THE WOBBLY DIVISORS OF THE MODULI SPACE OF RANK-2 VECTOR BUNDLES

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Abstract. Let X be a smooth projective complex curve of genus $g \geq 2$ and let $\mathrm{M}_X(2,\Lambda)$ be the moduli space of semi-stable rank-2 vector bundles over X with fixed determinant Λ. We show that the wobbly locus, i.e., the locus of semi-stable vector bundles admitting a non-zero nilpotent Higgs field is a union of divisors $\mathcal{W}_k \subset \mathrm{M}_X(2,\Lambda)$. We show that on one wobbly divisor the set of maximal subbundles is degenerate. We also compute the class of the divisors $\mathcal{W}_k$ in the Picard group of $\mathrm{M}_X(2,\Lambda)$.

1. Introduction

Let X be a smooth projective complex curve of genus $g \geq 2$ and let K be its canonical line bundle. Fixing a line bundle Λ we consider the coarse moduli space $\mathrm{M}_X(2,\Lambda)$ parameterizing semi-stable rank-2 vector bundles of fixed determinant Λ over X. In this note we study the locus in $\mathrm{M}_X(2,\Lambda)$ of wobbly, or non-very stable, vector bundles over X. We recall that a vector bundle E is called very stable if E has no non-zero nilpotent Higgs field $\phi \in \mathrm{H}^0(X, \mathrm{End}(E) \otimes K)$. Laumon [Lau, Proposition 3.5] proved, assuming $g \geq 2$, that a very stable vector bundle is stable and that the locus of very stable bundles is a non-empty open subset of $\mathrm{M}_X(2,\Lambda)$. Hence the locus of wobbly bundles is a closed subset

$$\mathcal{W} \subset \mathrm{M}_X(2,\Lambda).$$

It was announced in Laumon [Lau] Remarque 3.6 (ii) that $\mathcal{W}$ is of pure codimension 1. The term "wobbly" was introduced in the paper [DP].

Our first result proves this claim. Since the isomorphism class of the moduli space $\mathrm{M}_X(2,\Lambda)$ depends only on the parity of the degree $\lambda = \deg \Lambda$, it will be enough to study two cases, $\lambda = 0$ and $\lambda = 1$. For $1 \leq k \leq g - \lambda$ we define $\mathcal{W}_k$ to be the closure in $\mathrm{M}_X(2,\Lambda)$ of the locus of all semi-stable vector bundles arising as extensions

$$0 \longrightarrow L \longrightarrow E \longrightarrow \Lambda L^{-1} \longrightarrow 0,$$

with $\deg L = 1 - k$ and $\dim H^0(X, KL^2\Lambda^{-1}) > 0$. We denote by $\lceil x \rceil$ the ceiling of the real number x. With this notation we have the following results.

Theorem 1.1. *The wobbly locus $\mathcal{W} \subset \mathrm{M}_X(2,\Lambda)$ is of pure codimension 1 and we have the following decomposition for $\lambda = 0$ and $\lambda = 1$*

$$\mathcal{W} = \mathcal{W}_{\lceil \frac{g-\lambda}{2} \rceil} \cup \ldots \cup \mathcal{W}_{g-\lambda}.$$

In particular, all loci $\mathcal{W}_k$ appearing in the above decomposition are divisors. They are all irreducible, except $\mathcal{W}_g$ for $\lambda = 0$, which is the union of 2^{2g} irreducible divisors.

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This theorem completes the results obtained in [P] showing that $\mathcal{W}$ is of codimension 1 for $\lambda = 1$. The idea of the proof is to consider the rational forgetful map (forgetting the non-zero Higgs field) from the equidimensional nilpotent cone in the moduli space of semi-stable Higgs bundles to $\mathrm{M}_X(2,\Lambda)$. It turns out that roughly half of the irreducible components of the nilpotent cone gets contracted by the forgeful map with one-dimesional fibers to the above mentioned divisors $\mathcal{W}_k$, and the other half gets contracted with fibers of dimension > 1 to subvarieties of these disisors $\mathcal{W}_k$.

Our second result studies the relationship between very stable vector bundles E and the loci of maximal line subbundles of E. We recall here the main results on maximal line subbundles of rank-2 bundles (see e.g. [O], [LN]). Under the assumption that $g + \lambda$ is odd, i.e., g odd if $\lambda = 0$ and g even if $\lambda = 1$, the Quot-scheme

$$M(E) := \mathrm{Quot}^{1,1-\lceil \frac{g}{2} \rceil}(E)$$

parameterizing subsheaves of rank 1 and degree $1 - \lceil \frac{g}{2} \rceil$ of E is a zero-dimensional, reduced scheme of length 2^g for a *general* bundle $E \in \mathrm{M}_X(2,\Lambda)$. In that case $M(E)$ parameterizes line subbundles of E of maximal degree. We say that $M(E)$ is non-degenerate if $\dim M(E) = 0$ and $M(E)$ is reduced, and degenerate if the opposite holds.

Theorem 1.2. *Under the assumption that $g + \lambda$ is odd, the following holds.*

1. *If E is very stable, then $M(E)$ is non-degenerate.*
2. *The subscheme $M(E)$ is degenerate for any $E \in \mathcal{W}_{\lceil \frac{g-\lambda}{2} \rceil}$.*

The case $g = 2, \lambda = 1$ was already worked out in [P]. In Remark 3.1 we show that part (2) does not hold on other components of the wobbly locus.

Our last result computes the class $cl(\mathcal{W}_k)$ of the wobbly divisors $\mathcal{W}_k$ in the Picard group of the moduli space $\mathrm{M}_X(2,\Lambda)$, which is isomorphic to $\mathbb{Z}$ (see e.g. [DN]).

Theorem 1.3. *We have the following equality for $\lambda = 0$ and $\lambda = 1$*

$$cl(\mathcal{W}_k) = 2^{2k} \binom{g}{2g-2k-\lambda} \quad \text{for} \quad \left\lceil \frac{g-\lambda}{2} \right\rceil \leq k \leq g - \lambda.$$

In the case $\lambda = 0$ the computations of the class $cl(\mathcal{W}_k)$ were already carried out in [Fa], but due to some typos the final result in loc.cit. is not correct. For the convenience of the reader we include a detailed presentation of the computations in the case $\lambda = 1$.

