

# Self-duality of Coble's quartic hypersurface and applications

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## Abstract

The moduli space  $\mathcal{M}_0$  of semi-stable rank 2 vector bundles with fixed trivial determinant over a non-hyperelliptic curve  $C$  of genus 3 is isomorphic to a quartic hypersurface in  $\mathbb{P}^7$  (Coble's quartic). We show that  $\mathcal{M}_0$  is self-dual and that its polar map associates to a stable bundle  $E \in \mathcal{M}_0$  a bundle  $F$  which is characterized by  $\dim H^0(C, E \otimes F) = 4$ . The projective space  $\mathbb{P}H^0(C, E \otimes F)$  is equipped with a net of quadrics  $\Pi$  and it is shown that the map which associates to  $E \in \mathcal{M}_0$  the isomorphism class of the plane quartic Hessian curve of  $\Pi$  is a dominant map to the moduli space of genus 3 curves.

## 1 Introduction

In his book [C] A.B. Coble constructs for any non-hyperelliptic curve  $C$  of genus 3 a quartic hypersurface in  $\mathbb{P}^7$  which is singular along the Kummer variety  $\mathcal{K}_0 \subset \mathbb{P}^7$  of  $C$ . It is shown in [NR] that this hypersurface is isomorphic to the moduli space  $\mathcal{M}_0$  of semi-stable rank 2 vector bundles with fixed trivial determinant. For many reasons Coble's quartic hypersurface may be viewed as a genus-3-analogue of a Kummer surface, i.e. a quartic surface  $S \subset \mathbb{P}^3$  with 16 nodes. For example the restriction of  $\mathcal{M}_0$  to an eigenspace  $\mathbb{P}_\alpha^3 \subset \mathbb{P}^7$  for the action of a 2-torsion point  $\alpha \in JC[2]$  is isomorphic to a Kummer surface (of the corresponding Prym variety). It is classically known (see e.g. [GH]) that a Kummer surface  $S \subset \mathbb{P}^3$  is self-dual.

In this paper we show that this property also holds for the Coble quartic  $\mathcal{M}_0$  (Theorem 3.1). The rational polar map  $\mathcal{D} : \mathbb{P}^7 \rightarrow (\mathbb{P}^7)^*$  maps  $\mathcal{M}_0$  birationally to  $\mathcal{M}_\omega \subset (\mathbb{P}^7)^*$ , where  $\mathcal{M}_\omega$  ( $\cong \mathcal{M}_0$ ) is the moduli space parametrizing vector bundles with fixed canonical determinant. More precisely we show that the embedded tangent space at a stable bundle  $E$  to  $\mathcal{M}_0$  corresponds to a semi-stable bundle  $\mathcal{D}(E) = F \in \mathcal{M}_\omega$ , which is characterized by the condition  $\dim H^0(C, E \otimes F) = 4$  (its maximum). We also show that  $\mathcal{D}$  resolves to a morphism  $\tilde{\mathcal{D}}$  by two successive blowing-ups, and that  $\mathcal{D}$  contracts the trisecant scroll of  $\mathcal{K}_0$  to the Kummer variety  $\mathcal{K}_\omega \subset \mathcal{M}_\omega$ .

The condition which relates  $E$  to its "tangent space bundle"  $F$ , namely  $\dim H^0(C, E \otimes F) = 4$ , leads to many geometric properties. First we observe that  $\mathbb{P}H^0(C, E \otimes F)$  is naturally equipped with a net of quadrics  $\Pi$  whose base points (Cayley octad) correspond bijectively to the 8 line subbundles of maximal degree of  $E$  (and of  $F$ ). The Hessian curve  $\text{Hess}(E)$  of the net of quadrics  $\Pi \cong |\omega|^*$  is a plane quartic curve, which is everywhere tangent (Proposition 4.7) to the canonical curve  $C \subset |\omega|^*$ , i.e.  $\text{Hess}(E) \cap C = 2\Delta(E)$  for some divisor  $\Delta(E) \in |\omega^2|$ . Since these constructions are  $JC[2]$ -invariant, we introduce the quotient  $\mathcal{N} = \mathcal{M}_0/JC[2]$  parametrizing  $\text{PSL}_2$ -bundles over  $C$  and we show (Proposition 4.13) that the map  $\mathcal{N} \xrightarrow{\Delta} |\omega^2|$ ,  $E \mapsto \Delta(E)$  is the restriction of the projection from the projective space  $\mathcal{N} \subset |\bar{\mathcal{L}}|^* = \mathbb{P}^{13}$  ( $\bar{\mathcal{L}}$  is the ample generator of  $\text{Pic}(\mathcal{N})$ ) with

center of projection given by the linear span of the Kummer variety  $\mathcal{K}_0 \subset \mathcal{N}$  ( $\mathcal{K}_0$  parametrizes decomposable  $\mathbb{P}\mathrm{SL}_2$ -bundles).

We also show (Corollary 4.16) that the Hessian map  $\mathcal{N} \rightarrow \mathcal{R}$ ,  $E \mapsto \mathrm{Hess}(E)$  is finite of degree 72, where  $\mathcal{R}$  is the rational space parametrizing plane quartics everywhere tangent to  $C \subset |\omega|^* = \mathbb{P}^2$ . Considering the isomorphism class of  $\mathrm{Hess}(E)$ , we deduce that the map  $\mathrm{Hess} : \mathcal{N} \rightarrow \mathcal{M}_3$  is dominant, where  $\mathcal{M}_3$  is the moduli space of smooth genus 3 curves. We actually prove that some Galois-covers  $\tilde{\mathcal{N}} \rightarrow \mathcal{N}$  and  $\mathcal{P}_C \rightarrow \mathcal{R}$  are birational (Proposition 4.15). In particular we endow the space  $\tilde{\mathcal{N}}$ , parametrizing  $\mathbb{P}\mathrm{SL}_2$ -bundles  $E$  with an ordered set of 8 line subbundles of  $E$  of maximal degree, with an action of the Weyl group  $W(E_7)$  such that the action of the central element  $w_0 \in W(E_7)$  coincides with the polar map  $\mathcal{D}$ .

We hope that these results will be useful for dealing with several open problems, e.g. rationality of the moduli spaces  $\mathcal{M}_0$  and  $\mathcal{N}$ .

I would like to thank S. Ramanan for some inspiring discussions on Coble's quartic.

## 2 The geometry of Coble's quartic

In this section we briefly recall some known results related to Coble's quartic hypersurface, which can be found in the literature, e.g. [DO], [L2], [NR], [OPP]. We refer to [B1], [B2] for the results on the geometry of the moduli of rank 2 vector bundles.

### 2.1 Coble's quartic as moduli of vector bundles

Let  $C$  be a smooth non-hyperelliptic curve of genus 3 with canonical line bundle  $\omega$ . Let  $\mathrm{Pic}^d(C)$  be the Picard variety parametrizing degree  $d$  line bundles over  $C$  and  $JC := \mathrm{Pic}^0(C)$  be the Jacobian variety. We denote by  $\mathcal{K}_0$  the Kummer variety of  $JC$  and by  $\mathcal{K}_\omega$  the quotient of  $\mathrm{Pic}^2(C)$  by the involution  $\xi \mapsto \omega\xi^{-1}$ . Let  $\Theta \subset \mathrm{Pic}^2(C)$  be the Riemann Theta divisor and let  $\Theta_0 \subset JC$  be a symmetric Theta divisor, i.e. a translate of  $\Theta$  by a theta-characteristic. We also recall that the two linear systems  $|2\Theta|$  and  $|2\Theta_0|$  are canonically dual to each other via Wirtinger duality ([Mu2] p. 335), i.e. we have an isomorphism  $|2\Theta|^* \cong |2\Theta_0|$ .

Let  $\mathcal{M}_0$  (resp.  $\mathcal{M}_\omega$ ) denote the moduli space of semi-stable rank 2 vector bundles over  $C$  with fixed trivial (resp. canonical) determinant. The singular locus of  $\mathcal{M}_0$  is isomorphic to  $\mathcal{K}_0$  and points in  $\mathcal{K}_0$  correspond to bundles  $E$  whose  $S$ -equivalence class  $[E]$  contains a decomposable bundle of the form  $M \oplus M^{-1}$  for  $M \in JC$ . We have natural morphisms

$$\mathcal{M}_0 \xrightarrow{D} |2\Theta| = \mathbb{P}^7, \quad \mathcal{M}_\omega \xrightarrow{D} |2\Theta_0| = |2\Theta|^*,$$

which send a stable bundle  $E \in \mathcal{M}_0$  to the divisor  $D(E)$  whose support equals the set  $\{L \in \mathrm{Pic}^2(C) \mid \dim H^0(C, E \otimes L) > 0\}$  (if  $E \in \mathcal{M}_\omega$ , replace  $\mathrm{Pic}^2(C)$  by  $JC$ ). On the semi-stable boundary  $\mathcal{K}_0$  (resp.  $\mathcal{K}_\omega$ ) the morphism  $D$  restricts to the Kummer map. The moduli spaces  $\mathcal{M}_0$  and  $\mathcal{M}_\omega$  are isomorphic, although non-canonically (consider tensor product with a theta-characteristic). It is known that the Picard group  $\mathrm{Pic}(\mathcal{M}_0)$  is  $\mathbb{Z}$  and that  $|\mathcal{L}|^* = |2\Theta|$ , where  $\mathcal{L}$  is the ample generator of  $\mathrm{Pic}(\mathcal{M}_0)$ .

The main theorem of [NR] asserts that  $D$  embeds  $\mathcal{M}_0$  as a quartic hypersurface in  $|2\Theta| = \mathbb{P}^7$ , which was originally described by A.B. Coble [C] (section 33(6)). Coble's quartic is characterized

by a uniqueness property: it is the unique (Heisenberg-invariant) quartic which is singular along the Kummer variety  $\mathcal{K}_0$  (see [L2] Proposition 5).

We recall that Coble's quartic hypersurfaces  $\mathcal{M}_0 \subset |2\Theta|$  and  $\mathcal{M}_\omega \subset |2\Theta_0|$  contain some distinguished points. First ([C] section 48(4), [L1], [OPP]) there exists a unique stable bundle  $A_0 \in \mathcal{M}_\omega$  such that  $\dim H^0(C, A_0) = 3$  (its maximal dimension). We define for any theta-characteristic  $\kappa$  and for any 2-torsion point  $\alpha \in JC[2]$  the stable bundles, called *exceptional* bundles

$$A_\kappa := A_0 \otimes \kappa^{-1} \in \mathcal{M}_0 \quad \text{and} \quad A_\alpha := A_0 \otimes \alpha \in \mathcal{M}_\omega. \quad (2.1)$$

## 2.2 Global and local equations of Coble's quartic

Let  $F_4$  be the Coble quartic, i.e. the equation of  $\mathcal{M}_0 \subset |2\Theta| = \mathbb{P}^7$ . Then the eight partials  $C_i = \frac{\partial F_4}{\partial X_i}$  for  $1 \leq i \leq 8$  ( $X_i$  are coordinates for  $|2\Theta|$ ) define the Kummer variety  $\mathcal{K}_0$  scheme-theoretically ([L2] Theorem IV.6). We also need the following results ([L2] Theorem 6 bis).

- (i) The étale local equation (in affine space  $\mathbb{A}^7$ ) of Coble's quartic at the point  $[\mathcal{O} \oplus \mathcal{O}]$  is  $T^2 = \det [T_{ij}]$ , with coordinates  $T, T_{ij}$  with  $T_{ij} = T_{ji}$  and  $1 \leq i, j \leq 3$ .
- (ii) The étale local equation at the point  $[M \oplus M^{-1}]$  with  $M^2 \neq \mathcal{O}$  is a rank 4 quadric  $\det [T_{ij}] = 0$ , where  $T_{ij}$ ,  $1 \leq i, j \leq 2$  are four coordinates on  $\mathbb{A}^7$ .

Hence any point  $[M \oplus M^{-1}] \in \mathcal{K}_0$  has multiplicity 2 on  $\mathcal{M}_0$ .

## 2.3 Extension spaces

Given  $L \in \text{Pic}^1(C)$  we introduce the 3-dimensional space  $\mathbb{P}_0(L) := |\omega L^2|^* = \mathbb{P}\text{Ext}^1(L, L^{-1})$ . A point  $e \in \mathbb{P}_0(L)$  corresponds to an isomorphism class of extensions

$$0 \longrightarrow L^{-1} \longrightarrow E \longrightarrow L \longrightarrow 0 \quad (e) \quad (2.2)$$

and the composite of the classifying map  $\mathbb{P}_0(L) \rightarrow \mathcal{M}_0$  followed by the embedding  $D : \mathcal{M}_0 \rightarrow |2\Theta|$  is linear and injective ([B2] Lemme 3.6). It is shown that a point  $e \in \mathbb{P}_0(L)$  represents a stable bundle precisely away from  $\varphi(C)$ , where  $\varphi$  is the map induced by the linear system  $|\omega L^2|$ . A point  $e = \varphi(p)$  for  $p \in C$  is represented by the decomposable bundle  $L(-p) \oplus L^{-1}(p)$ .

We also introduce the projective spaces  $\mathbb{P}_\omega(L) := |\omega^2 L^{-2}|^* = \mathbb{P}\text{Ext}^1(\omega L^{-1}, L)$ . A point  $f \in \mathbb{P}_\omega(L)$  corresponds to an extension

$$0 \longrightarrow L \longrightarrow F \longrightarrow \omega L^{-1} \longrightarrow 0 \quad (f) \quad (2.3)$$

Similarly, we have an injective classifying map  $\mathbb{P}_\omega(L) \rightarrow \mathcal{M}_\omega$ . Although we will not use this fact, we observe that  $\mathbb{P}_0(L) = \mathbb{P}_\omega(\kappa L^{-1})$  for any theta-characteristic  $\kappa$ .

