VERY STABLE BUNDLES AND PROPERNESS OF THE HITCHIN MAP

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Abstract. Let X be a smooth complex projective curve of genus $g \geq 2$ and let K be its canonical bundle. In this note we show that a stable vector bundle E on X is very stable, i.e. E has no non-zero nilpotent Higgs field, if and only if the restriction of the Hitchin map to the vector space of Higgs fields $H^0(X, \mathrm{End}(E) \otimes K)$ is a proper map.

1. Introduction

Let X be a smooth complex projective curve of genus $g \geq 2$ and let K be its canonical bundle. We consider the moduli space $\mathrm{M}_X(n,d)$ of semi-stable degree-d rank-n vector bundles on X and the moduli space $\mathrm{Higgs}_X(n,d)$ of semi-stable degree-d rank-n Higgs bundles. Thanks to a result of N. Nitsure [N] it is known that the Hitchin fibration

$$h : \mathrm{Higgs}_X(n,d) \longrightarrow \mathrm{H} = \bigoplus_{i=1}^{n} \mathrm{H}^0(X, K^i),$$

defined by associating to a Higgs bundle (E,ϕ) the coefficients of the characteristic polynomial of ϕ, is a proper map. If E is a stable degree-d rank-n vector bundle, the vector space $\mathrm{V} = \mathrm{H}^0(X, \mathrm{End}(E) \otimes K)$ embeds naturally in the moduli space $\mathrm{Higgs}_X(n,d)$. So we can consider the restriction $h_V : \mathrm{V} \longrightarrow \mathrm{H}$ of the Hitchin map h to the vector space V and ask whether h_V is also proper.

In order to state the answer we need to consider very stable vector bundles introduced by Drinfeld. By definition a vector bundle E is very stable if it has no non-zero nilpotent Higgs field. Laumon [L, Proposition 3.5] proved, assuming $g \geq 2$, that a very stable vector bundle is stable and that the locus of very stable bundles is a non-empty open subset of $\mathrm{M}_X(n,d)$.

With these notation our main result is the following

Theorem 1.1. *Let E be a stable degree-d rank-n vector bundle over X. Then we have the following equivalences:*

$$\begin{aligned} E \text{ is very stable} \quad &\Longleftrightarrow \quad \mathrm{V} \text{ is closed in } \mathrm{Higgs}_X(n,d) \\ &\Longleftrightarrow \quad h_V \text{ is a proper map} \\ &\Longleftrightarrow \quad h_V \text{ is a quasi-finite map.} \end{aligned}$$

A few comments on the proof: the core of the result is to show that for a very stable E the vector space V is closed in $\mathrm{Higgs}_X(n,d)$, or equivalently, that if there exists a limit point $(F,\psi) \in \mathrm{Higgs}_X(n,d) \setminus V$, then E has a non-zero nilpotent Higgs field. In order to do that we use the $\mathbb{C}^*$-action on the one-dimensional family of Higgs bundles converging to (F,ψ) to contruct a rational map from a smooth surface to $\mathrm{Higgs}_X(n,d)$ whose indeterminacy locus is one point. Then,

Date: October 27, 2017.

2000 *Mathematics Subject Classification.* Primary 14H60, 14H70.

The second author was supported by a post-doctoral grant associated to the Marie Curie project GEOMODULI of the programme FP7/PEOPLE/2013/CIG, project number 618471.

Hironaka's theorem on the resolution of indeterminacies [Hi] gives a morphism from the exceptional divisor (a chain of projective lines) to $\mathrm{Higgs}_X(n,d)$ connecting the two Higgs bundles $(E,0)$ and (F,ψ).

We have the following

Corollary 1.2. *If E is very stable, then the restricted Hitchin map h_V is finite and surjective.*

We would like to thank T. Hausel, J. Heinloth, C. Simpson and H. Zelaci for useful discussions on these questions.

2. Preliminaries

In this section we recall basic definitions and prove some preliminary results used in the proof of Theorem 1.1.

By [S] section 6 there is an algebraic action of $\mathbb{C}^*$ on the coarse moduli space $\mathrm{Higgs}_X(n,d)$ given by multiplying the Higgs field by scalars

$$\lambda \cdot (E,\phi) \mapsto (E, \lambda \cdot \phi).$$

Clearly the subset $V \subset \mathrm{Higgs}_X(n,d)$ is invariant for the $\mathbb{C}^*$-action.

