The action of the Frobenius map on rank 2 vector bundles in characteristic 2

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1 Introduction

Let X be a smooth algebraic curve of genus g over a field k of characteristic $p > 0$. The behaviour of semi-stable bundles with respect to the absolute Frobenius F_a remains mysterious if $g \geq 2$. Let us briefly explain why this question should be of interest. Start with a continuous representation ρ of the algebraic fundamental group in $\mathrm{GL}_r(\bar{k})$, where $\bar{k}$ is the algebraic closure of k. Let E_ρ be the corresponding rank r bundle over X. Then all the bundles $F_a^{(n)*}E_\rho$, for $n > 0$, where $F_a^{(n)}$ denotes the n-fold composite $F_a \circ \cdots \circ F_a$, are semi-stable. Conversely, assuming that k is finite, let E be a semi-stable rank r bundle defined over $\bar{k}$. Because the set of isomorphism classes of semi-stable bundles of degree 0 over $X_{k'}$, where k' is any finite extension of k, is finite, one observes (see [LS]) that some twist of E comes from a representation as above. Therefore, if one is interested in unramified continuous representations of the Galois group over k of a global field $k(X)$ in characteristic p, it is natural to look at Frobenius semi-stable bundles, that is those whose pull-backs by $F_a^{(n)}$ are all semi-stable. This condition is stable by tensor product ([Mi] section 5), which is not usually the case for ordinary semi-stability in positive characteristic.

Assume that k is arbitrary of characteristic p. Let us emphasize that the locus of Frobenius semi-stable bundles of degree 0 is a countable intersection of open subsets of the coarse moduli scheme of semi-stable vector bundles of degree 0 over X. In particular, it is not clear at all if the Frobenius semi-stable points are dense in general. This could a priori depend on the arithmetic of the base field k. We would like to study the dynamics of the Frobenius, namely the sequence $n \longmapsto F_a^{(n)*}E$ for a given vector bundle E over X.

If k is a discrete valuation field of characteristic p and if X is a Mumford curve, the situation is well understood [F], [G]: among other results, it is shown for arbitrary genus and characteristic that there exist semi-stable bundles which are destabilized by the Frobenius F_a, that $E \longmapsto F_a^* E$ induces a surjective (rational) map on the moduli space of semi-stable rank r vector bundles of degree 0, and that the set of bundles coming from continuous representations of the algebraic fundamental group is dense. Another case, which is studied in the literature, are elliptic curves (see e.g. [O],[S]). For example, it can easily be shown that over an elliptic curve semi-stability is preserved under pull-back by Frobenius and that a stable bundle of rank r and degree d is Frobenius stable if and only if pd and r are coprime. But in general, not much seems to be known.

In this paper we study the action of the Frobenius map F_a in a very particular case: X is an ordinary curve of genus 2 defined over an algebraically closed field k of characteristic 2. In that case the coarse moduli space M_X of semi-stable rank 2 vector bundles of trivial determinant is known to be the projective space $\mathbb{P}^3$ and we show (Proposition 6.1) that the Frobenius map identifies with a rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ given by quadratic polynomials, which are explicitly computed in terms of generalized theta constants of the curve X. This result allows us to deduce that, if the determinant is fixed, the Frobenius map is not surjective and that the set of Frobenius semi-stable bundles is Zariski dense in $\mathbb{P}^3$ (Proposition 8.1). On the other hand, the Frobenius map becomes surjective if the determinant is not fixed (Proposition 6.4).

As a by-product of our computations, we also get the explicit equation of Kummer's quartic surface (section 4) and a description of the Frobenius map acting on the moduli space of rank 2 vector bundles of fixed odd degree determinant (section 7).

For other aspects of this problem see the recent article [JX].

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2 Review of Theta groups and Theta divisors in characteristic p

Let k be a perfect field of characteristic $p > 0$. By schemes we implicitly mean k-scheme.

2.1 Relative Frobenius

Let A be an ordinary abelian variety of dimension g. We shall denote by F the relative Frobenius morphism

$$F : A \longrightarrow A_1$$

which is a purely inseparable k-isogeny of degree p^g. Its kernel $\hat{G}$ is a subgroup scheme of the group scheme $A[p]$ of p-torsion points of A and F identifies to the quotient morphism $A \longrightarrow A/\hat{G}$. Because

$$\ker F = \hat{G} \subset A[p] = \ker [p],$$

the diagram

$$A \longrightarrow A/\hat{G} \longrightarrow A/A[p]$$

becomes

$$[p] : A \xrightarrow{F} A_1 \xrightarrow{V} A.$$

2.1. Remark. In the case where A is a Jacobian, the map V, called Verschiebung, is simply the pull-back by the relative Frobenius $F : X \to X_1$.

Let G be the reduced part of $A[p]$. Because A is ordinary, G intersects $\hat{G}$ at the origin and one gets a canonical decomposition

$$A[p] = G \times \hat{G} \tag{2.1}$$

and the relative Frobenius induces an isomorphism

$$F : \ \ker V = A[p]/\hat{G} \xrightarrow{\sim} G.$$

2.2 Theta group scheme

We will use the basic results (and notations) of Theta groups associated to line bundles on abelian varieties as in [Mu2]. We assume that A_1 is principally polarized and that k is big enough in order to have an isomorphism $G \cong (\mathbb{Z}/p\mathbb{Z})^g$ (however, we do not choose such an isomorphism). Let Θ_1 be a (not necessarily symmetric) Theta divisor representing the polarization. We denote by M_1 (resp. L_1) the line bundle $\mathcal{O}(\Theta_1)$ (resp. $\mathcal{O}(p\Theta_1)$). We consider the Theta group $G(L_1)$ associated to the line bundle L_1 (see [Mu2], page 221). This group scheme is a central extension

$$0 \longrightarrow \mathbb{G}_m \longrightarrow G(L_1) \longrightarrow K(L_1) \longrightarrow 0$$

and $K(L_1) = A_1[p]$. We denote by $e^{L_1} : K(L_1) \times K(L_1) \to \mathbb{G}_m$ the skew-symmetric form on $K(L_1)$ given by the commutator taken in the Theta group $G(L_1)$.

2.2. Lemma. *There exists a unique ample line bundle M on A such that $V^*M = L_1$.*

Proof. Because $K(L_1)$ is self-dual and contains $\hat{G}$, it is the whole $A[p]$. One has to show (theorem 2, page 231 of [Mu2]) that the restriction of $G(L_1)$ to $G = \ker V$ is split (the uniqueness is obvious). Since G is reduced, it is enough to define the splitting at the level of k-points. Let $g \in G(k)$. Since $\mu_p(k) = \{1\}$ and k^* is divisible, there exists a unique $\tilde{g} \in G(L_1)$ over g satisfying $\tilde{g}^p = 1$. Since $\mu_p(k) = \{1\}$ again, the restriction of e^{L_1} to $G(k) \times G(k)$ is trivial and therefore G is isotropic for e^{L_1}. This implies that any two elements $\tilde{g}, \tilde{h}$ commute and the map $g \mapsto \tilde{g}$ is a morphism of group schemes. □

The bundle M defines a principal polarization on A and the G-invariant morphism

$$V^*M^p = V^*[(V_*L_1)^G]^{\otimes p} \longrightarrow L_1^p = M_1^{p^2} = V^*F^*M_1$$

defines an isomorphism

$$M^p \xrightarrow{\sim} F^*M_1. \tag{2.2}$$

We denote by $G(L)$ the Theta group scheme associated to the line bundle $L := F^*M_1 = M^p$.

2.3. Lemma. *The restriction of $G(L)$ to both G and $\hat{G}$ is canonically split.*

Proof. Because L is the pull-back of M_1 by F, the restriction of $G(L)$ to both $\hat{G}$ splits (see Theorem 2, page 231 of [Mu2] again). Observe however that the splitting is determined not only by L but by M_1. For the splitting over G, proceed as in lemma 2.2. □

Therefore, the decomposition $A[p] = K(L) = \hat{G} \times G$ of (2.1) is symplectic in the sense that both G and $\hat{G}$ are isotropic for e^L. In particular, e^L identifies $\hat{G}$ and the Cartier dual of G.

2.4. Remark. One can interpret more geometrically the action of $\hat{G}$ as follows. By [K], the line bundle L comes with a canonical p-integrable connection ∇. The Lie algebra $\mathfrak{g}$ of $\hat{G}$ is the whole tangent space $\mathbf{T}_0A$. The link between the infinitesimal action of $\mathfrak{g}$ on L and ∇ is simply given by

$$v.l = \nabla_v l$$

where l is a local section of L, v a point of $\mathfrak{g}$ and $\nabla_v l$ is the ∇-derivative of l with respect to the invariant vector field defined by v.