In the last section we give a description of these divisors for low genus.

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2. Proof of Theorem 1.1

Let $\mathrm{Higgs}_X(2,\Lambda)$ be the moduli space of semi-stable Higgs bundles of rank 2 with fixed determinant Λ. The Hitchin map defined by mapping a Higgs field (E,ϕ) to its determinant $\det(\phi)$ is a proper surjective map

$$h : \mathrm{Higgs}_X(2,\Lambda) \to H^0(X, K^2).$$

It is easy to see that the nilpotent cone decomposes as

$$h^{-1}(0) = M_X(2,\Lambda) \cup \tilde{\mathcal{E}},$$

where $\mathrm{M}_X(2,\Lambda)$ denotes here pairs $(E,0)$ with zero Higgs field and $\tilde{\mathcal{E}}$ consists of semi-stable pairs (E,ϕ) with non-zero nilpotent Higgs field — note that the underlying bundle E is not necessarily semi-stable. In other words, the image of $\tilde{\mathcal{E}}$ under the rational forgetful map $h^{-1}(0) \dashrightarrow M_X(2,\Lambda)$ is the locus of wobbly bundles. In the case $\lambda = 1$ the nilpotent cone was already described in [T]. For the convenience of the reader we recall now the description.

By [P] Lemma 3.1 a vector bundle E admits a non-zero nilpotent Higgs field if and only if it contains a line subbundle L with $H^0(X, KL^2\Lambda^{-1}) \neq 0$. Thus any such bundle can be written as an extension

$$0 \to L \xrightarrow{i} E \xrightarrow{\pi} L^{-1}\Lambda \to 0, \tag{2.1}$$

where Λ is a line bundle of degree $\lambda \in \{0,1\}$ and the nilpotent Higgs field is given as the composition $\phi = i \circ u \circ \pi$ with a non-zero $u \in H^0(X, KL^2\Lambda^{-1}) = \mathrm{Hom}(L^{-1}\Lambda, LK)$. Then we have the following inequalities :

- Since $\phi(L) = 0, L$ is invariant under ϕ and by semi-stability of the pair (E,ϕ), we have $\deg(L) = d \leq \frac{\lambda}{2}$.

- We also have $u \neq 0$. This implies that $\deg(KL^2\Lambda^{-1}) \geq 0 \Leftrightarrow d \geq \frac{\lambda}{2} + 1 - g$.

Hence we obtain the inequalities

$$\frac{\lambda}{2} + 1 - g \leq d \leq \frac{\lambda}{2}.$$

We set $k = 1 - d$ and we distinguish two cases:

- $\lambda = 0 : 1 - g \leq d \leq 0 \Leftrightarrow 1 \leq k \leq g$.

- $\lambda = 1 : \frac{1}{2} + 1 - g \leq d \leq \frac{1}{2} \Leftrightarrow 1 \leq k \leq g - 1$.

We introduce the subloci

$$\mathcal{W}_k^0 := \{E \in \mathcal{W} : E \text{ contains a line subbundle } L \text{ of degree 1-k} \text{ with } H^0(X, KL^2\Lambda^{-1}) \neq 0\}$$

and denote by $\mathcal{W}_k$ the Zariski closure of $\mathcal{W}_k^0$ in $\mathrm{M}_X(2,\Lambda)$. We therefore deduce from the above considerations the following decompositions $\mathcal{W} = \bigcup_{k=1}^{g} \mathcal{W}_k$ for $\lambda = 0$ and $\mathcal{W} = \bigcup_{k=1}^{g-1} \mathcal{W}_k$ for $\lambda = 1$.

Remark 2.1. We observe that for $\lambda = 0$ the locus $\mathcal{W}_1$ coincides with the semi-stable boundary of $\mathrm{M}_X(2,\Lambda)$, which equals the Kummer variety of X.

Now we decompose $\tilde{\mathcal{E}}$ as $\bigcup_{k=1}^{g} \mathcal{E}_k$ for $\lambda = 0$ and $\bigcup_{k=1}^{g-1} \mathcal{E}_k$ for $\lambda = 1$ such that the image of $\mathcal{E}_k$ under the forgetful map is $\mathcal{W}_k$. The construction goes as follows (we omit the construction of $\mathcal{E}_1$ for $\lambda = 0$ — see Remark 2.1) :
We introduce the subvarieties $Z_k \subset \mathrm{Pic}^{1-k}(X)$ for $1 \leq k \leq g - \lambda$ defined by

$$Z_k := \{L \in \mathrm{Pic}^{1-k}(X) \text{ such that } h^0(X, KL^2\Lambda^{-1}) \neq 0\}.$$

Then one can construct Z_k as the pre-image of the Brill-Noether locus $W_{2g-2k-\lambda}(X)$ under the map

$$\mu_k : \mathrm{Pic}^{1-k}(X) \to \mathrm{Pic}^{2g-2k-\lambda}(X)$$

taking L to $KL^2\Lambda^{-1}$. Then

$$\begin{aligned} \dim Z_k &= 2g - 2k - \lambda && \text{if } 2g - 2k - \lambda \leq g \\ &= g && \text{if } 2g - 2k - \lambda \geq g. \end{aligned}$$

Note that in the latter case $Z_k = \mathrm{Pic}^{1-k}(X)$. Consider the fiber product $\tilde{Z}_k = Z_k \times_{W_{2g-2k-\lambda}} S^{2g-2k-\lambda}(X)$

$$\begin{array}{ccc} \tilde{Z}_k & \longrightarrow & S^{2g-2k-\lambda}(X) \\ \downarrow{\scriptstyle q} & & \downarrow \\ Z_k & \xrightarrow{\mu_k} & W_{2g-2k-\lambda} \end{array} \tag{2.2}$$

where the right vertical map is the natural map from the symmetric product of the curve to its Picard variety. Then the projection map $p : \tilde{Z}_k \to S^{2g-2k-\lambda}(X)$ is a 2^{2g}-fold étale covering of $S^{2g-2k-\lambda}(X)$ and $\tilde{Z}_k$ parameterizes line bundles L and effective divisors in $|KL^2\Lambda^{-1}|$. There exists a unique line bundle $\mathcal{L}$ over $S^{2g-2k-\lambda}(X)$ whose fiber at a divisor D is canonically isomorphic to the space of sections of the line bundle determined by D which vanish precisely on D. Furthermore, it can be shown that this line bundle is trivial.