It is well-known that the Kummer variety  $\mathcal{K}_0 \subset |2\Theta|$  admits a 4-dimensional family of trisecant lines (e.g. [Mu3]). It follows from [OPP] Theorem 1.4 and Theorem 2.1 that any trisecant line to  $\mathcal{K}_0$  is contained in some space  $\mathbb{P}_0(L)$ , where it is a trisecant to the curve  $\varphi(C) \subset \mathbb{P}_0(L)$ . We denote by  $\mathcal{T}_0$  the trisecant scroll, which is a divisor in  $\mathcal{M}_0$ . Similarly we define  $\mathcal{T}_\omega \subset \mathcal{M}_\omega$ .

The main tool for the proof of the self-duality is that  $\mathcal{M}_0$  (resp.  $\mathcal{M}_\omega$ ) can be covered by the projective spaces  $\mathbb{P}_0(L)$  (resp.  $\mathbb{P}_\omega(L)$ ). This is expressed by the following result [NR](see

also [OP2]): there exist rank 4 vector bundles  $\mathcal{U}_0$  and  $\mathcal{U}_\omega$  over  $\text{Pic}^1(C)$  such that  $\forall L \in \text{Pic}^1(C)$ ,  $(\mathbb{P}\mathcal{U}_0)_L \cong \mathbb{P}_0(L)$ ,  $(\mathbb{P}\mathcal{U}_\omega)_L \cong \mathbb{P}_\omega(L)$  and their associated classifying morphisms  $\psi_0$  and  $\psi_\omega$ ,

$$\begin{array}{ccc} \mathbb{P}\mathcal{U}_0 & \xrightarrow{\psi_0} & \mathcal{M}_0 \subset |2\Theta| \\ \downarrow & & \downarrow \\ \text{Pic}^1(C) & & \text{Pic}^1(C) \end{array} \quad \begin{array}{ccc} \mathbb{P}\mathcal{U}_\omega & \xrightarrow{\psi_\omega} & \mathcal{M}_\omega \subset |2\Theta_0| \\ \downarrow & & \downarrow \\ \text{Pic}^1(C) & & \text{Pic}^1(C) \end{array}$$

are surjective (Nagata's theorem) and of degree 8 (see section 4.1).

## 2.4 Tangent spaces to Theta-divisors

Following [B2] section 2, we associate to any  $[F] \in \mathcal{M}_\omega \subset |2\Theta_0|$  the divisor  $\Delta(F) \subset \mathcal{M}_0 \subset |2\Theta|$  which has the properties

1.  $\text{supp } \Delta(F) = \{[E] \in \mathcal{M}_0 \mid \dim H^0(C, E \otimes F) > 0\}$ ,
2.  $\Delta(F) \in |\mathcal{L}| \cong |2\Theta|^*$  is mapped to  $[F]$  under the canonical duality  $|2\Theta|^* \cong |2\Theta_0|$ .

Symmetrically, we associate to any  $E \in \mathcal{M}_0$  the divisor  $\Delta(E) \subset \mathcal{M}_\omega$  with the analogous properties.

For any  $E, F$  with  $[E] \in \mathcal{M}_0$  and  $[F] \in \mathcal{M}_\omega$ , the rank 4 vector bundle  $E \otimes F = \mathcal{H}\text{om}(E, F)$  is equipped with a  $\omega$ -valued non-degenerate quadratic form (given by the determinant of local sections), hence, by Mumford's parity theorem [Mu1], the parity of  $\dim H^0(C, E \otimes F)$  is constant under degeneration. Considering e.g. a degeneration of either  $E$  or  $F$  to a decomposable bundle, we obtain that  $\dim H^0(C, E \otimes F)$  is even. The divisor  $\Delta(F)$  is defined as the Pfaffian divisor associated to a family  $\mathcal{E} \otimes F$  of orthogonal bundles [LS] and satisfies the equality

$$2\Delta(F) = \text{detdiv}(\mathcal{E} \otimes F),$$

where  $\text{detdiv}(\mathcal{E} \otimes F)$  is the determinant divisor of the family  $\mathcal{E} \otimes F$ . Thus for any stable bundle  $E \in \mathcal{M}_0$ , we have

$$\text{mult}_{[E]}\Delta(F) = \frac{1}{2}\text{mult}_{[E]}\text{detdiv}(\mathcal{E} \otimes F) \geq \frac{1}{2}\dim H^0(C, E \otimes F).$$

The last inequality is Corollaire II.3 [L1].

**2.1 Lemma.** *Suppose that  $E$  is stable and that  $\dim H^0(C, E \otimes F) \geq 4$ . Then  $\Delta(F) \subset \mathcal{M}_0$  is singular at  $E$  and the embedded tangent space  $\mathbb{T}_E\mathcal{M}_0 \in |2\Theta|^* \cong |2\Theta_0|$  corresponds to the point  $[F] \in |2\Theta_0|$ .*

*Proof.* The first assertion is an immediate consequence of the previous inequality. To show the second, it is enough to observe that, since  $E$  is a singular point of the divisor  $\Delta(F)$ , we have equality between the Zariski tangent spaces  $T_E\Delta(F) = T_E\mathcal{M}_0$  and  $T_E\Delta(F)$  coincides with the hyperplane cutting out the divisor  $\Delta(F)$ , which corresponds to the point  $[F]$  by property (2).  $\square$

We will also need the dual version.

**2.2 Lemma.** *Suppose that  $F$  is stable and that  $\dim H^0(C, E \otimes F) \geq 4$ . Then  $\Delta(E) \subset \mathcal{M}_\omega$  is singular at  $F$  and the embedded tangent space  $\mathbb{T}_F\mathcal{M}_\omega \in |2\Theta_0|^* \cong |2\Theta|$  corresponds to the point  $[E] \in |2\Theta|$ .*

### 3 Self-duality

#### 3.1 Statement of the main theorem

Let  $\mathcal{D}$  be the rational map defined by the polars of Coble's quartic  $F_4$ , i.e. the eight cubics  $C_i$ ,

$$\begin{array}{ccc} \mathcal{D} : |2\Theta| & \longrightarrow & |2\Theta|^* \cong |2\Theta_0| \\ \cup & & \cup \\ \mathcal{M}_0 & & \mathcal{M}_\omega \end{array}$$

Note that  $\mathcal{D}$  is defined away from  $\mathcal{K}_0$ . Geometrically,  $\mathcal{D}$  maps a stable bundle  $E \in \mathcal{M}_0$  to the hyperplane defined by the embedded tangent space  $\mathbb{T}_E \mathcal{M}_0$  at the smooth point  $E$ . The main theorem of this paper is the following

**3.1 Theorem (Self-duality).** *The moduli space  $\mathcal{M}_0$  is birationally mapped by  $\mathcal{D}$  to  $\mathcal{M}_\omega$ , i.e.  $\mathcal{M}_\omega$  is the dual hypersurface of  $\mathcal{M}_0$ . More precisely, we have*

1.  $\mathcal{D}$  restricts to an isomorphism  $\mathcal{M}_0 \setminus \mathcal{T}_0 \xrightarrow{\sim} \mathcal{M}_\omega \setminus \mathcal{T}_\omega$ .
2.  $\mathcal{D}$  contracts the divisor  $\mathcal{T}_0$  ( $\in |\mathcal{L}^8|$ ) to  $\mathcal{K}_\omega$ .
3. For any stable  $E \in \mathcal{M}_0$ , the moduli point  $\mathcal{D}(E) \in \mathcal{M}_\omega$  can be represented by a semi-stable bundle  $F$ , which satisfies  $\dim H^0(C, E \otimes F) \geq 4$ . Moreover, if  $E \in \mathcal{M}_0 \setminus \mathcal{T}_0$  then there exists a unique stable bundle  $F = \mathcal{D}(E)$  for which  $\dim H^0(C, E \otimes F)$  has its maximal value 4.
4.  $\mathcal{D}$  resolves to a morphism  $\tilde{\mathcal{D}}$  from a blowing-up  $\tilde{\mathcal{M}}_0$ ,

$$\begin{array}{ccc} \mathcal{E} & \subset & \tilde{\mathcal{M}}_0 \\ \downarrow & & \downarrow \\ \tilde{\mathcal{K}}_0 & \subset & \text{Bl}_s(\mathcal{M}_0) \quad \searrow^{\tilde{\mathcal{D}}} \\ \downarrow & & \downarrow \\ \mathcal{K}_0 & \subset & \mathcal{M}_0 \quad \xrightarrow{\mathcal{D}} \quad \mathcal{M}_\omega \end{array}$$

where  $\tilde{\mathcal{M}}_0$  is obtained by two successive blowing-ups: first we blow-up the singular points of  $\mathcal{K}_0$  and secondly we blow-up  $\text{Bl}_s(\mathcal{M}_0)$  along the smooth proper transform  $\tilde{\mathcal{K}}_0$  of  $\mathcal{K}_0$ . The exceptional divisor  $\mathcal{E}$  is mapped by  $\tilde{\mathcal{D}}$  onto the divisor  $\mathcal{T}_\omega$ .

#### 3.2 Restriction of $\mathcal{D}$ to the extension spaces

The strategy of the proof is to restrict  $\mathcal{D}$  to the extension spaces  $\mathbb{P}_0(L)$ . We start by defining a map

$$\mathcal{D}_L : \mathbb{P}_0(L) \longrightarrow \mathcal{M}_\omega$$

as follows: consider a point  $e \in \mathbb{P}_0(L)$  (2.2) and denote by  $W_e \subset H^0(C, \omega L^2)$  the corresponding 3-dimensional linear subspace of divisors. If we suppose that  $e \notin \varphi(C)$ , then the evaluation map  $\mathcal{O}_C \otimes W_e \xrightarrow{ev} \omega L^2$  is surjective and we define  $F_e = \mathcal{D}_L(e)$  to be the rank 2 vector bundle such that  $\ker(ev) \cong (F_e L)^*$ , i.e. we have an exact sequence

$$0 \longrightarrow (F_e L)^* \longrightarrow \mathcal{O}_C \otimes W_e \xrightarrow{ev} \omega L^2 \longrightarrow 0. \quad (3.1)$$

If there is no ambiguity, we will drop the subscript  $e$ .

**3.2 Lemma.** *Suppose that  $e \notin \varphi(C)$ . Then*

1. The bundle  $F_e$  has canonical determinant, is semi-stable and  $F_e L$  is generated by global sections.
2. There exists a nonzero map  $L \rightarrow F_e$ , hence  $[F_e]$  defines a point in  $\mathbb{P}_\omega(L)$ .
3. We have  $\dim H^0(C, E \otimes F_e) \geq 4$ , where  $E$  is the stable bundle associated to  $e$  (2.2).

*Proof.* (1) The first assertion is immediately deduced from the exact sequence (3.1). We take the dual of (3.1)

$$0 \longrightarrow \omega^{-1}L^{-2} \longrightarrow \mathcal{O}_C \otimes W^* \longrightarrow FL \longrightarrow 0. \quad (3.2)$$

Taking global sections leads to the inclusion  $W^* \subset H^0(FL)$ , which proves the last assertion. Let us check semi-stability: suppose that there exists a line subbundle  $M$  which destabilizes  $FL$  (assume  $M$  saturated), i.e.  $0 \rightarrow M \rightarrow FL \rightarrow \omega L^2 M^{-1} \rightarrow 0$ . Then  $\deg M \geq 4$ , which implies that  $\deg \omega L^2 M^{-1} \leq 2$ . Hence  $\dim H^0(\omega L^2 M^{-1}) \leq 1$ , so the subspace  $H^0(M) \subset H^0(FL)$  has codimension  $\leq 1$ , which contradicts that  $FL$  is globally generated.

(2) Since  $\det F = \omega$ , we have  $(FL)^* = FL^{-1}\omega^{-1}$ . Taking global sections of the exact sequence (3.1) tensored with  $\omega$  leads to

$$0 \longrightarrow H^0(FL^{-1}) \longrightarrow H^0(\omega) \otimes W \longrightarrow H^0(\omega^2 L^2) \longrightarrow \dots$$

Now we observe that  $\dim H^0(\omega) \otimes W = 9$  and  $\dim H^0(\omega^2 L^2) = 8$  (Riemann-Roch), which implies that  $\dim H^0(FL^{-1}) \geq 1$ .