Let θ be a non-zero global section of M. By the remark above, since the section θ^p is annihilated by ∇, it defines a $\hat{G}$-invariant section of $F^*\Theta_1$, namely a non-zero section θ_1 of M_1 such that

$$\theta^p = F^*\theta_1. \tag{2.3}$$

The main theorem on the group scheme $G(L)$ and its representations is the following structure theorem due to Mumford (for the characteristic p version we need, we refer to [Sek]):

2.5. Theorem. *The space of global sections $H^0(A, L)$ is the unique irreducible representation of weight 1 (i.e. $\mathbb{G}_m$ acts by its natural character) of the Theta group scheme $G(L)$.*

2.3 Canonical Theta basis of $|p\Theta|$ and $|p\Theta_1|$

In [Mu1] Mumford constructs canonical bases for any linear system $\mathbb{P}H^0(A, L)$ where L is a line bundle of separable type. Because of Theorem 2.5, we can adapt his construction to line bundles L not of separable type. In our situation, namely $L = \mathcal{O}_A(p\Theta)$, where Θ is not necessarily symmetric, we get the following

2.6. Lemma. *(i) There exists a basis $\{X_g\}_{g\in G}$ of $H^0(A, L)$, unique up to a multiplicative scalar, which satisfies the following relations*

$$a.X_g = X_{g+a} \qquad \alpha.X_g = e^L(\alpha, g)X_g \qquad \forall a, g \in G,\ \forall \alpha \in \hat{G} \tag{2.4}$$

(ii) For every $g \in G$, there exists a unique section $Y_g \in H^0(A_1, L_1)$ such that $X_g^p = F^*Y_g$. The family $\{Y_g\}_{g\in G}$ is a basis of $H^0(A_1, L_1)$.

Proof. Let us construct geometrically the basis $\{X_g\}_{g\in G}$. We define X_g by

$$X_g = g.F^*\theta_1 = g.\theta^p.$$

By construction, one has

$$a.X_g = X_{g+a} \qquad \forall a, g \in G.$$

Because X_0 comes from $A_1 = A/\hat{G}$, it is $\hat{G}$ invariant. The relations (2.4) follow. Because $H^0(A, L)$ is irreducible, $\{X_g\}_{g\in G}$ is a basis. If we have another such basis X'_g, the endomorphism given by $X_g \longmapsto X'_g$ is $G(L)$-equivariant and therefore a scalar by Schur's lemma, which proves (i).

The sections X_g^p live in $H^0(A, L^p) = H^0(A, F^*L_1)^{\hat{G}}$ by (i) and (ii) follows. □

We will identify $H^0(A, L)$ and $H^0(A_1, L_1)$ with their duals (more precisely $\mathbb{P}H^0(A, L)$ and $\mathbb{P}H^0(A, L)$) using these bases.

2.7. Remark. More generally, we can construct Theta bases for any power $L = \mathcal{O}(p^l\Theta)$ with $l \geq 1$. We denote by G_l the reduced part of ker $[p^l]$. Note that we have $G_l = (\mathbb{Z}/p^l\mathbb{Z})^g$, $G_1 = G$ and, for any $l \geq 1$, we have an exact sequence $0 \longrightarrow G_1 \longrightarrow G_{l+1} \xrightarrow{p} G_l \longrightarrow 0$. As above, we prove that there exist canonical bases $\{X_g^{(l)}\}_{g\in G_l}$ of $H^0(A, L)$ and $\{Y_g^{(l)}\}_{g\in G_l}$ of $H^0(A_1, L_1)$ satisfying relations (2.4).

2.4 Addition formula in characteristic 2

From now we assume that M is symmetric and that $p = 2$. We will explain how to obtain an addition formula in this context. The method essentially goes as in the proof of Lemma 1.2 of [Sek], which is a generalization of Mumford's arguments.

Consider the homomorphism

$$\xi : \begin{cases} A \times A & \longrightarrow & A \times A \\ (x, y) & \longmapsto & (x - y, x + y) \end{cases} \tag{2.5}$$

(in the notation of *loc. cit.*, $a = b = 1$). By the See-Saw theorem, we have an isomorphism

$$\xi^*(M \boxtimes M) \cong M^2 \boxtimes M^2 \tag{2.6}$$

hence we get an injection

$$\xi^* : H^0(A, M) \otimes H^0(A, M) \hookrightarrow H^0(A, M^2) \otimes H^0(A, M^2)$$

Let θ be the unique (up to a scalar) section of $H^0(A,M)$ and consider the Theta basis $\{X_g\}_{g\in G}$ (2.4) of $H^0(A,L) = H^0(A,M^2)$. We have an exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow G(M^2 \boxtimes M^2) \longrightarrow A[2]\times A[2] \longrightarrow 1.$$

The kernel $K = \ker \xi = A[2]$ sits diagonally in $A[2]\times A[2]$ and the isomorphism (2.6) determines a lift $K^* \subset G(M^2 \boxtimes M^2)$ of K.

The main point is that $K = A[2]$ has a Göpel system in classical terminology of theta functions, namely

$$K \cong G \times \hat{G}$$

where both G and $\hat{G}$ are isotropic. In *loc. cit.*, this existence follows from the condition $p \nmid a+b$, which is the only reason to put this arithmetic condition, which of course is not fulfilled here. Via the projection map

$$G(M^2)\times G(M^2) \longrightarrow G(M^2 \boxtimes M^2)$$

the lift K^* induces lifts of G and $\hat{G}$ into $G(M^2)$. Because the $\hat{G}$-invariant part of $H^0(A,M^2)$ is generated by X_0, Sekiguchi's result gives

2.8. Lemma (Sekiguchi). *Normalizing ξ suitably, one has the formula*

$$\xi^*(\theta\boxtimes\theta) = \sum_{h\in G} X_h \boxtimes X_h \qquad (2.7)$$

In other words, we have

$$\theta(x-y)\theta(x+y) = \sum_{g\in G} X_g(x)X_g(y), \;\; x,y\in A.$$

Let us define the Kummer morphism $\mathrm{Kum}_A : A \to |2\Theta|$, $y \mapsto T_y^*\Theta + T_{-y}^*\Theta$, where T_y denotes translation by y. Then we can write

$$\mathrm{div}(\sum_g X_g(y)X_g) = T_y^*\Theta + T_{-y}^*\Theta = \mathrm{Kum}_A(y), y\in A.$$

Using the relation $F^*\Theta_1 = 2\Theta$, one gets the analogous relation

$$\mathrm{div}(\sum_g Y_g(y)Y_g) = T_y^*\Theta_1 + T_{-y}^*\Theta_1 = \mathrm{Kum}_{A_1}(y), y\in A_1.$$

The element (2.7) induces a linear isomorphism $H^0(A,M^2)^* \xrightarrow{\sim} H^0(A,M^2)$, which allows us to identify both spaces.

2.9. Corollary. *With the identification above, the complete linear system φ_L (resp. φ_{L_1}) is the Kummer morphism* Kum_A *(resp.* Kum_{A_1}*).*

2.5 The Theta divisor Θ in characteristic 2

From now we assume that $p = 2$ and that X is an ordinary curve of genus g, whose Jacobian is denoted by J. Let B be the theta-characteristic of X_1 defined by the exact sequence ([R] section 4)

$$0 \longrightarrow \mathcal{O}_{X_1} \longrightarrow F_*\mathcal{O}_X \longrightarrow B \longrightarrow 0 \qquad (2.8)$$

and we denote by $\Theta_1 \subset J_1$ the symmetric Theta divisor determined by B, i.e.,

$$\mathrm{supp}\,\Theta_1 = \{N \in J_1 :\ h^0(X_1, B \otimes N) > 0\}. \tag{2.9}$$

We denote by Θ the Theta divisor on J obtained from Θ_1 (Lemma 2.2). Note that we have $\Theta_1 = \iota^*\Theta$, where $\iota : J_1 \to J$ is the k-semi-linear isomorphism. Let R be the ring of dual numbers $k[\epsilon]$ with $\epsilon^2 = 0$. We recall that the Lie algebra $\mathfrak{g} = \hat{G}(R)$ identifies with the tangent space $\mathbf{T}_0 J$. For any tangent vector $v \in \mathfrak{g}$ and $g \in G$ we still denote by $e^L(v, g) \in k$ the derivative of

$$\hat{G} \longrightarrow \mathbb{G}_m, \qquad \bar{g} \longmapsto e^L(\bar{g}, g).$$

Writing $\bar{g} = 1 + \epsilon v$, with $v \in \mathfrak{g}$, we obtain $\forall v \in \mathfrak{g}, \forall g, h \in G$, $e^L(v, g+h) = e^L(v, g) + e^L(v, h)$. We recall some well-known facts about Θ.

