EXERCICES ALGEBRAIC GEOMETRY
MASTER 2

Affine varieties

(1) Let R be a Noetherian ring.
  (a) Let $I \subset R$ be an ideal. Show that the quotient ring R/I is also Noetherian.
  (b) Suppose that R is an integral domain and let $R \subset Q(R)$ be its field of fractions. Let $0 \notin S \subset R$ be a subset, and define
$$T = R[S^{-1}] = \{\frac{a}{b} \in Q(R) \mid a \in R, b = 1 \text{ or a product of elements in } S\}.$$
  Show that T is also a Noetherian ring. *Hint: show that an ideal in T is determined by its intersection with the subring $R \subset T$.*

(2) Show that the radical of an ideal $I \subset R$ defined by
$$\sqrt{I} = \{r \in R \mid \exists n \in \mathbb{N}^* \quad r^n \in I\}$$
is an ideal in R.

(3) Find the radicals of the following ideals
- $(x^3) \subset k[x]$
- $(xy) \subset k[x, y]$
- $(x^3y^5) \subset k[x, y]$
- $(x^2(x^2 - y^2)) \subset k[x, y]$

(4) Give an example of a radical ideal, which is not prime.

(5) Consider the following ideals in $k[x, y, z]$
$$J = (xy, xz, yz) \text{ and } J' = (xy, (x - y)z).$$
  (a) Determine the two sets $Z(J)$ and $Z(J')$.
  (b) Are they irreducible ?
  (c) Show that $J = I(Z(J))$.
  (d) Determine the radical $\sqrt{J'}$. Give the explicit expression of the powers of generators of $\sqrt{J'}$ in terms of generators of J'.

(6) Consider the ideal $J = (xz - y^2, x^3 - yz) \subset k[x, y, z]$.
  (a) Show that J is not prime. *Hint: find an expression of $x(z^2 - x^2y)$ in terms of the generators of J.*
  (b) Show that there is a decomposition into closed subsets
$$Z(J) = (Z(J) \cap Z(x)) \cup \left(Z(J) \cap Z(z^2 - x^2y)\right).$$
  (c) Determine $Z(J) \cap Z(x)$.
  (d) We denote $C = Z(J) \cap Z(z^2 - x^2y)$. Show that there is a surjective map
$$\varphi : \mathbb{A}^1 \to C \subset \mathbb{A}^3, \quad \varphi(t) = (t^3, t^4, t^5).$$
  (e) Deduce that C is irreducible.

Date: September 2018.

(7) (***) A unique factorization domain (UFD) is defined to be an integral domain R such that every non-zero $r \in R$ can be written uniquely as

$$r = up_1 \dots p_n, \text{ with } u \text{ unit and } p_i \text{ irreducible.}$$

(a) Let R be a UFD and let $r \in R$ be irreducible. Then the ideal (r) generated by r is a prime ideal.
(b) If K is a field, then the ring $K[X]$ is a UFD. *Hint: use Euclidian division in* $K[X]$
(c) (**) Let R be a UFD. A polynomial $f \in R[X]$ is said to be primitive if the greatest common divisor of its coefficients is 1. Then (Gauss Lemma) a product of primitive polynomials is primitive.
(d) (***) If R is a UFD, then the ring $R[X]$ is a UFD. *Hint: use unique factorization in* $Q(R)[X]$ *and Gauss lemma on primitive polynomials.*
(e) Deduce that the ring $K[X_1, \dots, X_n]$ is a UFD.

(8) (**) Let K denote the field of fractions of the ring $k[X]$ and consider $k[X,Y] = k[X][Y]$ as a subring of $K[Y]$. Recall that $k[Y]$ is a principal ideal domain (every ideal in $k[Y]$ is principal).
(a) Let $I \subset K[Y]$ be an ideal and put $J = I \cap k[x,y]$. Show that J is an ideal in $k[x,y]$ and that the ideal generated by J in $K[Y]$ equals I.
(b) Let $p \in K[Y]$ and define

$$L = \{d \in k[X] \mid d \cdot p \in k[X,Y]\}.$$

Show that L is an ideal in $k[X]$.
(c) Deduce that for any $p \in K[Y]$ there exists a $\overline{p} \in k[X,Y]$ such that $(p) \cap k[X,Y] = (\overline{p})$.
(d) Deduce that if $f, g \in k[X,Y]$ have a common factor in $K[Y]$, then f, g have a common factor in $k[X,Y]$.

