THE SPACE OF GENERALIZED G_2-THETA FUNCTIONS OF LEVEL ONE

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Abstract. Let C be a smooth projective complex curve of genus at least 2. For a simply-connected complex Lie group G the vector space of global sections $H^0(\mathcal{M}(G), \mathcal{L}_G^{\otimes l})$ of the l-th power of the ample generator $\mathcal{L}_G$ of the Picard group of the moduli stack of principal G-bundles over C is commonly called the space of generalized G-theta functions or Verlinde space of level l. In the case $G = \mathrm{G}_2$, the exceptional Lie group of automorphisms of the complex Cayley algebra, we study natural linear maps between the Verlinde space $H^0(\mathcal{M}(G_2), \mathcal{L}_{\mathrm{G}_2})$ of level one and some Verlinde spaces for SL_2 and SL_3. We deduce that the image of the monodromy representation of the WZW-connection for $G = \mathrm{G}_2$ and $l = 1$ is infinite.

1. Introduction

Let C be a smooth projective complex curve of genus $g \geq 2$. For a complex semi-simple Lie group G we denote by $\mathcal{M}(G)$ the moduli stack of principal G-bundles over C. If G is simply-connected, the Picard group of the stack $\mathcal{M}(G)$ is infinite cyclic and we denote by $\mathcal{L}$ its ample generator. The finite-dimensional vector spaces of global sections $H^0(\mathcal{M}(G), \mathcal{L}^{\otimes l})$, the so-called spaces of generalized G-theta functions or Verlinde spaces of level l, have been intensively studied from different perspectives, e.g. gauge theory, mathematical theory of conformal blocks, and quantization. Note that much of the literature deals with the vector bundle case $G = \mathrm{SL}_r$.

In this note we will study the Verlinde space $H^0(\mathcal{M}(\mathrm{G}_2), \mathcal{L}_{\mathrm{G}_2})$ for the smallest exceptional Lie group G_2 and at level 1. The starting point of our investigation was the striking numerical relation between the dimensions of the Verlinde spaces for G_2 at level 1 and for SL_2 at level 3

$$(1) \qquad \dim H^0(\mathcal{M}(\mathrm{G}_2), \mathcal{L}_{\mathrm{G}_2}) = \frac{1}{2^g} \dim H^0(\mathcal{M}(\mathrm{SL}_2), \mathcal{L}_{\mathrm{SL}_2}^{\otimes 3}) = \left(\frac{5+\sqrt{5}}{2}\right)^{g-1} + \left(\frac{5-\sqrt{5}}{2}\right)^{g-1}.$$

These dimensions are computed by the Verlinde formula (see e.g. [B5] Corollary 9.8). It turns out that linear maps between these Verlinde spaces arise in a natural way by restricting to some distinguished substacks in $\mathcal{M}(\mathrm{G}_2)$. The group G_2 contains the subgroups SL_3 and SO_4 as maximal reductive subgroups of maximal rank. These group inclusions induce maps

$$i : \mathcal{M}(\mathrm{SL}_3) \longrightarrow \mathcal{M}(\mathrm{G}_2) \qquad \text{and} \qquad j : \mathcal{M}(\mathrm{SL}_2) \times \mathcal{M}(\mathrm{SL}_2) \longrightarrow \mathcal{M}(\mathrm{G}_2)$$

via the étale double cover $\mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}_4$.

Our main results are the following.

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Theorem I. For any smooth curve C of genus $g \geq 2$ the linear map obtained by pull-back by the map j of global sections of $\mathcal{L}_{\mathrm{G}_2}$

$$j^* : H^0(\mathcal{M}(\mathrm{G}_2), \mathcal{L}_{\mathrm{G}_2}) \longrightarrow \left[H^0(\mathcal{M}(\mathrm{SL}_2), \mathcal{L}_{\mathrm{SL}_2}^{\otimes 3}) \otimes H^0(\mathcal{M}(\mathrm{SL}_2), \mathcal{L}_{\mathrm{SL}_2}) \right]_0$$

is an isomorphism.

Theorem II. For any smooth curve C of genus $g \geq 2$ without vanishing theta-null the linear map obtained by pull-back by the map i of global sections of $\mathcal{L}_{\mathrm{G}_2}$

$$i^* : H^0(\mathcal{M}(\mathrm{G}_2), \mathcal{L}_{\mathrm{G}_2}) \longrightarrow H^0(\mathcal{M}(\mathrm{SL}_3), \mathcal{L}_{\mathrm{SL}_3})_+$$

is surjective.

The subscripts 0 and + denote subspaces of invariant sections for the group of 2-torsion line bundles over C and for the duality involution respectively.

The first example of isomorphism between Verlinde spaces was given in [B1] for the embedding $\mathbb{C}^* \subset \mathrm{SL}_2$ at level 1. More recently, the rank-level dualities provide series of isomorphisms between Verlinde spaces (and their duals) for special pairs of structure groups. In this context Theorem I can be viewed as a new example.

Most of the constructions presented in this paper are valid for the coarse moduli spaces of semi-stable G-bundles over C. However, the generator $\mathcal{L}_{\mathrm{G}_2}$ of the Picard group of the moduli stack $\mathcal{M}(\mathrm{G}_2)$ does not descend [LS] to the moduli space $M(\mathrm{G}_2)$ because the Dynkin index of G_2 is 2. This forces us to use the moduli stack.