Excluding the case $\lambda = 0$ and $k = 1$ (see Remark 2.1), we observe that the dimension of $\mathrm{Ext}^1(\Lambda L^{-1}, L) = H^1(\Lambda^{-1}L^2)$ depends only on the degree of the line bundle L. Therefore there exists a vector bundle $\mathcal{V}_k$ over Z_k whose fiber at a point $L \in Z_k$ is canonically isomorphic to $\mathrm{Ext}^1(\Lambda L^{-1}, L)$. The rank of the vector bundle $\mathcal{V}_k$ is $g + 2k + \lambda - 3$ and a general extension class v in the fiber $(\mathcal{V}_k)_L$ defines a stable rank-2 vector bundle E_v.

Proposition 2.2. *We have the following :*

1. *The total space of the vector bundle $q^*\mathcal{V}_k \oplus p^*\mathcal{L}$ over $\tilde{Z}_k$ parameterizes triples (L, v, u), where L is a line bundle in Z_k, v is an extension class in the fiber $(\mathcal{V}_k)_L$ and u is a global section of $KL^2\Lambda^{-1}$.*
2. *There exists a rational map ϕ_k from the projectivized bundle $\mathbb{P}(q^*\mathcal{V}_k \oplus p^*\mathcal{L})$ to $\mathrm{Higgs}_X(2, \Lambda)$ defined by sending (L, v, u) to the Higgs bundle $(E_v, i \circ u \circ \pi)$ as defined by the exact sequence (2.1).*
3. *We have a commutative diagram*

$$\begin{array}{ccc} \mathbb{P}(q^*\mathcal{V}_k \oplus p^*\mathcal{L}) & \xrightarrow{\phi_k} & \mathrm{Higgs}_X(2, \Lambda) \\ \downarrow & & \downarrow \\ \mathbb{P}(\mathcal{V}_k) & \xrightarrow{\psi_k} & \mathrm{M}_X(2, \Lambda), \end{array} \tag{2.3}$$

   where all arrows are rational maps. The vertical maps are forgetful maps of the global section of $KL^2\Lambda^{-1}$ and of the Higgs field respectively.
4. *The restriction of ϕ_k to the vector bundle $q^*\mathcal{V}_k \subset \mathbb{P}(q^*\mathcal{V}_k \oplus p^*\mathcal{L})$ is an injective morphism.*

Proof. Part (1) follows immediately from the previous description of $\mathcal{V}_k$ and $\mathcal{L}$. As for part (2) it will be enough to show that the Higgs bundle associated to the triple $(L, \lambda v, \lambda u)$ does not depend on the scalar $\lambda \in \mathbb{C}^*$. But this follows from the observation that the extension class of the exact sequence obtained from (2.1) by replacing either i by $\frac{1}{\lambda}i$ or π by $\frac{1}{\lambda}\pi$ equals $\lambda v \in \mathrm{Ext}^1(\Lambda L^{-1}, L)$ for any $\lambda \in \mathbb{C}^*$. Part (3) and part (4) are straightforward. □

Now we define $\mathcal{E}_k^0$ to be the image of the rational map ϕ_k and $\mathcal{E}_k$ its Zariski closure. Then clearly $\mathcal{E}_k \subset \mathcal{E}$ and $\mathcal{W}_k^0$ is the image of $\mathcal{E}_k^0$ under the forgetful map. Clearly $\mathcal{E}_k$ is irreducible, except if $\dim Z_k = 0$ which is equivalent to $\lambda = 0$ and

$k = g$, and since ϕ_k is generically injective

$$\begin{aligned} \dim \mathcal{E}_k &= \dim \tilde{Z}_k + \mathrm{rk}\ \mathcal{V}_k \\ &= (2g - 2k - \lambda) + (g + 2k + \lambda - 3) \\ &= 3g - 3. \end{aligned}$$

We note that the fiber over a general element $E \in \mathcal{W}_k^0$ of the forgetful map $\mathcal{E}_k^0 \to \mathcal{W}_k^0$ is $H^0(X, KL^2\Lambda^{-1})$, where L is a line bundle of degree $-k+1$ contained in E. If $2g - 2k - \lambda \leq g$ and L is a general line bundle of degree $-k+1$ with $H^0(X, KL^2\Lambda^{-1}) \neq 0$, then $h^0(X, KL^2\Lambda^{-1}) = 1$. Therefore $\dim \mathcal{W}_k^0 = 3g - 3 - 1 = 3g - 4$ for $2g - 2k - \lambda \leq g \Leftrightarrow \frac{g-\lambda}{2} \leq k$. Thus $\mathcal{W}_k$ is an irreducible divisor in $M_X(2, \Lambda)$ for $k \geq \frac{g-\lambda}{2}$ and $k \neq g$.

Proposition 2.3. *We have the inclusions*

$$\bigcup_{k=1}^{\lceil \frac{g-\lambda}{2} \rceil - 1} \mathcal{W}_k \subset \mathcal{W}_{\lceil \frac{g-\lambda}{2} \rceil}.$$

Proof. We put $k_0 = \left\lceil \frac{g-\lambda}{2} \right\rceil$ and consider the rational map

$$\psi_{k_0} : \mathbb{P}(\mathcal{V}_{k_0}) \dashrightarrow \mathrm{M}_X(2, \Lambda)$$

introduced in Proposition 2.2 (3). Let $L \in Z_{k_0}$. The restriction of ψ_{k_0} on the fiber of $\mathbb{P}(\mathcal{V}_{k_0})_L$ over L is not defined at the points where the associated bundles are not semi-stable. Let

$$\psi_L : \mathbb{P}_L := \mathbb{P}(H^1(X, L^2\Lambda^{-1})) \dashrightarrow \mathrm{M}_X(2, \Lambda)$$

be the restriction of ψ_{k_0} at the fiber over L. Then by [B] Theorem 1 there is a natural sequence σ of blow-ups along smooth centers resolving ψ_L into a morphism $\tilde{\psi}_L : \tilde{\mathbb{P}}_L \to M_X(2, \Lambda)$. The image of $\tilde{\psi}_L$ is contained in closure of the image of the rational map ψ_L. Here σ is the blow-up morphism $\sigma : \tilde{\mathbb{P}}_L \to \mathbb{P}_L$.

