$SU(2)$-Verlinde spaces as theta spaces on Pryms

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1 Introduction

By an ($SU(r)$-)Verlinde space one means a complex vector space $H^0(M, \mathcal{L}^k)$ of sections of a line bundle $\mathcal{L}^k$ on a moduli space $M = M(r, d)$ of semistable vector bundles of rank r and fixed determinant, degree $d \in \mathbf{Z}/r$, over a smooth complex projective curve C. One knows that the projective variety M has infinite cyclic Picard group [DN], and $\mathcal{L}$ denotes the ample generator. These spaces,whose dimensions are given by the celebrated Verlinde formulae, are of considerable interest and have been much studied in recent years.

The purpose of the present article is to explain the following coincidence in the Verlinde formulae for the anticanonical line bundles of the varieties $M_0 = M(2, 0)$ and $M_1 = M(2, 1)$. We should first remark that there are already a number of well-known 'coincidences' linking the rank 2 Verlinde spaces for low values of k to spaces of theta functions on the Jacobian of the curve. For example, the simplest Verlinde formula

$$h^0(M_0, \mathcal{L}) = 2^g$$

(where g is the genus of C) was first shown directly [B1] by noting that $H^0(M_0, \mathcal{L})$ is naturally isomorphic to the 2-theta space $V = H^0(J^{g-1}(C), 2\Theta)$. The next case is the pair of formulae

$$h^0(M_0, \mathcal{L}^2) = 2^{g-1}(2^g + 1), \qquad h^0(M_1, \mathcal{L}) = 2^{g-1}(2^g - 1);$$

that is, these give the numbers of even and odd theta-characteristics on the curve. This also was explained by Beauville [B2]: for generic C the respective Verlinde spaces are canonically isomorphic to the 4-theta spaces

$S^2V = H^0_+(J,4\Theta)$ and $\bigwedge^2 V = H^0_-(J,4\Theta)$, the subscript $\pm$ denoting even/odd theta functions. One point of interest of these results is their connection with the classical Schottky-Jung geometry of the Jacobian and its associated Prym varieties of unramified double covers of C. Indeed, the Kummers of all these abelian varieties, 2-theta embedded in $\mathbf{P}(V^\vee)$, lie naturally inside the image of M_0,which lives, via the Beauville isomorphism, in the same projective space by means of the linear system $|\mathcal{L}|$. This observation has been used [vGP1] to give a vector bundle theoretic proof of the Schottky-Jung relations between the theta-nulls of the Jacobian and those of its Pryms.

It is therefore interesting to explore further this sort of relationship between the 'Verlinde' geometry of the curve, and its classical theta geometry. In [vGP1] the space $H^0(M_0,\mathcal{L}^4)$ has been studied by restricting sections to all the embedded Kummers; this gives an injection of $H^0(M_0,\mathcal{L}^4)$ into the direct sum of the spaces $H^0_+(P_x, 8\Xi_x)$ as x ranges through the group $J[2]$ of 2-torsion points in the Jacobian. The notation here is: (P_x,Ξ_x) denotes the canonically principally polarised Prym variety associated to the 2-torsion point x when this is nonzero, and equals $(J(C),\Theta)$ when $x=0$.

In the present work we attempt to improve on this. One observes the following: first, the Verlinde formulae for $h^0(M_0,\mathcal{L}^4)$ and $h^0(M_1,\mathcal{L}^2)$, respectively, are the following numbers, differing by a sign change:

$$\begin{aligned} & 3^{g-1}2^{2g-1} \pm 2^{2g-1} + 3^{g-1} \\ = & \frac{3^g \pm 1}{2} + (2^{2g}-1)\Big(\frac{3^{g-1}\pm 1}{2}\Big). \end{aligned}$$

On the other hand, for any principally polarised abelian variety (P,Ξ) of dimension n one has $h^0_\pm(P,3\Xi) = (3^n \pm 1)/2$. Finally, M_0 and M_1 are Fano varieties, with anticanonical sheaf given (by [DN]) by $K^{-1}_{M_0} = \mathcal{L}^4$ and $K^{-1}_{M_1} = \mathcal{L}^2$.

Taken together, these observations motivate the following result:

1.1 Theorem. *For any curve C there are natural homomorphisms:*

$$\begin{aligned} S_0 : H^0(M_0, K^{-1})^\vee &\longrightarrow \bigoplus_{x\in J[2]} H^0_+(P_x, 3\Xi_x), \\ S_1 : H^0(M_1, K^{-1})^\vee &\longrightarrow \bigoplus_{x\in J[2]} H^0_-(P_x, 3\Xi_x). \end{aligned}$$

Moreover, S_0 is an isomorphism for all C without vanishing theta-nulls.

1.2 *Remark.* In fact we also show that S_1 is an isomorphism whenever the multiplication map
$$S^2H^0(M_1, \mathcal{L}) \to H^0(M_1, \mathcal{L}^2)$$
is surjective. We conjecture that this is true for all C without vanishing theta-nulls, but this is not proved.

Roughly speaking, each homomorphism is dual to the pull-back of hyperplane sections via a collection of rational maps

$$\begin{array}{rcl} g : M_d & \to & |3\Xi_x| \\ V & \mapsto & \{L \in P_x | H^0(\widetilde{C}_x, L \otimes \mathrm{ad}\ p^*V) \neq 0\} \end{array}$$

where $p : \widetilde{C}_x \to C$ is the double cover associated to $x \in J[2]$ and ad V is the bundle of tracefree endomorphisms of V.

After collecting together some preliminaries concerning Pryms and theta characteristics in §2, we explain the main construction for $x = 0$ in §3, and generalise it to $x \neq 0$ in §4. In §5 we verify that this construction yields theta functions of the correct parity $(-)^d$ and prove theorem 1.1.

Finally we should remark that there exists a natural generalisation of theorem 1.1: one can replace the level 3 on the right by higher odd numbers, if one replaces M_0 and M_1 on the left by moduli spaces of spin bundles. This is explained in [O2].

2 Theta characteristics and Prym varieties

2.1 Theta characteristics

Let A be a principally polarised abelian variety of dimension g with symplectic form
$$\langle\ ,\ \rangle : A[2] \times A[2] \to \{\pm 1\};$$
and let $S(A)$ denote the $A[2]$-torsor of symmetric divisors Θ representing the polarisation. The line bundle $L = \mathcal{O}_A(2\Theta)$ is independent of $\Theta \in S(A)$, and we shall write $V = H^0(A, L)$.

