RANK FOUR VECTOR BUNDLES WITHOUT THETA DIVISOR OVER A CURVE OF GENUS TWO

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Abstract. We show that the locus of stable rank four vector bundles without theta divisor over a smooth projective curve of genus two is in canonical bijection with the set of theta-characteristics. We give several descriptions of these bundles and compute the degree of the rational theta map.

1. Introduction

Let C be a complex smooth projective curve of genus 2 and let $\mathcal{M}_r$ denote the coarse moduli space parametrizing semi-stable rank-r vector bundles with trivial determinant over the curve C. Let $C \cong \Theta \subset \mathrm{Pic}^1(C)$ be the Riemann theta divisor in the degree 1 component of the Picard variety of C. For any $E \in \mathcal{M}_r$ we consider the locus

$$\theta(E) = \{L \in \mathrm{Pic}^1(C) \mid h^0(C, L \otimes E) > 0\},$$

which is either a curve linearly equivalent to $r\Theta$ or $\mathrm{Pic}^1(C)$, in which case we say that E has no theta divisor. We obtain thus a rational map, the so-called theta map

$$\theta : \mathcal{M}_r \dashrightarrow |r\Theta|,$$

between varieties having the same dimension $r^2 - 1$. We denote by $\mathcal{B}_r$ the closed subvariety of $\mathcal{M}_r$ parametrizing semi-stable bundles without theta divisor. It is known [R] that $\mathcal{B}_2 = \mathcal{B}_3 = \emptyset$ and that $\mathcal{B}_r \neq \emptyset$ for $r \geq 4$. We refer to the survey papers [B1] and [Po] for a detailed exposition of some results and some open problems related to the theta map θ and the loci $\mathcal{B}_r$.

Is was recently shown that θ is generically finite; see [B2] Theorem A. Moreover the cases of low rank r have been studied in the past: if $r = 2$ the theta map is an isomorphism $\mathcal{M}_2 \cong \mathbb{P}^3$ [NR] and if $r = 3$ the theta map realizes $\mathcal{M}_3$ as a double covering of $\mathbb{P}^8$ ramified along a sextic hypersurface [O].

In this note we study the next case $r = 4$ and give a complete description of the locus $\mathcal{B}_4$. Our main result is the following

Theorem 1.1. *Let C be a curve of genus 2.*

1. *The locus $\mathcal{B}_4$ is of dimension 0, reduced and of cardinality 16.*
2. *There exists a canonical bijection between $\mathcal{B}_4$ and the set of theta-characteristics of C. Let $E_\kappa \in \mathcal{B}_4$ denote the stable vector bundle associated with the theta-characteristic κ. Then*
$$\Lambda^2 E_\kappa = \bigoplus_{\alpha \in S(\kappa)} \alpha, \qquad \mathrm{Sym}^2 E_\kappa = \bigoplus_{\alpha \in J[2] \setminus S(\kappa)} \alpha,$$
where $S(\kappa)$ is the set of 2-torsion line bundles $\alpha \in J[2]$ such that $\kappa\alpha \in \Theta \subset \mathrm{Pic}^1(C)$.
3. *If κ is odd, then E_κ is a symplectic bundle. If κ is even, then E_κ is an orthogonal bundle with non-trivial Stiefel-Whitney class.*
4. *For each theta-characteristic κ, the vector bundle E_κ is invariant under the tensor product with the group $J[2]$.*

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The 16 vector bundles E_κ already appeared in Raynaud's paper [R] as Fourier-Mukai transforms and were further studied in [Hi] and [He] — see section 2.2. We note that Theorem 1.1 completes the main result of [Hi] which describes the restriction of $\mathcal{B}_4$ to *symplectic* rank-4 bundles. The method of this paper is different and is partially based on [Pa].

As an application of Theorem 1.1 we obtain the degree of the theta map for $r = 4$. We refer to [BV] for a geometric interpretation of the general fiber of θ in terms of certain irreducible components of a Brill-Noether locus of the curve $\theta(E) \subset \mathrm{Pic}^1(C)$.

Corollary 1.2. *The degree of the rational theta map* $\theta : \mathcal{M}_4 \dashrightarrow |4\Theta|$ *equals* 30.