Now X is embedded in $\mathbb{P}_L$ via the natural map. Let $x \in X$ and E be the bundle associated to x. Then E fits into the exact sequence (2.1) and by [B] Observation (2) page 451 the bundle E is not semi-stable. Furthermore by [B] Theorem 1 (2) there is a natural isomorphism $\sigma^{-1}(x) \cong \tilde{\mathbb{P}}_{L(x)}$ and, when restricted to $\sigma^{-1}(x)$, the morphism $\tilde{\psi}_L$ coincides with the morphism

$$\tilde{\psi}_{L(x)} : \tilde{\mathbb{P}}_{L(x)} \to \mathrm{M}_X(2, \Lambda).$$

Now $\tilde{\mathbb{P}}_{L(x)}$ is the blow-up of $\mathbb{P}_{L(x)}$ and the bundles corresponding to extension classes in $\mathbb{P}_{L(x)}$ fit in the exact sequence of the form

$$0 \to L(x) \to V \to L^{-1}(-x)\Lambda \to 0.$$

Hence we deduce that the image of $\tilde{\psi}_{L(x)}$ is contained in $\mathcal{W}_{k_0}$. Next we observe that if $L \in Z_{k_0}$, then $L(x) \in Z_{k_0 - 1}$ for any $x \in X$. Hence we obtain a morphism

$$\mu : Z_{k_0} \times X \longrightarrow Z_{k_0-1}, \quad \mu(L, x) = L(x).$$

It will be enough to show that μ is surjective to conclude that $\mathcal{W}_{k_0-1}^0 \subset \mathcal{W}_{k_0}$, hence $\mathcal{W}_{k_0-1} \subset \mathcal{W}_{k_0}$ since $\mathcal{W}_{k_0}$ is closed.

If $g - \lambda$ is even, then $Z_{k_0-1} = \mathrm{Pic}^{2-k_0}(X)$ et $Z_{k_0} = \mathrm{Pic}^{1-k_0}(X)$ and μ is obviously surjective. If $g - \lambda$ is odd, then $Z_{k_0-1} = \mathrm{Pic}^{2-k_0}(X)$ and Z_{k_0} is an irreducible divisor in $\mathrm{Pic}^{1-k_0}(X)$. If on the contrary μ is not surjective, then Z_{k_0} would be invariant by a translation by an line bundle of the form $\mathcal{O}_X(x - y)$ for $x, y \in X$, hence by any translation. This is a contradiction.

More generally let D be a general effective divisor of degree $1 \leq d \leq k_0 - 1$. Then again by [B] Observation 2 any point $x \in \overline{D}$ corresponds to a non-semi-stable bundle, except if $\lambda = 0$ and $d = k_0 - 1$ (see below for a discussion of this exceptional case). Furthermore if x is general in $\overline{D}$ then by [B] Theorem 1 (2) there is a natural isomorphism $\sigma^{-1}(x) \cong \tilde{\mathbb{P}}_{L(D)}$ and the restriction of $\tilde{\psi}_L$ to $\sigma^{-1}(x)$ coincides with the map

$$\tilde{\psi}_{L(D)} : \tilde{\mathbb{P}}_{L(D)} \to M_X(2, \Lambda).$$

As before, the natural multiplication morphism

$$\mu : Z_{k_0} \times S^d(X) \longrightarrow Z_{k_0 - d}$$

is easily seen to be surjective, which implies that $\mathcal{W}^0_{k_0-d} \subset \mathcal{W}_{k_0}$, hence $\mathcal{W}_{k_0-d} \subset \mathcal{W}_{k_0}$ since $\mathcal{W}_{k_0}$ is closed.

Finally, the case $\lambda = 0$ and $d = k_0 - 1$ corresponds to $\mathcal{W}_1$, which is the image of the $(k_0 - 1)$-th secant variety to $X \subset \mathbb{P}_L$ under the rational map ψ_L, when L varies in Z_{k_0}. □

We will need the following lemma in section 4.

Lemma 2.4. *With the notation of Proposition 2.2 and for* $\left\lceil \frac{g-\lambda}{2} \right\rceil \leq k \leq g - \lambda$, *a* general *line bundle* $L \in Z_k$ *and a* general *extension class* $v \in (\mathcal{V}_k)_L = \mathrm{Ext}^1(\Lambda L^{-1}, L)$ *defining a bundle* E_v *as in (2.1), we have*

$$\dim \mathrm{Hom}(E_v, L^{-1}\Lambda) = 1.$$

Proof. Since L is general in Z_k we have $h^0(KL^2\Lambda^{-1}) = 1$, or equivalently by Riemann-Roch and Serre duality $h^1(KL^2\Lambda^{-1}) = h^0(L^{-2}\Lambda) = \lambda + 2k - g$. Applying the functor $\mathrm{Hom}(-, L^{-1}\Lambda)$ to the short exact sequence (2.1) we see that $\dim \mathrm{Hom}(E_v, L^{-1}\Lambda) = 1$ if and only if the coboundary map given by the cup product $\cup v$ with the extension class v

$$\cup v : H^0(L^{-2}\Lambda) = \mathrm{Hom}(L, L^{-1}\Lambda) \longrightarrow H^1(\mathcal{O}_X) = \mathrm{Ext}^1(L^{-1}\Lambda, L^{-1}\Lambda)$$

is injective. Given a non-zero section $s \in H^0(L^2\Lambda)$ it is well-known that $s \cup v = 0$ if and only if the extension class $v \in \mathrm{Ext}^1(\Lambda L^{-1}, L) = H^1(L^2\Lambda^{-1}) = H^0(KL^{-2}\Lambda)^*$ lies in the linear span $\langle D \rangle \subset |KL^{-2}\Lambda|^*$, where D is the zero divisor of s. But $\dim\langle D \rangle = 2k - 4 + \lambda$, so

$$\dim \bigcup_{D \in |L^{-2}\Lambda|} \langle D \rangle \leq (\lambda + 2k - g - 1) + (2k - 4 + \lambda) = 4k - g + 2\lambda - 5,$$

which is $< \dim \mathbb{P}(\mathcal{V}_k)_L = g + 2k + \lambda - 4$. So for a general extension class v we see that $s \cup v \neq 0$ for any non-zero $s \in H^0(L^{-2}\Lambda)$, which is equivalent to $\dim \mathrm{Hom}(E_v, L^{-1}\Lambda) = 1$. □

3. Proof of Theorem 1.2

We only consider the case when $\lambda = 1$ and g is even, i.e., $g = 2a$ for some integer a. The proof in the other case can be carried out similarly. If $g = 2a$, then $\left\lceil \frac{g-1}{2} \right\rceil = a$. In this situation, a general rank-2 vector bundle of degree 1 has a line subbundle of maximal degree $1 - a$ (see e.g.[O], [LN]).