A *theta characteristic* of A is a quadratic form $\kappa : A[2] \to \{\pm 1\}$ associated to $\langle\ ,\ \rangle$, i.e.

$$\kappa(x + y)\kappa(x)\kappa(y) = \langle x, y\rangle \qquad \text{for } x, y \in A[2].$$

The set of theta characteristics, which we shall denote by $\vartheta(A)$, is an $A[2]$-torsor with respect to the action $\kappa \mapsto x \cdot \kappa$, $x \in A[2]$, where

$$x \cdot \kappa(y) = \langle x, y \rangle \kappa(y) \qquad \text{for } y \in A[2].$$

There is a canonical identification of $A[2]$-torsors (which we shall implicitly make in what follows):

$$S(A) \xrightarrow{\sim} \vartheta(A).$$

This sends Θ to κ_Θ defined by $\kappa_\Theta(x) = (-1)^{\mathrm{mult}_x(\Theta)+\mathrm{mult}_0(\Theta)}$. The function

$$\begin{array}{rcl} \varepsilon : \vartheta(A) = S(A) & \to & \{\pm 1\} \\ \Theta & \mapsto & (-1)^{\mathrm{mult}_0(\Theta)} \end{array} \tag{1}$$

is called the *parity*, and has the property that any $\kappa \in \vartheta(A)$ takes the value $+1$ (resp. -1) at $2^{g-1}(2^g + \varepsilon(\kappa))$ points (resp. $2^{g-1}(2^g - \varepsilon(\kappa))$ points). It satisfies

$$\varepsilon(x \cdot \kappa) = \kappa(x)\varepsilon(\kappa). \tag{2}$$

We shall write $\vartheta^+(A)$ and $\vartheta^-(A)$ for the sets $\varepsilon^{-1}(\pm 1)$ of *even* and *odd* theta characteristics, respectively.

2.1 *Remark.* Suppose $A = J(C)$ is the Jacobian of a curve C, and denote by $\vartheta(C) \subset J^{g-1}(C)$ the set of theta characteristics of C, i.e. line bundles L such that $L^2 = K_C$. Then $\vartheta(C) \cong \vartheta(J(C)) \cong S(J(C))$ as $J[2]$-torsors by $L \mapsto \Theta_L = \{M \in J(C) | H^0(L \otimes M) \neq 0\}$; and ε is the usual parity function, by the Riemann singularity theorem.

Any given $\kappa = \kappa_\Theta \in \vartheta(A)$ can be used to identify, via the $A[2]$-action:

$$\begin{array}{rcl} A[2] & \xrightarrow{\sim} & \vartheta(A), \\ x & \mapsto & x \cdot \kappa_\Theta. \end{array} \tag{3}$$

Later on we shall make use of the following observation.

2.2 Lemma. *Let $k \geq 3$ be an odd integer. Then under the identification (3), the subset $\vartheta^+(A)$ (resp. $\vartheta^-(A)$) consists precisely of the base-points in $A[2]$ of the linear system $\mathbf{P}H^0_-(A, \mathcal{O}(k\Theta))$ (resp. $\mathbf{P}H^0_+(A, \mathcal{O}(k\Theta))$).*

Proof. Consider the natural involution $[-1] : A \leftrightarrow A$. This lifts to an action on the symmetric line bundle $\mathcal{O}(\Theta)$ and induces a linear involution $[-1]_x$ of the fibre of $\mathcal{O}(\Theta)$ at each $x \in A[2]$. It follows from the definition (1) that $[-1]_0 = +1$ (resp. -1) if and only if $\kappa_\Theta \in \vartheta^+(A)$ (resp. $\vartheta^-(A)$). Therefore under the identification (3), $\vartheta^\pm(A)$ consists of the points $x \in A[2]$ such that $[-1]_x = \pm 1$, respectively.

It follows that for any $s \in H^0_+(A, k\Theta)$ and $x \cdot \kappa_\Theta \in \vartheta^-(A)$ we have

$$\begin{aligned} s(x) &= ([-1]^* s)(x) \\ &= [-1]^*(s(x)) \\ &= -s(x) \end{aligned}$$

since $s(x)$ belongs to the fibre at x of $\mathcal{O}(k\Theta)$ on which $[-1]$ acts as -1, since k is odd. So $s(x) = 0$; while by the same argument $s(x) = 0$ whenever $s \in H^0_-(A, k\Theta)$ and $x \cdot \kappa_\Theta \in \vartheta^+(A)$.

Finally one notes that a point of A cannot be a base-point of both linear systems since by Lefschetz' theorem $|k\Theta|$ is base-point-free (and even very ample). □

2.2 Pryms

For each nonzero half-period $x \in J_2(C) - \{\mathcal{O}\}$ we have an associated unramified double cover

$$\pi : \widetilde{C}_x \to C.$$

We shall denote by σ the involution of $\widetilde{C}_x$ given by sheet-interchange over C; and by abuse of notation it will denote also the induced involution of $\mathrm{Pic}(\widetilde{C}_x)$. The kernel of the norm map on divisors has two isomorphic connected components:

$$\ker \mathrm{Nm} = P_x \cup P_x^-,$$

where $P_x = (1-\sigma)J^0(\widetilde{C}_x)$ and $P_x^- = (1-\sigma)J^1(\widetilde{C}_x)$.

A canonical principal polarisation is induced on P_x from $J(\widetilde{C}_x)$ as follows. Writing $\widetilde{J}^{2g-2} = J^{2g-2}(\widetilde{C}_x)$ we have

$$\mathrm{Nm}^{-1}(K_C) = \widehat{P}_x \cup \widehat{P}_x^- \subset \widetilde{J}^{2g-2},$$

where the two components are distinguished by:

$$h^0(\widetilde{C}_x, L) \equiv \begin{cases} 0 \mod 2 \text{ if } L \in \widehat{P}_x, \\ 1 \mod 2 \text{ if } L \in \widehat{P}_x^-, \end{cases}$$

for $L \in \mathrm{Nm}^{-1}(K_C)$. Then the theta-divisor $\widetilde{\Theta}$ in $\widetilde{J}^{2g-2}$ cuts $\widehat{P}_x$ with multiplicity two in a divisor Ξ_x which defines a principal polarisation. This induces by translation an identification of $\widehat{P}_x$ with the dual of P_x, and hence also a polarisation on P_x.

Notation. It will be convenient to follow [vGP1,2] and write $P_0 = J(C)$, and $\widehat{P}_0 = J^{g-1}(C)$, $\Xi_0 = \Theta_C$.

We can describe the theta characteristics of P_x in terms of those on C. Recall that $\ker \pi^* = \langle x \rangle$ and that there is a canonical symplectic isomorphism

$$\pi^* : x^{\perp}/\langle x \rangle \xrightarrow{\sim} P_x[2].$$

2.3 Proposition. *There are canonical bijections (for ϑ^+ and ϑ^- respectively)*

$$\coprod_{x \in J[2]} \vartheta^{\pm}(P_x) \cong S^2 \vartheta^{\pm}(C).$$

Proof. Pick theta characteristics $\kappa, \kappa' \in \vartheta(C)$ of the same parity, i.e. $\kappa' = x \cdot \kappa$ where $\kappa(x) = 1$. We may assume that $x \neq 0$, as the case $x = 0$ is trivial—$\vartheta^{\pm}(P_0)$ identifies with the diagonal of $S^2\vartheta^{\pm}(C)$ in the obvious way.