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Notations: If E is a vector bundle over C, we will write $H^i(E)$ for $H^i(C, E)$ and $h^i(E)$ for $\dim H^i(C, E)$. We denote the slope of E by $\mu(E) := \frac{\deg E}{\mathrm{rk} E}$, the canonical bundle over C by K and the degree d component of the Picard variety of C by $\mathrm{Pic}^d(C)$. We denote by $J := \mathrm{Pic}^0(C)$ the Jacobian of C and by $J[n]$ its group of n-torsion points. The divisor $\Theta_\kappa \subset J$ is the translate of the Riemann theta divisor $C \cong \Theta \subset \mathrm{Pic}^1(C)$ by a theta-characteristic κ. The line bundle $\mathcal{O}_J(2\Theta_\kappa)$ does not depend on κ and will be denoted by $\mathcal{O}_J(2\Theta)$. The tensor product $M \otimes N$ of two line bundles M and N will simply be denoted by MN.

2. Proof of Theorem 1.1

2.1. **The 16 vector bundles E_κ.** We first show that the set-theoretical support of $\mathcal{B}_4$ consists of 16 stable vector bundles E_κ, which are canonically labelled by the theta-characteristics of C.

We note that $\mathcal{B}_4 \neq \emptyset$ by [R], see also [Pa] Theorem 1.1. We consider a vector bundle $\mathcal{E} \in \mathcal{B}_4$. First we will show that $\mathcal{E}$ is stable. Assume that $\mathcal{E}$ is strictly semi-stable. Then $\mathcal{E}$ is S-equivalent to a direct sum $\oplus_i E_i$ with rk $E_i \leq 3$ and E_i stable of degree 0. Moreover we have the inequalities

$$0 < h^0(L \otimes \mathcal{E}) \leq \sum_i h^0(L \otimes E_i) \qquad \text{for any } L \in \mathrm{Pic}^1(C),$$

which implies that there exists an index i such that $h^0(C, L \otimes E_i) > 0$ for any $L \in \mathrm{Pic}^1(C)$. But this means that the bundle E_i has no theta divisor, which contradicts $\mathcal{B}_2 = \mathcal{B}_3 = \emptyset$.

We introduce $\mathcal{E}' = \mathcal{E}^* \otimes K$. Then $\mu(\mathcal{E}') = 2$ and since $\mathcal{E} \in \mathcal{B}_4$, we obtain that $h^0(\mathcal{E}' \otimes \lambda^{-1}) = h^1(\mathcal{E} \otimes \lambda) = h^0(\mathcal{E} \otimes \lambda) > 0$ for any $\lambda \in \mathrm{Pic}^1(C)$. In particular for any $x \in C$ we have $h^0(\mathcal{E}' \otimes \mathcal{O}_C(-x)) > 0$. On the other hand stability of $\mathcal{E}$ implies that $h^0(\mathcal{E}) = h^1(\mathcal{E}') = 0$. Hence $h^0(\mathcal{E}') = 4$ by Riemann-Roch. Thus we obtain that the evaluation map of global sections

$$\mathcal{O}_C \otimes H^0(\mathcal{E}') \xrightarrow{ev} \mathcal{E}'$$

is not of maximal rank. Let us denote by $I :=$ im ev the subsheaf of $\mathcal{E}'$ given by the image of ev. Then clearly $h^0(I) = 4$. The cases of rk $I \leq 2$ are easily ruled out using stability of $\mathcal{E}'$. Hence we conclude that rk $I = 3$. We then consider the natural exact sequence

$$0 \longrightarrow L^{-1} \longrightarrow \mathcal{O}_C \otimes H^0(\mathcal{E}') \xrightarrow{ev} I \longrightarrow 0, \tag{1}$$

where L is the line bundle such that $L^{-1} :=$ ker ev.

Proposition 2.1. *We have* $h^0(I^*) = 0$.

Proof. Suppose on the contrary that there exists a non-zero map $I \to \mathcal{O}_C$. Its kernel $S \subset I$ is a rank-2 subsheaf of $\mathcal{E}'$ and by stability of $\mathcal{E}'$ we obtain $\mu(S) < \mu(\mathcal{E}') = 2$, hence $\deg S \leq 3$. Moreover $h^0(S) \geq h^0(I) - 1 = 3$.

Assume that $\deg S = 3$. Then S is stable and S can be written as an extension

$$0 \longrightarrow \mu \longrightarrow S \longrightarrow \nu \longrightarrow 0,$$

with $\deg \mu = 1$ and $\deg \nu = 2$. The condition $h^0(S) \geq 3$ then implies that $\mu = \mathcal{O}_C(x)$ for some $x \in C$, $\nu = K$ and that the extension has to be split, i.e., $S = K \oplus \mathcal{O}_C(x)$. This contradicts stability of S.

