-1}\mathcal{G}_*^{par},$$

where $\mathcal{G}_*^{par}$ is the parabolic Gunning bundle associated to the vector $\underline{\beta} \in \mathbb{R}^n$. Moreover, under this isomorphism the filtration $E_{\bullet *}$ corresponds to the natural filtration of $\mathrm{Sym}^{r-1}\mathcal{G}_*^{par}$.

Proof. In order to simplify the notation we introduce the parabolic line bundles $Q_{i*} = E_{i*}/E_{i-1*}$ for $i = 1, \ldots, r$. Then the quasi-isomorphisms $Q_{i*} \to Q_{i+1*} \otimes K[-1]_*$ correspond to isomorphisms

$$Q_{i*} = Q_{i+1*} \otimes K[-\underline{\beta}]_*$$

for $i = 1, \ldots, r-1$. Iterating these formulae we can express all line bundles in terms of Q_{r*}

$$Q_{r-i*} = Q_{r*} \otimes K^i[-i\underline{\beta}].$$

Since $\det E_* = \mathcal{O}_*$, we obtain that the parabolic tensor product of all Q_{i*} equals $\mathcal{O}_*$, which leads to the isomorphism

$$Q_{r*}^r = K^{-\frac{r(r-1)}{2}}[\frac{r(r-1)}{2}\underline{\beta}].$$

We choose a parabolic theta-characteristic θ_*^{par}, i.e., a parabolic line bundle satisfying $(\theta_*^{par})^2 = K^{par} = K[-\underline{\beta}]$. Then the above equality is equivalent to saying that Q_{r*} and $(\theta_*^{par})^{-(r-1)}$ differ by an r-torsion line bundle. So, after tensorizing E_* and consequently all quotient line bundles Q_{i*} by this r-torsion line bundle, we can assume that $Q_{r*} = (\theta_*^{par})^{-(r-1)}$. From the above formulae, we immediately obtain that

$$Q_{r-i*} = (\theta_*^{par})^{-(r-1)+2i}$$

for $i = 0, \ldots, r-1$. Next we will show that the natural exact sequence

$$0 \longrightarrow Q_{i*} \longrightarrow E_{i+1*}/E_{i-1*} \longrightarrow Q_{i+1*} \longrightarrow 0$$

is the unique non-split parabolic extension of Q_{i+1*} by Q_{i*}. By section A.7 parabolic extensions are parameterized by $\mathrm{Ext}^1(Q_{i+1}*, Q_{i*})$ and we have

$$\begin{aligned} \dim \mathrm{Ext}^1(Q_{i+1*}, Q_{i*}) &= \dim H^1(Q_{i+1*}^\vee \otimes Q_{i*}) \\ &= \dim H^1(K[-\underline{\beta}]_*) = \dim H^1(K) = 1. \end{aligned}$$

We now show that E_{i+1*}/E_{i-1*} is non-split. Suppose on the contrary that we have a direct sum decomposition

$$E_{i+1*}/E_{i-1*} = Q_{i*} \oplus Q_{i+1*}.$$

We claim that this splitting implies that the exact sequence

$$0 \longrightarrow E_{i*} \longrightarrow E_{i+1*} \longrightarrow Q_{i+1*} \longrightarrow 0 \tag{A.4}$$

also splits. To see that, we consider the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^1(Q_{i+1*}, E_{i-1*}) \longrightarrow \mathrm{Ext}^1(Q_{i+1*}, E_{i*}) \xrightarrow{\mu} \mathrm{Ext}^1(Q_{i+1*}, Q_{i*}) \longrightarrow \cdots$$

where μ is induced by the push-out under the map $E_{i*} \to Q_{i*}$. Thus, to show that the exact sequence (A.4) splits, it will be enough to show that μ is injective. But $\mathrm{Ext}^1(Q_{i+1*}, E_{i-1*}) = 0$, since E_{i-1*} can be constructed by a series of a successive extensions of Q_{j*}'s for $j \leq i-1$ and we have

$$\mathrm{Ext}^1(Q_{i+1*}, Q_{j*}) = 0 \qquad \text{for all } j \leq i-1.$$

Thus $E_{i+1*} = E_{i*} \oplus Q_{i+1*}$ and after projecting from E_{i+1*} onto E_{i*} the connection ∇_* restricts to a connection on E_{i*}. But, for $i \leq r-1$ we have $\mathrm{pardeg}(E_{i*}) > 0$, which is the desired contradiction by Theorem A.5.

Finally, we invoke Theorem 4.7 [JP] to conclude that, since the rank-2 parabolic bundles E_{i+1*}/E_{i-1*} are the unique non-split parabolic extensions of Q_{i+1*} by Q_{i*} for all $i = 1, \ldots, r-1$, the underlying parabolic vector bundle E_* is unique up to isomorphism. On the other hand, it is easily checked that the parabolic symmetric power $\mathrm{Sym}^{r-1}\mathcal{G}_*^{par}$ also satisfies these properties, hence by uniqueness both parabolic vector bundles E_* and $\mathrm{Sym}^{r-1}\mathcal{G}_*^{par}$ are isomorphic.

Note that Theorem 4.7 [JP] deals with non-parabolic opers, but its extension to parabolic opers is straightforward. □

Remark A.9. The last theorem shows that the above definition of parabolic $\mathrm{SL}(r)$-oper coincides with [BDP] Definition 5.2.

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