Theorem I has an application to the flat projective connection on the bundle of conformal blocks associated to the Lie algebra $\mathfrak{g}_2$ at level 1. Let $\pi : \mathcal{C} \to S$ be a family of smooth projective curves and consider the vector bundle $\mathbb{V}_1^*(\mathfrak{g}_2)$ over S whose fiber over the curve $C = \pi^{-1}(s)$ equals the conformal block $\mathcal{V}_1^*(\mathfrak{g}_2)$. Note that this conformal block is canonically (up to homothety) isomorphic to our space $H^0(\mathcal{M}(\mathrm{G}_2), \mathcal{L}_{\mathrm{G}_2})$ by the general Verlinde isomorphism [LS]. By [U] the vector bundle $\mathbb{V}_1^*(\mathfrak{g}_2)$ is equipped with a flat projective connection, the so-called WZW-connection. Then we have the

Corollary. There exist families of smooth curves of any genus $g \geq 2$ for which the projective monodromy representation of the projective WZW-connection on $\mathbb{V}_1^*(\mathfrak{g}_2)$ has infinite image.

In section 2 we review the properties of the exceptional group G_2 and of its subgroups, as well as some results on the Verlinde spaces for SL_2 at low levels. In section 3 we give the proof of the main theorems.

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2. Moduli spaces and moduli stacks of principal G_2-bundles

In this section we review some results on the exceptional group G_2 and on the moduli of principal G_2-bundles over a smooth projective curve C.

2.1. **The exceptional group G_2 and its rank-2 subgroups.**

The complex exceptional group G_2 is given by one of the following equivalent definitions (see e.g. [Br] section 2 Theorem 3) :

- as the automorphism group $G_2 = \mathrm{Aut}(\mathbb{O})$ of the complex 8-dimensional Cayley algebra or algebra of octonions $\mathbb{O}$ (see e.g. [Ba]).
- as the connected component of the stabilizer in $\mathrm{GL}(V)$ of a non-degenerate alternating trilinear form $\omega : \Lambda^3 V \to \mathbb{C}$ on a complex 7-dimensional vector space V (see e.g. [SK])

We recall the following facts:

(a) For a generic trilinear form ω we have $\mathrm{Stab}_{\mathrm{GL}(V)}(\omega) = G_2 \times \mu_3$ and $\mathrm{Stab}_{\mathrm{SL}(V)}(\omega) = G_2$. Note that non-degenerate alternating forms form the unique dense $\mathrm{GL}(V)$-orbit in $\Lambda^3 V^*$.

(b) Introducing G_2 as $\mathrm{Aut}(\mathbb{O})$ there is a natural non-degenerate G_2-invariant trilinear form on the space of purely imaginary octonions $V = \mathrm{Im}(\mathbb{O})$ given by $\omega(x, y, z) = \mathrm{Re}(xyz)$, as well as a non-degenerate symmetric G_2-invariant bilinear form given by $q(x, y) = \mathrm{Re}(xy)$. This shows that G_2 is a subgroup of SO_7.

(c) The complex Lie group G_2 is connected, simply-connected, has no center and is of dimension 14.

According to [BD] the group G_2 has, up to conjugation, two maximal Lie subgroups of maximal rank, i.e. of rank 2. These two subgroups are of type A_2 and $A_1 \times A_1$ respectively. As we could not find a reference in the literature, we will give for the reader's convenience an explicit realization of these subgroups in G_2.

- $\mathrm{SL}_3 \subset G_2$. We consider a non-degenerate alternating trilinear form $\omega \in \Lambda^3 V^*$ and define $G_2 = \mathrm{Stab}_{\mathrm{SL}(V)}(\omega)$. We associate to ω the quadratic form

$$q_\omega : \mathrm{Sym}^2 V \to \mathbb{C}, \qquad q_\omega(x, y) = L_x\omega \wedge L_y\omega \wedge \omega \in \Lambda^7 V^* \cong \mathbb{C},$$

where $L_x : \Lambda^3 V^* \to \Lambda^2 V^*$ denotes the contraction operator with the vector $x \in V$. Note that ω is non-degenerate iff q_ω is non-degenerate. We now choose a 3-dimensional subspace $W \subset V$ such that W is isotropic for q_ω and such that the restriction $\omega_0 = \omega_{|W} \neq 0$. The following gives a description of SL_3 as a subgroup of G_2.

Proposition 2.1. *With the above notation we have*

$$\mathrm{SL}_3 = \mathrm{Stab}_{G_2}(W) = \{g \in G_2 \mid g(W) = W\}.$$

More precisely, the subspace $W \subset V$ induces a natural decomposition

$$V = W \oplus \Lambda^2 W \oplus \mathbb{C}, \tag{2}$$

which coincides with the decomposition of V as SL_3-module.