(3) We tensor the exact sequence (2.2) defined by  $e$  with  $F$  and take global sections

$$0 \longrightarrow H^0(FL^{-1}) \longrightarrow H^0(E \otimes F) \longrightarrow H^0(FL) \xrightarrow{\cup e} H^1(FL^{-1}) \longrightarrow \dots$$

The coboundary map is the cup-product with the extension class  $e \in H^1(L^{-2})$  and, since  $\det F = \omega$ , the coboundary map  $\cup e$  is skew-symmetric (by Serre-duality  $H^1(FL^{-1}) = H^0(FL)^*$ ). Hence the linear map  $\epsilon \mapsto \cup \epsilon$  factorizes as follows

$$H^0(\omega L^2)^* \longrightarrow \Lambda^2 H^0(FL)^* \subset \mathcal{H}om(H^0(FL), H^1(FL)), \quad (3.3)$$

and its dual map  $\Lambda^2 H^0(FL) \xrightarrow{\mu} H^0(\omega L^2)$  coincides with exterior product of global sections (see e.g. [L1]). On the other hand, it is easy to check that the image under  $\mu$  of the subspace  $\Lambda^2 W^* \subset \Lambda^2 H^0(FL)$  equals  $W \subset H^0(\omega L^2)$  and  $\mu$  restricts to the canonical isomorphism  $\Lambda^2 W^* = W$ . Therefore the linear map  $\cup e$  is zero on  $W^* \subset H^0(FL)$ , from which we deduce that  $\dim H^0(E \otimes F) = \dim H^0(FL^{-1}) + \dim \ker(\cup e) \geq 4$ .  $\square$

It follows that the map  $\mathcal{D}_L$  factorizes

$$\mathcal{D}_L : \mathbb{P}_0(L) \longrightarrow \mathbb{P}_\omega(L) \subset \mathcal{M}_\omega. \quad (3.4)$$

Moreover, by Lemma 3.2(3) and Lemma 2.1 the point  $\mathcal{D}_L(e)$  corresponds to the embedded tangent space at  $e \in \mathbb{P}_0(L)$ , hence  $\mathcal{D}_L$  is the restriction of  $\mathcal{D}$  to  $\mathbb{P}_0(L)$ . In particular,  $\mathcal{D}_L$  is given by a linear system of cubics through  $\varphi(C)$ .

We recall from section 2.3 that the restriction of the trisecant scroll  $\mathcal{T}_0$  to  $\mathbb{P}_0(L)$  is the surface, denoted by  $\mathcal{T}_0(L)$ , ruled out by the trisecants to  $\varphi(C) \subset \mathbb{P}_0(L)$ .

**3.3 Lemma.** *Given a point  $e \in \mathbb{P}_0(L)$  such that  $e \notin \varphi(C)$ . The bundle  $F_e$  is stable if and only if  $e \notin \mathcal{T}_0$ . Moreover*

- if  $\dim H^0(L^2) = 0$ , then the trisecant  $\overline{pqr}$  to  $\varphi(C)$  is contracted to the semi-stable point  $[L(u) \oplus \omega L^{-1}(-u)] = \varphi(u) \in \mathbb{P}_\omega(L)$  for some point  $u \in C$ , which satisfies  $p+q+r \in |L^2(u)|$ .

- if  $\dim H^0(L^2) > 0$ , then  $\omega L^{-2} = \mathcal{O}_C(u+v)$  for some points  $u, v \in C$  and any trisecant  $\overline{pqr}$  is contracted to the semi-stable point  $[L(u) \oplus L(v)]$ .

*Proof.* The bundle  $F$  fits into an exact sequence  $0 \rightarrow L \rightarrow F \rightarrow \omega L^{-1} \rightarrow 0$ . Suppose that  $F$  has a line subbundle  $M$  of degree 2 and consider the composite map  $\alpha : M \rightarrow F \rightarrow \omega L^{-1}$ .

First we consider the case  $\alpha = 0$ : then  $M = L(u) \hookrightarrow F$  for some  $u \in C$ , or equivalently  $\dim H^0(FL^{-1}(-u)) > 0$ . We tensor (3.1) with  $\omega(-u)$  and take global sections

$$0 \longrightarrow H^0(FL^{-1}(-u)) \longrightarrow H^0(\omega(-u)) \otimes W \xrightarrow{m} H^0(\omega^2 L^2(-u)) \longrightarrow \dots$$

The second map  $m$  is the multiplication map of global sections. Since  $W \subset H^0(\omega L^2)$ , let us consider for a moment the extended multiplication map  $\tilde{m} : H^0(\omega(-u)) \otimes H^0(\omega L^2) \longrightarrow H^0(\omega^2 L^2(-u))$ . By the base-point-free-pencil-trick applied to the pencil  $|\omega(-u)|$ , we have  $\ker \tilde{m} = H^0(L^2(u))$  and a tensor in  $\ker \tilde{m}$  is of the form  $s \otimes t\alpha - t \otimes s\alpha$ , with  $\{s, t\}$  a basis of  $H^0(\omega(-u))$  and  $\alpha \in H^0(L^2(u))$ . We denote by  $p+q+r$  the zero divisor of  $\alpha$ . Then we see that  $\ker m \neq \{0\}$  if and only if  $W$  contains the linear space spanned by  $t\alpha$  and  $s\alpha$ . Dually, this means that  $e \in \overline{pqr}$ , the trisecant through the points  $p, q, r$ . Conversely, any  $e \in \overline{pqr}$  is mapped by  $\mathcal{D}_L$  to  $[L(u) \oplus \omega L^{-1}(-u)]$ .

Secondly we consider the case  $\alpha \neq 0$ : then  $M = \omega L^{-1}(-u) \hookrightarrow F$  for some  $u \in C$ , or equivalently  $\dim H^0(F\omega^{-1}L(u)) > 0$ . As in the first case we take global sections of (3.1) tensored with  $L^2(u)$  and we obtain that  $H^0(F\omega^{-1}L(u))$  is the kernel of the multiplication map  $H^0(L^2(u)) \otimes W \xrightarrow{m} H^0(\omega L^4(u))$ . Then  $\ker \tilde{m} \neq \{0\}$  implies that  $\dim H^0(L^2(u)) = 2$ . Hence  $L^2(u) = \omega(-v)$  for some point  $v \in C$ , i.e.  $\omega L^{-2} = \mathcal{O}_C(u+v)$ , which implies that  $\dim H^0(\omega L^{-2}) = \dim H^0(L^2) > 0$ . Furthermore the multiplication map becomes  $H^0(\omega(-v)) \otimes W \xrightarrow{m} H^0(\omega^2 L^2(-v))$ . We can now conclude exactly as in the first case, with the additional observation that any trisecant  $\overline{pqr}$  is contracted to the point  $[L(v) \oplus \omega L^{-1}(-v)] = [L(v) \oplus L(u)]$ .  $\square$

Now we are going to construct along the same lines an inverse map to  $\mathcal{D}_L$  (3.4)

$$\mathcal{D}'_L : \mathbb{P}_\omega(L) \longrightarrow \mathbb{P}_0(L).$$

Given an extension class  $f \in \mathbb{P}_\omega(L)$  such that  $f \notin \varphi(C)$ , we denote by  $W_f \subset H^0(C, \omega^2 L^{-2})$  the corresponding 3-dimensional linear space of divisors and we define  $E_f = \mathcal{D}'_L(f)$  to be the rank 2 vector bundle which fits in the exact sequence

$$0 \longrightarrow E_f \omega^{-1} L \longrightarrow W_f \otimes \mathcal{O}_C \xrightarrow{ev} \omega^2 L^{-2} \longrightarrow 0.$$

Exactly as in Lemma 3.2 we show that  $E_f$  has the following properties.

**3.4 Lemma.** *Suppose that  $f \notin \varphi(C)$ . Then*

1. *The bundle  $E_f$  has trivial determinant, is semi-stable and  $E_f \omega L^{-1}$  is generated by global sections.*
2. *There exists a nonzero map  $L^{-1} \rightarrow E_f$ , hence  $[E_f]$  defines a point in  $\mathbb{P}_0(L)$ .*
3. *We have  $\dim H^0(C, E_f \otimes F) \geq 4$ , where  $F$  is the stable bundle associated to  $f$  (2.3).*

Similarly the analogue of Lemma 3.3 holds for the bundle  $E_f$ .

**3.5 Lemma.** *The map  $\mathcal{D}'_L$  is the birational inverse of  $\mathcal{D}_L$ , i.e.*

$$\mathcal{D}'_L \circ \mathcal{D}_L = Id_{\mathbb{P}_0(L)} \quad \text{and} \quad \mathcal{D}_L \circ \mathcal{D}'_L = Id_{\mathbb{P}_\omega(L)}.$$

*Proof.* Start with  $e \in \mathbb{P}_0(L)$  with  $e \notin \mathcal{T}_0(L)$ . Then (Lemma 3.3)  $\mathcal{D}_L(e) = F_e$  is stable and (Lemma 3.2(3))  $\dim H^0(C, E \otimes F_e) \geq 4$ . Now the stable bundle  $F_e$  determines an extension class  $f \in \mathbb{P}_\omega(L)$  with  $f \notin \varphi(C)$ . Let us denote  $E_f = \mathcal{D}'_L(f)$ . We know (Lemma 3.4(3)) that  $\dim H^0(C, E_f \otimes F_e) \geq 4$  and since  $F$  is stable we deduce from Lemma 2.2 that the embedded tangent space  $\mathbb{T}_F \mathcal{M}_\omega$  corresponds to  $[E]$  and  $[E_f]$ . Hence  $[E] = [E_f]$  and since  $E$  is stable, we have  $E = E_f$ .  $\square$

We deduce that  $\mathcal{D}_L$  restricts to an isomorphism  $\mathbb{P}_0(L) \setminus \mathcal{T}_0(L) \xrightarrow{\sim} \mathbb{P}_\omega(L) \setminus \mathcal{T}_\omega(L)$ . Since  $\mathcal{M}_0$  is covered by the spaces  $\mathbb{P}_0(L)$  and since  $\mathcal{D}$  restricts to  $\mathcal{D}_L$  on  $\mathbb{P}_0(L)$ , we obtain that  $\mathcal{D}$  restricts to a birational bijective morphism from  $\mathcal{M}_0 \setminus \mathcal{T}_0$  to  $\mathcal{M}_\omega \setminus \mathcal{T}_\omega$ . Hence by Zariski's Main Theorem  $\mathcal{D}$  is an isomorphism on these open sets, which proves part (1) of Theorem 3.1. Lemma 3.3 implies part (2). As for part (3), we choose a  $\mathbb{P}_0(L)$  containing  $E \in \mathcal{M}_0$ . This determines a point  $e \in \mathbb{P}_0(L)$  and we consider  $F := F_e = \mathcal{D}_L(e)$ . By Lemma 3.2(3) and Lemma 2.1  $\mathcal{D}_L(e) = \mathcal{D}(e)$  - which shows that this construction does not depend on the choice of  $L$ . Moreover if  $e \notin \mathcal{T}_0$ , then  $F$  is stable and characterized by the property  $\dim H^0(C, E \otimes F) \geq 4$ . One easily shows that  $\dim H^0(C, E \otimes F) \geq 6$  cannot occur if  $e \notin \mathcal{T}_0$  (see also Remark 3.4(2)).

### 3.3 Blowing-up

Even if part (4) of Theorem 3.1 is a straight-forward consequence of the results obtained in [L2], we give the complete proof for the convenience of the reader. First we consider the blowing-up  $\mathcal{Bl}_s(\mathbb{P}^7)$  of  $\mathbb{P}^7 = |2\Theta|$  along the 64 singular points of  $\mathcal{K}_0$ . Because of the invariance of  $\mathcal{K}_0$  and  $\mathcal{M}_0$  under the Heisenberg group, it is enough to consider the blowing-up at the "origin"  $O := [\mathcal{O} \oplus \mathcal{O}]$ . We denote by  $\widetilde{\mathcal{K}}_0$  (resp.  $\widetilde{\mathcal{Bl}}_s(\mathcal{M}_0)$ ) the proper transform of  $\mathcal{K}_0$  (resp.  $\mathcal{M}_0$ ), and by  $\mathbb{P}(T_O \mathbb{P}^7) \subset \mathcal{Bl}_s(\mathbb{P}^7)$  the exceptional divisor (over  $O$ ).

By [L2] Remark 5 the Zariski tangent spaces  $T_O \mathcal{K}_0$  and  $T_O \mathcal{M}_0$  at the origin  $O$  to  $\mathcal{K}_0$  and  $\mathcal{M}_0$  satisfy the relations

$$\mathrm{Sym}^2 H^0(\omega)^* \cong T_O \mathcal{K}_0 \subset T_O \mathcal{M}_0 = T_O \mathbb{P}^7 \quad \text{and} \quad T_O \mathcal{M}_0 / T_O \mathcal{K}_0 \cong \Lambda^3 H^0(\omega)^*.$$

Moreover with the notation of section 2.2 the equation of the hyperplane  $T_O \mathcal{K}_0 \subset T_O \mathcal{M}_0$  is  $T = 0$  and  $T_{ij}$  are coordinates on  $\mathrm{Sym}^2 H^0(\omega)^*$ . We deduce from the local equation of  $\mathcal{M}_0$  at the origin  $O$  (section 2.2(ii)) that  $\widetilde{\mathcal{K}}_0 \cap \mathbb{P}\mathrm{Sym}^2 H^0(\omega)^*$  is the Veronese surface  $S := \mathrm{Ver} H^0(\omega)^*$  and that  $\widetilde{\mathcal{K}}_0$  is smooth. Moreover the linear system spanned by the proper transforms of the cubics  $C_i$  is given by the six quadrics  $Q_{ij} := \frac{\partial}{\partial T_{ij}} (\det [T_{ij}])$  vanishing on  $S$ .