Then for any $y \in x^{\perp}$ we have

$$\kappa(x+y) = \kappa(x)\kappa(y)\langle x, y\rangle = \kappa(y),$$

so that κ induces a theta characteristic, which we shall denote by $\overline{\kappa}$, on $x^{\perp}/\langle x \rangle \cong P_x[2]$. Note that $\overline{\kappa'} = \overline{\kappa}$.

We now check that the parity of $\overline{\kappa}$ is the same as that of κ and $x \otimes \kappa$. Namely, $\overline{\kappa}$ takes the value $+1$ $N/2$ times, where N is the number of $y \in x^{\perp}$ such that $\kappa(y) = 1$. Let N' be the number of $y \in J[2]$ such that $\langle x, y \rangle = -1$ and $\kappa(y) = 1$. Then, on the one hand, for such y

$$\kappa(x+y) = \kappa(x)\kappa(y)\langle x, y\rangle = -\kappa(y),$$

so that the involution x partitions $J[2] - x^{\perp}$ into subsets of equal cardinality on which κ takes the value ± 1 respectively; so $N' = 2^{2g-2}$. On the other hand

$$N + N' = 2^{g-1}(2^g + \varepsilon(\kappa)).$$

Hence $N = 2^{g-1}(2^{g-1} + \varepsilon(\kappa))$, so the number of times $\overline{\kappa}$ takes the value $+1$ is $2^{g-2}(2^{g-1} + \varepsilon(\kappa))$, which shows that $\varepsilon(\overline{\kappa}) = \varepsilon(\kappa)$.

We have thus constructed canonical mappings

$$S^2\vartheta^{\pm}(C) \to \coprod_{x\in J[2]} \vartheta^{\pm}(P_x).$$

It is easy to see that these are injective; and hence bijections since both sets have the same cardinality.

□

2.4 *Remark.* In terms of line bundles, the proposition says that the theta characteristics of P_x are precisely the σ-invariant theta characteristics on $\widetilde{C}_x$, $\overline{\kappa} = \pi^*\kappa = \pi^*(x \otimes \kappa)$, such that κ and $x \otimes \kappa$ have the same parity.

It follows in particular that the symmetric divisors on P_x representing the principal polarisation are those of the form $t^*_{\overline{\kappa}}(\Xi_x)$, where $t_{\overline{\kappa}}$ denotes translation by $\overline{\kappa} \in \widehat{P}_x$.

3 The basic construction

We now set out to prove theorem 1.1 of the introduction. The main idea for comparing the anticanonical sections with level 3 theta functions is the following. Here and below, we shall always use M to mean either of the varieties M_0, M_1, whenever the same discussion applies to both.

There exists in the product $J^{g-1}(C) \times M$ a Brill-Noether type divisor $\mathcal{D}$ given set-theoretically by

$$\mathcal{D} = \{(L, V) \in J^{g-1}(C) \times M | h^0(L \otimes \text{ad } V) \geq 1\}.$$

More precisely, this is the support of $\mathcal{D}$ over stable bundles (in case of M_0), and we then take the closure over the projective variety M. The aim of the present section is to construct $\mathcal{D}$ and to prove:

3.1 Proposition. *$\mathcal{D} \in |3\Theta - K_M|$ where Θ is the pull-back to the product of the theta divisor on $J^{g-1}(C)$ and K_M is that of the canonical class on M.*

First one should note:

3.2 Lemma. *For any connected variety T one has*

$$\text{Pic } (T \times M) = \text{Pic } T \times \text{Pic } M.$$

Proof. We have $H^1(\mathcal{O}_M) = 0$ by [DN]. The lemma then follows as in [H], page 292. $\square$

Thus as in the statement of proposition 3.1 we shall freely use the same notation for line bundles on $J^{g-1}(C)$ or M, and their pull-backs to the product, whenever this is not likely to cause confusion.

To construct the divisor $\mathcal{D}$ we begin by fixing on the curve a smooth canonical divisor $D = x_1 + \cdots + x_{2g-2} \in |K_C|$, a Poincaré line bundle $\mathcal{L} \to C \times J^{g-1}$ and a universal adjoint bundle ad $\mathcal{V} \to C \times M^{st}$. Here M^{st} denotes the open set of stable bundles in M, and ad $\mathcal{V}$ restricted to any $C \times \{V\}$ is isomorphic to the bundle ad V of tracefree endomorphisms of the stable bundle V. That such a universal bundle ad $\mathcal{V}$ exists is well-known—see for example [B2] section 1.6. (Note that we use the notation ad $\mathcal{V}$ for convenience even though $\mathcal{V}$ does *not* denote a bundle on the product.)

We shall let $D, \mathcal{L}, \text{ad } \mathcal{V}$ denote also the pull-backs of these objects to the product $C \times J^{g-1} \times M^{st}$ via the obvious projections; and on this product we shall consider the exact sequence:

$$(4) \qquad 0 \to \mathcal{L}(-D) \otimes \text{ad } \mathcal{V} \to \mathcal{L} \otimes \text{ad } \mathcal{V} \to \mathcal{O}_D \otimes \mathcal{L} \otimes \text{ad } \mathcal{V} \to 0.$$

Taking the direct image of (4) under the projection

$$\pi : C \times J^{g-1} \times M^{st} \to J^{g-1} \times M^{st};$$

we get a sequence

$$(5) \qquad 0 \to \pi_*(\mathcal{L} \otimes \text{ad } \mathcal{V}) \to B \to A^\vee \to R^1_\pi(\mathcal{L} \otimes \text{ad } \mathcal{V}) \to 0.$$

Note that for any semistable bundle V the bundle ad V is semistable and so by [NR] lemma 2.1 we have $H^0(C, L \otimes \text{ad } V) = 0$ whenever $\deg L < 0$. Hence $\pi_*(\mathcal{L}(-D) \otimes \text{ad } \mathcal{V}) = 0$ in (5); also we have written

$$A^\vee = R^1_\pi(\mathcal{L}(-D) \otimes \text{ad } \mathcal{V}) \qquad \text{and} \qquad B = \bigoplus_{i=1}^{2g-2} (\mathcal{L} \otimes \text{ad } \mathcal{V})_{x_i}.$$

A and B are vector bundles on $J^{g-1} \times M^{st}$ of rank $6g-6$. We shall denote the central coboundary map in (5) by

$$\delta : B \to A^\vee.$$

The reason for using A to denote the dual of the bundle occuring in the exact sequence will become clear in due course (see (13)).

As in classical Brill-Noether theory one can show that (5) behaves functorially under base change, and it follows from this that the determinantal divisor

$$\mathcal{D}^{st} = \text{zero-scheme of } \det \delta$$

has support

$$\text{supp } \mathcal{D}^{st} = \{(L, V) \in J^{g-1} \times M^{st} | h^0(L \otimes \text{ad } V) \geq 1\}. \tag{6}$$

We now define the divisor $\mathcal{D}$ appearing in proposition (3.1) to be the closure of $\mathcal{D}^{st}$ in $J^{g-1} \times M$.