Proof. We consider the composite map

$$\iota : \Lambda^2 W \hookrightarrow \Lambda^2 V \xrightarrow{L_\omega} V^*,$$

where L_ω is contraction with $\omega \in \Lambda^3 V^*$. If we further compose with the projection $V^* \to W^*$, we obtain the isomorphism $\Lambda^2 W \xrightarrow{\sim} W^*$ induced by the non-zero restricted form ω_0. Hence ι is injective and we also denote by $\Lambda^2 W \subset V$ its image in V, which we identify with V^* via the non-degenerate quadratic form q_ω. Next we observe that $W \cap \Lambda^2 W = \{0\}$, since the composite map $W \to V^* \to W^*$ is zero — W is isotropic. This shows that $W \oplus \Lambda^2 W$ is a hyperplane in V. Then we take the orthogonal complement to obtain the decomposition (2). We observe that any $g \in \mathrm{Stab}_{\mathrm{G}_2}(W)$ also preserves the subspace $\Lambda^2 W \subset V$, hence the decomposition (2). Moreover, since $g(\omega_0) = \omega_0$, we have $g \in \mathrm{SL}_3 = \mathrm{SL}(W)$. Hence $\mathrm{Stab}_{\mathrm{G}_2}(W) \subset \mathrm{SL}_3$. On the other hand we consider the action of G_2 on the Grassmannian of isotropic subspaces $W \subset V$, which is of dimension 6. Hence $\dim \mathrm{Stab}_{\mathrm{G}_2}(W) \geq 8$, which leads to the equality $\mathrm{Stab}_{\mathrm{G}_2}(W) = \mathrm{SL}_3$. □

- $\mathrm{SO}_4 \subset \mathrm{G}_2$. We need to recall some basic facts on quarternions and octonions. We begin by recalling that the complex octonion algebra $\mathbb{O}$ is generated as $\mathbb{C}$-vector space by the 8 basis vectors $e_0 = 1, e_1, \ldots, e_7$ satisfying the relations given by the Fano plane (see e.g. [Ba]). Then the algebra $\mathbb{O}$ contains as a subalgebra the complex quaternion algebra $\mathbb{H} = \mathbb{C}1 \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$ and we have a vector space decomposition

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}e_4. \tag{3}$$

We recall that the subgroup $U = \{p \in \mathbb{H} \mid p\overline{p} = 1\}$ of unit quarternions can be identified with the complex Lie group SL_2 and that there is a surjective group homomorphism

$$\varphi : U \times U \longrightarrow \mathrm{SO}(\mathbb{H}) = \mathrm{SO}_4, \qquad \varphi(p,q) = [x \mapsto px\overline{q}]$$

with kernel $\mathbb{Z}/2$ generated by $(-1,-1)$. Using the decomposition (3) we consider the map

$$\psi : U \times U \longrightarrow \mathrm{SO}(\mathbb{O}), \qquad \psi(p,q) = (\varphi(p,p), \varphi(p,q)).$$

One easily checks that $\mathrm{im}\,\psi \subset \mathrm{G}_2$ and that $\ker \psi = \ker \varphi$. This gives a realization of SO_4 as subgroup of G_2. We also note that the center $Z(\mathrm{SO}_4)$ is generated by $\varphi(-1,1) = -\mathrm{Id}_{\mathbb{H}}$ and that SO_4 is the centralizer of the element $\psi(-1,1) = (\mathrm{Id}_{\mathbb{H}}, -\mathrm{Id}_{\mathbb{H}}) \in \mathrm{G}_2$ of order 2 (see [BD]).

2.2. **The moduli space $\mathrm{M}(\mathrm{G}_2)$ and the moduli stack $\mathcal{M}(\mathrm{G}_2)$.** Because of the equality $\mathrm{Stab}_{\mathrm{SL}(V)}(\omega) = \mathrm{G}_2$, a principal G_2-bundle E_{G_2} is equivalent to a rank-7 vector bundle $\mathcal{V}$ with trivial determinant equipped with a non-degenerate alternating trilinear from $\eta : \Lambda^3 \mathcal{V} \to \mathcal{O}_C$. The correspondence is given by sending E_{G_2} to $(\mathcal{V} = E_{\mathrm{G}_2}(V), \eta)$ via the embedding $\mathrm{G}_2 \subset \mathrm{SL}(V)$. Moreover, it is shown in [S] that E_{G_2} is semi-stable if and only if $\mathcal{V}$ is semi-stable. We therefore obtain a map between coarse moduli spaces of semi-stable bundles $\mathrm{M}(\mathrm{G}_2) \to \mathrm{M}(\mathrm{SL}_7)$.

Although the embeddings of SL_3 and SO_4 in G_2 are defined only up to conjugation, the induced maps between coarse moduli spaces of semi-stable principal bundles

$$i : \mathrm{M}(\mathrm{SL}_3) \to \mathrm{M}(\mathrm{G}_2) \qquad \text{and} \qquad j : \mathrm{M}(\mathrm{SL}_2) \times \mathrm{M}(\mathrm{SL}_2) \to \mathrm{M}(\mathrm{SO}_4) \to \mathrm{M}(\mathrm{G}_2)$$

are well-defined. We find it more convenient to work with the simply-connected group $\mathrm{SL}_2 \times \mathrm{SL}_2$, which is a double cover of the subgroup SO_4. Abusing notation we also denote by i and j their composites with the map $\mathrm{M}(\mathrm{G}_2) \to \mathrm{M}(\mathrm{SL}_7)$. It follows from the description of the subgroups SL_3 and SO_4 in the previous section that

$$i(E) = E \oplus E^* \oplus \mathcal{O}_C \qquad \text{and} \qquad j(F,G) = \mathrm{End}_0(F) \oplus F \otimes G. \tag{4}$$

Here E is an SL_3-bundle and F, G are SL_2-bundles. Note that $i(E)$ and $j(F,G)$ are semi-stable if E, F and G are semi-stable.