Given a smooth point  $x = [M \oplus M^{-1}] \in \mathcal{K}_0$  with  $M^2 \neq \mathcal{O}$ , the Zariski tangent spaces  $T_x \mathcal{K}_0$  and  $T_x \mathcal{M}_0$  satisfy the relations

$$H^0(\omega)^* \cong T_x \mathcal{K}_0 \subset T_x \mathcal{M}_0 = T_x \mathbb{P}^7 \quad \text{and} \quad T_x \mathcal{M}_0 / T_x \mathcal{K}_0 \cong H^0(\omega M^2)^* \otimes H^0(\omega M^{-2})^*.$$

The tangent space  $T_x \mathcal{K}_0 \subset T_x \mathcal{M}_0$  is cut out by the four equations  $T_{ij} = 0$ , where the  $T_{ij}$  are natural coordinates on  $H^0(\omega M^2)^* \otimes H^0(\omega M^{-2})^*$ . Let  $\widetilde{\mathcal{E}}$  be the exceptional divisor of the blowing-up of  $\mathcal{Bl}_s(\mathbb{P}^7)$  along the smooth variety  $\widetilde{\mathcal{K}}_0$  and let  $\mathcal{E}$  be its restriction to the proper transform  $\widetilde{\mathcal{M}}_0$ . We denote by  $\widetilde{\mathcal{E}}_x$  and  $\mathcal{E}_x$  the fibres of  $\widetilde{\mathcal{E}}$  and  $\mathcal{E}$  over a point  $x \in \mathcal{K}_0$ . Then for a smooth point  $x$ , it follows from the local equation at  $x$  (section 2.2 (ii)) that  $\mathcal{E}_x$  is the Segre embedding  $\mathbb{P}^1 \times \mathbb{P}^1 = |\omega M^2|^* \times |\omega M^{-2}|^* \hookrightarrow \mathbb{P} H^0(\omega M^2)^* \otimes H^0(\omega M^{-2})^* = \widetilde{\mathcal{E}}_x$  and the linear system spanned by the proper transforms of the cubics  $C_i$  is given by the four linear forms  $T_{ij}$ .



At a singular point (we take  $x = O$ ), it follows from the preceding discussion that  $\mathcal{E}_O$  is the exceptional divisor of the blowing-up of  $\mathbb{P}\mathrm{Sym}^2 H^0(\omega)^*$  along the Veronese surface  $S$ , i.e. the projectivized normal bundle over  $S$ . It is a well-known fact (duality of conics) that the rational map given by the quadrics  $Q_{ij}$  resolves by blowing-up  $S$ .

It remains to show that  $\widetilde{\mathcal{D}}$  maps  $\mathcal{E}$  onto the trisecant scroll  $\mathcal{T}_\omega$ . Since  $\mathcal{E}$  is irreducible, it will be enough to check this on an open subset of  $\mathcal{E}$ . We consider again the extension spaces  $\mathbb{P}_0(L) \subset \mathcal{M}_0$ . For simplicity we choose  $L$  such that

- (1)  $\mathbb{P}_0(L)$  does not contain a singular point of  $\mathcal{K}_0$ ,
- (2) the morphism  $\varphi : C \rightarrow \mathbb{P}_0(L)$  is an embedding, or equivalently  $\dim H^0(L^2) = 0$ .

Let  $\widetilde{\mathbb{P}_0(L)}$  be the blowing-up of  $\mathbb{P}_0(L)$  along the curve  $C$ , with exceptional divisor  $\mathcal{E}_L$ . Because of assumptions (1) and (2), we have an embedding  $\widetilde{\mathbb{P}_0(L)} \hookrightarrow \widetilde{\mathcal{M}_0}$ ,  $\mathcal{E}$  restricts to  $\mathcal{E}_L$ , and  $\mathcal{E}_L$  is the projectivized normal bundle  $N$  of the embedded curve  $C \subset \mathbb{P}_0(L)$ . We have the following commutative diagram

$$\begin{array}{ccccc} \mathbb{P}(N) = \mathcal{E}_L & \subset & \widetilde{\mathbb{P}_0(L)} & & \\ & & \downarrow & \searrow & \widetilde{\mathcal{D}}_L \\ & & C & \subset & \mathbb{P}_0(L) & \xrightarrow{\mathcal{D}_L} & \mathbb{P}_\omega(L) \end{array}$$

In order to study the image  $\widetilde{\mathcal{D}}_L(\mathcal{E}_L)$  we introduce, for a point  $u \in C$ , the rank 2 bundle  $E_u$  which is defined by the exact sequence

$$0 \rightarrow E_u^* \rightarrow \mathcal{O}_C \otimes H^0(\omega L^2(-u)) \xrightarrow{ev} \omega L^2(-u) \rightarrow 0.$$

Note that  $H^0(\omega L^2(-u))$  corresponds to the hyperplane defined by  $u \in C \subset \mathbb{P}_0(L)$ . Then exactly as in Lemma 3.2(1) we show that  $\det E_u = \omega L^2(-u)$ ,  $E_u$  is stable and globally generated with  $H^0(E_u) \cong H^0(\omega L^2(-u))^*$ . We introduce the Hecke line  $\mathcal{H}_u$  defined as the set of bundles which are (negative) elementary transformations of  $E_u L^{-1}(u)$  at the point  $u$ , i.e. the set of bundles which fit into the exact sequence

$$0 \rightarrow F \rightarrow E_u L^{-1}(u) \rightarrow \mathbb{C}_u \rightarrow 0. \quad (3.5)$$

Since  $E_u$  is stable, any  $F$  is semi-stable (and  $\det F = \omega$ ) and we have a linear map (see [B2])  $\mathbb{P}^1 \cong \mathcal{H}_u \rightarrow \mathcal{M}_\omega$ .

**3.6 Lemma.** *Given a point  $u \in C$ , the fibre  $\mathbb{P}(N_u) = \mathcal{E}_{L,u}$  is mapped by  $\widetilde{\mathcal{D}}_L$  to the Hecke line  $\mathcal{H}_u \subset \mathbb{P}_\omega(L)$ . Moreover  $\mathcal{H}_u$  coincides with the trisecant line  $\overline{pqr}$  to  $C \subset \mathbb{P}_\omega(L)$ , with  $p + q + r \in |\omega L^{-2}(u)|$ .*

*Proof.* The Zariski tangent space  $T_u \mathbb{P}_0(L)$  at the point  $u$  is identified with  $H^0(\omega L^2(-u))^* \cong H^0(E_u)$ . Under this identification the tangent space  $T_u C$  corresponds to the subspace  $H^0(E_u(-u))$ . Hence we obtain a canonical isomorphism of  $\mathbb{P}(N_u)$  with the projectivized fibre over the point  $u$  of the bundle  $E_u$ , i.e. the Hecke line  $\mathcal{H}_u$ . It is straight-forward to check that  $\widetilde{\mathcal{D}}_L$  restricts to the isomorphism  $\mathbb{P}(N_u) \cong \mathcal{H}_u$ . To show the last assertion, it is enough to observe that the Hecke line  $\mathcal{H}_u$  intersects the curve  $C \subset \mathbb{P}_\omega(L)$  at a point  $p$  if and only if  $\dim H^0(E_u L^{-1}(u-p)) > 0$  and to continue as in the proof of Lemma 3.3.  $\square$

Since the union of those  $\mathcal{E}_L$  such that  $L$  satisfies assumptions (1) and (2) form an open subset of  $\mathcal{E}$ , we conclude that  $\widetilde{\mathcal{D}}(\mathcal{E}) = \mathcal{T}_\omega$ . This completes the proof of Theorem 3.1.

### 3.4 Some remarks

(1) The divisor  $\mathcal{T}_\omega \in |\mathcal{L}^8|$ . This is seen as follows: it suffices to restrict  $\mathcal{T}_\omega$  to a general  $\mathbb{P}_\omega(L) \subset \mathcal{M}_\omega$  and to compute the degree of the trisecant scroll  $\mathcal{T}_\omega(L) \subset \mathbb{P}_\omega(L)$ . By Lemma 3.6  $\mathcal{T}_\omega(L)$  is the image of  $\mathcal{E}_L = \mathbb{P}(N)$  under the morphism  $\tilde{\mathcal{D}}_L$ . The hyperplane bundle over  $\mathbb{P}_\omega(L)$  pulls-back under  $\tilde{\mathcal{D}}_L$  to  $\mathcal{O}_{\mathbb{P}}(1) \otimes \pi^*(\omega^3 L^6)$  over the ruled surface  $\mathbb{P}(N)$ . Since  $\tilde{\mathcal{D}}_{L|\mathcal{E}_L}$  is birational, we obtain that  $\deg \mathcal{T}_\omega(L) = \deg \pi_* \mathcal{O}_{\mathbb{P}}(1) \otimes \omega^3 L^6 = \deg N^* \omega^3 L^6 = 8$ .

(2) Using the same methods as before, one can show a refinement of Theorem 3.1(3). Consider  $E$  stable with  $E \in \mathcal{M}_0$  and  $F$  semi-stable with  $[F] \in \mathcal{M}_\omega$ .

- The only pairs  $(E, F)$  for which  $\dim H^0(C, E \otimes F) = 6$  are the 64 exceptional pairs  $E = A_\kappa$  and  $F = \kappa \oplus \kappa$  for a theta-characteristic  $\kappa$  (2.1). We note that  $\mathcal{D}(A_\kappa) = [\kappa \oplus \kappa]$ .
- Suppose  $\mathcal{D}(E) = [M \oplus \omega M^{-1}]$  for some  $M$  and  $E \neq A_\kappa$ , i.e.  $M^2 \neq \omega$ . Then there are exactly three semi-stable bundles  $F$  such that  $\mathcal{D}(E) = [F]$  and  $\dim H^0(C, E \otimes F) = 4$ , namely
  - (1) the decomposable bundle  $F = M \oplus \omega M^{-1}$  (note that  $\dim H^0(EM) = 2$ ).
  - (2) two indecomposable bundles with extension classes in  $\text{Ext}^1(M, \omega M^{-1}) = H^0(M^2)^*$  and  $\text{Ext}^1(\omega M^{-1}, M) = H^0(\omega^2 M^{-2})^*$  defined by the images of the exterior product maps

$$\Lambda^2 H^0(EM) \longrightarrow H^0(M^2) \quad \text{and} \quad \Lambda^2 H^0(E\omega M^{-1}) \longrightarrow H^0(\omega^2 M^{-2}).$$

(3) As a corollary of Lemma 3.6 we obtain that the morphism  $\tilde{\mathcal{D}}$  maps the exceptional divisor  $\tilde{\mathcal{E}}$  onto the dual hypersurface  $\mathcal{K}_0^*$  of the Kummer variety  $\mathcal{K}_0$  (more precisely  $\tilde{\mathcal{D}}$  maps  $\tilde{\mathcal{E}}_x = \mathbb{P}^3$  isomorphically to the subsystem of divisors singular at  $x \in \mathcal{K}_0^{sm}$ ) and that the hypersurface  $\tilde{\mathcal{D}}(\tilde{\mathcal{E}}) = \mathcal{K}_0^*$  intersects (set-theoretically)  $\mathcal{M}_\omega$  along the trisecant scroll  $\mathcal{T}_\omega$ . It is worthwhile to figure out the relationship with other distinguished hypersurfaces in  $|2\Theta|$ , e.g. the octic  $G_8$  defined by the equation  $\mathcal{D}^{-1}(F_4) = F_4 \cdot G_8$  and the Hessian  $H_{16}$  of Coble's quartic  $F_4$ .

## 4 Applications

### 4.1 The 8 maximal line subbundles of $E \in \mathcal{M}_0$

In this section we recall the results of [LN] (see also [OPP], [OP2]) on line subbundles of stable bundles  $E \in \mathcal{M}_0$  and  $F \in \mathcal{M}_\omega$ . We introduce the closed subsets  $\mathbf{M}_0(E)$  and  $\mathbf{M}_\omega(F)$  of  $\text{Pic}^1(C)$  parametrizing line subbundles of maximal degree of  $E$  and  $F$ ,

$$\mathbf{M}_0(E) := \{L \in \text{Pic}^1(C) \mid L^{-1} \hookrightarrow E\} \quad \text{and} \quad \mathbf{M}_\omega(F) := \{L \in \text{Pic}^1(C) \mid L \hookrightarrow F\}.$$

The next lemma follows from [LN] section 5 and Nagata's theorem. For simplicity we assume that  $C$  is not bi-elliptic.

**4.1 Lemma.** *The subsets  $\mathbf{M}_0(E)$  and  $\mathbf{M}_\omega(F)$  are non-empty and 0-dimensional, unless  $E$  and  $F$  are exceptional (see (2.1)). In these cases we have*

$$\mathbf{M}_0(A_\kappa) = \{\kappa(-p) \mid p \in C\} \cong C \quad \text{and} \quad \mathbf{M}_\omega(A_\alpha) = \{\alpha(p) \mid p \in C\} \cong C.$$

Note that  $A_\kappa \in \mathcal{T}_0$  and  $A_\alpha \in \mathcal{T}_\omega$  (see [OPP] Theorem 5.3) and that in the bi-elliptic case, we additionally have a  $JC[2]$ -orbit in  $\mathcal{M}_0$  (resp.  $\mathcal{M}_\omega$ ) of bundles  $E$  (resp.  $F$ ) with 1-dimensional  $\mathbf{M}_0(E)$  (resp.  $\mathbf{M}_0(F)$ ).

Since  $\mathbf{M}_0(E)$  is non-empty, any stable  $E \in \mathcal{M}_0$  lies in at least one extension space  $\mathbb{P}_0(L)$  for some  $L \in \text{Pic}^1(C)$  with extension class  $e \notin \varphi(C)$ . Now [LN] Proposition 2.4 says that there exists a bijection between the sets of

- (1) effective divisors  $p + q$  on  $C$  such that  $e$  lies on the secant line  $\overline{pq}$
- (2) line bundles  $M \in \text{Pic}^1(C)$  such that  $M^{-1} \hookrightarrow E$  and  $M \neq L$ .