2.10. Lemma. *(i) The Theta divisor Θ_1 (and Θ) passes through any non-zero $g \in G$.*

(ii) A non-zero g is a smooth point of Θ_1 (and Θ) if and only if $h^0(X_1, B \otimes g) = 1$.

(iii) Assuming (ii), the tangent space at g $\mathbf{T}_g\Theta_1 \subset \mathbf{T}_g J_1$ is defined by the linear form $e^L(., g)$ on $\mathfrak{g}$. Alternatively, identifying the tangent space $\mathbf{T}_g J$ with $H^1(X, \mathcal{O}_X)$, the tangent space $\mathbf{T}_g\Theta$ is the image of the injective k-linear map induced by the relative Frobenius,i.e.,

$$\mathbf{T}_g\Theta = \mathrm{im}\ \ F : H^1(X_1, g) \hookrightarrow H^1(X, \mathcal{O}_X).$$

Proof. Part (i) follows immediately from (2.8) and (2.9). Part (ii) is a special case of Riemann's singularity theorem (see e.g. [Ke1]). The differential at the origin of the separable isogeny $V : J_1 \to J$ is an isomorphism $dV : \mathbf{T}_0 J_1 \xrightarrow{\sim} \mathbf{T}_0 J$, which identifies with the Hasse-Witt map $F : H^1(X_1, \mathcal{O}_{X_1}) \to H^1(X, \mathcal{O}_X)$. Given $g \neq 0$, it will be enough to compute the tangent space to the divisor $V^*(T_g^*\Theta)$ at the origin. Let $\{Y_g\}_{g \in G}$ be the canonical Theta basis of $H^0(J_1, \mathcal{O}(\Theta_1 + T_g^*\Theta_1))$ (Apply Lemma 2.6(ii) to $L_1 = \mathcal{O}(\Theta_1 + T_g^*\Theta_1) = \mathcal{O}(2T_h^*\Theta_1)$ with $2h = g$). Then, by the isogeny formula (2.11), we have (up to a scalar) $V^*(T_g^*\Theta) = \sum_{g \in G} Y_g$. Let $\phi_v : \mathrm{Spec}(R) \to J_1$ be a tangent vector to J_1 at the origin. Then we compute, using $v.Y_g = e^L(v, g) Y_g$, $\forall v \in \mathfrak{g}$

$$\phi_v^*(\sum_{g \in G} Y_g) = \epsilon e^L(v, g) \sum_{h \in G/\langle g \rangle} Y_h(0)$$

Form this we deduce that Θ_1 is singular at g if and only if $\sum_{h \in G/\langle g \rangle} Y_h(0) = 0$ and, assuming g smooth, the equation of $\mathbf{T}_g\Theta_1$. The second description of $\mathbf{T}_g\Theta$ is a consequence of [Ke1]. □

2.11. Question. Are there other principally polarized abelian varieties (A, Θ) than Jacobians which have property (i) of Lemma 2.10?

2.12. Definition. *We say that X has* no vanishing theta-null *if X is ordinary and if all theta characteristics κ different from B satisfy $h^0(X, \kappa) = 1$, or equivalently (Lemma 2.10(ii)) all non-zero 2-torsion points are smooth points of Θ.*

In the next section we will see (Proposition 3.1(1)) that a generic curve has no vanishing theta-null.

2.6 Isogeny formulae

Given an isogeny $f : X \to Y$ and a line bundle L on Y, the isogeny formula gives the linear map $f^* : H^0(Y, L) \to H^0(X, f^*L)$ in terms of the canonical Theta bases. Although originally proved for separable isogenies and line bundles of separable type, we can extend the isogeny formula to more general line bundles. We assume $p = 2$ and we present (without proof) the three cases needed in this paper.

1. The separable isogeny $V : J_1 \to J$ with kernel G. Let $\{X_g\}_{g\in G}$ be the Theta basis of $H^0(J, 2\Theta)$ and $\{Y_u^{(2)}\}_{u\in G_2}$ be the Theta basis of $H^0(J_1, 4\Theta_1)$ (Remark 2.7). Then we have
$$\forall g \in G \qquad V^*X_g = \sum_{\substack{u\in G_2\\ 2u=g}} Y_u^{(2)}. \tag{2.10}$$

2. Isogeny V as in 1, with $V^* : H^0(J, T_g^*\Theta) \to H^0(J_1, \mathcal{O}(\Theta_1 + T_g^*\Theta_1))$, for $g \in G$. Then
$$V^*X_0 = \sum_{g\in G} Y_g. \tag{2.11}$$

3. The inseparable isogeny $\xi : J_1 \times J_1 \longrightarrow J_1 \times J_1$ defined in (2.5) with kernel $J_1[2] = G \times \hat{G}$. Let A be the quotient $J_1 \times J_1/\hat{G}$. Then ξ factorizes through a separable isogeny (with kernel G) $\bar{\xi} : A \to J_1 \times J_1$ and we identify $H^0(A, \bar{\xi}^*(2\Theta_1 \boxtimes 2\Theta_1))$ with the $\hat{G}$-invariant subspace of $H^0(J_1 \times J_1, 4\Theta_1 \boxtimes 4\Theta_1)$. A canonical basis of the latter space is given by the tensors $\{Y_u^{(2)} \boxtimes Y_v^{(2)}\}$ with $u, v \in G_2$ such that $u + v \in G_1 \subset G_2$. The isogeny formula, applied to $\bar{\xi}$, gives
$$\forall g, h \in G \qquad \xi^*(Y_h \boxtimes Y_{h+g}) = \sum_{\substack{u,v\in G_2\\ \xi(u,v)=(h,h+g)}} Y_u^{(2)} \boxtimes Y_v^{(2)}. \tag{2.12}$$

3 Extending Frobenius to $|2\Theta|$ in characteristic 2

We consider a principally polarized Jacobian (J, Θ) of an ordinary curve X, with Θ the symmetric Theta divisor defined by B (section 2.5), and the morphism φ induced by the linear system $|2\Theta|$ on J (resp. φ_1 on J_1), which we identify with the Kummer morphism Kum_J (Corollary 2.9).

3.1 Factorization

3.1. Proposition. *With the notation as above, we have*

1. *There exists a non-empty open set of the moduli space of genus g curves parametrizing curves X with no vanishing theta-null.*

2. *Suppose X has no vanishing theta-null. Then the isogeny V can be "extended" to a rational map $\tilde{V}$ such that $\varphi \circ V = \tilde{V} \circ \varphi_1$, i.e., the diagram*
$$\begin{array}{ccc} J_1 & \xrightarrow{V} & J \\ \big\downarrow{\scriptstyle\varphi_1} & & \big\downarrow{\scriptstyle\varphi} \\ |2\Theta_1|^* & \xrightarrow{\tilde{V}} & |2\Theta|^* \end{array} \tag{3.1}$$
is commutative.

3. *In terms of the canonical Theta bases of* $|2\Theta|$ *and* $|2\Theta_1|$ *the equations of* $\tilde{V}$ *are given by* 2^g *quadrics*

$$\tilde{V} : |2\Theta_1|^* \longrightarrow |2\Theta|^* \qquad x := (x_0 : \cdots : x_g : \cdots) \longmapsto (\lambda_0 P_0(x) : \cdots : \lambda_g P_g(x) : \cdots)$$

where

   (i) *the constants* $\{\lambda_g\}_{g\in G}$*,* $\lambda_g \in k$*, satisfy* $\forall g \in G, g \neq 0$

$$\lambda_g = 0 \qquad \Longleftrightarrow \qquad g \text{ is a singular point of } \Theta$$

   In particular, if X *has no vanishing theta-null, the* λ_g*'s are all non-zero.*

   (ii) *the polynomials* P_g *are given by*

$$P_g(x) = \sum_{h\in G/\langle g\rangle} x_{g+h}x_h. \tag{3.2}$$

Proof. In order to show commutativity of the diagram (3.1) it suffices to show that the image of the injection $V^* : H^0(J, 2\Theta) \hookrightarrow H^0(J_1, 4\Theta_1)$ is contained in the image of the multiplication map

$$\mathrm{Sym}^2 H^0(J_1, 2\Theta_1) \longrightarrow H^0(J_1, 4\Theta_1). \tag{3.3}$$

Let $\{X_g\}_{g\in G}$ and $\{Y_g\}_{g\in G}$ be the canonical Theta bases of $H^0(J, 2\Theta)$ and $H^0(J_1, 2\Theta_1)$ and consider the pull-back of equality (2.7), written for $A = J_1$, by the morphism $\psi_g : J_1 \to J_1 \times J_1$, $\psi_g(x) = (x + g, x)$,

$$\psi_g^*\xi^*(\theta_1 \boxtimes \theta_1) = \theta_1(g)V^*X_g, \qquad \psi_g^*(\sum_{h\in G} Y_h \boxtimes Y_h) = \sum_{h\in G} Y_{h+g}Y_h. \tag{3.4}$$