(9) (**) Same notation as in the previous exercise. Let $f, g \in k[X,Y]$ be irreducible elements, not multiples of one another.
(a) Show that f and g have no common factor in $K[Y]$, where $K = k(X)$.
(b) Show that there exist polynomials $h \in k[X]$ and $a, b \in k[X,Y]$ such that

$$h = af + bg.$$

(c) Deduce that the zero set $Z(f,g)$ is a finite set.

Topological spaces and Zariski topology

(10) (**) Consider $\mathbb{A}^n$ as a topological space with the Zariski topology. Let $\overline{X}$ denote the closure of a set X in $\mathbb{A}^n$. Show the equality

$$Z(I(X)) = \overline{X}.$$

(11) (**) Consider $\mathbb{P}^n$ as a topological space with the Zariski topology. Let $\overline{X}$ denote the closure of a set X in $\mathbb{P}^n$. Show the equality

$$Z_p(I_H(X)) = \overline{X}.$$

(12) (*) Let X be a topological space and let $(U_i)_{i \in I}$ be an open cover of X. Show that $Y \subset X$ is closed if and only if $Y \cap U_i$ is closed in U_i for all $i \in I$.

(13) (*) Let X be an irreducible topological space and let U_1, U_2 be two non-empty open subsets of X. Then $U_1 \cap U_2$ is non-empty.

(14) (*) Let $f : X \to Y$ be a continuous map between two topological spaces. Suppose that the image of f is dense in Y and that X is irreducible. Then Y is also irreducible.

(15) (*) A topological space is quasi-compact if one can extract from any open covering a finite open covering. Show that a Noetherian topological space is quasi-compact. *Hint: Use descending chain of closed subsets.*

(16) (*) Describe the closed sets of the topology on $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ which is the product of the Zariski topologies on the two factors. Find a closed subset in the Zariski topology of $\mathbb{A}^2$ not of this form.

Projective varieties

(17) (*) For $i = 0, \ldots, n$ show that the two maps

$$u_i : \mathbb{A}^n \to U_i = \{[a_0, \ldots, a_n] \in \mathbb{P}^n \mid a_i \neq 0\} \subset \mathbb{P}^n \text{ and } \varphi_i : U_i \to \mathbb{A}^n,$$

as defined in the lecture, are inverse to each other.

(18) (**) Show that an ideal $I \subset k[X_0, \ldots, X_n]$ is homogeneous if and only if it can is generated by homogeneous polynomials.

(19) (**) Show that a non-zero polynomial $f \in k[X_0, \ldots X_n]$ is homogeneous if and only if the ideal (f) contains at least one non-zero homogeneous polynomial.

(20) (*) Use Hilbert Basis Theorem to show that a homogeneous ideal $I \subset k[X_0, \ldots, X_n]$ can be generated by a finite set of homogeneous polynomials.

(21) (**) Let $I \subset k[X_0, \ldots, X_n]$ be a homogeneous ideal and let $X = Z_p(I) \subset \mathbb{P}^n$ be its projective zero set and $C(X) \subset \mathbb{A}^{n+1}$ be its associated cone.
   (a) If $X \neq \emptyset$, then $C(X) = Z_a(I) \subset \mathbb{A}^{n+1}$ and $I(C(X)) = I_H(X)$
   (b) If $X = \emptyset$, then $C(X) = \{0\}$ and $Z_a(I) = \{0\}$ or $\emptyset$.

(22) (**) Let $X \subset \mathbb{A}^n$ be a non-empty affine algebraic set defined by an ideal $X = Z(I)$. Define the ideal $\tilde{I} \subset k[X_0, \ldots, X_n]$ by defining its homogeneous part of degree d as

$$\tilde{I}_d = \{X_0^d f(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}) \mid f \in I, \deg(f) \leq d\}.$$

   (a) Show that $\tilde{I} = \bigoplus_{d \geq 0} \tilde{I}_d$ is a homogeneous ideal.
   (b) Show that $X_0 \notin \sqrt{\tilde{I}}$.
   (c) Show that if I is prime, then $\tilde{I}$ is also prime.
   (d) Given a homogeneous ideal $J \subset k[X_0, \ldots, X_n]$ such that $X_0 \notin \sqrt{J}$ we define $\overline{J} \subset k[X_1, \ldots, X_n]$ as

$$\overline{J} = \{f(1, X_1, \ldots, X_n) \in k[X_1, \ldots, X_n] \mid f \in J\}.$$

   Show that $\overline{J}$ is an ideal in $k[X_1, \ldots, X_n]$.
   (e) Show that $Z(\overline{J}) \neq \emptyset$.
   (f) Show that if J is prime, then $\overline{J}$ is also prime.
   (g) For any proper ideal $I \subset k[X_1, \ldots, X_n]$, we have $\overline{\tilde{I}} = I$.
   (h) For any prime ideal J with $X_0 \notin \sqrt{J}$, we have $\tilde{\overline{J}} = J$.