Proof of (1): Let E be a very stable rank-2 vector bundle of degree 1. Suppose on the contrary that dim $M(E) > 0$ or $M(E)$ is non-reduced. Let $L_0 \in M(E)$. If $L_0 \to E$ is not saturated, then we have a sequence of maps

$$L_0 \to L \to E \to L^{-1}\Lambda \to L_0^{-1}\Lambda,$$

where $\deg L \geq \deg L_0 + 1 = 2 - a$. Then $\chi(L^{-2}\Lambda) \leq -2$. Therefore $h^1(X, L^{-2}\Lambda) = h^0(X, KL^2\Lambda^{-1}) > 0$, which implies that E contains a line subbundle L with

$h^0(X, KL^2\Lambda^{-1}) \neq 0$. Then by [P] Lemma 3.1 the bundle E is not very stable, a contradiction.

On the other hand, if $L_0 \to E$ is saturated, then E fits in the exact sequence

$$0 \to L_0 \to E \to L_0^{-1}\Lambda \to 0.$$

Note that the Zariski tangent space at L_0 is given by $\mathrm{Hom}(L_0, L_0^{-1}\Lambda)$. Therefore if $\dim M(E) \geq 1$ or if L_0 is a non-reduced point in $M(E)$, then $\dim \mathrm{Hom}(L_0, L_0^{-1}\Lambda) > 0$. But $\chi(L_0^{-2}\Lambda) = 0$. Thus if $\dim \mathrm{Hom}(L_0, L_0^{-1}\Lambda) > 0$, then $h^1(X, L_0^{-2}\Lambda) = h^0(X, KL^2\Lambda) > 0$. Therefore E is not very stable, a contradiction.

Proof of (2): Let $E \in \mathcal{W}_a^0$. Then E contains a line subbundle L of degree $1-a$ such that $h^0(X, KL^2\Lambda^{-1}) \neq 0$ and we have the exact sequence (2.1). Since $\chi(L^{-2}\Lambda) = 0$, we obtain that $h^0(X, L^{-2}\Lambda) = h^0(X, KL^2\Lambda^{-1}) > 0$. Therefore the dimension of the tangent space at L to the Quot-scheme $M(E)$ is $h^0(X, L^{-2}\Lambda) > 0$. Therefore $M(E)$ is degenerate at L. Finally, since being degenerate is a closed condition, $M(E)$ is degenerate for any $E \in \mathcal{W}_a$.

Remark 3.1. The statement in Theorem 1.2 (2) is not valid for the points in the other components $\mathcal{W}_k$ for $k > \left\lceil \frac{g-\lambda}{2} \right\rceil$.

Example: Take $\lambda = 0$ and $g = 2a+1 = 3$, i.e., $a = 1$. For simplicity we assume that the curve X is non-hyperelliptic. Then the wobbly locus has two components, $\mathcal{W}_2$ and $\mathcal{W}_3$, where $\mathcal{W}_2$ is an irreducible divisor, but $\mathcal{W}_3$ is a union of 64 hyperplane sections. The 64 hyperplane sections are indexed by the 64 theta-characteristics θ, i.e. line bundles satisfying $\theta^2 = K$. We claim that a general extension class $\xi \in \mathbb{P}(H^1(X, \theta^{-2})) \dashrightarrow \mathcal{W}_3 \subset M_X(2, \mathcal{O})$ is such that $M(E_\xi)$ is non-degenerate. Note that $\mathbb{P}(H^1(X, \theta^{-2})) = \mathbb{P}^5 = \mathbb{P}(H^0(X, K^2)^*) = |K^2|^*$ contains the following secant varieties to the curve $X \hookrightarrow |K^2|^*$

$$X = \mathrm{Sec}^1(X) \hookrightarrow |K^2|^*, \quad \mathrm{Sec}^2(X) \hookrightarrow |K^2|^*,$$

which are of dimension 1 and 3 respectively, and we have an equality (see e.g. [Lan])

$$\mathrm{Sec}^3(X) = |K^2|^*. \tag{3.1}$$

It is well-known, see e.g. [LN] Proposition 1.1, that an extension class $\xi \in |K^2|^*$ lies on $\mathrm{Sec}^i(X)$ for $1 \leq i \leq 3$ if and only if the associated vector bundle E_ξ contains a line subbundle of degree $2-i$. Let $\xi \in \mathrm{Sec}^3(X) \setminus \mathrm{Sec}^2(X)$. Then the maximal degree of line subbundles of E_ξ is -1. Let $D = x_1 + x_2 + x_3$ be a general effective divisor on X of degree 3 such that $\xi \in \langle D \rangle$, the linear span of three points x_1, x_2, x_3 of D in $|K^2|^*$. Then E_ξ contains the line subbundle $\theta(-D)$. Note that E_ξ is an extension

$$0 \to \theta^{-1} \to E_\xi \to \theta \to 0.$$

We also note that $L = \theta(-D)$ is a reduced point of the Quot-scheme $M(E_\xi)$ if and only if $\mathrm{Hom}(L, E_\xi/L) = \mathrm{Hom}(L, L^{-1}) = H^0(X, L^{-2}) = \{0\}$.