The two data are related by the equation

$$L \otimes M = \mathcal{O}_C(p + q). \quad (4.1)$$

Let us count secant lines to  $\varphi(C)$  through a *general* point  $e \in \mathbb{P}_0(L)$ : composing  $\varphi$  with the projection from  $e$  maps  $C$  birationally to a plane nodal sextic  $S$ . By the genus formula, we obtain that the number of nodes of  $S$  (= number of secants) equals 7. Hence, for  $E$  general, the cardinal  $|\mathbf{M}_0(E)|$  of the finite set  $\mathbf{M}_0(E)$  is 8. We write

$$\mathbf{M}_0(E) = \{L_1, \dots, L_8\}.$$

From now on we assume that  $E$  is sufficiently general in order to have  $|\mathbf{M}_0(E)| = 8$ . Since  $E \in \mathbb{P}_0(L_i)$  for  $1 \leq i \leq 8$ , we deduce from relation (4.1) that

$$1 \leq i < j \leq 8, \quad L_i \otimes L_j = \mathcal{O}_C(D_{ij}), \quad (4.2)$$

where  $D_{ij}$  is an effective degree two divisor on  $C$ .

**4.2 Lemma.** *The 8 line bundles  $L_i$  satisfy the relation  $\bigotimes_{i=1}^8 L_i = \omega^2$ .*

*Proof.* We represent  $E$  as a point  $e \in \mathbb{P}_0(L_8)$  and assume that the plane sextic curve  $S \subset \mathbb{P}^2$  obtained by projection with center  $e$  has 7 nodes as singularities. It will be enough to prove the equality for such a bundle  $E$ . Then  $C \xrightarrow{\pi} S$  is the normalization of  $S$  and, by the adjunction formula, we have  $\omega = \pi^* \mathcal{O}_S(3) \otimes \mathcal{O}_C(-\Delta)$ , where  $\Delta$  is the divisor lying over the 7 nodes of  $S$ , i.e.  $\Delta = \sum_{i=1}^7 D_{i8}$ . Hence

$$\omega = \omega^3 L_8^6 \left( - \sum_{i=1}^7 D_{i8} \right) = \omega^3 L_8^{-1} \otimes \bigotimes_{i=1}^7 (L_8(-D_{i8})) = \omega^3 \otimes \bigotimes_{i=1}^8 L_i^{-1},$$

where we used relations (4.2). □

*4.3 Remark.* Conversely, suppose we are given 8 line bundles  $L_i$  which satisfy the 28 relations (4.2). Then there exists a unique stable bundle  $E \in \mathcal{M}_0$  such that  $\mathbf{M}_0(E) = \{L_1, \dots, L_8\}$ . This is seen as follows: take e.g.  $L_8$  and consider any two secant lines  $\overline{D}_{i8}$  and  $\overline{D}_{j8}$  ( $i < j < 8$ ) in  $\mathbb{P}_0(L_8)$ . Then relations (4.2) imply that these two lines intersect in a point  $e$ . It is straight-forward to check that the bundle  $E$  associated to  $e$  does not depend on the choices we made.

## 4.2 Nets of quadrics

We consider  $E \in \mathcal{M}_0$  and we assume that  $E \notin \mathcal{T}_0$  and  $|\mathbf{M}_0(E)| = 8$ . Then  $F = \mathcal{D}(E)$  is stable and  $\dim H^0(C, E \otimes F) = 4$ . We recall that the rank 4 vector bundle  $E \otimes F$  is equipped with a non-degenerate quadratic form (we note that  $E = E^*$ )

$$\det : E \otimes F = \mathcal{H}om(E, F) \longrightarrow \omega.$$

Taking global sections on both sides endows the projective space  $\mathbb{P}^3 := \mathbb{P}H^0(C, \mathcal{H}om(E, F))$  with a net  $\Pi = |\omega|^*$  of quadrics. We denote by  $Q_x \subset \mathbb{P}^3$  the quadric associated to  $x \in \Pi$  and, identifying  $C$  with its canonical embedding  $C \subset |\omega|^* = \Pi$ , we see that (the cone over) the quadric  $Q_p$  for  $p \in C$  corresponds to the sections

$$Q_p := \{\phi \in H^0(C, \mathcal{H}om(E, F)) \mid E_p \xrightarrow{\phi_p} F_p \text{ not surjective}\}, \quad (4.3)$$

where  $E_p, F_p$  denote the fibres of  $E, F$  over  $p \in C$ . It follows from Lemma 3.2(2) that  $\mathbf{M}_0(E) = \mathbf{M}_\omega(F)$ , or equivalently any line bundle  $L_i \in \mathbf{M}_0(E)$  fits into a sequence of maps

$$x_i : E \longrightarrow L_i \longrightarrow F.$$

We denote by  $x_i \in \mathbb{P}^3$  the composite map (defined up to a scalar).

**4.4 Lemma.** *The base locus of the net of quadrics  $\Pi$  consists of the 8 distinct points  $x_i \in \mathbb{P}^3$ .*

*Proof.* A base point  $x$  corresponds to a vector bundle map  $x : E \rightarrow F$  such that  $\text{rk } x \leq 1$  (since  $x \in Q_p, \forall p$ ). Hence there exists a line bundle  $L$  such that  $E \rightarrow L \rightarrow F$  and since  $E$  and  $F$  are stable, of slope 0 and 2, we obtain that  $\deg L = 1$  and  $L \in \mathbf{M}_0(E) = \mathbf{M}_\omega(F)$ .  $\square$

The set of base points  $\bar{x} = \{x_1, \dots, x_8\}$  of a net of quadrics in  $\mathbb{P}^3$  is *self-associated* (for the definition of (self-)association of point sets we refer to [DO] chapter 3) and called a *Cayley octad*. We recall ([DO] chapter 3 example 6) that ordered Cayley octads  $\bar{x} = \{x_1, \dots, x_8\}$  are in 1-to-1 correspondence with ordered point sets  $\bar{y} = \{y_1, \dots, y_7\}$  in  $\mathbb{P}^2$  (note that we consider here general ordered point sets up to projective equivalence). The correspondence goes as follows: starting from  $\bar{x}$  we consider the projection with center  $x_8, \mathbb{P}^3 \xrightarrow{pr_{x_8}} \mathbb{P}^2$ , and define  $\bar{y}$  to be the projection of the remaining seven points. Conversely, given  $\bar{y}$  in  $\mathbb{P}^2$ , we obtain by association 7 points  $x_1, \dots, x_7$  in  $\mathbb{P}^3$ . The missing 8-th point  $x_8$  of  $\bar{x}$  is the additional base point of the net of quadrics through the 7 points  $x_1, \dots, x_7$ .

Consider a general  $E \in \mathcal{M}_0$  and choose a line subbundle  $L_8 \in \mathbf{M}_0(E)$ . We denote by  $x_8$  the corresponding base point of the net  $\Pi$ . We consider the two (different) projections onto  $\mathbb{P}^2$

- (1) projection with center  $x_8$  of  $\mathbb{P}^3 = \mathbb{P}H^0(C, \mathcal{H}om(E, F)) \xrightarrow{pr_{x_8}} \mathbb{P}^2$ . Let  $\bar{y} = \{y_1, \dots, y_7\} \subset \mathbb{P}^2$  be the projection of the 7 base points  $x_1, \dots, x_7$ .
- (2) projection with center  $e$  of  $\mathbb{P}_0(L_8) \xrightarrow{pr_e} \mathbb{P}^2$ . Let  $\bar{z} = \{z_1, \dots, z_7\} \subset \mathbb{P}^2$  be the images of the 7 secant lines to  $\varphi(C)$  through  $e$ . Note that  $z_1, \dots, z_7$  are the 7 nodes of the plane sextic  $S$ .

**4.5 Lemma.** *The two target  $\mathbb{P}^2$ 's of the projections (1) and (2) are canonically isomorphic (to  $\mathbb{P}W_e^*$ ) and the two point sets  $\bar{y}$  and  $\bar{z}$  coincide.*

*Proof.* First we recall from the proof of Lemma 3.2 that we have an exact sequence

$$0 \longrightarrow H^0(FL_8^{-1}) \xrightarrow{i} H^0(E \otimes F) \xrightarrow{\pi} H^0(FL_8) \longrightarrow 0,$$

and that  $H^0(FL_8) \cong W_e^*$  and  $\dim H^0(FL_8^{-1}) = 1$ . Moreover it is easily seen that  $\mathbb{P}(\text{im } i) = x_8 \in \mathbb{P}^3$ , hence the projectivized map  $\pi$  identifies with  $pr_{x_8}$ . The images  $pr_{x_8}(x_i)$  for  $1 \leq i \leq 7$  are given by the sections  $s_i \in H^0(FL_8)$  vanishing at the divisor  $D_{i8}$  (since  $L_i L_8 = \mathcal{O}_C(D_{i8}) \hookrightarrow FL_8$ ). It remains to check that the section  $s_i \in H^0(FL_8) \cong W_e^*$  corresponds to the 2-dimensional subspace  $H^0(\omega L^2(-D_{i8})) \subset W_e \subset H^0(\omega L^2)$ , which is standard.  $\square$

We introduce the non-empty open subset  $\mathcal{M}_0^{reg} \subset \mathcal{M}_0$  of stable bundles  $E$  which satisfy  $E \notin \mathcal{T}_0$ ,  $|\mathbf{M}_0(E)| = 8$  and for any  $L \in \mathbf{M}_0(E)$  the point set  $\bar{z} \subset \mathbb{P}^2$  is such that no three points in  $\bar{z}$  are collinear.

### 4.3 The Hessian construction

It is classical (see e.g. [DO] chapter 9) to associate to a net of quadrics  $\Pi$  on  $\mathbb{P}^3$  its Hessian curve parametrizing singular quadrics, i.e.

$$\text{Hess}(E) := \{x \in \Pi = |\omega|^* \mid Q_x \text{ singular}\}.$$

Note that  $C$  and  $\text{Hess}(E)$  lie in the same projective plane.

**4.6 Lemma.** *We suppose that  $E \in \mathcal{M}_0^{reg}$ . Then the curve  $\text{Hess}(E)$  is a smooth plane quartic.*

*Proof.* It follows from [DO] chapter 9 Lemma 5 that  $\text{Hess}(E)$  is smooth if and only if every 4 points of  $\bar{x} = \{x_1, \dots, x_8\}$  span  $\mathbb{P}^3$ . Projecting from one of the  $x_i$ 's and using Lemma 4.5 we see that this condition holds for  $E \in \mathcal{M}_0^{reg}$ .  $\square$

First we determine for which bundles  $E \in \mathcal{M}_0^{reg}$  the Hessian curve  $\text{Hess}(E)$  equals the base curve  $C$ . We need to recall some facts about nets of quadrics and Cayley octads [DO]. The net  $\Pi$  determines an even theta-characteristic  $\theta$  over the smooth curve  $\text{Hess}(E)$ , such that the Steinerian embedding

$$\text{Hess}(E) \xrightarrow{\text{St}} \mathbb{P}^3 = |\omega\theta|^*, \quad x \longmapsto \text{Sing}(Q_x),$$

is given by the complete linear system  $|\omega\theta|$ . The image  $\text{St}(E)$  is called the *Steinerian curve*. Given two distinct base points  $x_i, x_j \in \mathbb{P}^3$  of the net  $\Pi$ , the pencil  $\Lambda_{ij}$  of quadrics of the net  $\Pi$  which contain the line  $\overline{x_i x_j}$  is a bitangent to the curve  $\text{Hess}(E)$ . In this way we obtain all the  $28 = \binom{8}{2}$  bitangents to  $\text{Hess}(E)$ . Let  $u, v$  be the two intersection points of the bitangent  $\Lambda_{ij}$  with  $\text{Hess}(E)$ . Then the secant line to the Steinerian curve  $\text{St}(E)$  determined by  $\text{St}(u)$  and  $\text{St}(v)$  coincides with  $\overline{x_i x_j}$ .

Conversely, given a smooth plane quartic  $X \subset \mathbb{P}^2$  with an even theta characteristic  $\theta$ , taking the symmetric resolution over  $\mathbb{P}^2$  of the sheaf  $\theta$  supported at the curve  $X$  gives a net of quadrics  $\Pi$  whose Hessian curve equals  $X$ . Thus the correspondence between nets of quadrics  $\Pi$  and the data  $(X, \theta)$  is 1-to-1.

This correspondence allows us to construct some more distinguished bundles in  $\mathcal{M}_0$ . We consider a triple  $(\theta, L, x)$  consisting of an even theta-characteristic  $\theta$  over  $C$ , a square-root  $L \in \text{Pic}^1(C)$ , i.e.  $L^2 = \theta$ , and a base point  $x$  of the net of quadrics  $\Pi$  associated to  $(C, \theta)$ . We denote by

$$A(\theta, L, x) \in \mathcal{M}_0 \tag{4.4}$$

the stable bundle defined by the point  $x \in \mathbb{P}_0(L) = |\omega\theta|^*$ . Since  $C$  is smooth, we have  $A(\theta, L, x) \in \mathcal{M}_0^{reg}$ . These bundles will be called *Aronhold bundles* (see Remark 4.12). We leave it to the reader to deduce the following characterization:  $E$  is an Aronhold bundle if and only if the 28 line bundles  $L_i L_j$  ( $1 \leq i < j \leq 8$ ) are the odd theta-characteristics, with  $L_i \in \mathbf{M}_0(E)$ .