The assumption $\deg S \leq 2$ similarly leads to a contradiction. We leave the details to the reader. □

Now we take the cohomology of the dual of the exact sequence (1) and we obtain — using $h^0(I^*) = 0$ — an inclusion $H^0(\mathcal{E}')^* \subset H^0(L)$. Hence $h^0(L) \geq 4$, which implies $\deg L \geq 5$. On the other hand $\deg L = \deg I$ and by stability of $\mathcal{E}'$, we have $\mu(I) < 2$, i.e., $\deg L \leq 5$. So we can conclude that $\deg L = 5$, that $H^0(\mathcal{E}')^* = H^0(L)$ and that $I = W_L$, where W_L is the *evaluation bundle* associated to L defined by the exact sequence

$$0 \longrightarrow W_L^* \longrightarrow H^0(L) \otimes \mathcal{O}_C \xrightarrow{ev} L \longrightarrow 0. \tag{2}$$

Moreover the subsheaf $W_L \subset \mathcal{E}'$ is of maximal degree, hence W_L is a subbundle of $\mathcal{E}'$ and we have an exact sequence

$$0 \longrightarrow W_L \longrightarrow \mathcal{E}' \longrightarrow K^4L^{-1} \longrightarrow 0, \tag{3}$$

with extension class $e \in \mathrm{Ext}^1(K^4L^{-1}, W_L) = H^1(W_L \otimes K^{-4}L) = H^0(W_L^* \otimes K^5L^{-1})^*$. Using Riemann-Roch and stability of W_L (see e.g. [Bu]) one shows that

$$h^0(W_L^* \otimes K^5L^{-1}) = 7, \qquad h^0(W_L^* \otimes K^5L^{-1}(-x)) = 4, \qquad h^0(W_L^* \otimes K^5L^{-1}(-x-y)) = 1$$

for *general* points $x, y \in C$. In that case we denote by $\mu_{x,y} \in \mathbb{P}H^0(W_L^* \otimes K^5L^{-1})$ the point determined by the 1-dimensional subspace $H^0(W_L^* \otimes K^5L^{-1}(-x-y))$. We also denote by

$$\mathbb{S} \subset \mathbb{P}H^0(W_L^* \otimes K^5L^{-1})$$

the linear span of the points $\mu_{x,y}$ when x and y vary in C and by $H_e \subset \mathbb{P}H^0(W_L^* \otimes K^5L^{-1})$ the hyperplane determined by the non-zero class e.

Tensoring the sequence (3) with $K^{-4}L(x+y)$ and taking cohomology one shows that $\mu_{x,y} \in H_e$ if and only if $h^0(\mathcal{E}' \otimes K^{-4}L(x+y)) > 0$. Since we assume $\mathcal{E} \in \mathcal{B}_4$, we obtain

$$\mathbb{S} \subset H_e.$$

We consider a *general* point $x \in C$ such that $h^0(W_L^* \otimes K^5L^{-1}(-x)) = 4$ and denote for simplicity

$$A := W_L^* \otimes K^5L^{-1}(-x).$$

Then A is stable with $\mu(A) = \frac{7}{3}$. We consider the evaluation map of global sections

$$ev_A : \mathcal{O}_C \otimes H^0(A) \longrightarrow A$$

and consider the set S_A of points $p \in C$ for which $(ev_A)_p$ is not surjective, i.e.

$$S_A = \{p \in C \mid h^0(A(-p)) \geq 2\}.$$

Then we have the following

Lemma 2.2. *We assume that x is general.*

1. *If $L^2 \neq K^5$, then the set S_A consists of the 2 distinct points p_1, p_2 determined by the relation $\mathcal{O}_C(p_1 + p_2) = K^4L^{-1}(-x)$.*

(2) *If $L^2 = K^5$, then the set S_A consists of the 2 distinct points p_1, p_2 introduced in (1) and the conjugate $\sigma(x)$ of x under the hyperelliptic involution σ.*

Proof. Given a point $p \in C$, we tensorize the exact sequence (2) with $K^5L^{-1}(-x-p)$ and take cohomology:

$$0 \longrightarrow H^0(A(-p)) \longrightarrow H^0(L) \otimes H^0(K^5L^{-1}(-x-p)) \longrightarrow H^0(K^5(-x-p)) \longrightarrow \cdots$$

We note that $h^0(K^5L^{-1}(-x-p)) = 2$. We distinguish two cases.