Now consider the map

$$\Phi : S^3(X) \to \mathrm{Pic}^2(X)$$

which takes a point (x_1, x_2, x_3) to $\mathcal{O}(2x_1 + 2x_2 + 2x_3) \otimes \theta^{-2}$. Clearly the map Φ is surjective. Let Z denote the divisor $\Phi^{-1}(\Theta)$, where Θ denotes the theta divisor in $\mathrm{Pic}^2(X)$. Then $D \in Z$ if and only if $h^0(X, \mathcal{O}(2D) \otimes \theta^{-2}) \neq 0$. We denote by $\tilde{Z}$ the span of all projective planes $\langle D \rangle \subset |K^2|^*$ when D varies in Z. Then $\tilde{Z}$ is a divisor in $|K^2|^*$ and we have $\xi \in \tilde{Z}$ if and only if E_ξ contains a line subbundle L such that $H^0(X, L^{-2}) \neq \{0\}$. So for general $\xi \notin \tilde{Z}$ the set $M(E_\xi)$ is reduced and consists of 8 line subbundles.

A similar computation can be done for $\lambda = 1$ and $g = 4$ to show that the statement in Theorem 1.2 (2) is not valid for the points in the component $\mathcal{W}_3$.

4. Proof of Theorem 1.3

In this section we compute the class $cl(\mathcal{W}_k)$ of the wobbly divisor $\mathcal{W}_k$ in the case $\lambda = 1$ for $\lceil \frac{g-1}{2} \rceil \leq k \leq g-1$ following closely the method used in [Fa] Section 5 Example 1. Note that in [Fa] the case $\lambda = 0$ is worked out.

Let S be a smooth connected variety and let $\mathcal{E}$ be a rank-2 vector bundle over $S \times X$ such that $\det \mathcal{E} = \pi_X^*(\Lambda)$, where π_S and π_X denote the projections onto S and X respectively, and such that $\mathcal{E}_s := \mathcal{E}_{|\{s\} \times X}$ is stable for any $s \in S$. Then the family $\mathcal{E}$ determines a classifying map

$$f : S \longrightarrow \mathrm{M}_X(2, \Lambda).$$

Our first task is to compute the first Chern class of the pull-back under f of the ample generator $\mathcal{D}$ of the Picard group of $\mathrm{M}_X(2, \Lambda)$, i.e.,

$$\Theta_S := c_1(f^*\mathcal{D}) \in H^2(S),$$

in terms of Chern classes of $\mathcal{E}$. We recall [DN] Théorème B that the line bundle $f^*\mathcal{D}$ is defined as the inverse of the determinant line bundle

$$\det R\pi_{S*}(\mathcal{E} \otimes \pi_X^* H),$$

where H is a rank-2 vector bundle of degree $2g-3$. Note that the condition on the degree is equivalent to $\chi(\mathcal{E}_s \otimes H) = 0$. Then the Grothendieck-Riemann-Roch theorem gives the equalities

$$\begin{aligned} \Theta_S &= -c_1(\det R\pi_{S*}(\mathcal{E} \otimes \pi_X^* H)) \\ &= -\tfrac{1}{2}\pi_{S*}\left[c_1(\mathcal{E} \otimes \pi_X^* H)^2 - 2c_2(\mathcal{E} \otimes \pi_X^* H) - c_1(\mathcal{E} \otimes \pi_X^* H) \cdot \pi_X^*(c_1(K))\right] \\ &= \pi_{S*}c_2(\mathcal{E} \otimes \pi_X^* H) \\ &= 2\pi_{S*}c_2(\mathcal{E}) \in H^2(S). \end{aligned}$$

Since we need to compute the class in $H^2(S)$ of the k-th wobbly divisor $f^{-1}(\mathcal{W}_k) \subset S$ in terms of Θ_S, it will be enough to do the computations modulo classes in $H^i(S)$ for $i \geq 3$. Hence the above relation allows to write

$$c_2(\mathcal{E}) = \frac{1}{2}\Theta_S \otimes \eta \in H^4(S \times X),$$

where $\eta \in H^2(X)$ denotes the class of a point in X — note that we omit classes in $H^4(S) \otimes H^0(X)$ and $H^3(S) \otimes H^1(X)$. Since $c_1(\mathcal{E}) = 1 \otimes \eta \in H^0(S) \otimes H^2(X) \subset H^2(S \times X)$ we get the following expression for the Chern character of $\mathcal{E}$

$$ch(\mathcal{E}) = 2 + 1 \otimes \eta - \frac{1}{2}\Theta_S \otimes \eta + h.o.t.,$$

where all h.o.t. are contained in $\oplus_{i \geq 3} H^i(S) \otimes H^*(X)$.

We also need to recall some standard facts on the first Chern class of a Poincaré bundle $\mathcal{L}$ over $P \times X$, with $P := \mathrm{Pic}^{1-k}(X)$ (see e.g. [ACGH] page 335). We have

$$c_1(\mathcal{L}) = (1-k)1 \otimes \eta + \gamma \in H^2(P \times X),$$

where γ denotes a class in $H^1(P) \otimes H^1(X)$ — for a more precise description of γ see [ACGH] — with the property

$$\gamma^2 = -2\Theta_P \otimes \eta.$$

Here $\Theta_P \in H^2(P)$ denotes the class of a theta divisor in P. The rest of the computations goes exactly as in the case $\lambda = 0$. For the convenience of the reader we include the details.

The main idea is to realize the k-th wobbly divisor

$$f^{-1}(\mathcal{W}_k) = \pi_S(\Delta_k \cap (S \times Z_k)) \subset S$$

as the projection onto S of the intersection of $S \times Z_k$ with the determinantal subvariety $\Delta_k \subset S \times P$ defined by

$$\Delta_k = \{(s, L) \in S \times P \mid \mathrm{Hom}(\mathcal{E}_s, L^{-1}\Lambda) \neq 0\},$$

and which is constructed by the standard technique as follows. We fix a reduced divisor D_0 of degree d_0 sufficiently large such that $h^1(X, Hom(\mathcal{E}_s, L^{-1}\Lambda(D_0))) = 0$ for all $s \in S$ and $L \in P$. We consider the exact sequence over $S \times P \times X$

$$0 \to Hom(\mathcal{E}, \mathcal{L}^{-1}\Lambda) \to Hom(\mathcal{E}, \mathcal{L}^{-1}\Lambda(D_0)) \to Hom(\mathcal{E}, \mathcal{L}^{-1}\Lambda(D_0))_{|D_0} \to 0.$$