Before proceeding we need to mention that $\mathcal{D}$ is indeed a proper divisor, ie. that δ has maximal rank at the generic point. To see this it suffices to exhibit a pair $(L, V) \in J^{g-1} \times M^{st}$ for which $h^0(L \otimes \text{ad } V) = 0$; and for this, by the results of [O1], we can take L to be a theta-characteristic of the same parity (mod 2) as the degree of the bundles in M.

3.1 Proof of proposition 3.1

In view of lemma 3.2, we just have to compute the line bundle $\det A \otimes \det B$ along the fibres of $J^{g-1} \times M$. We consider first the restriction to $\{L\} \times M$ for some $L \in J^{g-1}$. The restriction of

$$(\mathcal{L} \otimes \text{ad } V)_{x_i} = (pr_J^* \mathcal{L}_{x_i}) \otimes pr_M^* (\text{ad } \mathcal{V})_{x_i}$$

is just $(\text{ad } \mathcal{V})_{x_i}$, which has zero first Chern class and therefore trivial determinant. Thus

$$\det B|_{\{L\} \times M} = \mathcal{O}_M.$$

For $\det A$ we shall take $L = \mathcal{O}_C(\Lambda)$ where Λ is a smooth effective divisor on C. By Grothendieck-Serre duality (since D is a canonical divisor)

$$A = R^0_\pi (K_C^2 \otimes \mathcal{L}^{-1} \otimes \text{ad } \mathcal{V});$$

whilst on $C \times J^{g-1} \times M^{st}$ we have an exact sequence

$$0 \to K_C^2 \otimes \mathcal{L}^{-1} \otimes \text{ad } \mathcal{V} \to K_C^2 \otimes \text{ad } \mathcal{V} \to \mathcal{O}_\Lambda \otimes K_C^2 \otimes \text{ad } \mathcal{V} \to 0,$$

which shows on applying π_* that

$$\det A|_{\{L\}\times M} = \det R^0_\pi(K_C^2 \otimes \mathrm{ad}\ \mathcal{V})|_{\{L\}\times M}.$$

On the other hand, from

$$0 \to K_C \otimes \mathrm{ad}\ \mathcal{V} \to K_C^2 \otimes \mathrm{ad}\ \mathcal{V} \to \mathcal{O}_D \otimes K_C^2 \otimes \mathrm{ad}\ \mathcal{V} \to 0$$

we see that

$$\begin{aligned}\det R^0_\pi(K_C^2 \otimes \mathrm{ad}\ \mathcal{V})|_{\{L\}\times M} &= \det R^0_\pi(K_C \otimes \mathrm{ad}\ \mathcal{V})\\ &= \det \Omega^1_M\\ &= K_M.\end{aligned}$$

Hence $\det A \otimes \det B$ restricts to K_M along $\{L\} \times M$.

So we now consider $\det A \otimes \det B$ along the fibres $J^{g-1} \times \{V\}$; we know in fact that this restriction is some power of the line bundle $\mathcal{O}(-3\Theta)$ since by [BNR] the divisor $\mathcal{D}$ restricts on the Jacobian to one supported on a 3-theta divisor. To verify that it is exactly $\mathcal{O}(-3\Theta)$, therefore, it is sufficient to check that it has the right cohomology class.

For this, first note that $\det B$ restricts to $\bigotimes_{i=1}^{2g-2} \mathcal{L}^3_{x_i}$ on the Jacobian, and this is trivial [ACGH] page 309. Second, we shall show that the restriction of $\det A^\vee$ has first Chern class 3θ where θ is the cohomology class of the theta divisor. We apply Grothendieck-Riemann-Roch to the projection

$$pr_J : C \times J^{g-1} \to J^{g-1}.$$

Let us write

$$E = A^\vee|_{J^{g-1}\times\{V\}} = R^1_{pr_J}(\mathcal{L} \otimes pr_C^*(\mathcal{O}(-D) \otimes \mathrm{ad}\ V)).$$

Then, since $R^0_{pr_J}(\mathcal{L} \otimes pr_C^*(\mathcal{O}(-D) \otimes \mathrm{ad}\ V)) = 0$, we have

$$-\,\mathrm{ch(E)} = \mathrm{pr_{J*}}\Big(\mathrm{ch}(\mathcal{L}(-\mathrm{D}) \otimes \mathrm{ad\ V}) \cdot \mathrm{td(C)}\Big). \tag{7}$$

Also $\mathrm{td(C)} = 1 - (\mathrm{g}-1)\eta$ where $\eta \in H^2(C, \mathbf{Z})$ is the class of a point; and it is easy to check that

$$\mathrm{ch(ad\ V)} = 1 + 2\cosh \mathrm{d}\eta$$

where $d = \deg V$. So

$$\mathrm{ch}(\mathcal{L}(-\mathrm{D}) \otimes \mathrm{ad\ V}) = \mathrm{e}^{\mathrm{c_1}(\mathcal{L})-(2\mathrm{g}-2)\eta}(1 + 2\cosh \mathrm{d}\eta).$$

Following [ACGH] chapter 8, one has

$$e^{c_1(\mathcal{L})} = 1 + (g-1)\eta + \gamma - \eta\theta$$

where γ is a class in $H^1(C) \otimes H^1(J^{g-1})$ satisfying

$$\gamma^3 = \eta\gamma = \eta^2 = 0, \qquad \gamma^2 = -2\eta\theta.$$

Thus $\mathrm{ch}(\mathcal{L}(-\mathrm{D}) \otimes \mathrm{ad\ V})$ reduces to

$$3(1-(2g-2)\eta)(1+(g-1)\eta+\gamma-\eta\theta) = 3 - 3(g-1)\eta + 3\gamma - 3\eta\theta;$$

and (7) yields

$$\begin{aligned} -\mathrm{ch}(\mathrm{E}) &= pr_{J*}(1-(g-1)\eta)(3-3(g-1)\eta+3\gamma-3\eta\theta) \\ &= pr_{J*}(-6(g-1)\eta - 3\eta\theta), \end{aligned}$$

from which $\mathrm{ch}(\mathrm{E}) = 6\mathrm{g} - 6 + 3\theta$ and so $c_1(\det A^\vee) = 3\theta$. □

3.3 *Remark.* An alternative, and direct, proof that $\det A \otimes \det B$ restricts to $\mathcal{O}(-3\Theta)$ on J^{g-1} can be given as follows. As already noted,

$$\det B|_{J^{g-1}} = \mathcal{N}^3, \qquad \text{where } \ \mathcal{N} = \bigotimes_{i=1}^{2g-2} \mathcal{L}_{x_i} \in \mathrm{Pic}^0(J^{g-1}).$$

It is not hard to show that

$$\det R^1_{pr_J}(\mathcal{L} \otimes pr_C^* \mathcal{O}(-D)) = \mathcal{N}(\Theta).$$