If $g = 0$, we get

$$\sum_{h\in G} Y_h^2 = \theta_1(0)V^*X_0 \tag{3.5}$$

with $\theta_1(0) \neq 0$, and define $\lambda_0 = \theta_1(0)$. If $g \neq 0$, we see that both members of (3.4) are zero. In order to get a meaningful statement we restrict to a first infinitesimal neighbourhood of $\psi_g(J_1) \subset J_1 \times J_1$. The notation is as in section 2.5. We pull-back the morphisms ξ and ψ_g to $\mathrm{Spec}(R)$ replacing g by $(1 + \epsilon v).g$ and keeping the same notation for the object over k and its pull-back to R. Pulling back equality (2.7) by ψ_g, for $g \neq 0$, we get the following two elements in $H^0(J_1, 4\Theta_1) \otimes R$

$$\psi_g^*\xi^*\theta_1 \boxtimes \theta_1 = \epsilon e^L(v, g)\lambda_g V^*X_g$$

where λ_g is a scalar which vanishes if and only if Θ_1 is singular at the point g (Lemma 2.10), and

$$\begin{aligned}
\psi_g^*(\sum_{h\in G} Y_h \boxtimes Y_h) &= \sum_{h\in G} [(1 + \epsilon v).Y_{g+h}]\, Y_h \\
&= \sum_{h\in G} Y_{g+h}Y_h + \epsilon \sum_{h\in G} (v.Y_{g+h})Y_h \\
&= \epsilon \sum_{h\in G} e^L(v, g + h)Y_{g+h}Y_h \\
&= \epsilon e^L(v, g) \sum_{h\in G/\langle g\rangle} Y_{g+h}Y_h
\end{aligned}$$

where h runs over a set of representatives of $G/\langle g\rangle$. Since these elements are equal $\forall v \in \mathfrak{g}$ and there exists $v \in \mathfrak{g}$ such that $e^L(v,g) \neq 0$, we obtain (up to a multiplicative non-zero scalar)

$$\sum_{h \in G/\langle g\rangle} Y_{g+h} Y_h = \lambda_g V^* X_g. \tag{3.6}$$

In order to complete the proof it suffices to show that for a general curve X the λ_g's are non-zero, for all $g \neq 0$. Assume that the contrary holds. Then the equality in $\mathrm{Sym}^2 H^0(J_1, 2\Theta_1)$ (which we leave as an exercise)

$$\left(\sum_{h \in G_g} Y_h\right)\left(\sum_{h \notin G_g} Y_h\right) = \sum_{h \notin G_g} P_h$$

where G_g is any index 2 subgroup of G not containing g and the P_h are the quadrics (3.2), shows that either $\sum_{h \in G_g} Y_h$ or $\sum_{h \notin G_g} Y_h$ is zero, since the RHS is mapped to zero in $H^0(J_1, 4\Theta_1)$. But this is impossible, since $\{Y_h\}$ is a basis. Hence for any curve X there exists $g \neq 0$ such that $\lambda_g \neq 0$ and we conclude by a monodromy argument found in [E].

□

3.2. Remark. We notice that $\tilde{V}$ is uniquely defined only up to a degree 2 equation of the image $\varphi_1(J_1) \subset |2\Theta_1|^*$. We will show uniqueness of $\tilde{V}$ (Proposition 4.1) for an ordinary genus 2 curve. We expect that there are no quadrics containing $\varphi_1(J_1)$ for a curve of genus $g \geq 2$ with no vanishing theta-null.

Equalities (3.5) and (3.6) can be used to define the vector $(\lambda_g)_{g \in G}$ up to a scalar. In the next proposition we give a more direct definition in terms of theta-constants. Let $\{Y_u^{(2)}\}_{u \in G_2}$ be the Theta basis of $H^0(J_1, 4\Theta_1)$ and we denote by $Y_u^{(2)}(0) \in k$ the value of $Y_u^{(2)}$ at the origin (after having chosen an isomorphism $\mathcal{O}(4\Theta_1)_0 \xrightarrow{\sim} k$).

3.3. Proposition. *With the notation as above, we have*

$$\forall g \in G \qquad \lambda_g = \sum_{u \in S_g} Y_u^{(2)}(0) \tag{3.7}$$

where S_g is a set of representatives of $\{u \in G_2 \ : \ 2u = g\}/\langle g\rangle$.

Proof. Let i be the inclusion in the first factor $i : J_1 \hookrightarrow J_1 \times J_1$. A standard computation modelled on [Ke2] Proposition 6, which involves (2.12), gives the equality in $\mathrm{Sym}^2 H^0(J_1, 4\Theta_1)$

$$i^*\xi^*\left(\sum_{h \in G/\langle g\rangle} Y_h \boxtimes Y_{h+g}\right) = \left(\sum_{u \in S_g} Y_u^{(2)}(0)\right) \cdot \left(\sum_{\substack{u \in G_2 \\ 2u = g}} Y_u^{(2)}\right)$$

We observe that the composite $\xi \circ i$ is the diagonal map, hence the LHS can be rewritten as $\sum Y_h Y_{h+g}$. Applying the isogeny formula (2.10), we get $\forall g \in G$

$$\sum_{h \in G/\langle g\rangle} Y_h Y_{h+g} = \left(\sum_{u \in S_g} Y_u^{(2)}(0)\right) \cdot V^* X_g.$$

Comparing with (3.5) and (3.6) we can conclude. Finally we observe that the expression (3.7) is well-defined since $\forall u$ such that $2u = g$ we have $Y_u^{(2)}(0) = Y_{-u}^{(2)}(0) = Y_{u+g}^{(2)}(0)$ ($Y_0^{(2)}$ is symmetric).
□

4 Kummer's quartic surface in characteristic 2

As an application of Proposition 3.1 we shall deduce the equation of the image $\varphi(J) \subset |2\Theta|^*$ for a Jacobian of an ordinary genus 2 curve X. We observe that by Clifford's theorem $h^0(X_1, Bg) \leq 1$, $\forall g \in G$, hence X has no vanishing theta-null and we can apply Proposition 3.1(2). Somewhat surprisingly, invariance under the Theta group $G(L)$ and the "Frobenius" map $\tilde{V}$ are sufficient to determine the equation. We fix an isomorphism $G \cong (\mathbb{Z}/2\mathbb{Z})^2$.

4.1. Proposition. *In the canonical Theta basis $\{X_g\}_{g\in G}$ of $H^0(J, 2\Theta)$ the equation of the Kummer surface is (up to a non-zero scalar)*

$$\lambda_{10}^2(x_{00}^2x_{10}^2 + x_{01}^2x_{11}^2) + \lambda_{01}^2(x_{00}^2x_{01}^2 + x_{10}^2x_{11}^2) + \lambda_{11}^2(x_{00}^2x_{11}^2 + x_{01}^2x_{10}^2) + \frac{\lambda_{10}\lambda_{01}\lambda_{11}}{\lambda_{00}}x_{00}x_{10}x_{01}x_{11} \tag{4.1}$$

In particular, $\varphi(J)$ is not contained in a quadric and therefore the map $\tilde{V}$ is uniquely defined.

Proof. First we observe that the image $\varphi(J) \subset |2\Theta|$ is invariant under the Theta group $G(L)$. Hence the equation of $\varphi(J)$ has to be $G(L)$-invariant. Since $(2\Theta)^2 = 8$ and $\deg \varphi = 2$ (φ is separable), we have $\deg \varphi(J) = 4$. The unique $G(L)$-invariant quadric is $x_{00}^2 + x_{01}^2 + x_{10}^2 + x_{11}^2 = (x_{00} + x_{01} + x_{10} + x_{11})^2$. Since $\varphi(J)$ is non-degenerate, we see that it is not contained in a quadric. A straightforward computation shows that a basis of $G(L)$-invariant quartics is given by the 5 polynomials

$$S := x_{00}^4 + x_{01}^4 + x_{10}^4 + x_{11}^4 \qquad R := x_{00}x_{01}x_{10}x_{11}$$

$$Q_{10} := x_{00}^2x_{10}^2 + x_{01}^2x_{11}^2 \qquad Q_{01} := x_{00}^2x_{01}^2 + x_{10}^2x_{11}^2 \qquad Q_{11} := x_{00}^2x_{11}^2 + x_{01}^2x_{10}^2$$