Remark: It is shown in [G] that the singular locus of the moduli space $\mathrm{M}(\mathrm{G}_2)$ coincides with the union of the images $i(\mathrm{M}(\mathrm{SL}_3)) \cup j(\mathrm{M}(\mathrm{SO}_4))$.

We also denote by i and j the maps between the corresponding moduli stacks. Let $\mathcal{L}_G$ denote the ample generator of the Picard group $\mathrm{Pic}(\mathcal{M}(G))$ when G is a simply-connected group.

Lemma 2.2. *With the above notation we have*

$$i^*\mathcal{L}_{\mathrm{G}_2} = \mathcal{L}_{\mathrm{SL}_3} \qquad \textit{and} \qquad j^*\mathcal{L}_{\mathrm{G}_2} = \mathcal{L}_{\mathrm{SL}_2}^{\otimes 3} \boxtimes \mathcal{L}_{\mathrm{SL}_2}.$$

Proof. This follows straightforwardly from a Dynkin index computation using the tables in [LS]. ☐

We consider the involution $\sigma : \mathcal{M}(\mathrm{SL}_3) \to \mathcal{M}(\mathrm{SL}_3)$ given by taking the dual $\sigma(E) = E^*$. Then the line bundle $\mathcal{L}_{\mathrm{SL}_3}$ is invariant under the involution σ. We consider the linearisation $\sigma^*\mathcal{L}_{\mathrm{SL}_3} \xrightarrow{\sim} \mathcal{L}_{\mathrm{SL}_3}$ which restricts to the identity over the fixed points of σ and denote by $H^0(\mathcal{M}(\mathrm{SL}_3), \mathcal{L}_{\mathrm{SL}_3})_+$ the subspace of invariant sections.

The group of 2-torsion line bundles $JC[2]$ acts on $\mathcal{M}(\mathrm{SL}_2)$ by tensor product and the Mumford group $\mathcal{G}(\mathcal{L}_{\mathrm{SL}_2})$, a central extension of $JC[2]$, acts linearly on $H^0(\mathcal{M}(\mathrm{SL}_2), \mathcal{L}_{\mathrm{SL}_2})$ with level 1. The $\mathcal{G}(\mathcal{L}_{\mathrm{SL}_2})$-representation $H^0(\mathcal{M}(\mathrm{SL}_2), \mathcal{L}_{\mathrm{SL}_2}^{\otimes 3}) \otimes H^0(\mathcal{M}(\mathrm{SL}_2), \mathcal{L}_{\mathrm{SL}_2})$ is of level 4 and therefore admits a linear $JC[2]$-action.

Proposition 2.3. *The induced maps between Verlinde spaces*

$$\begin{aligned} i^* : H^0(\mathcal{M}(\mathrm{G}_2), \mathcal{L}_{\mathrm{G}_2}) &\longrightarrow H^0(\mathcal{M}(\mathrm{SL}_3), \mathcal{L}_{\mathrm{SL}_3})_+ \\ j^* : H^0(\mathcal{M}(\mathrm{G}_2), \mathcal{L}_{\mathrm{G}_2}) &\longrightarrow \left[H^0(\mathcal{M}(\mathrm{SL}_2), \mathcal{L}_{\mathrm{SL}_2}^{\otimes 3}) \otimes H^0(\mathcal{M}(\mathrm{SL}_2), \mathcal{L}_{\mathrm{SL}_2})\right]_0 \end{aligned}$$

take values in the subspace invariant under the involution σ and the $JC[2]$-action respectively.

Proof. First we show that the map $i : \mathcal{M}(\mathrm{SL}_3) \to \mathcal{M}(\mathrm{G}_2)$ is σ-invariant. There is a natural inclusion between Weyl groups $W(\mathrm{SL}_3) \subset W(\mathrm{G}_2)$. Consider an element $g \in \mathrm{G}_2$ which lifts an element in $W(\mathrm{G}_2) \setminus W(\mathrm{SL}_3)$. Then $g \notin \mathrm{SL}_3$. As the subalgebra $\mathfrak{sl}_3$ of $\mathfrak{g}_2$ corresponds to the long roots and as $W(\mathrm{G}_2)$ preserves the Cartan-Killing form, the inner automorphism $C(g)$ of G_2 induced by g preserves the subgroup SL_3. The restriction of $C(g)$ to SL_3 is an outer automorphism, which permutes its two fundamental representations. It thus induces the involution σ on the moduli stack $\mathcal{M}(\mathrm{SL}_3)$. Since any inner automorphism of G_2 induces the identity on the moduli stack $\mathcal{M}(\mathrm{G}_2)$, we obtain that i is σ-invariant.