We now choose stable V such that $\mathrm{ad}\ V = L_1 \oplus L_2 \oplus L_3$ with $L_1, L_2, L_3 \in \mathrm{Pic}^0(C)$, $L_1 \otimes L_2 \otimes L_3 = \mathcal{O}_C$. (This is always possible—see, for example, [R].) By the theorem of the square any line bundle $\mathcal{M}$ on J^{g-1} satisfies

$$\mathcal{M}^3 = T^*_{L_1}\mathcal{M} \otimes T^*_{L_2}\mathcal{M} \otimes T^*_{L_3}\mathcal{M}.$$

Applying this to $\mathcal{M} = \mathcal{N}(\Theta)$ gives

$$\begin{aligned} \det A^\vee|_{J^{g-1}} &= \det \textstyle\bigoplus_{i=1}^3 R^1_{pr_J}(\mathcal{L} \otimes pr_C^*(L_i \otimes \mathcal{O}(-D))) \\ &= \textstyle\bigotimes_{i=1}^3 T^*_{L_i}(\det R^1_{pr_J}(\mathcal{L} \otimes pr_C^* \mathcal{O}(-D))) \\ &= \mathcal{N}(\Theta)^{\otimes 3}. \end{aligned}$$

So again we find $\det A \otimes \det B|_{J^{g-1}} = \mathcal{O}(-3\Theta)$. □

4 Extending the construction to the Pryms

The next step is, by analogy with proposition 3.1, to construct a divisor $\mathcal{D}_x$ in the product $\widehat{P}_x \times M$ such that

$$\text{supp } \mathcal{D}_x^{st} = \{(L,V) \in \widehat{P}_x \times M^{st} | h^0(\widetilde{C}_x, L \otimes \text{ad } p^*(V)) \geq 1\},$$

$$\mathcal{D}_x \in |3\Xi_x - K_M|.$$

We first need:

4.1 Lemma. *Let $\pi : \widetilde{C} = \widetilde{C}_x \to C$ be an unramified double cover and V a rank 2 vector bundle on C. Then if V is semistable, so is $\widetilde{V} = \pi^* V$; while if V is stable then either $\widetilde{V}$ is stable or $V^\vee = \pi_* N$ for some line bundle $N \to \widetilde{C}$.*

Proof. Let $L \subset \widetilde{V}$ be any line subbundle. So

$$0 \neq H^0(\widetilde{C}, L^{-1} \otimes \widetilde{V}) = H^0(C, V \otimes W^\vee) \tag{8}$$

where $W^\vee = \pi_* L^{-1}$; it is easy to check that $\deg W = \deg L$.

We first note that that W is semistable: if $N \subset W$ is a line subbundle then

$$0 \neq H^0(C, N^{-1} \otimes W) = H^0(\widetilde{C}, L \otimes \pi^* N^{-1}),$$

which implies that $\deg W - 2 \deg N = \deg L \otimes \pi^* N^{-1} \geq 0$.

Moreover, W fails to be stable only if equality holds, in which case $L = \pi^* N$, so that

$$W = (\pi_* L^{-1})^\vee = N \oplus x \otimes N.$$

Now suppose that V is semistable. Then by (8) and [NR], lemma 2.1 we have

$$\deg L = \deg W \leq \deg V = \frac{1}{2} \deg \widetilde{V},$$

so $\widetilde{V}$ is semistable.

Finally suppose that V is stable. Then the preceding argument shows that $\widetilde{V}$ is stable unless *either* W fails to be stable *or* V is isomorphic to $W = (p_* L^{-1})^\vee$. In the first case $W = N \oplus x \otimes N$ for some line bundle N; then (8) implies that either $H^0(C, N^{-1} \otimes V) \neq 0$ or $H^0(C, x \otimes N^{-1} \otimes V) \neq 0$; both of which imply $\deg N < \frac{1}{2} \deg V$ by stability. But then

$$\deg L = \deg W = 2 \deg N < \deg V = \frac{1}{2} \deg \widetilde{V}$$

so $\widetilde{V}$ is stable. $\square$

As a consequence of the above lemma we see that the pull-back $u = \pi^*$ defines a morphism

$$u : M \to \widetilde{M}_0$$

where $\widetilde{M}_0$ denotes the moduli space of rank 2 bundles with fixed determinant of even degree on $\widetilde{C}$. In addition we have:

4.2 Lemma. *$u^*\widetilde{K} = K_M^2$ where $\widetilde{K}$ denotes the canonical line bundle on $\widetilde{M}_0$.*

Proof. We consider the pull-back $u^*\Omega^1_{\widetilde{M}_0}$ of the cotangent bundle of $\widetilde{M}_0$. At a point $V \in M^{st}$ this has fibre

$$H^0(\widetilde{C}_x, \widetilde{K} \otimes \text{ad } \widetilde{V}) = H^0(C, K \otimes \text{ad } V) \oplus H^0(C, K \otimes x \otimes \text{ad } V),$$

using the projection formula and $\pi_* \mathcal{O}_{\widetilde{C}} = \mathcal{O}_C \oplus x$. Note that the second summand has dimension

$$h^0(C, K \otimes x \otimes \text{ad } V) = 3g - 3 + h^0(C, x \otimes \text{ad } V)$$

and that $h^0(C, x \otimes \text{ad } V) = 0$ away from the image $S \subset M$ of a rational map $P_x \to M$, and that this image has codimension at least 2.

Using the Poincaré bundle $\text{ad } \mathcal{V} \to C \times M^{st}$ we can globalise the above description: we have

$$u^*\Omega^1_{\widetilde{M}_0}|_{M^{st}} = \Omega^1_{M^{st}} \oplus R^0_{pr}(K \otimes x \otimes \text{ad } \mathcal{V})$$

where $pr : C \times M^{st} \to M^{st}$ is the natural projection. Applying Grothendieck-Riemann-Roch to pr, we see that away from $S \subset M^{st}$ the second summand has the same Chern character as $\Omega^1_{M^{st}}$, and hence, since $\text{Pic } M = \text{Pic } M^{st} = \mathbf{Z}$, the same determinant bundle. Since $\text{codim } S \geq 2$ this shows that

$$u^*\widetilde{K} = \det u^*\Omega^1_{\widetilde{M}_0} = K_M^2.$$

$\square$

We can now set about constructing the divisor $\mathcal{D}_x$ mentioned at the beginning of this section. We already have a divisor $\mathcal{D}^{st} \subset \widetilde{J}^{2g-2} \times \widetilde{M}_0^{st}$ supported on pairs (L,W) for which $H^0(\widetilde{C}_x, L \otimes \text{ad } W) \neq 0$, as constructed in proposition 3.1. We shall denote by

$$\mathcal{E}_x^{st} \subset \widehat{P}_x \times M^{st}$$

its pull-back under the map

$$\overline{u} = \text{inclusion} \times u : \widehat{P}_x \times M^{st} \to \widetilde{J}^{2g-2} \times \widetilde{M}_0^{st}.$$

First it is necessary to check:

4.3 Lemma. *$\mathcal{E}_x^{st}$ is a divisor. In other words for the generic pair $(L, V) \in \widehat{P}_x \times M^{st}$ we have $H^0(\widetilde{C}_x, L \otimes \text{ad } \widetilde{V}) = 0$, where $\widetilde{V} = \pi^* V$.*