(i) Show that the projective zero set $Z_p(\tilde{I})$ is the closure $\overline{X} \subset \mathbb{P}^n$ of X in the Zariski topology of $\mathbb{P}^n$. We view $\mathbb{A}^n$ as the open subset in $\mathbb{P}^n$ given as the complement of the hyperplane at infinity $H_\infty = Z_p(X_0)$. *Hint: Show that one can reduce to the case I radical. If I is radical, show the equality $I_H(X) = \tilde{I}$ and apply Exercise (11).*

(23) (*) Use the notation of the previous exercise.
(a) Show that there is a natural isomorphism of k-algebras $S(\overline{X}) \to A(X)$.
(b) Let $K(\overline{X})$ be the field of rational functions of the projective variety $\overline{X} \subset \mathbb{P}^n$ and let $K(X)$ be the field of rational functions of the affine variety $X \subset \mathbb{A}^n$. Show that there is a natural k-linear isomorphism $K(\overline{X}) \cong K(X)$.

Regular functions and morphisms

(24) (*) Consider the affine curve defined as the zero set $C = Z(f) \subset \mathbb{A}^2$ of the polynomial $f(X,Y) = Y^2 - X^3$. Consider the polynomial map

$$\varphi : \mathbb{A}^1 \to C, \quad T \mapsto (T^2, T^3).$$

(a) Show that φ is a bijective map. Describe the inverse map $\varphi^{-1} : C \to \mathbb{A}^1$.
(b) Show that φ is a morphism, but not an isomorphism.
(c) Consider the restriction $\varphi_0 : \mathbb{A}^1 \setminus \{0\} \to C \setminus \{(0,0)\}$. Show that φ_0 is an isomorphism.

(25) (**) Show that the two maps u_i and φ_i of Exercise (13) are isomorphisms.

(26) (*) Consider the polynomial map $\varphi : \mathbb{A}^1 \to \mathbb{A}^3$ given by $T \mapsto (T, T^2, T^3)$.
(a) Show that the image of φ is an algebraic subset $C \subset \mathbb{A}^3$.
(b) Show that $\varphi : \mathbb{A}^1 \to C$ is an isomorphism.
(c) Try to generalize.

(27) (**) Let n be a positive integer. Consider the polynomial map $\varphi_n : \mathbb{A}^1 \to \mathbb{A}^2$ given by $T \mapsto (T^2, T^n)$.
(a) Show that if n is even, then the image of φ_n is isomorphic to $\mathbb{A}^1$ and φ_n is 2-to-1 outside $(0,0)$.
(b) Show that if n is odd, the φ_n is a bijective map. Give a rational inverse to φ_n.

(28) (**) Let C denote the closed subset $Z(X^3 + X^2 - Y^2) \subset \mathbb{A}^2$.
(a) Show that C is irreducible.
(b) Consider the polynomial map $\varphi : \mathbb{A}^1 \to C$ given by $T \mapsto (T^2 - 1, T^3 - T)$. Is φ an isomorphism ?
(c) Is the restriction $\varphi_0 : \mathbb{A}^1 \setminus \{1\} \to C \setminus \{(0,0)\}$ an isomorphism ?

(29) (*) (Projective transformation) Let A be an invertible $(n+1) \times (n+1)$ matrix with coefficients in the field k. Define a map $[A] : \mathbb{P}^n \to \mathbb{P}^n$ by $[b_0, \ldots, b_n] \mapsto [A(b_0, \ldots b_n)^t]$, where $(b_0, \ldots b_n)^t$ denotes the column vector of $(b_0, \ldots b_n)$.
(a) Show that the map $[A]$ is well-defined.
(b) Show that $[A]$ is an isomorphism with inverse $[A]^{-1} = [A^{-1}]$.

(30) (*) (Projection) Consider $H_0, \ldots, H_{n-l-1}$ independent linear forms on $\mathbb{P}^n$, i.e. the H_i are homogeneous polynomials of degree 1 in $k[X_0, \ldots X_n]$.

(a) Define $W = Z_p(H_0, \ldots, H_{n-l-1}) \subset \mathbb{P}^n$. Show that W is isomorphic to $\mathbb{P}^l$.