Let us denote by

$$F = \alpha_{10}Q_{10} + \alpha_{01}Q_{01} + \alpha_{11}Q_{11} + \beta R + \gamma S$$

the equation of $\varphi(J)$, where $\alpha_{10}, \alpha_{01}, \alpha_{11}, \beta, \gamma \in k$ need to be determined. First $\varphi(e) = (1 : 0 : 0 : 0)$, where $e \in J$ is the origin, implies that $\gamma = 0$. Since the image is reduced, $\beta \neq 0$. Next, the equation of $\varphi_1(J_1) \subset |2\Theta_1|$ is given in a Theta basis by

$$F_1 = \alpha_{10}^2Q_{10} + \alpha_{01}^2Q_{01} + \alpha_{11}^2Q_{11} + \beta^2R$$

Since $\varphi_1(J_1)$ is not contained in a quadric, the rational map $\tilde{V}$ (3.1) is uniquely defined. Since $\tilde{V}(\varphi_1(J_1)) = \varphi(J)$, there exists a $G(L)$-invariant quartic A, such that (up to a non-zero scalar)

$$\tilde{V}^*(F) = F_1 \cdot A \tag{4.2}$$

We shall use the Theta coordinates $\{x_g\}_{g\in G}$ on both spaces. In order to determine the equation of A, we restrict equality (4.2) to the hyperplane $H : \sum_{g\in G} x_g = 0$ and write the expressions as degree 8 polynomials in x_{01}, x_{10}, x_{11}. A straightforward computation, which we omit, leads to

$$\tilde{V}^*(F)_{|H} = \lambda_{01}^2\lambda_{10}^2\lambda_{11}^2 \qquad (x_{01} + x_{11})^2(x_{01} + x_{10})^2(x_{11} + x_{10})^2$$
$$\left[x_{01}^2\left(\frac{\alpha_{10}}{\lambda_{10}^2} + \frac{\alpha_{11}}{\lambda_{11}^2}\right) + x_{10}^2\left(\frac{\alpha_{01}}{\lambda_{01}^2} + \frac{\alpha_{11}}{\lambda_{11}^2}\right) + x_{11}^2\left(\frac{\alpha_{10}}{\lambda_{10}^2} + \frac{\alpha_{01}}{\lambda_{01}^2}\right)\right]$$

Suppose that $A_{|H} \neq 0$, i.e. $\tilde{V}^*(F)_{|H} \neq 0$. Then at least one of the factors $x_{01}+x_{10}, x_{01}+x_{11}, x_{11}+x_{10}$ has to divide $F_{1|H}$. Again elementary computation shows that this can only happen when $\beta = 0$, which is impossible. Hence A is a multiple of H and, since A is also $G(L)$-invariant, we get $A = H^4 = \sum_{g\in G} x_g^4$. Moreover $\tilde{V}^*(F)_{|H} = 0$ implies that there exists a $\mu \in k^*$ such that $\alpha_g = \mu\lambda_g^2$ for all $g \in G^*$.

It remains to determine the constant β. We compute the degree 8 polynomial $\tilde{V}^*(F)$ (again we omit details)

$$\begin{aligned}\mu\lambda_{00}^2 H^4(\lambda_{10}^4 Q_{10} + \lambda_{01}^4 Q_{01} + \lambda_{11}^4 Q_{11})\\ +\mu\lambda_{01}^2\lambda_{10}^2\lambda_{11}^2 H^2(x_{00}^2x_{01}^2x_{10}^2 + x_{00}^2x_{01}^2x_{11}^2 + x_{00}^2x_{10}^2x_{11}^2 + x_{01}^2x_{10}^2x_{11}^2)\\ +\beta\lambda_{00}\lambda_{01}\lambda_{10}\lambda_{11}H^2(RH^2 + x_{00}^2x_{01}^2x_{10}^2 + x_{00}^2x_{01}^2x_{11}^2 + x_{00}^2x_{10}^2x_{11}^2 + x_{01}^2x_{10}^2x_{11}^2)\end{aligned}$$

This expression being equal to $F_1 \cdot H^4$, we get the equation mentioned in the proposition. □

4.2. Remark. If the characteristic is different from 2, Kummer's quartic surface is a much studied object (see e.g.[H]). Among many other results, let us just mention that the coefficients of the quartic equation (in a Theta basis) satisfy a cubic equation and are themselves polynomials of degree 12 in the theta-constants. Since we could not find any treatment of the characteristic 2 case in the literature, we decided to include it in our paper.

5 The moduli space M_X of rank 2 vector bundles

We assume $p = 2$. Let M_X denote the moduli space of rank 2 semi-stable vector bundles over X with trivial determinant. As was observed in [MR], the theta divisor

$$\tilde{\Theta} = \{[E] \in \mathrm{M}_X \mid h^0(E \otimes B) \neq 0\}$$

is Cartier (and not only $\mathbb{Q}$-Cartier) and is ample by GIT. We denote by $\mathcal{L}_0$ the line bundle $\mathcal{O}_{\mathrm{M}_X}(\tilde{\Theta})$. By [R] there exists a regular morphism

$$D : \ \mathrm{M}_X \longrightarrow \mathbb{P}H^0(J, 2\Theta) = |2\Theta|$$

which maps the class of the semi-stable bundle E to the divisor $D(E)$

$$\mathrm{supp}\ D(E) = \{L \in J \mid H^0(E \otimes B \otimes L) \neq 0\}.$$

As in the complex case [NR] one has

5.1. Proposition (V. Balaji). *If $g = 2$, the morphism $D : \mathrm{M}_X \longrightarrow \mathbb{P}^3 = |2\Theta|$ is an isomorphism.*

Proof. (sketch) We proceed in two steps. First we consider a flat family of curves $\mathcal{X} \longrightarrow T = \mathrm{Spec}(A)$, where A is a discrete valuation ring such that its residue field at the closed point 0 is k, its field of fraction K is of characteristic zero, and $\mathcal{X}_0 = X$. We consider the moduli scheme $\mathcal{M} \longrightarrow T$ of semi-stable rank 2 vector bundles of trivial determinant over the family $\mathcal{X} \to T$. Let $\mathcal{M}_0$ be the fibre of $\mathcal{M} \longrightarrow T$ over 0. Then, by GIT over arbitrary base [Ses], we have a canonical bijective morphism $i : \mathrm{M}_X \longrightarrow \mathcal{M}_0$. Moreover on the open set of stable points i is an isomorphism since the action of the projective group on the Quot scheme is free. We conclude that $i : \mathrm{M}_X \longrightarrow \mathcal{M}_0$ is an isomorphism by Zariski's Main Theorem.

Secondly, we extend the morphism D [NR] to the family $\mathcal{M} \longrightarrow T$,i.e., we construct a morphism over the base T

$$\mathcal{D} : \mathcal{M} \longrightarrow \mathbb{P}^3_T$$

such that $\mathcal{D}_0 = D$. In order to show the proposition, it will be enough (again by Zariski's Main Theorem) to show that $\mathcal{D}$ is birational and bijective. We consider the fibre $\mathcal{D}_\xi : \mathcal{M}_\xi \longrightarrow \mathbb{P}^3_\xi$ over the generic point $\xi \in T$. Working over an algebraic closure of K (of characteristic 0), we see [NR] that D_ξ is an isomorphism. Hence $\mathcal{D}$ is birational. It remains to show that $\mathcal{D}_0 = D$ is bijective. Surjectivity is obvious. Let H be the hyperplane $|2\Theta|$ of divisors passing through $\mathcal{O}$. The inverse image of H is Θ showing $D^*\mathcal{O}(1) \cong \mathcal{L}_0$. It follows that D is finite because $D^*\mathcal{O}(1)$ is ample. Let $\mathcal{L} = D^*\mathcal{O}_{\mathbb{P}^3_T}(1)$ be the relatively ample line bundle over $\mathcal{M} \longrightarrow T$. Consider the canonical inclusion of $\mathcal{O}_{\mathbb{P}^3_T}$ in $\mathcal{D}_*\mathcal{O}_\mathcal{M}$ with cokernel $\mathcal{Q}$.

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_T} \longrightarrow \mathcal{D}_*\mathcal{O}_\mathcal{M} \longrightarrow \mathcal{Q} \longrightarrow 0$$

We twist by $\mathcal{O}_{\mathbb{P}^3_T}(n)$. If n is large enough, we have $\forall t \in T$ and $\forall i > 0$, $h^i(\mathcal{M}_t, \mathcal{L}_t^n) = 0$. Hence, since $\mathbb{P}^3_T \longrightarrow T$ is flat, we see that

$$h^0(\mathbb{P}^3_t, \mathcal{O}_{\mathbb{P}^3_t}(n)) - h^0(\mathcal{M}_t, \mathcal{L}_t^n) = \chi(\mathbb{P}^3_t, \mathcal{O}_{\mathbb{P}^3_t}(n)) - \chi(\mathcal{M}_t, \mathcal{L}_t^n)$$

is constant. If $t = \xi$, this number is zero [NR]. Hence $h^0(\mathbb{P}^3_0, \mathcal{Q}_0(n)) = 0$ and $Q_0 = 0$. So $\mathcal{O}_{\mathbb{P}^3_0} = D_*\mathcal{O}_{\mathrm{M}_X}$ and D is injective.