Since $i^*\mathcal{L}_{\mathrm{G}_2} = \mathcal{L}_{\mathrm{SL}_3}$ and since i is σ-invariant, the line bundle $\mathcal{L}_{\mathrm{SL}_3}$ carries a natural σ-linearisation, namely the one which restricts to the identity over fixed points of σ. It is now clear that $\mathrm{im}(i^*) \subset H^0(\mathcal{M}(\mathrm{SL}_3), \mathcal{L}_{\mathrm{SL}_3})_+$.

The second statement follows immediately from the fact that j is invariant under the diagonal $JC[2]$-action on the moduli stack $\mathcal{M}(\mathrm{SL}_2) \times \mathcal{M}(\mathrm{SL}_2)$. ☐

2.3. **A family of divisors in** $\mathbb{P}H^0(\mathcal{M}(\mathrm{G}_2), \mathcal{L}_{\mathrm{G}_2})$**.** Let $\theta(C)$ resp. $\theta^+(C)$ denote the set of theta-characteristics resp. even theta-characteristics over the curve C. The moduli stack $\mathcal{M}(\mathrm{SO}_7)$ has two connected components $\mathcal{M}^+(\mathrm{SO}_7)$ and $\mathcal{M}^-(\mathrm{SO}_7)$ distinguished by the second Stiefel-Whitney class. Since $\mathcal{M}(\mathrm{G}_2)$ is connected, the homomorphism $\mathrm{G}_2 \subset \mathrm{SO}_7$ induces a map

$$\rho : \mathcal{M}(\mathrm{G}_2) \to \mathcal{M}^+(\mathrm{SO}_7).$$

For each $\kappa \in \theta(C)$ we introduce the Pfaffian line bundle $\mathcal{P}_\kappa$ over $\mathcal{M}^+(\mathrm{SO}_7)$ (see e.g. [BLS] section 5). We have

$$\rho^* \mathcal{P}_\kappa = \mathcal{L}_{\mathrm{G}_2}.$$

Moreover, for $\kappa \in \theta^+(C)$, there exists a Cartier divisor $\Delta_\kappa \in \mathbb{P}H^0(\mathcal{M}^+(\mathrm{SO}_7), \mathcal{P}_\kappa)$ with support

$$\mathrm{supp}(\Delta_\kappa) = \{E \in \mathcal{M}^+(\mathrm{SO}_7) \mid \dim H^0(C, E(\mathbb{C}^7) \otimes \kappa) > 0\},$$

where $E(\mathbb{C}^7)$ denotes the rank-7 vector bundle associated to E. We also denote by $\Delta_\kappa \in \mathbb{P}H^0(\mathcal{M}(\mathrm{G}_2), \mathcal{L}_{\mathrm{G}_2})$ the pull-back $\rho^*(\Delta_\kappa)$ to $\mathcal{M}(\mathrm{G}_2)$. We will show later (Corollary 3.2) that the family of divisors $\{\Delta_\kappa\}_{\kappa \in \theta^+(C)}$ spans the linear system $\mathbb{P}H^0(\mathcal{M}(\mathrm{G}_2), \mathcal{L}_{\mathrm{G}_2})$. Abusing notation we also write Δ_κ for a section of $H^0(\mathcal{M}(\mathrm{G}_2), \mathcal{L}_{\mathrm{G}_2})$ vanishing at the divisor Δ_κ.

2.4. **Verlinde spaces for** SL_2 **at level** 1**,** 2 **and** 3**.** We denote $V_n = H^0(\mathcal{M}(\mathrm{SL}_2), \mathcal{L}_{SL_2}^{\otimes n})$ for $n \geq 1$. We now review some results from [B2] describing special bases of the vector spaces $V_1 \otimes V_1$ and V_2.

First we recall that the Mumford group $\mathcal{G}(\mathcal{L}_{\mathrm{SL}_2})$ acts linearly on the space V_n with level n, i.e., the center $\mathbb{C}^* \subset \mathcal{G}(\mathcal{L}_{\mathrm{SL}_2})$ acts via $\lambda \mapsto \lambda^n$. For n odd, there exists a unique (up to isomorphism) irreducible $\mathcal{G}(\mathcal{L}_{\mathrm{SL}_2})$-module H_n of level n. Note that $\dim H_n = 2^g$. If n is divisible by 4, any $\mathcal{G}(\mathcal{L}_{\mathrm{SL}_2})$-module Z of level n admits a linear $JC[2]$-action. We denote by Z_0 the $JC[2]$-invariant subspace of Z.

We now list the results needed in the proof of Theorem II.

Lemma 2.4. *We have*

$$\dim (V_1 \otimes V_3)_0 = \frac{1}{|JC[2]|} \dim V_1 \otimes V_3.$$

Proof. By the general representation theory of Heisenberg groups, the $\mathcal{G}(\mathcal{L}_{\mathrm{SL}_2})$-module $V_1 \otimes V_3$ decomposes into a direct sum of factors, which are all isomorphic to $H_1 \otimes H_3$. It is then straightforward to show that the space of $JC[2]$-invariants $(H_1 \otimes H_3)_0$ is 1-dimensional. □

Proposition 2.5 ([B2])**.** *The two $\mathcal{G}(\mathcal{L}_{\mathrm{SL}_2})$-modules $V_1 \otimes V_1$ and V_2 of level 2 decompose as direct sum of 1-dimensional character spaces for $\mathcal{G}(\mathcal{L}_{\mathrm{SL}_2})$*

$$V_1 \otimes V_1 = \bigoplus_{\kappa \in \theta(C)} \mathbb{C}\xi_\kappa, \qquad V_2 = \bigoplus_{\kappa \in \theta^+(C)} \mathbb{C} d_\kappa.$$