(a) The pencil $|K^5L^{-1}(-x-p)|$ has a base-point, i.e. there exists a point $q \in C$ such that $K^5L^{-1}(-x-p) = K(q)$, or equivalently $K^4L^{-1}(-x) = \mathcal{O}_C(p+q)$. Since x is general, we have $h^0(K^4L^{-1}(-x)) = 1$, which determines p and q, i.e., $\{p,q\} = \{p_1,p_2\}$. In this case $|K^5L^{-1}(-x-p)| = |K(q)| = |K|$ and $h^0(A(-p)) = h^0(K^{-1}L) = 2$. This shows that $p_1, p_2 \in S_A$.

(b) The pencil $|K^5L^{-1}(-x-p)|$ is base-point-free. By the base-point-free-pencil-trick, we have $H^0(A(-p)) \cong H^0(L^2K^{-5}(x+p))$. Since $\deg L^2K^{-5}(x+p) = 2$, we have $h^0(L^2K^{-5}(x+p)) = 2$ if and only if $L^2K^{-5}(x+p) = K$, or equivalently $\mathcal{O}_C(p) = K^6L^{-2}(-x)$. If $K^6L^{-2} \neq K$, then for general $x \in C$ the line bundle $K^6L^{-2}(-x)$ is not of the form $\mathcal{O}_C(p)$. If $K^6L^{-2} = K$, then for any $x \in C$, $K^6L^{-2}(-x) = \mathcal{O}_C(\sigma(x))$, which implies that $\sigma(x) \in S_A$.

This shows the lemma. □

Proposition 2.3. *If $L^2 \neq K^5$, then $\mathbb{S} = \mathbb{P}H^0(W_L^* \otimes K^5L^{-1})$.*

Proof. We consider a general point $x \in C$ and the rank-3 bundle A. Let $B \subset A$ denote the subsheaf given by the image of ev_A. By Lemma 2.2 (1) we have $\deg B = \deg A - 2 = 5$. Moreover $H^0(B) = H^0(A)$ and there is an exact sequence

$$0 \longrightarrow M^{-1} \longrightarrow \mathcal{O}_C \otimes H^0(B) \xrightarrow{ev_A} B \longrightarrow 0, \tag{4}$$

with $M \in \mathrm{Pic}^5(C)$. It follows that the rational map

$$\phi_x : C \dashrightarrow \mathbb{P}H^0(B) = \mathbb{P}H^0(A) = \mathbb{P}^3, \qquad y \mapsto \mu_{x,y}$$

factorizes through

$$C \xrightarrow{\varphi_M} |M|^* \longrightarrow \mathbb{P}H^0(B),$$

where φ_M is the morphism given by the linear system $|M|$ and the second map is linear and identifies with the projectivization of the dual of δ, which is given by the long exact sequence obtained from (4) by dualizing and taking cohomology:

$$0 \longrightarrow H^0(B^*) \longrightarrow H^0(B)^* \xrightarrow{\delta} H^0(M) \longrightarrow H^1(B^*) \longrightarrow \cdots$$

We obtain that the linear span of $\mathrm{im}\ \phi_x$ is non-degenerate if and only if $h^0(B^*) = 0$.

We now show that $h^0(B^*) = 0$. Suppose on the contrary that there exists a non-zero map $B \to \mathcal{O}_C$. Its kernel $S \subset B$ is a rank-2 subsheaf of A with $\deg S \geq \deg B = 5$, hence $\mu(S) \geq \frac{5}{2}$, which contradicts stability of A — recall that $\mu(A) = \frac{7}{3}$.

This shows that $\mathrm{im}\ \phi_x$ spans $\mathbb{P}H^0(A) \subset \mathbb{P}H^0(W_L^* \otimes K^5L^{-1})$ for general $x \in C$. We now take 2 general points $x, x' \in C$ and deduce from $\dim H^0(A) \cap H^0(A') = \dim H^0(W_L^* \otimes K^5L^{-1}(-x-x')) = 1$ that the linear span of the union $\mathbb{P}H^0(A) \cap \mathbb{P}H^0(A')$ equals the full space $\mathbb{P}H^0(W_L^* \otimes K^5L^{-1})$. This shows the proposition. □

We deduce from the proposition that the line bundle L satisfies the relation $L^2 = K^5$, i.e.

$$L = K^2\kappa$$

for some theta-characteristic κ of C. In that case we note that $H^0(W_L^* \otimes K^5L^{-1})$ equals $H^0(W_L^* \otimes L)$ and we can consider the exact sequence

$$0 \longrightarrow H^0(W_L^* \otimes L) \longrightarrow H^0(L) \otimes H^0(L) \xrightarrow{\mu} H^0(L^2) \longrightarrow 0,$$

obtained from (2) by tensoring with L and taking cohomology. We also note that there is a natural inclusion $\Lambda^2 H^0(L) \subset H^0(W_L^* \otimes L)$, see e.g. [Pa] section 2.1. More precisely we can show