We introduce the following two vector bundles over $S \times P$

$$\mathcal{F} := (\pi_{S\times P})_* \left(Hom(\mathcal{E}, \mathcal{L}^{-1}\Lambda(D_0))\right) \text{ and } \mathcal{A} := \bigoplus_{x \in D_0} Hom(\mathcal{E}, \mathcal{L}^{-1})_{|S\times P \times \{x\}}$$

of ranks $2(d_0 + k - g) + 1$ and $2d_0$ respectively. Taking the direct image of the above exact sequence under the projection $\pi_{S\times P}$ onto $S \times P$ we obtain a map $\phi : \mathcal{F} \to \mathcal{A}$ over $S \times P$. Let us denote by $q : \mathbb{P}(\mathcal{F}) \to S \times P$ the projection from the projectivized bundle $\mathbb{P}(\mathcal{F})$ onto the base variety $S \times P$. Then the composition of the tautological section over $\mathbb{P}(\mathcal{F})$ with $q^*\phi$

$$\mathcal{O}(-1) \to q^*\mathcal{F} \to q^*\mathcal{A}$$

defines a global section $s \in H^0(\mathbb{P}(\mathcal{F}), q^*\mathcal{A} \otimes \mathcal{O}(1))$ whose zero set equals

$$\tilde{\Delta}_k = \{(s, L, \overline{\varphi}) \mid (s, L) \in \Delta_k \text{ and } \overline{\varphi} \in \mathbb{P}(\mathrm{Hom}(\mathcal{E}_s, L^{-1}\Lambda))\}.$$

By Lemma 2.4 the map $\tilde{\Delta}_k \cap q^{-1}(S \times Z_k) \to \Delta_k \cap (S \times Z_k)$ induced by the projection q is birational, which implies that

$$\dim \tilde{\Delta}_k \cap q^{-1}(S \times Z_k) = \dim \Delta_k \cap (S \times Z_k) = \dim S - 1.$$

Hence

$$\dim \tilde{\Delta}_k \leq \dim S - 1 + \mathrm{codim}\ Z_k = \dim \mathbb{P}(\mathcal{F}) - 2d_0,$$

which shows that codim $\tilde{\Delta}_k = 2d_0$. Hence we can conclude that its fundamental class is given by the top Chern class

$$[\tilde{\Delta}_k] = c_{2d_0}(q^*\mathcal{A} \otimes \mathcal{O}(1)).$$

Moreover, by the projection formula we have

$$\begin{aligned} [\Delta_k] &= q_*[\tilde{\Delta}_k] \\ &= \sum_{i=0}^{2d_0} q_* \left(c_i(q^*\mathcal{A}) c_1(\mathcal{O}(1))^{2d_0 - i}\right) \\ &= \sum_{i=0}^{2d_0} c_i(\mathcal{A}) q_* \left(c_1(\mathcal{O}(1))^{2d_0 - i}\right) \\ &= q_* \left(c_1(\mathcal{O}(1))^{2d_0}\right) \ \mathrm{mod}\ H^i(S)\ i \geq 3. \end{aligned}$$

The last equality follows from the facts that

$$c_1(Hom(\mathcal{E}, \mathcal{L}^{-1})_{|S\times P\times\{x\}}) = 0 \text{ and } c_2(Hom(\mathcal{E}, \mathcal{L}^{-1})_{|S\times P\times\{x\}}) \in H^4(S) \otimes H^0(P),$$

which imply that

$$c_1(\mathcal{A}) = 0 \text{ and } c_k(\mathcal{A}) \in H^{2k}(S) \otimes H^0(P) \text{ for } k \geq 2.$$

In order to compute the class $q_*\left(c_1(\mathcal{O}(1))^{2d_0}\right)$ we compute by the Grothendieck-Riemann-Roch theorem the first terms of the Chern character of $\mathcal{F}$.

$$\begin{aligned} ch(\mathcal{F}) &= (\pi_{S\times P})_*\left(ch(\mathcal{E}^*)\cdot ch(\mathcal{L}^{-1}\Lambda(D_0))\cdot Td(X)\right) \\ &= (\pi_{S\times P})_*\big((2-1\otimes\eta-\tfrac{1}{2}\Theta_S\otimes\eta+h.o.t)\cdot(1+(d_0+k)1\otimes\eta+\gamma \\ &\quad -\Theta_P\otimes\eta)\cdot(1+(1-g)1\otimes\eta) \\ &= (\pi_{S\times P})_*(2+(2d_0+2k-2g+1)1\otimes\eta+2\gamma-2\Theta_P\otimes\eta-\tfrac{1}{2}\Theta_S\otimes\eta \\ &\quad +h.o.t.) \\ &= r-\tfrac{1}{2}\Theta_S-2\Theta_P+h.o.t., \end{aligned}$$

where all h.o.t. of the last line are contained in $\oplus_{i\geq 3}H^i(S)\otimes H^*(P)$ and $r = 2(d_0+k-g)+1$ denotes the rank of $\mathcal{F}$. So we obtain

$$c_1(\mathcal{F}) = -(\frac{1}{2}\Theta_S+2\Theta_P)$$

and working modulo the ideal $\oplus_{i\geq 3}H^i(S)\otimes H^*(P)$ we easily show the following relations for $k\geq 1$

$$c_k(\mathcal{F}) = \frac{1}{k!}(c_1(\mathcal{F}))^k \bmod H^i(S)\ i\geq 3.$$

Hence we can write the Chern polynomial of $\mathcal{F}$ as

$$c_t(\mathcal{F}) = 1+c_1(\mathcal{F})t+\cdots+c_r(\mathcal{F})t^r = \exp(c_1(\mathcal{F})t).$$

The class $q_*\left(c_1(\mathcal{O}(1))^{2d_0}\right)$ is by definition (see e.g. [Fu] Chapter 3) the $(2d_0-r+1)$-th Segre class of $\mathcal{F}$ and is computed as the coefficient of t^{2d_0-r+1} of the inverse of the Chern polynomial

$$c_t(\mathcal{F})^{-1} = \exp(-c_1(\mathcal{F})t),$$

which equals

$$\frac{(\frac{1}{2}\Theta_S+2\Theta_P)^{e+1}}{(e+1)!},$$

where we put $e = 2d_0-r = 2g-2k-1 = \dim Z_k$. We only need to compute the component in $H^2(S)\otimes H^{2e}(P)$ of this class, which equals

$$\frac{(e+1)\frac{1}{2}\Theta_S\otimes 2^e\Theta_P^e}{(e+1)!} = \Theta_S\otimes\frac{2^{e-1}}{e!}\Theta_P^e.$$

In order to conclude we will need the following fact.