(b) Let $X \subset \mathbb{P}^n$ be a subvariety such that $X \cap W = \emptyset$. Consider the morphism $\pi_W : X \to \mathbb{P}^{n-l-1}$ defined by

$$p \mapsto \pi_W(p) = [H_0(p), \ldots, H_{n-l-1}(p)].$$

Show that π_W is well-defined and that π_W is a morphism. Show that π_W only depends on the linear subspace W and not on the linear forms H_i.

(c) Let $l = 0$ and suppose that $W = \{p\}$ with $p = [0, \ldots, 0, 1]$ and $H_i = X_i$ for $i = 0, \ldots, n-1$. Then $\pi_p : \mathbb{P}^n \setminus \{p\} \to \mathbb{P}^{n-1}$ defined by $X_0, \ldots, X_{n-1}$ is a well-defined morphism. Show that $\pi_p(q)$ is the intersection point of the line $\overline{pq}$ with the projective space $Z(X_n) \cong \mathbb{P}^{n-1}$.

(31) (**) (product of affine subvarieties) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be closed subvarieties. Show that the product

$$X \times Y = \{(x, y) \in \mathbb{A}^n \times \mathbb{A}^m \mid x \in X,\ y \in Y\}$$

is a closed subvariety in $\mathbb{A}^{n+m}$.

Hint: To show that $X \times Y$ is irreducible, suppose that $X \times Y = S_1 \cup S_2$ with S_i closed and show that $T_i = \{p \in X \mid \{p\} \times Y \subset S_i\}$ is closed in X.

(32) (*) Show the following generalization of the previous statement. Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be locally closed subvarieties. Then $X \times Y$ is a locally closed subvariety in $\mathbb{A}^{n+m}$.

(33) (*) (universal property of products) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be closed subvarieties.

(a) Show that the projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are morphisms.

(b) Let Z be any variety. Show that a morphism $\phi : Z \to X \times Y$ is the same as two morphisms $f : Z \to X$ and $g : Z \to Y$.

(34) (*) The quadric surface in $\mathbb{P}^3$.

(a) Show that the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ gives an isomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ with the quadric

$$Q = Z(X_0X_3 - X_1X_2) \subset \mathbb{P}^3.$$

(b) What are the images in Q of the two families of lines $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{p\}$?

(c) Show that there are two lines of Q passing through the point $p = [1, 0, 0, 0]$ and that the complement U of these two lines is the image of $\mathbb{A}^1 \times \mathbb{A}^1$ under the Segre embedding.

(d) Show that under the projection (exercise 30) $\pi_p : Q \dashrightarrow \mathbb{P}^2$ the open subset U maps isomorphically to a copy of $\mathbb{A}^2$ and the two lines through p are mapped to points of $\mathbb{P}^2$.

(35) Show that a morphism $f : \mathbb{P}^n \to \mathbb{P}^m$ with $m \geq n$ is a polynomial map, which means that f is given by $m + 1$ homogeneous polynomials of the same degree without common zeros. *Hint: Use the local description of a morphism and show that given two open subsets the local description of f on each open extends to their union.*

(36) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine subvarieties. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ be coordinates on $\mathbb{A}^n$ and $\mathbb{A}^m$. Then

(a) The affine coordinate ring of the product $X \times Y$ equals
$$A(X \times Y) = A(X) \otimes_k A(Y).$$
Hint: view $A(X)$ as ring of polynomial functions on X

(b) Let $I(X) \subset k[x_1, \ldots, x_n]$ and $I(Y) \subset k[y_1, \ldots, y_m]$ be the ideals of polynomials vanishing on X and Y. Then
$$I(X \times Y) = I(X) \otimes k[y_1, \ldots, y_m] + k[x_1, \ldots, x_n] \otimes I(Y).$$
Here $+$ denotes the sum of the two ideals.

Rational maps

(37) (**) (Cremona transformation) Let Φ be the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by
$$[x_0, x_1, x_2] \mapsto [x_0x_1, x_0x_2, x_1x_2].$$
(a) Determine the domain of Φ.
(b) Show that Φ is a dominant rational map.
(c) Show that Φ is birational by giving an inverse map.

(38) (**) Let $Q = Z(F) \subset \mathbb{P}^n$ be a quadric, i.e. F is a homogeneous polynomial of degree 2. Show that Q is rational. *Hint: Use projection π with center $p = [0, \ldots, 0, 1]$ and write $F = F_2 + F_1X_n + F_0X_n^2$ with $F_i \in k[X_0, \ldots, X_{n-1}]$ of degree i. Assume $p \in Q$. Give the birational inverse map of π in terms of the F_i*