Proposition 2.4. *The linear span* $\mathbb{S}$ *equals*

$$\mathbb{S} = \mathbb{P}\Lambda^2 H^0(L) \subset \mathbb{P}H^0(W_L^* \otimes L).$$

Proof. Using the standard exact sequences and the base-point-free-pencil-trick, one easily works out that for general points $x, y \in C$

$$\mu_{x,y} = \mathbb{P}\Lambda^2 H^0(L(-x-y)) \subset \mathbb{P}\Lambda^2 H^0(L) \subset \mathbb{P}H^0(W_L^* \otimes L).$$

This implies that $\mathbb{S} \subset \mathbb{P}\Lambda^2 H^0(L)$. In order to show equality one chooses 4 general points $x_i \in C$ such that their images $C \to |L|^* = \mathbb{P}^3$ linearly span the $\mathbb{P}^3$. We denote by $s_i \in H^0(L)$ the global section vanishing on the points x_j for $j \neq i$ and not vanishing on x_i. Then one checks that for any choice of the indices i, j, k, l such that $\{i, j, k, l\} = \{1, 2, 3, 4\}$ one has $s_i \wedge s_j = \mu_{x_k, x_l}$. Since the 6 tensors $s_i \wedge s_j$ are a basis of $\Lambda^2 H^0(L)$, we obtain equality. $\square$

The hyperplane $\mathbb{S} = \mathbb{P}\Lambda^2 H^0(L) \subset \mathbb{P}H^0(W_L^* \otimes L)$ determines a unique (up to a scalar) non-zero extension class $e \in H^0(W_L^* \otimes L)^*$ by $\mathbb{S} = H_e$, which in turn determines a unique stable vector bundle $\mathcal{E} \in \mathcal{B}_4$, which we will denote by E_κ. For the convenience of the reader we recall the exact sequence

$$0 \longrightarrow W_L \otimes K^{-1} \longrightarrow E_\kappa^* \longrightarrow \kappa \longrightarrow 0, \qquad \text{with } L = K^2\kappa. \tag{5}$$

We will see in the next section that E_κ is self-dual, i.e. $E_\kappa^* = E_\kappa$.

This shows that $\mathcal{B}_4$ is of dimension 0 and of cardinality 16.

2.2. **The Raynaud bundles.** In this subsection we recall the construction of the Raynaud bundles introduced in [R] as Fourier-Mukai transforms. We refer to [Hi] section 9.2 for the details and the proofs.

The rank-4 vector bundle $\mathcal{O}_J(2\Theta) \otimes H^0(J, \mathcal{O}_J(2\Theta))^*$ over J admits a canonical $J[2]$-linearization and descends therefore under the duplication map $[2] : J \to J$, i.e., there exists a rank-4 vector bundle M over J such that

$$[2]^* M \cong \mathcal{O}_J(2\Theta) \otimes H^0(J, \mathcal{O}_J(2\Theta))^*.$$

Proposition 2.5. *For any theta-characteristic* κ *of* C *there exists an isomorphism*

$$\xi_\kappa : M \xrightarrow{\sim} M^* \otimes \mathcal{O}_J(\Theta_\kappa).$$

Moreover if κ *is even (resp. odd), then* ξ_κ *is symmetric (resp. skew-symmetric).*

Let $\gamma_\kappa : C \to J$ be the Abel-Jacobi map defined by $\gamma_\kappa(p) = \kappa^{-1}(p)$. We define the Raynaud bundle

$$R_\kappa := \gamma_\kappa^* M \otimes \kappa^{-1}.$$

Then by [R] the bundle $R_\kappa \in \mathcal{B}_4$. Since $\gamma_\kappa^* \mathcal{O}_J(\Theta_\kappa) = K$ we see that the isomorphism ξ_κ induces an orthogonal (resp. symplectic) structure on the bundle R_κ, if κ is even (resp. odd). In particular the bundle R_κ is self-dual, i.e., $R_\kappa = R_\kappa^*$. The pull-back $\gamma_\kappa^*(\xi'_\kappa)$ for a theta-characteristic $\kappa' = \kappa\alpha$ with $\alpha \in J[2]$ gives an isomorphism

$$R_\kappa \xrightarrow{\sim} R_\kappa^* \otimes \alpha,$$

hence a non-zero section in $H^0(\Lambda^2 R_\kappa \otimes \alpha)$ (resp. $H^0(\mathrm{Sym}^2 R_\kappa \otimes \alpha)$) if $h^0(\kappa\alpha) = 1$ (resp. $h^0(\kappa\alpha) = 0$). We deduce that there are isomorphisms

$$\Lambda^2 R_\kappa = \bigoplus_{\alpha \in S(\kappa)} \alpha, \qquad \mathrm{Sym}^2 R_\kappa = \bigoplus_{\alpha \in J[2]\setminus S(\kappa)} \alpha. \tag{6}$$