Proposition 2.1. *Let E be a stable bundle. If h_V is quasi-finite, then E is very stable.*

Proof. Suppose that E is stable, but not very stable, and let $\phi \in \mathrm{V}$ be a non-zero nilpotent Higgs field. Then $h_V^{-1}(0)$ contains the line $\mathbb{C}\phi$, a contradiction. □

Proposition 2.2. *Let E be a stable bundle. Then*

$$\mathrm{V} \textit{ is closed in } \mathrm{Higgs}_X(n,d) \iff h_V \textit{ is a proper map.}$$

Proof. This is a consequence of the valuative criterion of properness applied to the morphism $i_V : V \to \mathrm{Higgs}_X(n,d)$ and its composite $h_V = h \circ i_V$ with the proper map h, see e.g. [Ha] Corollary II.4.8 (a),(b) and (e). □

Proposition 2.3. *Let E be a stable bundle and let C be a smooth curve with a morphism*

$$\varphi : C \to \mathrm{Higgs}_X(n,d),$$

such that $\varphi(C \setminus \{c\}) \subset V$ for some point $c \in C$. Denote $\varphi(c) = (F,\psi)$. If $F \neq E$, then F is not semi-stable.

Proof. Suppose on the contrary that F is semi-stable. By passing to an étale cover of C we can assume that there is a family of vector bundles $\mathcal{E}$ over X parameterized by C such that $\mathcal{E}_{|X\times\{p\}} = E$ for $p \neq c$ and $\mathcal{E}_{|X\times\{c\}} = F$. Now the classifying map φ' associated to $\mathcal{E}$ maps C to the coarse moduli space $\mathrm{M}_X(n,d)$, which is separated. Hence φ' is constant, which contradicts $F \neq E$. □

The next lemma is probably well-known, but as we have not found a reference in the literature we include a full proof.

Lemma 2.4. *Given a morphism $f : T \to Y$ to a quasi-projective variety Y. Let $U = f(T)$ be the image of f in Y. Suppose that there exists a point $z \in \overline{U} \setminus U$, where $\overline{U}$ is the Zariski closure of U in Y. Then there exists a smooth (not necessarily complete) curve C, a point $c \in C$ and a morphism $\varphi : C \to \overline{U}$ such that*

$$\varphi(C \setminus \{c\}) \subset U \textit{ and } \varphi(c) = z.$$

Proof. By Chevalley's theorem (see e.g. [Ha] Ex II.3.18 and Ex II.3.19) we know that the image $U = f(T)$ is a constructible set, hence a finite disjoint union $U = \bigcup_i U_i$ of locally closed subsets U_i. Hence $z \in \overline{U}_i \setminus U_i$ for some integer i. To simplify notation we will write U instead of U_i. Clearly $\dim U \geq 1$.

We choose an embedding of $U \subset \overline{U} \subset Y \hookrightarrow \mathbb{P}^N$ in projective space. We denote by Z the irreducible component of $\overline{U} \setminus U$ which contains z. Then $\delta = \dim Z < \dim U$. We choose a point $u \in U$ and consider the linear system $\Gamma = |\mathcal{J}_{u,z}(m)|$ of hypersurfaces in $\mathbb{P}^N$ of fixed degree $m \geq 2$ through the two points u and z. Our strategy is to cut out a curve through the limit point z by intersecting divisors in the linear system Γ. For that we observe that the base locus of Γ is reduced to $\{u, z\}$. Therefore we can choose δ divisors $D_1, D_2, \ldots, D_\delta$ in Γ which cut out on Z a finite set of points containing z. We denote by W the intersection $D_1 \cap D_2 \cap \ldots \cap D_\delta \cap U \subset U$. Then W is non-empty, since $u \in W$, and $\dim W \geq 1$. Moreover, since W is closed in U, we have the inclusion $\overline{W} \setminus W \subset D_1 \cap D_2 \cap \ldots \cap D_\delta \cap Z$, which shows that $\overline{W} \setminus W$ is also a finite set of points containing z. If $\dim W > 1$ we intersect W with divisors in Γ till we obtain a curve C passing through z. Since $\overline{W} \setminus W$ is a finite set, there is a neighboorhood Ω of z in C which does not intersect this finite set, hence $\Omega \setminus \{z\}$ is contained in U. If the curve C has singular points, we take its normalization. □

3. Proof of Theorem 1.1

Because of Propositions 2.1 and 2.2 it will be enough to show that:

i) If h_V is proper, then it is quasi-finite.

ii) V is closed in $\mathrm{Higgs}_X(n,d)$ if E is very stable, or equivalently, that if the Zariski closure $\overline{\mathrm{V}}$ of V in $\mathrm{Higgs}_X(n,d)$ properly contains V, then E admits a non-zero nilpotent Higgs field.