**4.7 Proposition.** *Given a bundle  $E \in \mathcal{M}_0^{reg}$ . Then*

1. *We have  $\text{Hess}(E) = C$  if and only if  $E$  is an Aronhold bundle.*
2. *Assuming  $\text{Hess}(E) \neq C$ , the curves  $C$  and  $\text{Hess}(E)$  are everywhere tangent. More precisely, the scheme-theoretical intersection  $C \cap \text{Hess}(E)$  is non-reduced of the form  $2\Delta(E)$ , with  $\Delta(E) \in |\omega^2|$ .*

*Proof.* We deduce from (4.3) that the intersection  $C \cap \text{Hess}(E)$  corresponds (set-theoretically) to the sets of points where the evaluation map of global sections

$$\mathcal{O}_C \otimes H^0(C, \mathcal{H}\text{om}(E, F)) \xrightarrow{ev} \mathcal{H}\text{om}(E, F) \quad (4.5)$$

is not surjective.

Let us suppose that  $C = \text{Hess}(E)$ . Then  $ev$  is not generically surjective ( $\text{rk } ev \leq 3$ ). We choose a line subbundle  $L_8 \in \mathbf{M}_0(E)$  and we consider (as in Lemma 4.5) the exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(FL_8^{-1}) & \longrightarrow & H^0(\mathcal{H}\text{om}(E, F)) & \longrightarrow & H^0(FL_8) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow^{ev} & & \downarrow^{ev'} & & \\ 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{H}\text{om}(E, F) & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \end{array}$$

where the vertical arrows are evaluation maps. Note that  $\mathcal{O}_C \hookrightarrow FL_8^{-1} \hookrightarrow \mathcal{H}\text{om}(E, F)$  corresponds to the section of  $H^0(FL_8^{-1})$ . We denote by  $\mathcal{E}$  the rank 3 quotient. Then  $ev' : H^0(FL_8) \longrightarrow \mathcal{E}$  is not generically surjective either. But  $\mathcal{E}$  has a quotient  $E \rightarrow FL_8$  with kernel  $\omega L_8^{-2}$ . Now since  $H^0(FL_8) \xrightarrow{ev} FL_8$  is surjective, we obtain a direct sum decomposition  $\mathcal{E} = \omega L_8^{-2} \oplus FL_8$ . Furthermore since  $E \otimes F$  is poly-stable (semi-stable and orthogonal) and of slope 2, we obtain that  $\omega L_8^{-2}$  is an orthogonal direct summand. Hence  $\omega L_8^{-2} = \theta$  for some theta-characteristic  $\theta$ . Now we can do this reasoning for any line bundle  $L_i \in \mathbf{M}_0(E)$ , establishing that all  $\omega L_i^{-2}$  are theta-characteristics contained in  $\mathcal{H}\text{om}(E, F)$ . Projecting to  $FL_8$  shows that  $L_i^2 = L_8^2 = \theta$  for all  $i$  and therefore the 28 line bundles  $L_i L_j$  are the odd theta-characteristics. It follows that  $E$  is an Aronhold bundle.

Assuming  $C \neq \text{Hess}(E)$ , the evaluation map (4.5) is injective

$$0 \longrightarrow \mathcal{O}_C \otimes H^0(C, \mathcal{H}\text{om}(E, F)) \xrightarrow{ev} \mathcal{H}\text{om}(E, F) \longrightarrow \mathbb{C}_{\Delta(E)} \longrightarrow 0.$$

The cokernel is a sky-scraper sheaf supported at a divisor  $\Delta(E)$ . Since  $\det \mathcal{H}\text{om}(E, F) = \omega^2$ , we have  $\Delta(E) \in |\omega^2|$ . This shows that set-theoretically we have  $C \cap \text{Hess}(E) = \Delta(E)$ . Let us determine the local equation of  $\text{Hess}(E)$  at a point  $p \in \Delta(E)$ . We denote by  $m$  the multiplicity of  $\Delta(E)$  at the point  $p$ . Then, since there is no section of  $\mathcal{H}\text{om}(E, F)$  vanishing twice at  $p$  (stability of  $E$  and  $F$ ), we have  $\dim H^0(\mathcal{H}\text{om}(E, F)(-p)) = m$ . We choose a basis  $\phi_1, \dots, \phi_m$  of sections of the subspace  $H^0(\mathcal{H}\text{om}(E, F)(-p)) \subset H^0(\mathcal{H}\text{om}(E, F))$  and complete it (if necessary) by  $\phi_{m+1}, \dots, \phi_4$ . Let  $z$  be a local coordinate in an analytic neighbourhood centered at the point  $p$ . With these notations the quadrics  $Q_z$  of the net can be written

$$Q_z(\lambda_1, \dots, \lambda_4) = \det \left( \sum_{i=1}^4 \lambda_i \phi_i(z) \right),$$

where the  $\phi_i(z)$  are a basis of the fibre  $\mathcal{H}\text{om}(E, F)_z$  for  $z \neq 0$ . By construction we have for  $1 \leq i \leq m$ ,  $\phi_i(z) = z\psi_i(z)$ , and the local equation of  $\text{Hess}(E)$  is the determinant of the symmetric  $4 \times 4$  matrix

$$\text{Hess}(E)(z) = \det [B(\phi_i(z), \phi_j(z))]_{1 \leq i, j \leq 4},$$

where  $B$  is the polarization of the determinant. We obtain that  $\text{Hess}(E)(z)$  is of the form  $z^{2m}R(z)$ . Hence  $\text{mult}_p(\text{Hess}(E)) \geq 2m$ , proving the statement.  $\square$

**4.8 Definition.** We call the divisor  $\Delta(E)$  the discriminant divisor of  $E$  and the rational map  $\Delta : \mathcal{M}_0 \rightarrow |\omega^2|$  the discriminant map.

In the sequel of this paper we will show that the bundle  $E$  and its Hessian curve  $\text{Hess}(E)$  are in bijective correspondence (modulo some discrete structure, which will be defined in section 4.5.2). A first property is the following: given  $E \in \mathcal{M}_0^{\text{reg}}$ , we associate to the 28 degree two effective divisors  $D_{ij}$  (4.2) on the curve  $C$  their corresponding secant lines  $\overline{D}_{ij} \subset |\omega|^*$ .

**4.9 Proposition.** The secant line  $\overline{D}_{ij}$  to the curve  $C$  coincides with the bitangent  $\Lambda_{ij}$  to the smooth quartic curve  $\text{Hess}(E)$ .

*Proof.* Since the bitangent  $\Lambda_{ij}$  to  $\text{Hess}(E)$  corresponds to the pencil of quadrics in  $\Pi$  containing the line  $\overline{x_i x_j}$ , it will be enough to show that  $Q_a$  and  $Q_b$  belong to  $\Lambda_{ij}$ , for  $D_{ij} = a + b$ , with  $a, b \in C$ . Consider the vector bundle map  $\pi_i \oplus \pi_j : E \rightarrow L_i \oplus L_j$ , where  $\pi_i$  and  $\pi_j$  are the natural projection maps. Since  $L_i L_j = \mathcal{O}(D_{ij})$ , the map  $\pi_i \oplus \pi_j$  has cokernel  $\mathbb{C}_a \oplus \mathbb{C}_b$ , which is equivalent to saying that the two linear forms  $\pi_{i,a} : E_a \rightarrow L_{i,a}$  and  $\pi_{j,a} : E_a \rightarrow L_{j,a}$  are proportional (same for  $b$ ). This implies that any map  $\phi \in \overline{x_i x_j}$  factorizes at the point  $a$  through  $\pi_{i,a} = \pi_{j,a}$ , hence  $\det \phi_a = 0$ . This means that  $\overline{x_i x_j} \subset Q_a$ , i.e.  $Q_a \in \Lambda_{ij}$  (same for  $b$ ).  $\square$

## 4.4 Moduli of $\mathbb{P}\text{SL}_2$ -bundles and the discriminant map $\Delta$

The finite group  $JC[2]$  of 2-torsion points of  $JC$  acts by tensor product on  $\mathcal{M}_0$  and  $\mathcal{M}_\omega$ . Since Coble's quartic is Heisenberg-invariant, it is easily seen that the polar map  $\mathcal{D} : \mathcal{M}_0 \rightarrow \mathcal{M}_\omega$  is  $JC[2]$ -equivariant, i.e.  $\mathcal{D}(E \otimes \alpha) = \mathcal{D}(E) \otimes \alpha$ ,  $\forall \alpha \in JC[2]$ . This implies that the constructions we made in sections 4.2 and 4.3, namely the projective space  $\mathbb{P}^3 = \mathbb{P}H^0(\mathcal{H}\text{om}(E, F))$ , the net of quadrics  $\Pi$ , its Hessian curve  $\text{Hess}(E)$  and discriminant divisor  $\Delta(E)$ , only depend on the class of  $E$  modulo  $JC[2]$ , which we denote by  $\overline{E}$ . It is therefore useful to introduce the quotient  $\mathcal{N} = \mathcal{M}_0/JC[2]$ , which can be identified with the moduli space of semi-stable  $\mathbb{P}\text{SL}_2$ -vector bundles with fixed trivial determinant. We observe that  $\mathcal{N}$  is canonically isomorphic to the quotient  $\mathcal{M}_\omega/JC[2]$ . Therefore the  $JC[2]$ -invariant polar map  $\mathcal{D}$  descends to a birational involution

$$\overline{\mathcal{D}} : \mathcal{N} \rightarrow \mathcal{N}. \quad (4.6)$$

We recall [BLS] that the generator  $\overline{\mathcal{L}}$  of  $\text{Pic}(\mathcal{N}) = \mathbb{Z}$  pulls-back under the quotient map  $q : \mathcal{M}_0 \rightarrow \mathcal{N}$  to  $q^* \overline{\mathcal{L}} = \mathcal{L}^4$  and that global sections  $H^0(\mathcal{N}, \overline{\mathcal{L}}^k)$  correspond to  $JC[2]$ -invariant sections of  $H^0(\mathcal{M}_0, \mathcal{L}^{4k})$ .

The Kummer variety  $\mathcal{K}_0$  is contained in the singular locus of  $\mathcal{N}$ : since the composite map  $JC \xrightarrow{i} \mathcal{M}_0 \xrightarrow{q} \mathcal{N}$ , with  $i(L) = [L \oplus L^{-1}]$ , is  $JC[2]$ -invariant, it factorizes  $JC \xrightarrow{[2]} JC \xrightarrow{\tilde{i}} \mathcal{N}$ , and the image  $\tilde{i}(JC) \cong \mathcal{K}_0 \subset \mathcal{N}$ .

We also recall from [OP1] that we have a morphism

$$\Gamma : \mathcal{N} \rightarrow |\mathbb{3}\Theta|_+ = \mathbb{P}^{13}, \quad \overline{E} \mapsto \Gamma(\overline{E}) = \{L \in \text{Pic}^2(C) \mid \dim H^0(C, \text{Sym}^2(E) \otimes L) > 0\},$$

which is well-defined since  $\Gamma(\overline{E})$  only depends on  $\overline{E}$ . The subscript  $+$  denotes invariant (w.r.t.  $\xi \mapsto \omega\xi^{-1}$ ) theta-functions. When restricted to  $\mathcal{K}_0$  the morphism  $\Gamma$  is the Kummer map, i.e. we

have a commutative diagram

$$\begin{array}{ccc} \mathcal{K}_0 & \xrightarrow{Kum} & |2\Theta| = \mathbb{P}^7 \\ \downarrow & & \downarrow +\Theta \\ \mathcal{N} & \xrightarrow{\Gamma} & |3\Theta|_+ = \mathbb{P}^{13} \end{array}$$

The main result of [OP1] is

**4.10 Proposition.** *The morphism  $\Gamma : \mathcal{N} \longrightarrow |3\Theta|_+$  is given by the complete linear system  $|\overline{\mathcal{L}}|$ , i.e. there exists an isomorphism  $|\overline{\mathcal{L}}|^* \cong |3\Theta|_+$ .*

*4.11 Remark.* Using the same methods as in [NR], one can show that  $\Gamma : \mathcal{N} \longrightarrow |3\Theta|_+$  is an embedding. We do not use that result.

Since the open subset  $\mathcal{M}_0^{reg}$  is  $JC[2]$ -invariant, we obtain that  $\mathcal{M}_0^{reg} = q^{-1}(\mathcal{N}^{reg})$ . By passing to the quotient  $\mathcal{N}$ , the Aronhold bundles (4.4) determine  $36 \cdot 8 = 288$  distinct points  $A(\theta, x) := \overline{A(\theta, L, x)} \in \mathcal{N}^{reg}$ , the exceptional bundles (2.1) determine one point in  $\mathcal{N}$ , denoted by  $A_0$ , and we obtain a (rational) discriminant map (4.8)

$$\Delta : \mathcal{N} \longrightarrow |\omega^2|$$

defined on the open subset  $\mathcal{N}^{reg} \setminus \{A(\theta, x)\}$ . We also note that the 28 line bundles  $L_i L_j$  for  $L_i \in \mathbf{M}_0(E)$  only depend on  $\overline{E}$ .

*4.12 Remark.* The 288 points  $A(\theta, x)$  are in 1-to-1 correspondence with unordered Aronhold sets (see [DO] page 167), i.e. sets of seven odd theta-characteristics  $\theta_i$  ( $1 \leq i \leq 7$ ) such that  $\theta_i + \theta_j - \theta_k$  is even  $\forall i, j, k$ . The seven  $\theta_i$  are cut out on the Steinerian curve by the seven lines  $\overline{xx_i}$ , where  $x, x_i$  are the base points of  $\Pi$ .