In particular the 16 bundles R_κ are non-isomorphic. Each R_κ is invariant under tensor product with $J[2]$. The isomorphisms (6) can be used to prove the relation

$$R_\kappa \otimes \beta = R_{\kappa\beta^2}, \qquad \forall \beta \in J[4]. \tag{7}$$

2.3. **Symplectic and orthogonal bundles.** In this subsection we give a third construction of the bundles in $\mathcal{B}_4$ as symplectic and orthogonal extension bundles. Let κ be a theta-characteristic.

If κ is odd, then $\kappa = \mathcal{O}_C(w)$ for some Weierstrass point $w \in C$. The construction outlined in [Pa] section 2.2 gives a unique symplectic bundle $\mathcal{E}_e \in \mathcal{B}_4$ with $e \in H^1(\mathrm{Sym}^2 G_\kappa)_+$. We denote this bundle by V_κ.

If κ is even, there is an analogue construction, which we briefly outline for the convenience of the reader. The proofs are similar to those given in [Hi]. Using the Atiyah-Bott-fixed-point formula one observes that among all non-trivial extensions

$$0 \longrightarrow \kappa^{-1} \longrightarrow G \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

there are 2 extensions (up to scalar), which are σ-invariant. We take one of them and call it G_κ. Then any non-zero class $e \in H^1(\Lambda^2 G_\kappa) = H^1(\kappa^{-1})$ determines an orthogonal bundle $\mathcal{E}_e$, which fits in the exact sequence

$$0 \longrightarrow G_\kappa \longrightarrow \mathcal{E}_e \longrightarrow G_\kappa^* \longrightarrow 0. \tag{8}$$

The composite map

$$D_{G_\kappa} : \mathbb{P}H^1(\Lambda^2 G_\kappa) \longrightarrow \mathcal{M}_4 \xrightarrow{\theta} |4\Theta|, \qquad e \mapsto \theta(\mathcal{E}_e)$$

is the projectivization of a linear map

$$\widetilde{D_{G_\kappa}} : H^1(\Lambda^2 G_\kappa) \longrightarrow H^0(\mathrm{Pic}^1(C), 4\Theta).$$

Moreover im $\widetilde{D_{G_\kappa}} \subset H^0(\mathrm{Pic}^1(C), 4\Theta)_-$, which can be seen as follows. By [Se] Thm 2 the second Stiefel-Whitney class $w_2(\mathcal{E}_e)$ of an orthogonal bundle $\mathcal{E}_e$ is given by the parity of $h^0(\mathcal{E}_e \otimes \kappa')$ for any theta-characteristic κ'. This parity can be computed by taking the cohomology of the exact sequence (8) tensorized with κ' and taking into account that the coboundary map is skew-symmetric. One obtains that $w_2(\mathcal{E}_e) \neq 0$ and one can conclude the above-mentioned inclusion by [B3] Lemma 1.4.

We now observe that by the Atiyah-Bott-fixed-point-formula $h^1(\Lambda^2 G_\kappa)_+ = h^1(\Lambda^2 G_\kappa)_- = 1$. By the argument given in [Pa] section 2.2 we conclude that one of the two eigenspaces $H^1(\Lambda^2 G_\kappa)_\pm$ is contained in the kernel ker $\widetilde{D_{G_\kappa}}$. We denote the corresponding bundle $\mathcal{E}_e$ by $V_\kappa \in \mathcal{B}_4$.

2.4. **Three descriptions of the same bundle.**

Proposition 2.6. *For any theta-characteristic κ the three bundles E_κ, R_κ and V_κ coincide.*

Proof. The proof of the identifications is worked out in detail in [Hi] sections 8 and 9 in the case κ odd. The case κ even is similar. For the convenience of the reader we briefly sketch the proof for any κ. Since the bundle V_κ is the unique rank-4 bundle without theta divisor which appears as an anti-symmetric (if κ even) or symmetric (if κ odd) extension of G_κ^* by G_κ, it will be enough to show that the rank-2 bundle G_κ is contained in E_κ and in R_κ. This is achieved by showing that G_κ is contained in the rank-3 bundle $W_{K^2\kappa} \otimes K^{-1} \subset E_\kappa$; see (5) and [Hi]. In order to show the inclusion $G_\kappa \subset R_\kappa$ we use the following result (see [Hi]): given an odd theta characteristic

$\kappa = \mathcal{O}_C(w)$, the fibers at the Weierstrass point w of any two degree -1 line subbundles N and σ^*N of the bundle R_κ coincide. We tensorize with a 4-torsion linebundle β in order to obtain an analogue result for any bundle $R_{\kappa\beta^2}$; see (7) □

This proposition shows all assertions of Theorem 1.1 except reducedness of $\mathcal{B}_4$.