To prove *i*), note that h_V is a proper map between affine spaces of the same dimension, and it is thus quasi-finite.

For *ii*), assume that there exists $(F, \psi) \in Z := \overline{V} \setminus V$, where $\overline{V}$ is the Zariski closure of V in $\mathrm{Higgs}_X(n,d)$. Since V is invariant for the $\mathbb{C}^*$-action, its Zariski closure $\overline{V}$ is also invariant for the $\mathbb{C}^*$-action. Therefore the limit point $(F_0, \psi_0) := \lim_{\lambda \to 0}(F, \lambda\psi) \in \overline{V}$ and satisfies $h(F_0, \psi_0) = 0$. By Lemma 2.4 and Proposition 2.3 we deduce that (F, ψ) and therefore (F_0, ψ_0) are not semi-stable, so $(F_0, \psi_0) \in Z$. Hence, replacing (F, ψ) by its limit (F_0, ψ_0) for the $\mathbb{C}^*$-action, we can assume that $h(F, \psi) = 0$ and that (F, ψ) is a fixed point for the $\mathbb{C}^*$-action.

By Lemma 2.4 there exists a curve C, a point $c \in C$ and a morphism $\varphi : C \to \overline{V}$ such that $\varphi(C^*) \subset V$ and $\varphi(c) = (F, \psi)$. Here C^* denotes the curve $C \setminus \{c\}$. The main idea of the proof is to consider the $\mathbb{C}^*$-action on the image of the curve C^* in $\mathrm{Higgs}_X(n,d)$. So we introduce the morphism

$$\Psi^* : \mathbb{C}^* \times C^* \longrightarrow \overline{\mathrm{V}} \quad \text{defined by} \quad \Psi^*(\lambda, p) = \lambda \cdot \varphi(p).$$

Proposition 3.1. *We can extend the morphism Ψ^* to a morphism*

$$\Psi : \mathbb{C} \times C \setminus \{(0, c)\} \to \overline{V},$$

such that $\Psi(0, p) = (E, 0)$ for $p \neq c$ and $\Psi(\lambda, c) = (F, \psi)$ for $\lambda \neq 0$.

Proof. It will be enough to extend Ψ^* to the two open subsets $\mathbb{C} \times C^*$ and $\mathbb{C}^* \times C$ of $\mathbb{C} \times C$. Note that the $\mathbb{C}^*$-action $\mathbb{C}^* \times V \to V$ is the scalar multiplication of the vector space V and hence naturally extends to an action $\mathbb{C} \times V \to V$. Since $\varphi(C^*) \subset V$, we therefore obtain a morphism $\Psi : \mathbb{C} \times C^* \to V$. Clearly, $\Psi(0, p) = (E, 0)$ for $p \neq c$. On the open subset $\mathbb{C}^* \times C$, we just take the definition of Ψ^* extended to C.

Clearly, $\Psi(\lambda, c) = \lambda \cdot (F, \psi) = (F, \psi)$ for $\lambda \neq 0$, as (F, ψ) is by assumption a fixed point. $\square$

So Ψ is a rational map from the surface $S := \mathbb{C} \times C$ to $\mathrm{Higgs}_X(n, d)$ whose indeterminacy locus is the point $(0, c)$. First, we consider the rational composite map

$$h' = h \circ \Psi : S \to \mathrm{H}.$$

Since H is a vector space, the morphism h' is given by holomorphic functions on a punctured smooth surface. By Hartog's theorem these functions extend to the surface S and by continuity $h'(0, c) = 0$ since $h'(0, p) = 0$ for $p \neq c$.