□

5.2. Remark. As we were told by C.S. Seshadri, the first part of the proof is completely worked out in the PhD thesis of Venkata Balaji (in preparation). A direct proof of this isomorphism along the lines of the original paper [NR] was obtained by M.S. Narasimhan.

6 Frobenius action on M_X for an ordinary genus 2 curve

The goal of this section is to describe the Frobenius map (more precisely, its separable part, the Verschiebung)

$$V : \mathrm{M}_{X_1} \longrightarrow \mathrm{M}_X, \qquad E \longmapsto F^*E.$$

We consider the morphism $\psi : J \to \mathrm{M}_X$, $L \mapsto [L \oplus L^{-1}]$ which, when composed with D, equals the Kummer morphism Kum_J. Because of Proposition 5.1 and Corollary 2.9 we can identify ψ and φ. Since $H^0(\mathrm{M}_X, \mathcal{L}_0) = H^0(J, 2\Theta)$ and since $\tilde{V}$ is uniquely defined (Proposition 4.1), we can identify (via D) the Verschiebung $V : \mathrm{M}_{X_1} \longrightarrow \mathrm{M}_X$ with the rational map $\tilde{V} : |2\Theta_1|^* \longrightarrow |2\Theta|^*$ given by the equations (3.2). We gather our results in the following proposition.

6.1. Proposition. *Let X be an ordinary genus 2 curve.*

1. *The semi-stable boundary of M_X (resp. M_{X_1}) is isomorphic to Kummer's quartic surface $\varphi(J)$ (resp. $\varphi_1(J_1)$) whose equation is given in (4.1). In particular, V maps $\varphi_1(J_1)$ onto $\varphi(J)$.*

2. *There exists a unique stable bundle $E_{BAD} \in \mathrm{M}_X$, which is destabilized by the Frobenius map, i.e. F^*E_{BAD} is not semi-stable. We have $E_{BAD} = F_*B^{-1}$ and its projective coordinates are $(1 : 1 : 1 : 1)$.*

3. *The set of bundles $\{[E] \in \mathrm{M}_{X_1} \mid \mathrm{Hom}(E, F_*B) \neq 0\}$ is the hyperplane $H_1 : \sum_{g\in G} x_g = 0$. In particular, $E_{BAD} \in H_1$. The restriction of V to H_1 contracts H_1 to the conic $\varphi(J) \cap H$, where H is the hyperplane $x_0 = 0$ in $|2\Theta|$. In particular, any stable bundle $E \in H_1$ is mapped into the semi-stable boundary of M_X.*

4. *The fiber of V over a point $[E] \in \mathrm{M}_X$ is*

   - *a non-degenerate G-orbit of a point $[E_1] \in \mathrm{M}_{X_1}$ (4 distinct points), if $[E] \notin H$*
   - *empty, if $[E] \in H \setminus (H \cap \varphi(J))$*

- *a projective line passing through E_{BAD}, if $[E] \in H \cap \varphi(J)$*

In particular, V is not surjective and the separable degree of V is 4.

Proof. 1. This follows immediately from [NR] and Proposition 4.1.

2. The base locus of $\tilde{V}$ is given by the intersection $\bigcap_{g \in G}\{P_g = 0\}$, which turns out (after some elementary computations) to be a unique point with projective coordinates $(1 : 1 : 1 : 1)$. In terms of vector bundles this point, denoted E_{BAD}, corresponds to the direct image F_*B^{-1}. Indeed, F_*B^{-1} is stable: a nonzero map $L \to F_*B^{-1}$ is equivalent, by adjunction, to a nonzero map $F^*L \to B^{-1}$, hence $2\deg L \leq -1$. Moreover, since we have a canonical nonzero map $F^*F_*B^{-1} \to B^{-1}$, this bundle is destabilized. Uniqueness can also be proved without the use of the equations: consider any stable bundle $E \in \mathrm{M}_X$, which is destabilized by V. By [LS] Corollary 2.6 there exists a theta-characteristic A which appears as a quotient $F^*E \to A^{-1}$. By adjunction this quotient map induces a map $E \to F_*A^{-1}$. Since the two bundles are stable with the same slope, we deduce that they have to be isomorphic. The determinant of F_*A^{-1} being trivial, we get $A = B$ and we are done.

3. First we observe that the hyperplane H_1 is mapped into the hyperplane H. After fixing an isomorphism $G \cong (\mathbb{Z}/2\mathbb{Z})^2$, we straightforwardly compute that a point in the image $V(H_1)$ satisfies the equations

$$\lambda_{10}x_{01}x_{11} + \lambda_{01}x_{10}x_{11} + \lambda_{11}x_{01}x_{10} = 0 \tag{6.1}$$

which is precisely (after squaring) the equation of Kummer's quartic surface (4.1) restricted to the hyperplane H. Moreover $V^*(H) = H_1^2$ and by adjunction $\mathrm{Hom}(E, F_*B) = \mathrm{Hom}(F^*E, B)$ which proves the first assertion.

4. Given a point $[E] \in \mathrm{M}_X$ with projective coordinates $(a_{00} : a_{01} : a_{10} : a_{11})$, we have to solve the system of quadratic equations

$$P_g = \frac{a_g}{\lambda_g} \qquad \forall g \in G \cong (\mathbb{Z}/2\mathbb{Z})^2 \tag{6.2}$$

where the quadrics P_g's are defined in (3.2). We write $b_g = \frac{a_g}{\lambda_g}$. Adding any two P_g's with $g \neq 0$, we find

$$P_{01} + P_{10} = x_{00}x_{01} + x_{10}x_{11} + x_{00}x_{10} + x_{01}x_{11} = (x_{00} + x_{11})(x_{01} + x_{10}) = b_{01} + b_{10} \tag{6.3}$$

and similarly

$$(x_{00} + x_{01})(x_{11} + x_{10}) = b_{11} + b_{10} \qquad (x_{00} + x_{10})(x_{01} + x_{11}) = b_{01} + b_{11} \tag{6.4}$$

and

$$x_{00} + x_{01} + x_{10} + x_{11} = c \tag{6.5}$$

with $c^2 = b_{00}$. We let $\alpha = x_{00}+x_{11}, \beta = x_{01}+x_{10}, \gamma = x_{00}+x_{01}, \delta = x_{11}+x_{10}$. Then the equations (6.3), (6.4), (6.5) imply that α, β (resp. γ, δ) are the roots of the polynomial

$$t^2 + ct + (b_{01} + b_{10}) \qquad \text{resp.} \quad t^2 + ct + (b_{11} + b_{10}) \tag{6.6}$$

Substituing x_{01}, x_{11}, x_{01} with $\gamma + x_{00}, \alpha + x_{00}$ and $\beta + \gamma + x_{00}$ respectively in the equation $P_{01} = b_{01}$, we find

$$cx_{00} = b_{10} + \gamma\alpha \tag{6.7}$$

Assuming $c \neq 0$,i.e. $[E] \notin H$, we see that the fiber $V^{-1}([E])$ consists of the point $[E_1] \in \mathrm{M}_{X_1}$ with projective coordinates $(x_{00} : x_{01} : x_{10} : x_{11})$ given by

$$x_{00} = \frac{1}{c}(b_{10} + \gamma\alpha) \qquad x_{01} = \frac{1}{c}(b_{10} + \gamma\beta) \qquad x_{10} = \frac{1}{c}(b_{10} + \beta\delta) \qquad x_{11} = \frac{1}{c}(b_{10} + \delta\alpha)$$

plus the other 3 points obtained by switching α and β as well as γ and δ. Since $c \neq 0$, one easily sees that these 4 points are distinct.