I am grateful to Olivier Serman for giving me the following fourth description of the bundle E_κ for an even theta-characteristic κ. We recall that an even theta-characteristic κ corresponds to a partition of the set of six Weierstrass points of C into two subsets of three points, which we denote by $\{w_1, w_2, w_3\}$ and $\{w_4, w_5, w_6\}$. With this notation we have

Proposition 2.7. *Let κ be an even theta-characteristic. We denote by A_κ (resp. B_κ) the unique stable rank-2 bundle with determinant κ and which contains the four 2-torsion line bundles $\mathcal{O}_X$, $\mathcal{O}_X(w_1 - w_2)$, $\mathcal{O}_X(w_1 - w_3)$ and $\mathcal{O}_X(w_2 - w_3)$ (resp. $\mathcal{O}_X$, $\mathcal{O}_X(w_4 - w_5)$, $\mathcal{O}_X(w_4 - w_6)$ and $\mathcal{O}_X(w_5 - w_6)$). Then the orthogonal rank-4 vector bundle E_κ is isomorphic to*

$$\mathrm{Hom}(A_\kappa, B_\kappa)$$

equipped with the quadratic form given by the determinant.

We refer to [S] section 5.5 for the proof. We note that the bundle G_κ introduced in section 2.3 is either A_κ^* or B_κ^*.

2.5. **Reducedness of $\mathcal{B}_4$.** We start with a description of the space of global sections $H^0(\mathcal{M}_4, \mathcal{L})$.

Proposition 2.8. *For any theta-characteristic κ there is a section $s_\kappa \in H^0(\mathcal{M}_4, \mathcal{L})$ with zero divisor*

$$\Delta_\kappa := \mathrm{Zero}(s_\kappa) = \{E \in \mathcal{M}_4 \mid h^0(\Lambda^2 E \otimes \kappa) > 0\}.$$

The 16 sections s_κ form a basis of $H^0(\mathcal{M}_4, \mathcal{L})$.

Proof. The Dynkin index of the second fundamental representation $\rho : \mathfrak{sl}_4(\mathbb{C}) \to \mathrm{End}(\Lambda^2\mathbb{C}^4)$ equals 2 (see e.g. [LS] Proposition 2.6). Moreover the bundle $\Lambda^2 E \otimes \kappa$ admits a K-valued non-degenerate quadratic form, which allows to construct the Pfaffian divisor s_κ, which is a section of $\mathcal{L}$ (see [LS]). The space $H^0(\mathcal{M}_4, \mathcal{L})$ is a representation of level 2 of the Heisenberg group $Heis(2)$, which is a central extension of $J[2]$ by $\mathbb{C}^*$. One can work out that the sections s_κ generate the 16 one-dimensional character spaces for the $Heis(2)$-action on $H^0(\mathcal{M}_4, \mathcal{L})$. This shows that the sections s_κ are linearly independent. □

Since $E_\kappa \in \mathcal{B}_4$, we have $E_\kappa \in \Delta_{\kappa'}$ for any theta-characteristic κ'. By the deformation theory of determinant and Pfaffian divisors (see e.g. [L], [LS]) the point $E_\kappa \in \mathcal{M}_4$ is a smooth point of the divisor $\Delta_{\kappa'} \subset \mathcal{M}_4$ if and only if the following two conditions hold

(1) $h^0(\Lambda^2 E_\kappa \otimes \kappa') = 2$,
(2) the natural linear form

$$\Phi_{\kappa'} : T_{E_\kappa}\mathcal{M}_4 = H^1(\mathrm{End}_0(E_\kappa)) \longrightarrow \Lambda^2 H^0(\Lambda^2 E_\kappa \otimes \kappa')^*$$

is non-zero.