The main result of this section is

**4.13 Proposition.** *We have a canonical isomorphism  $|3\Theta|_{|\Theta|_+} \cong |\omega^2|$ , which makes the right diagram commute*

$$\begin{array}{ccccccc} \mathcal{K}_0 & \subset & \mathcal{N} & \xrightarrow{\Delta} & |\omega^2| & & \\ & & \downarrow \Gamma & & \downarrow \cong & & \\ |2\Theta| & \xrightarrow{+\Theta} & |3\Theta|_+ & \xrightarrow{res_{\Theta}} & |3\Theta|_{|\Theta|_+} & & \end{array}$$

*In other words, considering  $\mathcal{N}$  (via  $\Gamma$ ) as a subvariety in  $|3\Theta|_+$ , the discriminant map  $\Delta$  identifies with the projection with center  $|2\Theta| = \text{Span}(\mathcal{K}_0)$ , or equivalently, with the restriction map of  $|3\Theta|_+$  to the Theta divisor  $\Theta \subset \text{Pic}^2(C)$ .*

*Proof.* First we show that the discriminant map  $\Delta$  is given by a linear subsystem of  $|\overline{\mathcal{L}}|$  ( $\cong |3\Theta|_+^*$ ). Consider a line bundle  $L \in \text{Pic}^1(C)$  and the composite map

$$\psi_L : \mathbb{P}^3 := \mathbb{P}_0(L) \longrightarrow \mathcal{M}_0 \xrightarrow{q} \mathcal{N} \xrightarrow{\Delta} |\omega^2|.$$

Then it will be enough to show that  $\psi_L^*(H) \in |\mathcal{O}_{\mathbb{P}^3}(4)|$  (since  $q^*\overline{\mathcal{L}} = \mathcal{L}^4$ ) for a hyperplane  $H$  in  $|\omega^2|$ . We denote by  $p$  (resp.  $q$ ) the projection of  $\mathbb{P}^3 \times C$  onto  $C$  (resp.  $\mathbb{P}^3$ ). There exists a universal extension bundle  $\mathbb{E}$  over  $\mathbb{P}^3 \times C$

$$0 \longrightarrow p^*L^{-1} \longrightarrow \mathbb{E} \longrightarrow p^*L \otimes q^*\mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow 0 \quad (4.7)$$



such that  $\forall e \in \mathbb{P}_0(L)$  the vector bundle  $\mathbb{E}_{|\{e\} \times C}$  corresponds to the extension class  $e$ . We denote by  $\mathbb{W} \hookrightarrow \mathcal{O}_{\mathbb{P}^3} \otimes H^0(\omega L^2)$  the universal rank 3 subbundle over  $\mathbb{P}^3$  and we define the family  $\mathbb{F}$  over  $U \times C$  by the exact sequence

$$0 \longrightarrow (\mathbb{F} \otimes p^*L)^* \longrightarrow q^*\mathbb{W} \xrightarrow{ev} p^*(\omega L^2) \longrightarrow 0, \quad (4.8)$$

where  $U$  is the open subset  $\mathbb{P}^3 \setminus C$ . We have  $\mathbb{F}_{|\{e\} \times C} \cong F_e$  (see (3.1)). Note that  $\text{Pic}(U) = \text{Pic}(\mathbb{P}^3)$ . It follows immediately from (4.7) and (4.8) that  $\det \mathbb{E} = q^*\mathcal{O}(-1)$ ,  $\det \mathbb{F} = q^*\mathcal{O}(1) \otimes p^*\omega$ , and that  $\det \mathbb{E} \otimes \mathbb{F} = p^*\omega^2$ . After removing (if necessary) the point  $A_0$  from  $U$  (see Remark 3.4(2)), we obtain that  $\forall e \in U$ ,  $\dim H^0(C, \mathbb{E} \otimes \mathbb{F}_{|\{e\} \times C}) = 4$ , hence by the base change theorems, the direct image sheaves  $q_*(\mathbb{E} \otimes \mathbb{F})$  and  $R^1q_*(\mathbb{E} \otimes \mathbb{F})$  are locally free over  $U$ . Suppose that the hyperplane  $H$  consists of divisors in  $|\omega^2|$  containing a point  $p \in C$ . Then  $\psi_L^*(H)$  is given by the determinant of the evaluation map over  $U$  (see (4.5))

$$q_*(\mathbb{E} \otimes \mathbb{F}) \xrightarrow{ev} \mathbb{E} \otimes \mathbb{F}_{|U \times \{p\}}.$$

Since  $\det(\mathbb{E} \otimes \mathbb{F}_{|U \times \{p\}}) = \mathcal{O}_U$ , the result will follow from the equality  $\det q_*(\mathbb{E} \otimes \mathbb{F}) = \mathcal{O}_U(-4)$ , which we prove by using some properties of the determinant line bundles [KM]. Given any family of bundles  $\mathcal{F}$  over  $U \times C$ , we denote the determinant line bundle associated to the family  $\mathcal{F}$  by  $\det Rq_*(\mathcal{F})$ . First we observe that by relative duality [K] we have

$$q_*(\mathbb{E} \otimes \mathbb{F}) \xrightarrow{\sim} (R^1q_*(\mathbb{E} \otimes \mathbb{F}))^*,$$

hence  $\det Rq_*(\mathbb{E} \otimes \mathbb{F}) = (\det q_*(\mathbb{E} \otimes \mathbb{F}))^{\otimes 2}$ . Next we tensor (4.7) with  $\mathbb{F}$

$$0 \longrightarrow \mathbb{F} \otimes p^*L^{-1} \longrightarrow \mathbb{E} \otimes \mathbb{F} \longrightarrow \mathbb{F} \otimes p^*L \otimes q^*\mathcal{O}(-1) \longrightarrow 0.$$

Since  $\det Rq_*$  is multiplicative, we obtain

$$\det Rq_*(\mathbb{E} \otimes \mathbb{F}) \cong \det Rq_*(\mathbb{F} \otimes p^*L^{-1}) \otimes \det Rq_*(\mathbb{F} \otimes p^*L \otimes q^*\mathcal{O}(-1)).$$

Again by relative duality we have  $\det Rq_*(\mathbb{F} \otimes p^*L^{-1}) \cong \det Rq_*(\mathbb{F} \otimes p^*L \otimes q^*\mathcal{O}(-1))$ , hence (since  $\text{Pic}(U) = \mathbb{Z}$ ) we can divide by 2 to obtain

$$\det q_*(\mathbb{E} \otimes \mathbb{F}) \cong \det Rq_*(\mathbb{F} \otimes p^*L \otimes q^*\mathcal{O}(-1)) \cong \det Rq_*(\mathbb{F} \otimes p^*L) \otimes \mathcal{O}(-2).$$

The last equation holds since  $\chi(F_e L) = 2$ . Finally, we apply the functor  $\det Rq_*$  to the dual of (4.8)

$$\begin{aligned} \det Rq_*(\mathbb{F} \otimes p^*L) &\cong \det Rq_*(q^*\mathbb{W}^*) \otimes \det Rq_*(p^*\omega L^2)^{-1} \\ &\cong (\det \mathbb{W}^*)^{\otimes \chi(\mathcal{O})} \cong \mathcal{O}(-2). \end{aligned}$$

which proves  $\det q_*(\mathbb{E} \otimes \mathbb{F}) = \mathcal{O}(-4)$ .

We also deduce from this construction that the exceptional locus of the rational discriminant map  $\Delta$  is the union of the Kummer variety  $\mathcal{K}_0$ , the exceptional bundle  $A_0$ , and the 288 Aronhold bundles  $A(\theta, x)$ . Therefore the map  $\Delta$  is given by the composite of  $\Gamma$  with a projection map  $\pi : |\overline{\mathcal{L}}|^* \cong |3\Theta|_+ \longrightarrow |\omega^2|$ , whose center of projection  $\ker \pi$  contains  $\text{Span}(\mathcal{K}_0) = |2\Theta|$ . In order to show that  $\ker \pi = |2\Theta|$ , it suffices (for dimensional reasons) to show that  $\Delta$  is dominant:

Consider a general divisor  $\delta = a_1 + \dots + a_8 \in |\omega^2|$  and choose  $M \in \text{Pic}^2(C)$  such that  $a_1 + \dots + a_4 \in |M^2|$ , or equivalently  $a_5 + \dots + a_8 \in |\omega^2 M^{-2}|$ . Using Lemma 3.3 we can find

a stable  $E \in \mathcal{T}_0$  such that  $[\mathcal{D}(E)] = [M \oplus \omega M^{-1}]$ . We easily deduce from Remark 3.4(2) that  $\Delta(E) = \delta$ .

Finally, we deduce from the natural exact sequence associated to the divisor  $\Theta \subset \text{Pic}^2(C)$

$$0 \longrightarrow H^0(\text{Pic}^2(C), 2\Theta) \xrightarrow{+\Theta} H^0(\text{Pic}^2(C), 3\Theta)_+ \xrightarrow{\text{res}_\Theta} H^0(\Theta, 3\Theta|_\Theta)_+ \longrightarrow 0$$

that the projectivized restriction map  $\text{res}_\Theta$  identifies with the projection  $\pi$ .  $\square$

*4.14 Remark.* Geometrically the assertion on the exceptional locus of  $\Delta$  given in the proof means that (we map  $\mathcal{N}$  via  $\Gamma$  into  $|3\Theta|_+$ )

$$\mathcal{N} \cap |2\Theta| = \mathcal{K}_0 \cup \{A_0\} \cup \{A(\theta, x)\},$$

or equivalently, that the  $3\theta$ -divisors  $\Gamma(A_0)$  and  $\Gamma(A(\theta, x))$  are reducible of the form

$$\Gamma(A_0) = \Theta + \Gamma^{\text{res}}(A_0), \quad \Gamma(A(\theta, x)) = \Theta + \Gamma^{\text{res}}(A(\theta, x)),$$

where the residual divisors  $\Gamma^{\text{res}}(A_0)$  and  $\Gamma^{\text{res}}(A(\theta, x))$  lie in  $|2\Theta|$ . This can be checked directly.

- (1) exceptional bundle  $A_0$ : since  $\Theta \cong \text{Sym}^2 C$ , the inclusion  $\Theta \subset \Gamma(A_0)$  is equivalent to  $\dim H^0(C, \text{Sym}^2(A_0) \otimes \omega^{-1}(p+q)) > 0$ ,  $\forall p, q \in C$  (here we take  $A_0 \in \mathcal{M}_\omega$  see (2.1)), or  $\dim H^0(C, \text{Sym}^2(A_0)(-u-v)) > 0$ ,  $\forall u, v \in C$ . But this follows immediately from  $\dim H^0(C, A_0) = 3$ , which implies that  $\forall u$  there exists a nonzero section  $s_u \in H^0(C, A_0(-u))$ . Taking the symmetric product, we obtain  $s_u \cdot s_v \in H^0(C, \text{Sym}^2(A_0)(-u-v))$ .
- (2) Aronhold bundles  $A(\theta, x)$ : similarly we have to show that  $\dim H^0(C, \text{Sym}^2(A) \otimes \omega(-p-q)) > 0$ ,  $\forall p, q \in C$  (take  $A = A(\theta, L, x) \in \mathcal{M}_0$ ). Since  $\mathbf{M}_0(A)$  is invariant under the involution  $L_i \mapsto \theta L_i^{-1}$ , we have  $\mathcal{D}(A) = A \otimes \theta$  and  $\dim H^0(C, A \otimes A \otimes \theta) = \dim H^0(C, \text{Sym}^2(A) \otimes \theta) = 4$ . Hence  $\forall p$  there exists a nonzero section  $s_p \in H^0(C, \text{End}_0(A) \otimes \theta(-p))$  (note that  $\text{End}_0(A) = \text{Sym}^2(A)$ ) and by taking the  $\text{End}_0$ -part of the composite section  $s_p \circ s_q$ , we obtain a nonzero element of  $H^0(C, \text{Sym}^2(A) \otimes \omega(-p-q))$ .

Moreover it can be shown by standard methods that  $\text{Sym}^2(A_0)$  and  $\text{Sym}^2(A(\theta, x))$  are stable bundles. It would be interesting to describe explicitly the  $2\theta$ -divisors  $\Gamma^{\text{res}}(A_0)$  and  $\Gamma^{\text{res}}(A(\theta, x))$ , which, we suspect, do not lie on the Coble quartic  $\mathcal{M}_0$ .

## 4.5 The action of the Weyl group $W(E_7)$

The aim of this section is to show that the Hessian map (section 4.3), which associates to a  $\text{PSL}_2$ -bundle  $\bar{E} \in \mathcal{N}^{\text{reg}}$  the isomorphism class of the smooth curve  $\text{Hess}(\bar{E}) \in \mathcal{M}_3$ , is dominant.