The supports of the zero divisors $Z(d_\kappa)$ and $Z(\xi_\kappa)$ equal

$$\begin{aligned} \mathrm{supp}\ Z(d_\kappa) &= \{E \in \mathcal{M}(\mathrm{SL}_2) \mid \dim H^0(C, \mathrm{End}_0(E) \otimes \kappa) > 0\}, \\ \mathrm{supp}\ Z(\xi_\kappa) &= \{(E,F) \in \mathcal{M}(\mathrm{SL}_2) \times \mathcal{M}(\mathrm{SL}_2) \mid \dim H^0(C, E \otimes F \otimes \kappa) > 0\}. \end{aligned}$$

Moreover, if C has no vanishing theta-null, ξ_κ is mapped to d_κ by the multiplication map $V_1 \otimes V_1 \to V_2$.

Proposition 2.6 ([A]). *For a general curve the multiplication map of global sections*

$$\mu : V_1 \otimes V_2 \longrightarrow V_3$$

is surjective.

3. Proof of the Main Theorems

In this section we give the proof of the two theorems and of the corollary stated in the introduction.

3.1. **Theorem I.** The first step is to show that the two spaces appearing at both ends of the map j^* have the same dimension. The dimension of the space on the RHS is computed by Lemma 2.4. The statement then follows from (1) and from the equalities $\dim V_1 = 2^g$ and $|JC[2]| = 2^{2g}$.

The next step is to show that j^* is surjective for a *general* curve, which will imply by the first step that j^* is an isomorphism for a general curve. We consider the following map

$$\alpha : V_1 \otimes V_1 \otimes V_2 \longrightarrow V_1 \otimes V_3, \qquad u \otimes v \otimes w \mapsto u \otimes \mu(v \otimes w),$$

where μ is the multiplication map introduced in Proposition 2.6. By Proposition 2.6 α is surjective for a general curve, hence its restriction to the subspace of $JC[2]$-invariant sections $\alpha_0 : (V_1 \otimes V_1 \otimes V_2)_0 \longrightarrow (V_1 \otimes V_3)_0$ remains surjective. Moreover, one easily works out that the family of tensors $\{\xi_\kappa \otimes d_\kappa\}_{\kappa \in \theta^+(C)}$ forms a basis of $(V_1 \otimes V_1 \otimes V_2)_0$.

We will use the family of divisors $\{\Delta_\kappa\}_{\kappa \in \theta^+(C)}$ introduced in section 2.3.

Lemma 3.1. *For all $\kappa \in \theta^+(C)$ we have the equality (up to a scalar)*

$$j^*(\Delta_\kappa) = \alpha_0(\xi_\kappa \otimes d_\kappa).$$

Proof. Using the description of j given in (4) and the description of the divisors $Z(d_\kappa)$ and $Z(\xi_\kappa)$ given in Proposition 2.5 we obtain the following decomposition as divisor in $\mathcal{M}(\mathrm{SL}_2) \times \mathcal{M}(\mathrm{SL}_2)$

$$j^*(\Delta_\kappa) = pr_1^* Z(d_\kappa) + Z(\xi_\kappa),$$

where pr_1 is the projection onto the first factor. This shows the lemma. □

Surjectivity (for a general curve) now follows as follows: since $\{\xi_\kappa \otimes d_\kappa\}_{\kappa \in \theta^+(C)}$ forms a basis of $(V_1 \otimes V_1 \otimes V_2)_0$ and since α_0 is surjective, we see by Lemma 3.1 that the family $\{j^*(\Delta_\kappa)\}_{\kappa \in \theta^+(C)}$ generates $(V_1 \otimes V_3)_0$.

Finally, we complete the proof by showing that j^* is an isomorphism for every smooth curve. We use the identification [LS] for any semi-simple, simply-connected complex Lie group G of the Verlinde space $H^0(\mathcal{M}(G), \mathcal{L}_G^{\otimes l})$ with the space of conformal blocks $\mathcal{V}_l^*(\mathfrak{g})$ at level l, where $\mathfrak{g}$ is the Lie algebra of G, for the two cases $G = \mathrm{G}_2$ and $G = \mathrm{SL}_2 \times \mathrm{SL}_2$. Then, [Be] Proposition 5.2 shows functoriality of the above isomorphism under group extensions. In our case $\mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{G}_2$ the linear map j^* can therfore be identified with the natural map

$$\beta_C : \mathcal{V}_1^*(\mathfrak{g}_2) \longrightarrow \mathcal{V}_3^*(\mathfrak{sl}_2) \otimes \mathcal{V}_1^*(\mathfrak{sl}_2).$$

We can define this linear map for a family of smooth curves $\pi : \mathcal{C} \to S$: by [U] there exist vector bundles of conformal blocks over the base S together with a homomorphism β, which specializes over a point $s \in S$ to the linear map $\beta_{\pi^{-1}(s)}$. These vector bundles are equipped with flat projective connections, the so-called WZW connections.