Moreover if these two conditions hold, then $T_{E_\kappa}\Delta_{\kappa'} = \ker \Phi_{\kappa'}$. The map $\Phi_{\kappa'}$ is built up as follows: the exceptional isomorphism of Lie algebras $\mathfrak{sl}_4 \cong \mathfrak{so}_6$ induces a natural vector bundle isomorphism

$$\mathrm{End}_0(E_\kappa) \xrightarrow{\sim} \Lambda^2(\Lambda^2 E_\kappa). \tag{9}$$

Then $\Phi_{\kappa'}$ is the dual of the linear map given by the wedge product of global sections

$$\Lambda^2 H^0(\Lambda^2 E_\kappa \otimes \kappa') \longrightarrow H^0(\Lambda^2(\Lambda^2 E_\kappa) \otimes K) = H^0(\mathrm{End}_0(E_\kappa) \otimes K).$$

Proposition 2.9. *The 0-dimensional scheme $\mathcal{B}_4$ is reduced.*

Proof. Since E_κ is a smooth point of $\mathcal{M}_4$ and $\dim T_{E_\kappa}\mathcal{M}_4 = 15$, it is sufficient to show that for any theta-characteristic $\kappa' \neq \kappa$ the divisor $\Delta_{\kappa'}$ is smooth at E_κ and that the 15 hyperplanes $\ker \Phi_{\kappa'} \subset T_{E_\kappa}\mathcal{M}_4$ are linearly independent: using the isomorphism (6) we obtain that for $\kappa' \neq \kappa$

$$h^0(\Lambda^2 E_\kappa \otimes \kappa') = \sharp S(\kappa) \cap S(\kappa') = 2$$

and using the isomorphism (9) we obtain that

$$\mathrm{End}_0(E_\kappa) = \bigoplus_{\alpha \in J[2]\setminus\{0\}} \alpha.$$

On the other hand one easily sees that if $\gamma, \delta \in J[2]$ are the two 2-torsion points in the intersection $S(\kappa) \cap S(\kappa')$, then $\kappa' = \kappa\gamma\delta$, hence $\Lambda^2 H^0(\Lambda^2 E_\kappa \otimes \kappa') \cong H^0(K\gamma\delta)$. This implies that the linear form

$$\Phi_{\kappa'} : \bigoplus_{\alpha \in J[2]\setminus\{0\}} H^1(\alpha) \longrightarrow H^0(K\gamma\delta)^* = H^1(\beta)$$

is projection onto the direct summand $H^1(\beta)$, where $\beta = \kappa^{-1}\kappa' \in J[2]$. This description of the linear forms $\Phi_{\kappa'}$ clearly shows that they are non-zero and linearly independent. ☐

This completes the proof of Theorem 1.1.

3. Proof of Corollary 1.2

Since by Theorem 1.1 $\mathcal{B}_4$ is a reduced 0-dimensional scheme of length 16, the degree of the theta map θ is given by the formula

$$\deg \theta + 16 = c_{15},$$

where $\frac{c_{15}}{15!}$ is the leading coefficient of the Hilbert polynomial

$$P(n) = \chi(\mathcal{M}_4, \mathcal{L}^n) = \frac{c_{15}}{15!} n^{15} + \text{lower degree terms.}$$

In order to compute the polynomial P we write

$$(10) \qquad P(X) = \sum_{k=0}^{15} \alpha_k Q_k(X), \qquad \text{with} \qquad Q_k(X) = \frac{1}{k!}(X+7)(X+6)\cdots(X+8-k)$$

and $Q_0(X) = 1$. Note that $\deg Q_k = k$ and that $c_{15} = \alpha_{15}$. The canonical bundle of $\mathcal{M}_4$ equals $\mathcal{L}^{-8}$. By the Grauert-Riemenschneider vanishing theorem we obtain that $h^i(\mathcal{M}_4, \mathcal{L}^n) = 0$ for any $i \geq 1$ and $n \geq -7$. Hence $P(n) = h^0(\mathcal{M}_4, \mathcal{L}^n)$ for $n \geq -7$. Moreover $P(n) = 0$ for $n = -7, -6, \ldots, -1$ and $P(0) = 1$. The values $P(n)$ for $n = 1, 2, \ldots, 8$ can be computed by the Verlinde formula and with the use of MAPLE. They are given in the following table.

n	1	2	3	4	5	6	7	8
$P(n)$	16	140	896	4680	21024	83628	300080	984539

Using the expression (10) of P one straightforwardly deduces the coefficients α_k by increasing induction on k: $\alpha_k = 0$ for $k = 0, 1, \ldots, 6$ and the values α_k for $k = 7, \ldots, 15$ are given in the following table.

k	7	8	9	10	11	12	13	14	15
α_k	1	8	32	96	214	328	324	184	46

Hence $\deg \theta = \alpha_{15} - 16 = 30$.

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