Assume $c = 0$, i.e. $[E] \in H$. We get $\alpha = \beta, \gamma = \delta, \alpha^2 = b_{01} + b_{10}, \gamma^2 = b_{11} + b_{10}$. Hence $(b_{10}+\gamma\alpha)^2 = b_{10}b_{11}+b_{01}b_{10}+b_{10}b_{11}$. Assume that $b_{10}b_{11}+b_{01}b_{10}+b_{10}b_{11} \neq 0$,i.e. $[E] \in H\backslash(H\cap\varphi(J))$. Because of (6.7) the system (6.2) does not have a solution. If $b_{10}b_{11} + b_{01}b_{10} + b_{10}b_{11} = 0$, i.e. $[E] \in H \cap \varphi(J)$, it is easy to check that $V^{-1}([E])$ is the $\mathbb{P}^1$, intersection of the two hyperplanes

$$H_1 : x_{00} + x_{10} + x_{01} + x_{11} = 0 \qquad H_{\alpha,\gamma} : (\alpha + \gamma)x_{00} + \alpha x_{01} + \gamma x_{11} = 0$$

□

6.2. Remark. Let $L_1 \in JX_1$ and $M = F^*L_1 \in JX$. We assume that $M^2 \neq \mathcal{O}_X$. Then obviously $F^*[L_1 \oplus L_1^{-1}] = [M \oplus M^{-1}]$ and, as one easily checks, if $h^0(M \otimes B) = 0$, the three isomorphism classes contained in $[L_1 \oplus L_1^{-1}]$, i.e. the decomposable bundle $L_1 \oplus L_1^{-1}$, the two non-split extensions of L_1 by L_1^{-1} and of L_1^{-1} by L_1 (which are interchanged by the hyperelliptic involution i), are mapped to the corresponding isomorphism class in $[M \oplus M^{-1}]$. On the other hand, if $h^0(M \otimes B) > 0$, all three isomorphism classes are mapped to $M \oplus M^{-1}$. Moreover, in that case, by Proposition 6.1 (4), there exist stable bundles E such that $[F^*E] = [M \oplus M^{-1}]$. Since i commutes with F and since any stable bundle on X_1 is i-invariant, we have $F^*E \cong M \oplus M^{-1}$. This shows that the non-split extension of M by M^{-1} is not of the form F^*E for some semi-stable bundle E.

6.3. Remark. It would be interesting to have a coordinate-free proof of the following fact, which follows from Proposition 6.1(3): If E_1 is stable over X_1 with $\mathrm{Hom}(E_1, F_*B) \neq 0$, then F^*E_1 is unstable.

Let N_X (resp. N_{X_1}) denote the moduli space of semi-stable rank 2 vector bundles of degree 0 over X (resp. X_1). As a corollary of Proposition 6.1 we have

6.4. Proposition. *For any ordinary curve X, the rational map $V : \mathrm{N}_{X_1} \longrightarrow \mathrm{N}_X$ given by $E \longmapsto F^*E$ is surjective.*

Proof. It will be enough to show that any point $[E] \in \mathrm{M}_X \subset \mathrm{N}_X$ is in the image, since, twisting by a degree zero line bundle, we can always assume the determinant to be trivial. For any non-zero $g \in G$, we choose a $h \in G_2$ such that $2h = g$ and we denote by $\mathrm{M}_{X_1}(g)$ the moduli of rank 2 vector bundles with fixed determinant equal to g. Then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{M}_{X_1}(g) & \xrightarrow{V} & \mathrm{M}_X \\ \downarrow{\scriptstyle T_h} & & \downarrow{\scriptstyle T_g} \\ \mathrm{M}_{X_1} & \xrightarrow{V} & \mathrm{M}_X \end{array}$$

Now since the vertical maps are isomorphisms, the image of the first horizontal map contains the complement to the hyperplane $x_g \neq 0$. Since the hyperplanes $\{x_g = 0\}_{g\in G}$ have no common point in M_X we get surjectivity. □

7 The action on the odd degree moduli space $\mathrm{M}_{X_1}(\Delta)$

In this section we briefly discuss the action of the Frobenius map on the moduli space $\mathrm{M}_{X_1}(\Delta)$ of semi-stable rank 2 vector bundles over X_1 with fixed determinant equal to Δ, with $\deg \Delta = 1$. We will study the rational map

$$V : \mathrm{M}_{X_1}(\Delta) \longrightarrow \mathrm{M}_X, \qquad E \longmapsto F^*E \otimes \Delta^{-1}$$

Note that we use the same letter $(\Delta, B, \dots)$ for the line bundles over X_1 and X, which correspond under the k-semi-linear isomorphism $\iota : X_1 \to X$.

The next proposition holds for any curve of genus $g \geq 2$.

7.1. Proposition. *The image of V is contained in the Theta divisor $\Theta \subset \mathrm{M}_X$, i.e.*

$$\forall E \in \mathrm{M}_{X_1}(\Delta) \qquad h^0(X, F^*E \otimes \Delta^{-1} \otimes B) > 0$$

In particular, V is not dominant.

Proof. We write E as an extension of line bundles

$$0 \longrightarrow L \longrightarrow E \longrightarrow \Delta L^{-1} \longrightarrow 0 \qquad (\epsilon)$$

for some line bundle L with $\deg L \leq 0$ and $\epsilon \in \mathbb{P}H^1(X_1, L^2\Delta^{-1})$. The extension class $F^*\epsilon$ of the exact sequence, gotten by pull-back under F, i.e.

$$0 \longrightarrow L^2 \longrightarrow F^*E \longrightarrow \Delta^2 L^{-2} \longrightarrow 0 \qquad (F^*\epsilon) \tag{7.1}$$

is obtained from ϵ via the linear map

$$H^1(X_1, L^2\Delta^{-1}) \xrightarrow{F^*} H^1(X, L^4\Delta^{-2}) = H^1(X_1, L^2\Delta^{-1} \otimes F_*\mathcal{O}_X) \tag{7.2}$$

The last map coincides with the induced map on cohomology of the canonical exact sequence

$$0 \longrightarrow L^2\Delta^{-1} \longrightarrow L^2\Delta^{-1} \otimes F_*\mathcal{O}_X \longrightarrow L^2\Delta^{-1}B \longrightarrow 0 \tag{7.3}$$

In order to prove that $h^0(X, F^*E \otimes \Delta^{-1}B) > 0$, we tensorize (7.1) with $\Delta^{-1}B$. If $h^0(X, L^2\Delta^{-1}B) > 0$, we are done. Therefore we assume $h^0(X, L^2\Delta^{-1}B) = 0$. It follows from (7.2) and (7.3) that

$$0 \longrightarrow H^1(X_1, L^2\Delta^{-1}) \xrightarrow{F^*} H^1(X, L^4\Delta^{-2}) \longrightarrow H^1(X_1, L^2\Delta^{-1}B) \longrightarrow 0 \tag{7.4}$$

On the other hand, we see that $h^0(X, F^*E \otimes \Delta^{-1} \otimes B) > 0$ if and only if the symmetric coboundary map

$$H^0(X, \Delta L^{-2}B) \xrightarrow{\cup F^*\epsilon} H^1(X, \Delta^{-1}L^2B) = H^0(X, \Delta L^{-2}B)^*$$

is degenerate. We write $V := H^0(X, \Delta L^{-2}B)$. It is well-known that the linear map

$$H^1(X, L^4\Delta^{-2}) = H^0(X, \Omega_X L^{-4}\Delta)^* \xrightarrow{m^*} V^* \otimes V^*, \qquad \delta \longmapsto \cup\delta \tag{7.5}$$

is the dual of the multiplication map of global sections $\mathrm{Sym}^2 V \xrightarrow{m} H^0(X, \Omega_X L^{-4}\Delta)$. Let us denote by $\mathcal{D}_2(V^*)$ the space of divided powers of $V^* \otimes V^*$,i.e. the subspace of tensors invariant under the involution $\phi \otimes \psi \longmapsto \psi \otimes \phi$. We have $\mathcal{D}_2(V^*) = (\mathrm{Sym}^2 V)^*$. We denote by $F(V)$ the subspace of $\mathrm{Sym}^2 V$ generated by the squares v^2 with $v \in V$. Dually, the kernel of the surjection $K := \ker\,(\mathcal{D}_2(V^*) \longrightarrow F(V)^*)$ coincides with the space of alternating maps $V \to V^*$. The main point, which will be used later, is that, since, by Riemann-Roch, $\dim V = \dim H^1(X, \Delta^{-1}L^2B) = 1 - 2\deg L$ is odd, any map in K is degenerate.