Lemma 4.1. *The fundamental class of Z_k in P equals*

$$[Z_k] = \frac{2^{2(g-e)}}{(g-e)!}\Theta_P^{g-e}.$$

Proof. We recall that the duplication map of an abelian variety A acts as multiplication by 2^n on the cohomology $H^n(A,\mathbb{C})$. We apply this fact to the map μ_k defining Z_k and we obtain

$$[Z_k] = \mu_k^*[W_e(X)] = \frac{2^{2(g-e)}}{(g-e)!}\Theta_P^{g-e},$$

where $W_e(X)\subset \mathrm{Pic}^e(X)$ denotes the Brill-Noether locus of line bundles L with $h^0(L)>0$, whose fundamental class equals $\frac{\Theta_P^{g-e}}{(g-e)!}$ by Poincaré's formula. □

We now combine the previous results and we obtain

$$[\Delta_k][Z_k] = \Theta_S\otimes\frac{2^{e-1}2^{2(g-e)}}{e!(g-e)!}\Theta_P^g = \Theta_S 2^{2g-e-1}\binom{g}{e} = \Theta_S 2^{2k}\binom{g}{2g-2k-1},$$

which gives the class $cl(\mathcal{W}_k)$ stated in Theorem 1.3.

Remark 4.2. We observe that in [Fa] page 350 the factor $\binom{g}{e}$ is missing in the formula giving "the integral of Θ_J^e over the preimage of C^e".

5. Examples

5.1. **Genus 2.**

5.1.1. $\lambda = 0$. It is known that $\mathrm{M}_X(2, \mathcal{O}_X)$ is isomorphic to $\mathbb{P}^3$. By Theorem 1.1 the wobbly locus has two components

$$\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2,$$

where $\mathcal{W}_k$ is the closure of the locus $\mathcal{W}_k^0$ for $k = 1, 2$.
Let $k = 1$. Note that for any line bundle L of degree zero $h^0(X, KL^2) \neq 0$ and any bundle which contains a line subbundle of degree zero is semi-stable. Therefore, $\mathcal{W}_1$ is precisely the locus of semi-stable bundles which are not stable. It is known that the strictly semi-stable locus is a quartic hypersurface (known as Kummer surface) in $\mathbb{P}^3$. Thus the class $\mathrm{cl}(\mathcal{W}_1)$ of the wobbly divisor $\mathcal{W}_1$ in the Picard group of $\mathrm{M}_X(2, \mathcal{O}_X)$ is 4Θ, where Θ is the ample generator of the Picard group of $\mathrm{M}_X(2, \mathcal{O}_X)$.

Let $k = 2$. Then for a line bundle L, $h^0(X, KL^2) \neq 0$ if and only if L is the inverse of a theta characteristic. There are precisely 16 such line bundles. If L is such a line bundle, then any nontrivial extension of L by L^{-1} is stable and for each such line bundle L the space of extensions gives a hyperplane in $\mathbb{P}^3$. Therefore $\mathcal{W}_2$ is the union of 16 hyperplanes in $\mathbb{P}^3$ and the class $\mathrm{cl}(\mathcal{W}_2)$ of the wobbly divisor $\mathcal{W}_2$ in the Picard group of $\mathrm{M}_X(2, \mathcal{O}_X)$ is 16Θ.

5.1.2. $\lambda = 1$. Let Λ be a line bundle of degree 1. It is known that $\mathrm{M}_X(2, \Lambda)$ is isomorphic to a smooth intersection Y of two quadrics in $\mathbb{P}^5$. By Theorem 1.1 the wobbly locus is irreducible. If a stable vector bundle E is in the wobbly locus, then under the identification of $\mathrm{M}_X(2, \Lambda)$ with Y, it corresponds to a point $P \in Y$ such that the intersection of Y with the projectivized embedded tangent space of Y at P contains fewer than 4 lines [P]. Classically, it is known that the locus of such points $P \in Y$ is isomorphic to a surface in $\mathbb{P}^5$ of degree 32. In other words, the irreducible wobbly divisor is isomorphic to a surface in $\mathbb{P}^5$ of degree 32. Thus the class $\mathrm{cl}(\mathcal{W}_1)$ of the wobbly divisor $\mathcal{W}_1$ in the Picard group of $\mathrm{M}_X(2, \Lambda)$ is 8Θ, where Θ is a hyperplane section (of degree 4) of $\mathrm{M}_X(2, \Lambda)$.

5.2. **Genus** $3, \lambda = 0, k = 2$**.** It is known that $\mathrm{M}_X(2, \mathcal{O}_X)$ is isomorphic to Coble's quartic hypersurface in $\mathbb{P}^7$ [NR]. On the other hand, by Theorem 1.3 we have that the class $\mathrm{cl}(\mathcal{W}_2)$ of the wobbly divisor $\mathcal{W}_2$ in the Picard group of $\mathrm{M}_X(2, \mathcal{O}_X)$ is 48Θ, where Θ is a hyperplane section of degree 4. Therefore, we can describe $\mathcal{W}_2$ as the cut out of Coble's quartic by a hypersurface of degree 48.

5.3. **Arbitrary genus** g**,** $\lambda = 0$**,** $k = g$**.** We recall that $\mathcal{W}_g^0 = \{E : E$ contains a line subbundle L of degree $1 - g$ with $h^0(KL^2) \neq 0\}$. For a line bundle L of degree $1 - g$, $h^0(KL^2) \neq 0$ if and only if L is the inverse of a theta characteristic. For each such line bundle L the space of non-trivial extension classes of L by L^{-1} gives a divisor of $\mathrm{M}_X(2, \mathcal{O}_X)$, whose class is the ample generator Θ of the Picard group of $\mathrm{M}_X(2, \mathcal{O}_X)$. Therefore the 2^{2g} irreducible divisors of $\mathcal{W}_g$ correspond to the 2^{2g} theta characteristics of X. Thus the class $\mathrm{cl}(\mathcal{W}_g)$ of the wobbly divisor $\mathcal{W}_g$ in the Picard group of $\mathrm{M}_X(2, \mathcal{O}_X)$ is $2^{2g}\Theta$.

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