In order to prove that the rational map Ψ can be resolved into a morphism, we will apply Main Theorem II in [Hi]. For it to apply, by the discussion following Question E in loc. cit. (page 140), it is enough to prove that the morphism π defined in the following commutative diagram is proper

$$\begin{array}{ccc} \Gamma & \hookrightarrow & S \times \mathrm{Higgs}_X(n, d) \\ & \searrow^{\pi} & \downarrow{\scriptstyle \pi_1} \\ & & S. \end{array}$$

Here Γ is the closure of the graph of Ψ. Let R be a discrete valuation ring with quotient field L. By the valuative criterion of properness, we need to prove that for any commutative diagram as below, the dashed arrow exists

$$\begin{array}{ccc} \mathrm{Spec}(L) & \longrightarrow & \Gamma \\ \downarrow & \nearrow & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}(R) & \longrightarrow & S. \end{array}$$

For each such commutative diagram, consider the extended commutative diagram

$$(3.1) \qquad \begin{array}{ccccc} \mathrm{Spec}(L) & \longrightarrow & \Gamma & \hookrightarrow & S \times \mathrm{Higgs}_X(n, d) \\ \downarrow & {\scriptstyle e_1} \nearrow & \downarrow{\scriptstyle \pi} & {\scriptstyle e_2} \nearrow & \downarrow{\scriptstyle Id \times h} \\ \mathrm{Spec}(R) & \longrightarrow & S & \xrightarrow[Id \times h']{} & S \times \mathrm{H}, \end{array}$$

where the morphism $h' : S \to \mathrm{H}$ was introduced above. By properness of the Hitchin map h, the map $Id \times h$ is also proper and therefore the dashed arrow e_2 in (3.1) exists. Moreover, since Γ is closed, its image is contained in Γ, so the dashed arrow e_1 also exists. So, by [Hi] Main Theorem II, this proves that Ψ resolves after a finite sequence of blow-ups along points to a morphism

$$\hat{\Psi} : \hat{S} \to \overline{V}.$$

First note that the exceptional divisor $D := \bigcup_{i=0}^{m} D_i$ is a connected union of projective lines D_i and that by restriction of $\hat{\Psi}$ we obtain a morphism

$$\hat{\Psi} : D := \bigcup_{i=0}^{m} D_i \longrightarrow \overline{V} \subset \mathrm{Higgs}_X(n, d)$$

whose image is a connected curve in $\overline{V}$. Let $p_0 \in D$ and $p_\infty \in D$ be the limit points in $\hat{S}$

$$p_0 = \lim_{p \to c} (0, p) \text{ and } p_\infty = \lim_{\lambda \to 0} (\lambda, c).$$

Then, by separability of the moduli space $\mathrm{Higgs}_X(n,d)$ [N, Theorem 5.10] (see also Remark 5.12 in loc.cit.), we clearly have

$$\hat{\Psi}(p_0) = (E,0) \text{ and } \hat{\Psi}(p_\infty) = (F,\psi).$$

Moreover, since $h'(0,c) = 0$ we have $\hat{\Psi}(D) \subset h^{-1}(0)$.

We can numerate the projective lines $D_0, D_1, \ldots, D_{m'}$ such that $p_0 \in D_0$ and $p_\infty \in D_{m'}$ and $D_i \cap D_{i+1} \neq \emptyset$ for all $i \leq m'-1$. Note that a priori m' can be smaller than m. Consider the smallest integer $i_0 \geq 0$ such that $\hat{\Psi}(D_{i_0})$ is not reduced to the point $(E,0)$. Such an integer exists since $\hat{\Psi}(p_\infty) = (F,\psi) \neq (E,0)$. Then we claim that $\hat{\Psi}(D_{i_0}) \cap V$ contains a Higgs bundle (E,ϕ_0) with non-zero Higgs field ϕ_0. Indeed, by definition of i_0 there exists a point $p \in D_{i_0}$ such that $\hat{\Psi}(p) = (E,0)$. Suppose on the contrary that for any point $q \neq p$ we have $\hat{\Psi}(q) \in \overline{V} \setminus V$. Then by Lemma 2.4 and Proposition 2.3 the underlying bundle of $\hat{\Psi}(q)$ is not semi-stable for all $q \neq p$. This contradicts the fact that the non-empty locus of stable bundles of the family parameterized by D_{i_0} is an open subset.

Since $\hat{\Psi}(D_{i_0}) \subset h^{-1}(0)$ the Higgs field ϕ_0 is nilpotent showing that E is not very stable.

4. Proof of Corollary 1.2

If E is a very stable bundle, then by Theorem 1.1 the map h_V is a proper map between affine spaces. Hence the fibers of h_V are affine and complete, so h_V is a quasi-finite map. But h_V proper and quasi-finite implies that h_V is a finite map. Surjectivity of h_V follows again from properness.

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