Proof. We choose an even theta characteristic $L \in \vartheta^+(P_x) \subset \widehat{P}_x$; by remark 2.4 this is a line bundle $L = \pi^* N$ on $\widetilde{C}_x$, where N and $x \otimes N$ are both even theta characteristics on C. For general $V \in M$ we have, using [O2],

$$\begin{aligned} H^0(\widetilde{C}_x, L \otimes \text{ad } \widetilde{V}) &= H^0(C, N \otimes \text{ad } V) \oplus H^0(C, N \otimes x \otimes \text{ad } V) \\ &= 0. \end{aligned}$$

□

We now define $\mathcal{E}_x$ to be the closure of $\mathcal{E}_x^{st}$. In view of 3.1 and 4.3 we have

$$\mathcal{E}_x \in |\overline{u}^*(3\Theta_{\widetilde{J}} - \widetilde{K})| = |6\Xi_x - 2K_M|.$$

The main point of the present section is the following fortuitous fact:

4.4 Proposition. *$\mathcal{E}_x = 2\mathcal{D}_x$ for some $\mathcal{D}_x \in |3\Xi_x - K_M|$.*

4.5 *Remark.* Proposition 4.4 is in fact exactly analogous to the fact that $\widetilde{\Theta} \cap \widehat{P}_x = 2\Xi_x$. The essential idea on which it is based is this: the Prym variety $\widehat{P}_x \subset \widetilde{J}^{2g-2}$ (together, of course, with $\widehat{P}_x^-$) is defined by the condition

$$L \otimes \sigma(L) = K_{\widetilde{C}_x} \tag{9}$$

(where σ is the sheet-involution over C). So $L \in \widehat{P}_x$ is formally analogous to a theta-characteristic on $\widetilde{C}_x$, and we shall use this idea to show (as in [M], [O1]) that $H^0(\widetilde{C}_x, L \otimes \text{ad } \widetilde{V})$ varies over $\widehat{P}_x \times M^{st}$ *as the intersection of a pair of maximal isotropic subbundles of an orthogonal bundle* (U, q).

First let us recall the determinantal construction of $\mathcal{E}_x^{st}$ from §6. We work with the maps:

$$\begin{array}{ccc} & & \widetilde{C}_x \times \widetilde{J}^{2g-2} \times \widetilde{M}_0^{st} \\ & & \downarrow pr \\ \widehat{P}_x \times M^{st} & \stackrel{\overline{u}}{\longrightarrow} & \widetilde{J}^{2g-2} \times \widetilde{M}_0^{st}. \end{array}$$

We let D, $\mathcal{L}$ and ad $\widetilde{V}$ denote the pull-backs to the product $\widetilde{C}_x \times \widetilde{J}^{2g-2} \times \widetilde{M}_0^{st}$ of, respectively, a smooth canonical divisor on $\widetilde{C}_x$, a Poincaré line bundle for $\widetilde{J}^{2g-2}$ and universal adjoint bundle for $\widetilde{M}_0$. We then consider the exact sequence (ie. (4)) on this product:

$$(10) \qquad 0 \to \mathcal{L}(-D) \otimes \text{ad } \widetilde{V} \to \mathcal{L} \otimes \text{ad } \widetilde{V} \to \mathcal{O}_D \otimes \mathcal{L} \otimes \text{ad } \widetilde{V} \to 0.$$

We take the direct image of this sequence under pr, and $\mathcal{E}_x^{st}$ is then the pull-back by $\overline{u}$ of the determinantal divisor associated to the coboundary map. If we write

$$A^\vee = \overline{u}^* R^1_{pr}(\mathcal{L}(-D) \otimes \text{ad } \widetilde{V}) \qquad \text{and} \qquad B = \overline{u}^* \bigoplus_{p \in \text{supp } D} (\mathcal{L} \otimes \text{ad } \widetilde{V})_p,$$

then $\mathcal{E}_x^{st} \subset \widehat{P}_x \times M^{st}$ is cut out by

$$\delta : B \to A^\vee,$$

the pull-back under $\overline{u}$ of the coboundary map.

At this point we introduce a second exact sequence on $\widetilde{C}_x \times \widetilde{J}^{2g-2} \times \widetilde{M}_0^{st}$:

$$(11) \; 0 \to \mathcal{L}(-D) \otimes \text{ad } \widetilde{V} \to \mathcal{L}(D) \otimes \text{ad } \widetilde{V} \to \mathcal{O}_{2D} \otimes \mathcal{L}(D) \otimes \text{ad } \widetilde{V} \to 0.$$

Pushing (11) down under pr and pulling back by $\overline{u}$ gives a short exact sequence on $\widehat{P}_x \times M^{st}$:

$$(12) 0 \to \overline{u}^* R^0_{pr}(\mathcal{L}(D) \otimes \text{ad } \widetilde{V}) \to U \to \overline{u}^* R^1_{pr}(\mathcal{L}(-D) \otimes \text{ad } \widetilde{V}) = A^\vee \to 0,$$

where $U = \overline{u}^* pr_*(\mathcal{O}_{2D} \otimes \mathcal{L}(D) \otimes \text{ad } \widetilde{V}$.

4.6 Lemma. $A \cong \overline{u}^* R^0_\pi(\mathcal{L}(D) \otimes \text{ad } \widetilde{V})$.

Proof. By Grothendieck-Serre duality, 9 and the fact that D is a canonical divisor, the bundle A has at $(L, V) \in \widehat{P}_x \times M^{st}$ fibre

$$\begin{aligned} A_{(L,V)} &= H^0(\widetilde{C}_x K^2_{\widetilde{C}_x} L^{-1} \otimes \mathrm{ad}\, \widetilde{V}) \\ &= H^0(\widetilde{C}_x, \sigma(L) \otimes K_{\widetilde{C}_x} \otimes \mathrm{ad}\, \widetilde{V}) \\ &= H^0(\widetilde{C}_x, \sigma(L(D) \otimes \mathrm{ad}\, \widetilde{V})). \end{aligned}$$

Thus the involution σ induces an isomorphism with $\overline{u}^* R^0_{pr}(\mathcal{L}(D) \otimes \mathrm{ad}\, \widetilde{V})$ as asserted. □

On the other hand, B is also—by definition—a subbundle of U, and in fact the outcome of the above discussion is that we now have a commutative diagram of vector bundles on $\widehat{P}_x \times M^{st}$ (completely analogous to (1.6) in [O1]):

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & B & & & & \\ & & & & \downarrow & \searrow \delta & & & \\ 0 & \rightarrow & A & \rightarrow & U & \rightarrow & A^\vee & \rightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & B^\vee & & & & \\ & & & & \downarrow & & & & \\ & & & & 0. & & & & \end{array} \qquad (13)$$

We have not justified writing $B^\vee$ for the vertical cokernel here, but this follows from the next observation: that the vector bundle U carries a quadratic form q for which both A and B are maximally isotropic.