### 4.5.1 Some group theory related to genus 3 curves

We recall here (see e.g. [A], [DO], [Ma]) the main results on root lattices and Weyl groups. Let  $\Gamma \subset \mathbb{P}^2$  be a smooth plane quartic and let  $V$  be its associated degree 2 del Pezzo surface, i.e. the degree 2 cover  $\pi : V \rightarrow \mathbb{P}^2$  branched along the curve  $\Gamma$ . We choose an isomorphism (called geometric marking of  $V$ ) of the Picard group  $\text{Pic}(V)$ ,

$$\varphi : \text{Pic}(V) \xrightarrow{\sim} H_7 = \bigoplus_{i=0}^7 \mathbb{Z}e_i, \tag{4.9}$$

with the hyperbolic lattice  $H_7$ , such that  $\varphi$  is orthogonal for the intersection form on  $\text{Pic}(V)$  and the quadratic form on  $H_7$  defined by  $e_0^2 = 1$ ;  $e_i^2 = -1$ , ( $i \neq 0$ );  $e_i \cdot e_j = 0$ , ( $i \neq j$ ). The anti-canonical class  $-k$  of  $V$  equals  $3e_0 - \sum_{i=1}^7 e_i$ . We put  $e_8 := \sum_{i=1}^7 e_i - 2e_0 = e_0 + k$ . Then the 63 positive roots of  $H_7$  are of two types

$$(1) \alpha_{ij} = e_i - e_j, \quad 1 \leq i < j \leq 8, \quad (2) \alpha_{ijk} = e_0 - e_i - e_j - e_k, \quad 1 \leq i < j < k \leq 7. \quad (4.10)$$

The 28 roots of type (1) correspond to the 28 positive roots of the Lie algebra  $\mathfrak{sl}_8$  seen as a subalgebra of the exceptional Lie algebra  $\mathfrak{e}_7$ . Similarly the 56 exceptional lines of  $H_7$  are of two types

$$(1) l_{ij} = e_i + e_j - e_8, \quad (2) l'_{ij} = e_0 - e_i - e_j, \quad 1 \leq i < j \leq 8. \quad (4.11)$$

The Weyl group  $W(\text{SL}_8)$  equals the symmetric group  $\Sigma_8$  and is generated by the reflections  $s_{ij}$  associated to the roots  $\alpha_{ij}$  of type (1). The action of the reflection  $s_{ij}$  on the exceptional lines  $l_{pq}$  and  $l'_{pq}$  is given by applying the transposition  $(ij)$  to the indices  $pq$ . The Weyl group  $W(E_7)$  is generated by the reflections  $s_{ij}$  and  $s_{ijk}$  (associated to  $\alpha_{ijk}$ ) and the reflection  $s_{ijk}$  acts on the exceptional lines as follows

- if  $|\{i, j, k, 8\} \cap \{p, q\}| = 1$ , then  $s_{ijk}(l_{pq}) = l_{pq}$ ,
- if  $|\{i, j, k, 8\} \cap \{p, q\}| = 0$  or  $2$ , then  $s_{ijk}(l_{pq}) = l'_{st}$  such that  $\{p, q, s, t\}$  equals  $\{i, j, k, 8\}$  or its complement in  $\{1, \dots, 8\}$ .

Let us consider the restriction map  $\text{Pic}(V) \xrightarrow{res} \text{Pic}(\Gamma)$  to the ramification divisor  $\Gamma \subset V$ . Then we have the beautiful fact (see [DO] Lemma 8 page 190) that *res* maps bijectively the 63 positive roots  $\{\alpha_{ij}, \alpha_{ijk}\}$  (4.10) to the 63 nonzero 2-torsion points  $J\Gamma[2] \setminus \{0\}$ , thus endowing the Jacobian  $J\Gamma$  with a level-2-structure, i.e. a symplectic isomorphism  $\psi : J\Gamma[2] \cong \mathbb{F}_2^3 \times \mathbb{F}_2^3$  (for the details, see [DO] chapter 9). We also observe that the partition of  $J\Gamma[2]$  into the two sets  $\{res(\alpha_{ij})\}$  (28 points) and  $\{res(\alpha_{ijk}), 0\}$  (36 points) corresponds to the partition into odd and even points (w.r.t. the level-2-structure  $\psi$ ). Moreover the images of the 56 exceptional lines (4.11) are the 28 odd theta-characteristics on  $\Gamma$ , which we denote by  $res(l_{ij}) = res(l'_{ij}) = \theta_{ij}$ . Moreover  $\pi(l_{ij}) = \pi(l'_{ij}) = \Lambda_{ij}$ , where  $\Lambda_{ij}$  is the bitangent to  $\Gamma$  corresponding to  $\theta_{ij}$ .

Two geometric markings  $\varphi, \varphi'$  (4.9) differ by an element  $g \in O(H_7) = W(E_7)$  and their induced level-2-structures  $\psi, \psi'$  differ by  $\bar{g} \in \text{Sp}(6, \mathbb{F}_2)$ . The restriction map  $W(E_7) \rightarrow \text{Sp}(6, \mathbb{F}_2)$ ,  $g \mapsto \bar{g}$  is surjective with kernel  $\mathbb{Z}/2 = \langle w_0 \rangle = \text{Center}(W(E_7))$ . The element  $w_0 \in W(E_7)$  acts as  $-1$  on the root lattice, leaves  $k$  invariant  $w_0(k) = k$  and exchanges the exceptional lines  $w_0(l_{ij}) = l'_{ij}$ .

We also note that  $w_0 \notin \Sigma_8 \subset W(E_7)$  and that the injective composite map  $\Sigma_8 \rightarrow W(E_7) \rightarrow \text{Sp}(6, \mathbb{F}_2)$  identifies  $\Sigma_8$  with the stabilizer of an even theta-characteristic.

#### 4.5.2 Two moduli spaces with $W(E_7)$ -action

We introduce the  $\Sigma_8$ -Galois cover  $\widetilde{\mathcal{M}}_0 \rightarrow \mathcal{M}_0^{reg}$  parametrizing stable bundles  $E \in \mathcal{M}_0^{reg}$  with an order on the 8 line subbundles  $\mathbf{M}_0(E) = \{L_1, \dots, L_8\}$ . The group  $J\mathcal{C}[2]$  acts on  $\widetilde{\mathcal{M}}_0$  and we denote the quotient  $\widetilde{\mathcal{M}}_0/J\mathcal{C}[2]$  by  $\widetilde{\mathcal{N}}$ , which is a  $\Sigma_8$ -Galois cover  $\widetilde{\mathcal{N}} \rightarrow \mathcal{N}^{reg}$ . The polar map  $\overline{\mathcal{D}} : \mathcal{N} \rightarrow \mathcal{N}$  (4.6) lifts to a  $\Sigma_8$ -equivariant birational involution  $\widetilde{\mathcal{D}} : \widetilde{\mathcal{N}} \rightarrow \widetilde{\mathcal{N}}$ .

We also consider the moduli space  $\mathcal{P}_C$  parametrizing pairs  $(\Gamma, \varphi)$ , with  $\Gamma \subset |\omega|^* = \mathbb{P}^2$  a smooth plane quartic curve which satisfies  $\Gamma \cap C = 2\Delta$  and  $\Delta \in |\omega^2|$ , and  $\varphi$  a geometric marking (4.9) for the Del Pezzo surface  $V$  associated to  $\Gamma$ . Then the forgetful map  $(\Gamma, \varphi) \mapsto \Gamma$  realizes  $\mathcal{P}_C$  as a  $W(E_7)$ -Galois cover of the space  $\mathcal{R}$  of smooth quartic curves  $\Gamma$  satisfying the above intersection

property. Since the general fibre  $f^{-1}(\Delta)$  of the projection map  $\mathcal{R} \xrightarrow{f} |\omega^2|$  corresponds to the pencil of curves spanned by the curve  $C$  and the double conic  $Q^2$  defined by  $Q \cap C = \Delta$ , we see that  $\mathcal{R}$  is an open subset of a  $\mathbb{P}^1$ -bundle over  $|\omega^2|$ , hence rational.

**4.15 Proposition.** *The Hessian map (section 4.3) induces a birational map*

$$\widetilde{\text{Hess}} : \widetilde{\mathcal{N}} \longrightarrow \mathcal{P}_C,$$

which endows  $\widetilde{\mathcal{N}}$  with a  $W(E_7)$ -action. The action of  $w_0$  corresponds to the polar map  $\widetilde{\mathcal{D}}$ .

*Proof.* Let  $\overline{E} \in \widetilde{\mathcal{N}}$  be represented by  $E \in \mathcal{M}_0^{reg}$  and by an ordered set  $\mathbf{M}_0(E) = \{L_1, \dots, L_8\}$ . In order to construct the data  $(\Gamma, \varphi)$ , we consider the Del Pezzo surface  $V \xrightarrow{\pi} \mathbb{P}^2$  associated to the Hessian curve  $\Gamma = \text{Hess}(E) \subset |\omega|^* = \mathbb{P}^2$ . Since  $\Gamma \cap C = 2\Delta(E)$ , the preimage  $\pi^{-1}(C) \subset V$  splits into two irreducible components  $C_1 \cup C_2$ , with  $C_1 = C_2 = C$ . More generally, it can be shown that the preimage  $\pi^{-1}(C \times \mathcal{R}) \subset \mathcal{V}$  has two irreducible components, where  $\mathcal{V} \rightarrow \mathcal{R}$  is the family of Del Pezzo's parametrized by  $\mathcal{R}$ . This allows us to choose uniformly a component  $C_1$ . Then by Proposition 4.9 the secant line  $\overline{D}_{ij}$  coincides with a bitangent to  $\Gamma$ . Therefore the preimage  $\pi^{-1}(\overline{D}_{ij})$  splits into two exceptional lines and we denote by  $l_{ij}$  the line which cuts out the divisor  $D_{ij}$  on the curve  $C_1 = C$ . Then the other line  $l'_{ij}$  cuts out the divisor  $D'_{ij}$  on  $C_1$  with  $D_{ij} + D'_{ij} \in |\omega|$ . Now it is immediate to check that the classes  $e_i = l_{i8}$  for  $1 \leq i \leq 7$  and  $e_0 = e_i + e_j - l_{ij} - k$  determine a geometric marking (4.9).

Conversely, given  $V$  and a geometric marking  $\varphi$ , we choose a line bundle  $L_8 \in \text{Pic}^1(C)$  such that  $\omega L_8^2 = e_0|_{C=C_1}$ . Next we define  $L_i$  for  $1 \leq i \leq 7$  by  $L_i L_8 = e_i|_{C=C_1}$ . Then one verifies that  $l_{ij}|_{C=C_1} = L_i L_j$  and therefore by Remark 4.3 there exists a bundle  $E \in \mathcal{M}_0$  such that  $\mathbf{M}_0(E) = \{L_1, \dots, L_8\}$ . Since  $L_8$  is defined up to  $JC[2]$ , this construction gives an element of  $\widetilde{\mathcal{N}}$ .

Since the element  $\overline{E} \in \widetilde{\mathcal{N}}$  is determined by the 28 line bundles  $L_i L_j$ , it will be enough to describe the action of  $\widetilde{\mathcal{D}}$  and  $w_0 \in W(E_7)$  on the  $L_i L_j$ 's. Suppose  $\widetilde{\mathcal{D}}(\overline{E}) = \overline{F}$  with  $\mathbf{M}_0(F) = \{M_1, \dots, M_8\}$ , then it follows from the equality  $\mathbf{M}_\omega(F) = \mathbf{M}_0(E)$  (assuming  $F = \mathcal{D}(E)$ ) that  $M_i M_j = \omega L_i^{-1} L_j^{-1}$ . On the other hand we have  $w_0(l_{ij}) = l'_{ij}$  and  $l_{ij} + l'_{ij} = -k$ . Restricting to  $C = C_1$  ( $-k|_C = \omega$ ), we obtain that  $w_0 = \widetilde{\mathcal{D}}$ .  $\square$

**4.16 Corollary.** *The morphism  $\text{Hess} : \mathcal{N}^{reg} \longrightarrow \mathcal{R}$ ,  $\overline{E} \longmapsto \text{Hess}(\overline{E})$  is finite of degree 72 and, if  $C$  is general, the map*

$$\mathcal{N}^{reg} \longrightarrow \mathcal{M}_3, \quad \overline{E} \longmapsto \text{iso class}(\text{Hess}(\overline{E}))$$

is dominant.

*Proof.* The first assertion follows from  $|W(E_7)/\Sigma_8| = 72$ . For the second it suffices to show that the forgetful map  $\mathcal{R} \rightarrow \mathcal{M}_3$  is dominant: let  $[C] \in |\mathcal{O}_{\mathbb{P}^2}(4)| = \mathbb{P}^{14}$  denote the quartic equation of  $C$ . Projection with center  $[C]$  maps  $|\mathcal{O}_{\mathbb{P}^2}(4)| \longrightarrow |\omega^4|$ . We immediately see that  $\mathcal{R}$  equals the cone with vertex  $[C]$  over the Veronese variety  $\text{Ver}|\omega^2| \hookrightarrow |\omega^4|$ . If  $C$  is general, one can show (e.g. by computing the differential of the natural map  $\mathbb{PGL}_3 \times \mathcal{R} \longrightarrow |\mathcal{O}_{\mathbb{P}^2}(4)|$ ) that the  $\mathbb{PGL}_3$ -orbit of the cone  $\mathcal{R}$  (note that  $\dim \mathcal{R} = 6$ ) in  $|\mathcal{O}_{\mathbb{P}^2}(4)| = \mathbb{P}^{14}$  is dense and since  $\mathcal{M}_3 = |\mathcal{O}_{\mathbb{P}^2}(4)|/\mathbb{PGL}_3$ , we obtain the result.  $\square$

**4.17 Remark.** The action of the reflection  $s_{ijk} \in W(E_7)$  on  $\widetilde{\mathcal{N}}$  is easily deduced from its action on the exceptional lines  $l_{pq}$  and  $l'_{pq}$  (see section 4.5.1). Representing an element  $\overline{E} \in \widetilde{\mathcal{N}}$  by  $e \in |\omega L_8^2|^*$ , it is easily checked that the restriction of  $s_{ijk}$  to  $|\omega L_8^2|^*$  is given by the linear system of quadrics on  $|\omega L_8^2|^*$  passing through the 6 points  $D_{ijk} = D_{i8} + D_{j8} + D_{k8}$ . In this way we can construct the  $72 = 2(1 + \binom{7}{3})$  bundles in the fibre of  $\text{Hess} : \mathcal{N}^{reg} \rightarrow \mathcal{R}$ .

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