We now observe that the Lie algebra embedding $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \subset \mathfrak{g}_2$ is conformal — by direct computation. Then, by [Be] Proposition 5.8, we obtain that the map β is projectively flat for the two WZW connections, hence its rank is constant in the family $\pi : \mathcal{C} \to S$. Since, by the previous step, β_C is injective for a general curve C (note that we do not take $JC[2]$-invariants on the conformal blocks), we conclude that β is injective for any smooth curve. Hence j^* is an isomorphism for any curve. This completes the proof.

From the above proof we immeadiately deduce the

Corollary 3.2. *For a general curve the family* $\{\Delta_\kappa\}_{\kappa \in \theta^+(C)}$ *linearly spans* $\mathbb{P}H^0(\mathcal{M}(\mathrm{G}_2), \mathcal{L}_{\mathrm{G}_2})$.

Remark: Note that Hitchin's connection [H1] is only defined on the vector bundle with fiber $H^0(\mathcal{M}(\mathrm{G}_2), \mathcal{L}_{\mathrm{G}_2}^{\otimes 2})$. We only get a connection for G_2 at level 1 via the isomorphism with the bundle of conformal blocks.

3.2. **Theorem II.** We consider the family of divisors $\{\Delta_\kappa\}_{\kappa \in \theta^+(C)}$ of $\mathbb{P}H^0(\mathcal{M}(\mathrm{G}_2), \mathcal{L}_{\mathrm{G}_2})$ introduced in section 2.3. A straightforward computation shows that $i^*(\Delta_\kappa) = H_\kappa$, where $H_\kappa \in \mathbb{P}H^0(\mathcal{M}(\mathrm{SL}_3), \mathcal{L})_+$ is the divisor with support

$$\mathrm{supp}(H_\kappa) = \{E \in \mathcal{M}(\mathrm{SL}_3) \mid \dim H^0(C, E \otimes \kappa) > 0\}.$$

Therefore, in order to show surjectivity of i^*, it is enough to show that the family $\{H_\kappa\}_{\kappa \in \theta^+(C)}$ linearly spans $\mathbb{P}H^0(\mathcal{M}(\mathrm{SL}_3), \mathcal{L})_+$. This is done as follows.

We introduce the Riemann Theta divisor $\Theta = \{L \in \mathrm{Pic}^{g-1}(C) \mid \dim H^0(C, L) > 0\}$ in the Picard variety $\mathrm{Pic}^{g-1}(C)$ parameterizing degree $g-1$ line bundles over C. We recall [BNR] that there is a canonical isomorphism

$$H^0(\mathrm{Pic}^{g-1}(C), 3\Theta)^* \xrightarrow{\sim} H^0(\mathcal{M}(\mathrm{SL}_3), \mathcal{L}), \tag{5}$$

which is invariant for the two involutions, i.e. $L \mapsto K_C \otimes L^{-1}$ on $\mathrm{Pic}^{g-1}(C)$ and σ on $\mathcal{M}(\mathrm{SL}_3)$ respectively. We thus obtain an isomorphism between subspaces of invariant divisors $|3\Theta|_+^* \cong \mathbb{P}H^0(\mathcal{M}(\mathrm{SL}_3), \mathcal{L})_+$. Is is easy to check that via this isomorphism $H_\kappa = \varphi_{3\Theta}(\kappa)$, where

$$\varphi_{3\Theta} : \mathrm{Pic}^{g-1}(C) \dashrightarrow |3\Theta|_+^*$$

is the rational map given by the linear system $|3\Theta|_+$. In order to show that the family of points $\{\varphi_{3\Theta}(\kappa)\}_{\kappa \in \theta^+(C)}$ linearly spans $|3\Theta|_+^*$, we factorize the map $\varphi_{3\Theta}$ as

$$\varphi_{4\Theta} : \mathrm{Pic}^{g-1}(C) \dashrightarrow |4\Theta|_+^* \dashrightarrow |3\Theta|_+^*,$$

where the first map is the rational map given by the linear system $|4\Theta|_+^*$ and the second is the projection induced by the inclusion $H^0(3\Theta)_+ \xrightarrow{+\Theta} H^0(4\Theta)_+$. The result then follows from the main statement in [KPS] saying that $\{\varphi_{4\Theta}(\kappa)\}_{\kappa\in\theta^+(C)}$ is a projective basis of $|4\Theta|_+^*$ if C has no vanishing theta-null.

Remark: For a curve of genus 2 we observe that both spaces have the same dimension, hence i^* is an isomorphism in that case — note that any genus-2 curve is without vanishing theta-null.

3.3. **Corollary.** The statement of the corollary is proved in [LPS] for the conformal block $\mathcal{V}_3^*(\mathfrak{sl}_2) = H^0(\mathcal{M}(\mathrm{SL}_2), \mathcal{L}_{SL_2}^{\otimes 3})$. Having already observed in the proof of Theorem I that the vector bundle map β is projectively flat for the WZW-connections, it suffices to show the statement for the $JC[2]$-invariants of $\mathcal{V}_3^*(\mathfrak{sl}_2) \otimes \mathcal{V}_1^*(\mathfrak{sl}_2)$. This follows from [Be] Corollary 4.2.

4. Some remarks

Here we collect some additional computations.