Before defining q let us give an explicit description of the local sections of U as follows. We can suppose that $D \in |K_{\widetilde{C}_x}|$ is the pull-back of a canonical divisor on C, so

$$D = \sum_{i=1}^{2g-2} (x_i + x_{\sigma i})$$

where $\sigma i = i + 2g - 2$ and $x_{\sigma i} = \sigma(x_i)$, $\sigma : \widetilde{C}_x \to \widetilde{C}_x$ being sheet-involution. Now for $j = 1, \ldots, 4g-4$ let z_j be a local holomorphic coordinate on $\widetilde{C}_x$ around the point x_j. Then a section of U over a sufficiently small open set $W \subset \widehat{P}_x \times M^{st}$ has the form:

$$\psi = (\frac{a_j}{z_j} + b_j)_{j=1,\ldots,4g-4}$$

where a_j, b_j are $sl_2(\mathbf{C})$-valued holomorphic functions on W.

We now define our quadratic form q as the sheaf homomorphism:

$$
\begin{array}{rcl}
q : U & \rightarrow & \mathcal{O}_{\widehat{P}_x \times M^{st}} \\
\psi & \mapsto & \sum_{j=1}^{4g-4} \mathrm{Res}_{x_j} \mathrm{trace}\ \psi\sigma(\psi).
\end{array} \tag{14}
$$

In other words

$$
q : (\frac{a_j}{z_j} + b_j)_{j=1,\ldots,4g-4} \mapsto \sum_{i=1}^{2g-2} 2\ \mathrm{trace}(a_i b_{\sigma i} + a_{\sigma i} b_i)
$$

in terms of the local description above.

The matrices a_j, b_j can be thought of as local fibre coordinates for the bundle U; then $B \subset U$ is the subbundle

$$
B = \{a_1 = \cdots = a_{4g-4} = 0\},
$$

from which one sees that it is totally isotropic with respect to q.

The subbundle $A \subset U$, on the other hand, consists of those ψ which extend (locally in $\widehat{P}_x \times M^{st}$) to families of global sections in $H^0(\widetilde{C}_x, L(D) \otimes \mathrm{ad}\ \widetilde{V})$. Thus by proposition 4.4, trace $\psi\sigma(\psi)$ is a family of global meromorphic differentials on $\widetilde{C}_x$, and hence by the residue theorem one sees that A also is totally isotropic.

Note that

$$
\mathrm{rank}\ A = \mathrm{rank}\ B = 6g - 6 = \frac{1}{2} \mathrm{rank}\ U;
$$

also that fibrewise over $\widehat{P}_x \times M^{st}$ one has

$$
A_{(L,V)} \cap B_{(L,V)} = H^0(\widetilde{C}_x, L \otimes \mathrm{ad}\ \widetilde{V}).
$$

As a consequence of this and lemma 4.3 one observes:

4.7 Corollary. *For all $L \in \widehat{P}_x$ and $V \in M^{st}$ one has $h^0(\widetilde{C}_x, L \otimes \mathrm{ad}\ \widetilde{V}) \equiv 0$ mod 2.*

Finally, proposition 4.4 follows from 4.7 and section 0.4 of [B2].

5 The 3-theta–anticanonical tensor

By 3.1 and 4.4 we have, for every 2-torsion point $x \in J[2]$, a naturally occuring Brill-Noether divisor $\mathcal{D}_x \in |3\Xi_x - K_M|$ in the product $\widehat{P}_x \times M$. By Künneth $\mathcal{D}_x$ is cut out by a tensor (up to scalar)

$$s_x \in H^0(\widehat{P}_x \times M, 3\Xi_x - K_M) \cong H^0(\widehat{P}_x, 3\Xi_x) \otimes H^0(M, K_M^{-1}).$$

5.1 *Remark.* One can also view the tensor s_x as follows. The intersections of $\mathcal{D}_x$ with the fibres of the product $\widehat{P}_x \times M$ define rational maps

$$f_x : \widehat{P}_x \to |K_M^{-1}| \qquad \text{and} \qquad g_x : M \to |3\Xi_x|.$$

These satisfy $f_x^*\mathcal{O}(1) = \mathcal{O}(3\Xi_x)$ and $g_x^*\mathcal{O}(1) = K_M^{-1}$, so pulling back global sections defines a pair of homomorphisms

$$f_x^* : H^0(K_M^{-1})^\vee \to H^0(3\Xi_x) \qquad \text{and} \qquad g_x^* : H^0(3\Xi_x)^\vee \to H^0(K_M^{-1});$$

either of which can be identified with s_x.

If we split the space of 3-thetas up into $\pm$ eigenspaces under the (-1)-involution of $\widehat{P}_x$, then

$$s_x \in (H^0_+(3\Xi_x) \otimes H^0(K_M^{-1})) \oplus (H^0_-(3\Xi_x) \otimes H^0(K_M^{-1})).$$

Of course, we have the two cases M_d to consider here, for even and odd degree semistable bundles. It turns out that s_x lives, respectively, in the $\pm$ summand:

5.2 Proposition.

$$\begin{array}{lcl} M = M_0 & \Rightarrow & s_x \in H^0_+(3\Xi_x) \otimes H^0(K_M^{-1}), \\ M = M_1 & \Rightarrow & s_x \in H^0_-(3\Xi_x) \otimes H^0(K_M^{-1}). \end{array}$$

In order to prove proposition 5.2 we shall consider the rational map $g_x : M \to |3\Xi_x|$ of remark 5.1. That is, for $V \in M^{st}$,

$$g_x(V) = \{L \in \widehat{P}_x | H^0(\widetilde{C}_x, L \otimes \operatorname{ad} \widetilde{V}) \neq 0\},$$

with $\widetilde{C}_x$ replaced by C and $\widetilde{V}$ by V in case $x = 0$.

We first observe that $g_x(V)$ is always a symmetric divisor on $\widehat{P}_x$: when $x = 0$ this can be seen set-theoretically from (6), by Riemann-Roch and Serre duality. Scheme-theoretically it follows from the functoriality of (5) under base change $[-1] : J \to J$. On the other hand, for $x \neq 0$ the symmetry of $\mathcal{E}_x$ follows from the case $x = 0$ and the fact that $[-1] : \widehat{P}_x \to \widehat{P}_x$ is induced by the Serre duality involution of $\widetilde{J}^{2g-2}$; hence $\mathcal{D}_x = \frac{1}{2}\mathcal{E}_x$ is also symmetric.

The subset of $|3\Xi_x|$ of symmetric divisors consists of the two disjoint projective subspaces $\mathbf{P}H^0_+(3\Xi_x)$ and $\mathbf{P}H^0_-(3\Xi_x)$; and the proposition now amounts to the assertion that

$$g_x : M_d \to \mathbf{P}H^0_\pm(3\Xi_x) \subset |3\Xi_x| \tag{15}$$

for $d = 0, 1$ respectively.