By Serre duality we have $H^1(X_1, L^2\Delta^{-1}B) = H^0(X_1, \Delta L^{-2}B)^*$. Put $V_1 := H^0(X_1, \Delta L^{-2}B)$. Then we have a k-linear isomorphism $\varphi : V_1 \xrightarrow{\sim} F(V)$ defined as follows: we have $V_1 = \iota^* V = V \otimes_k k$ and we put $\varphi(v \otimes t) = tv^2 \in F(V)$. We also note that $F(V)^* = F(V^*)$. Again by Serre duality, we observe that the linear maps (7.4) and (7.5) coincide, i.e. we have a commutative diagram

$$\begin{array}{ccccccc} H^1(X_1, L^2\Delta^{-1}) & \xrightarrow{F^*} & H^1(X, L^4\Delta^{-2}) & & \longrightarrow & H^1(X_1, L^2\Delta^{-1}B) & \longrightarrow 0 \\ & & \downarrow & & & \downarrow & \\ & & H^0(X, \Omega_X L^{-4}\Delta)^* & \xrightarrow{m^*} & \mathcal{D}_2(V^*) \longrightarrow & F(V)^* & \end{array}$$

where the vertical maps are k-linear isomorphisms. Now we can conclude as follows: by commutativity, any extension class $F^*\epsilon$, with $\epsilon \in H^1(X_1, L^2\Delta^{-1})$, is mapped by m^* into K. Hence the corresponding coboundary map is degenerate. □

We come back to an ordinary curve X of genus 2. Let us denote by $\mathrm{Gr} \subset \mathbb{P}\Lambda^2 H^0(J_1, 2\Theta_1) := \mathbb{P}^5$ the Grassmannian of projective lines in $\mathbb{P}H^0(J_1, 2\Theta_1) = \mathbb{P}^3 = \mathrm{M}_{X_1}$. Following [B] section 3.4 we consider, for a general point $q \in X$, the morphism

$$\mathrm{M}_{X_1}(\mathcal{O}(q)) \xrightarrow{\mathbb{L}} \mathrm{Gr}, \qquad E \longmapsto \mathbb{L}(E) = \{E' \in \mathrm{M}_{X_1} : \ E' \subset E\}.$$

We choose an isomorphism $G \cong (\mathbb{Z}/2\mathbb{Z})^2$ and consider the Plücker coordinates on $\Lambda^2 H^0(J_1, 2\Theta_1)$

$$\begin{array}{lll} z_1 = x_{00} \wedge x_{01}, & z_2 = x_{00} \wedge x_{10}, & z_3 = x_{00} \wedge x_{11}, \\ z_4 = x_{10} \wedge x_{11}, & z_5 = x_{01} \wedge x_{11}, & z_6 = x_{01} \wedge x_{10}. \end{array}$$

The equation of the Grassmannian Gr is $z_1 z_4 + z_2 z_5 + z_3 z_6 = 0$. and the G-invariant subspace $Z \subset \Lambda^2 H^0(J_1, 2\Theta_1)$ is given by the 3 linear equations

$$Z: \qquad z_1 + z_4 = z_2 + z_5 = z_3 + z_6 = 0$$

7.2. Proposition. *For a general point q, we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{M}_{X_1}(\mathcal{O}(q)) & \xrightarrow{V} & \mathrm{M}_X \\ \downarrow \mathbb{L} & & \downarrow \cong \\ \mathbb{P}\Lambda^2 H^0(J_1, 2\Theta_1) & \xrightarrow{\tilde{V}} & \mathbb{P}H^0(J, 2\Theta) = \mathbb{P}^3 \end{array}$$

where $\tilde{V}$ is the projection with center $\mathbb{P}Z = \mathbb{P}^2$. In terms of canonical coordinates on both spaces, we have

$$\tilde{V}^*(x_{00}) = 0, \qquad \tilde{V}^*(x_{10}) = z_2 + z_5, \qquad \tilde{V}^*(x_{11}) = z_3 + z_6, \qquad \tilde{V}^*(x_{01}) = z_1 + z_4.$$

There exists a unique bundle in $\mathrm{M}_{X_1}(\mathcal{O}(q))$ which is destabilized by F, namely $F_*(B^{-1}(q))$.

Proof. Since the proof is in the same spirit as the proof of Proposition 6.1, we just give a sketch. Let D_q be the vector field on J_1 associated to q. We also denote by D_q the endomorphism of $H^0(J_1, 4\Theta_1)$ obtained via the canonical p-integrable connection ∇ (Remark 2.4). We observe that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^2 H^0(J_1, 2\Theta_1) & \longrightarrow & \Lambda^2 H^0(J_1, 2\Theta_1) \\ \downarrow m & & \downarrow W_{D_q} \\ H^0(J_1, 4\Theta_1) & \xrightarrow{D_q} & H^0(J_1, 4\Theta_1) \end{array} \tag{7.6}$$

The first horizontal map is the canonical projection and W_{D_q} is the Wahl map associated to D_q ([B] section A.10). We consider the map $J_1 \to \mathrm{M}_{X_1}(\mathcal{O}(q))$ defined in [B] section 3 and compose with $\mathbb{L}$. For q general, the composite is non-degenerate and the induced (injective) map on global sections coincides with W_{D_q} [B]. Using (7.6) we can now deduce the equations of $\tilde{V}$. The last assertion can be proved as in Proposition 6.1 (2). □

8 Frobenius dynamics

Let F_a be the absolute Frobenius map of X. We write $F_a^{(n)}$ for the n-fold composite $F_a \circ \cdots \circ F_a$. We will study the set of Frobenius semi-stable bundles, i.e.

$$\Omega^{Frob} := \{[E] \in \mathrm{M}_X \mid F_a^{(n)*}E \text{ semi-stable } \forall n \geq 1\}$$

and the set of bundles coming from representations of the algebraic fundamental group of X,i.e. (see [LS] Satz 1.4)

$$\Omega^{Rep} := \{[E] \in \mathrm{M}_X \mid \exists n > 0 \ \ F_a^{(n)*}E \xrightarrow{\sim} E\}$$

We obviously have $\Omega^{Rep} \subset \Omega^{Frob}$ and we show

8.1. Proposition. *The set Ω^{Frob} is Zariski dense in $\mathrm{M}_X = \mathbb{P}^3$.*

Proof. Since $F_a = \iota \circ F$, the action of F_a on $\mathrm{M}_X = \mathbb{P}^3$ factorizes as

$$F_a^* : \mathrm{M}_X \xrightarrow{i^*} \mathrm{M}_{X_1} \xrightarrow{V} \mathrm{M}_X$$

Since $i^*(x_{00} : x_{01} : x_{10} : x_{11}) = (x_{00}^2 : x_{01}^2 : x_{10}^2 : x_{11}^2)$, we see that in terms of the canonical Theta coordinates

$$F_a^*(x) = (\ldots, \lambda_g P_g^2(x), \ldots).$$

Let k_0 be the subfield of k generated by the constants $(\lambda_g)_{g \in G}$ and $k_0^{(n)}$, for $n \geq 1$, be the finite field extension of k_0 generated by the coordinates $(x_g)_{g \in G}$ such that $F_a^{*(n)}(x) = (1 : 1 : 1 : 1)$ with $x = (x_g)_{g \in G}$. We obviously have a tower of extensions

$$k_0 = k_0^{(1)} \subset k_0^{(2)} \subset \cdots \subset k_0^{(n-1)} \subset k_0^{(n)} \subset \cdots$$

and, by the computations carried out in the proof of Proposition 6.1(4), we see that deg $[k_0^{(n)} : k_0^{(n-1)}]$ is a power of 2. Hence, by induction, any element $x \in \mathbb{P}^3 \setminus \Omega^{Frob}$ has coodinates $(x_g)_{g \in G}$, which lie in an extension of k_0 of degree 2^m for some m. But elements of odd degree over k_0 are evidently dense in $\mathbb{P}^3$. □

8.2. Question. Is Ω^{Rep} Zariski dense?

9 list of questions

1. For higher genus curves we no longer have a simple description of M_X as for $g = 2$ (Proposition 5.1). But we can ask whether the diagram

$$\begin{array}{ccc} \mathrm{M}_{X_1} & \xrightarrow{V} & \mathrm{M}_X \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D} \\ |2\Theta_1| & \xrightarrow{\tilde{V}} & |2\Theta| \end{array} \tag{9.1}$$

where $\tilde{V}$ is defined as in Proposition 3.1, is commutative for X general. Note that the "restriction" of V to J_1 commutes. Do we have $\dim H^0(\mathrm{M}_X, \mathcal{L}^2) = 2^{g-1}(2^g+1)$? Indeed, we could check that the latter equality implies commutativity of (9.1).

2. A straightforward computation shows that, for any genus g, the map $\tilde{V}$ defined by the 2^g quadrics (3.2) surjects on the complement of the hyperplane $H:\ x_0 = 0$. A priori this is not sufficient to deduce that the map $V : \mathrm{M}_{X_1} \to \mathrm{M}_X$ surjects on the complement of the divisor $\tilde{\Theta} = D^*(H) \subset \mathrm{M}_X$ as for $g = 2$. We optimistically conjecture

   9.1. Conjecture. *For any semi-stable bundle $E \in \mathrm{M}_X$ satisfying $h^0(X, E \otimes B) = 0$, there exists a semi-stable bundle $E_1 \in \mathrm{M}_{X_1}$ such that $F^*E_1 = E$.*

   As in the proof of Proposition 6.4, this conjecture implies that $V : \mathrm{N}_{X_1} \to \mathrm{N}_X$ is surjective.

3. What happens for non-ordinary curves?

We plan to return to these questions in a future work.

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