4.1. **Verlinde formula for $l = 2$ and $g = 2$.** We just record the computation of the Verlinde number $\dim H^0(\mathcal{M}(\mathrm{G}_2), \mathcal{L}^2) = 30$. Since the line bundle $\mathcal{L}^2$ descends to the coarse moduli space $\mathrm{M}(\mathrm{G}_2)$ we obtain a rational θ-map

$$\theta : \mathrm{M}(\mathrm{G}_2) \longrightarrow |\mathcal{L}^2|^* = \mathbb{P}^{29}.$$

We refer to the paper [B4] for some results on the θ-map on a genus-2 curve for vector bundles of small rank.

4.2. **Analogue for the exceptional group F_4.** There is a similar coincidence for the conformal embedding of Lie algebras $\mathfrak{sl}(2)\oplus\mathfrak{sp}(6) \subset \mathfrak{f}_4$. In fact, we observe that $\dim H^0(\mathcal{M}(\mathrm{F}_4), \mathcal{L}_{\mathrm{F}_4}) = \dim H^0(\mathcal{M}(\mathrm{G}_2), \mathcal{L}_{\mathrm{G}_2})$ and that $\dim H^0(\mathcal{M}(\mathrm{Sp}_6), \mathcal{L}_{\mathrm{Sp}_6}) = \dim H^0(\mathcal{M}(\mathrm{SL}_2), \mathcal{L}_{\mathrm{SL}_2}^{\otimes 3})$ (symplectic strange duality). Moreover, $\ker(\mathrm{SL}_2 \times \mathrm{Sp}_6 \to \mathrm{F}_4) = \mathbb{Z}/2$. These facts suggest a similar isomorphism for the Verlinde space $H^0(\mathcal{M}(\mathrm{F}_4), \mathcal{L}_{\mathrm{F}_4})$, but the method presented in this paper does to apply to that case.

References

[A] T. Abe: Projective normality of the moduli space of rank 2 vector bundles on a generic curve, Trans. Amer. Math. Soc. 362 (2010), no.1, 477-490

[Ba] J. C. Baez: The octonions, Bull. Amer. Math. Soc. (N.S.) 39 (2002), No. 2, 145 - 205

[B1] A. Beauville: Fibrés de rang 2 sur une courbe, fibré déterminant et fonctions thêta, Bull. Soc. Math. France 116 (1988), no. 4, 431-448

[B2] A. Beauville: Fibrés de rang 2 sur une courbe, fibré déterminant et fonctions thêta II, Bull. Soc. Math. France 119 (1991), no. 3, 259-291

[B3] A. Beauville: Conformal blocks, Fusion rings and the Verlinde formula. Proc. of the Hirzebruch 65 Conf. on Algebraic Geometry, Israel Math. Conf. Proc. 9 (1996), 75-96

[B4] A. Beauville: Vector bundles and theta functions on curves of genus 2 and 3, Amer. J. of Math. 128 (2006), 607-618

[B5] A. Beauville: Conformal blocks, Fusion rings and the Verlinde formula. Proc. of the Hirzebruch 65 Conf. on Algebraic Geometry, Israel Math. Conf. Proc. 9 (1996), 75-96

[BLS] A. Beauville, Y. Laszlo, C. Sorger: The Picard group of the moduli of G-bundles on a curve, Compositio Math. 112 (1998), 183-216

[BNR] A. Beauville, M.S. Narasimhan, S. Ramanan: Spectral curves and the generalised theta divisor, J. Reine Angew. Math. 398 (1989), 169-179

[Be] P. Belkale: Strange duality and the Hitchin/WZW connection. J. Differential Geom. 82 (2009), no. 2, 445465.

[BD] A. Borel, J. De Siebenthal: Les sous-groupes fermés de rang maximum des groupes de Lie clos, Comment. Math. Helv. 23 (1949), 200-221

[Br] R. Bryant: Metrics with exceptional holonomy, Ann. of Math. (2), vol. 126 (1987), no. 3, 525-576

[G] C. Grégoire: Espace de modules de G_2-fibrés principaux sur une courbe algébrique, PhD-Thesis, October 2010, Université de Montpellier II

[H1] N.J. Hitchin: Flat connections and geometric quantization, Comm. Math. Phys. 131 (1990), 347-380

[KPS] Y. Kopeliovich, C. Pauly, O. Serman: On theta functions of order 4, Bull. London Math. Soc. 41 (2009), no. 3, 423 - 428

[LS] Y. Laszlo, C. Sorger: The line bundles on the moduli of parabolic G-bundles over curves and their sections, Ann. Sci. Ecole Norm. Sup (4) 30 (1997), no. 4, 499 - 525

[LPS] Y. Laszlo, C. Pauly, C. Sorger: On the monodromy of the Hitchin connection, J. Geometry and Physics 64 (2013), 64-78

[S] S. Subramanian: On principal G_2 bundles, Asian J. Math. 3 (1999), no. 2, 353 - 357

[SK] M. Sato, T. Kimura: A classification of irreducible prehomogeneous vector spaces and their relative invariants, Nagoya Math. J. 65 (1977), 1 - 155

[U] K. Ueno: Introduction to conformal field theory with gauge symmetries, in Geometry and Physics, Lecture Notes in Pure and Applied Mathematics 184, Marcel Dekker (1996), 603-745

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