5.1 Proof of proposition 5.2

In other words of (15). We shall make use of the following fact from [O2] lemma (1.0): for any rank 2 vector bundle on C, and theta-characteristic $N \in \vartheta(C)$, we have:

$$h^0(C, N \otimes \mathrm{ad}\ V) \equiv \deg V + h^0(C, N) \mod 2. \tag{16}$$

First suppose $x = 0$. If $V \in M_0^{st}$ and N is an odd theta-characteristic then $h^0(C, N \otimes \mathrm{ad}\ V)$ is odd by (16); hence is nonzero. This means that the divisor $g_x(V) \subset J^{g-1}$ contains the point $N \in \vartheta^-(C)$ for every $V \in M_0^{st}$; and hence shows that $g_x(M_0)$ spans a linear system with base points along $\vartheta^-(C)$. Hence by remark 2.1 and lemma 2.2, it follows that since M is irreducible it lies in the subspace $\mathbf{P}H^0_+(J^{g-1}, 3\Theta)$.

The argument for $x \neq 0$ is similar. We suppose that $V \in M_0^{st}$ and that $L \in \vartheta^-(P_x)$. By proposition 2.3 $L = p^*N$ for some odd theta characteristic N on C. Then

$$h^0(\widetilde{C}_x, L \otimes \mathrm{ad}\ \widetilde{V}) = h^0(C, N \otimes \mathrm{ad}\ V) + h^0(C, N \otimes x \otimes \mathrm{ad}\ V)$$

which is again nonzero using (16). (Note, incidentally, that it is even—as it must be by 4.7.) So as before $g_x(M_0)$ spans a linear system with base points along $\vartheta^-(P_x)$ and is therefore contained in $\mathbf{P}H^0_+(\widehat{P}_x, 3\Xi_x)$, by lemma 2.2.

Finally, the argument for M_1 is exactly analogous. □

In summary, we have for each moduli space M_d a tensor

$$S_d = \sum_{x \in J[2]} s_x \in H^0(M_d, K^{-1}) \otimes \bigoplus_{x \in J[2]} H^0_{(-)^d}(\widehat{P}_x, 3\Xi_x). \tag{17}$$

As explained in the introduction, the two factors on the right have the *same dimension*: on the one hand

$$h^0_\pm(\widehat{P}_x, 3\Xi_x) = \frac{3^n \pm 1}{2} \qquad \text{where } n = \dim \widehat{P}_x.$$

Whilst by [DN], theorem F, $K^{-1}_{M_0} = \mathcal{L}_0^4$ and $K^{-1}_{M_1} = \mathcal{L}_1^2$ where $\mathcal{L}_d$ are the respective ample generators of Pic $M = \mathbf{Z}$; so by the Verlinde formulae (see [vGP2] or [O2]):

$$\begin{array}{rcl} h^0(M_d, K^{-1}) & = & 3^{g-1}2^{2g-1} \pm 2^{2g-1} + 3^{g-1} \\ & = & \displaystyle\sum_{x \in J[2]} h^0_\pm(\widehat{P}_x, 3\Xi_x). \end{array}$$

5.2 Proof of theorem 1.1

We would like to show that each tensor S_d is nondegenerate, and for this it suffices to show that the map

$$f = \coprod f_x : \coprod \widehat{P}_x \to |K_M^{-1}|$$

of remark 5.1 has, in each case M_0, M_1, nondegenerate image. In fact we shall show that the image of the finite set of even (resp. odd) theta characteristics

$$\coprod_{x \in J[2]} \vartheta^+(P_x) \subset \coprod_{x \in J[2]} \widehat{P}_x \qquad (\text{resp. } \coprod \vartheta^-(P_x))$$

is nondegenerate in $|K^{-1}_{M_0}|$ (resp. $|K^{-1}_{M_p}|$).

By proposition 2.3 and remark 2.4 a theta characteristic $\nu \in \vartheta^+(P_x)$ has the form $\nu = \pi^* N = \pi^*(x \otimes N)$ for some $N, x \otimes N \in \vartheta^+(C)$. If $x \neq 0$ then by the projection formula the divisor $f_x(\nu) \in |K^{-1}_{M_0}|$ is supported on (stable) bundles V such that

$$\begin{array}{rcl} 0 & \neq & H^0(\widetilde{C}_x, \pi^* N \otimes \text{ad } \pi^* V) \\ & = & H^0(C, \pi_* \mathcal{O}_{\widetilde{C}_x} \otimes N \otimes \text{ad } V) \\ & = & H^0(C, N \otimes \text{ad } V) \oplus H^0(C, x \otimes N \otimes \text{ad } V). \end{array}$$

In other words $f_x(\nu) = D_N + D_{x\otimes N}$, where $\{D_\kappa\}_{\kappa\in\vartheta^+(C)}$ is the basis of $|K_{M_0}^{-\frac{1}{2}}| = |\mathcal{L}^2|$ described in [B2], theorem 1.2: namely,

$$D_\kappa = \{V \in M | H^0(C, \kappa \otimes \text{ad } V) \neq 0\}.$$

If, on the other hand, $x = 0$ and $N = \kappa$ is an even theta characteristic on C then (by [B2], lemma 1.7) $f_x(N)$ is a divisor of multiplicity two supported on bundles V for which $H^0(C, N \otimes \text{ad } V) \neq 0$, i.e. $f_x(N) = 2D_N$.

Thus we have shown that as ν ranges through

$$\coprod \vartheta^+(P_x) \cong S^2\vartheta^+(C),$$

its image $f(\nu)$ ranges through the divisors

$$D_\kappa + D_{\kappa'} \in |K_{M_0}^{-1}|$$

where $\kappa, \kappa' \in \vartheta^+(C)$. In the odd degree case, the same argument shows that the odd theta characteristics

$$\coprod \vartheta^-(P_x) \cong S^2\vartheta^-(C)$$

map under f bijectively to the divisors

$$D_\kappa + D_{\kappa'} \in |K_{M_1}^{-1}|$$

where $\kappa, \kappa' \in \vartheta^-(C)$.

To conclude, it suffices to observe that these divisors span $|K_{M_0}^{-1}|$, $|K_{M_1}^{-1}|$ respectively. In each case if C has no vanishing theta-nulls then the divisors $\{D_\kappa\}$ span $|K_M^{-\frac{1}{2}}|$; thus nondegeneracy of the tensor S_d will follow from surjectivity of the multiplication map

$$S^2H^0(K_M^{-\frac{1}{2}}) \to H^0(K_M^{-1}).$$

For the odd degree case this is the surjectivity alluded to in remark 1.2; for the even degree case, on the other hand, surjectivity follows from the diagram

$$\begin{array}{ccc} S^2H^0(M_0, \mathcal{L}^2) & \xrightarrow{m_{2,4}} & H^0(M_0, \mathcal{L}^4) \\ S^2m_2 \uparrow & & \uparrow m_4 \\ S^2(S^2H^0(M_0, \mathcal{L})) & \xrightarrow{m_{2,2}} & S^4H^0(M_0, \mathcal{L}). \end{array}$$

Here $m_{2,2}$ is surjective, and for C without vanishing theta-nulls m_4 is also surjective by [vGP2]; hence $m_{2,4}$ is surjective and we are done. $\square$

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