

STRANGE DUALITY FOR VERLINDE SPACES OF EXCEPTIONAL GROUPS AT LEVEL ONE

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ABSTRACT. The moduli stack $\mathcal{M}_X(E_8)$ of principal E_8 -bundles over a smooth projective curve X carries a natural divisor Δ . We study the pull-back of the divisor Δ to the moduli stack $\mathcal{M}_X(P)$, where P is a semi-simple and simply connected group such that its Lie algebra $\mathrm{Lie}(P)$ is a maximal conformal subalgebra of $\mathrm{Lie}(E_8)$. We show that the divisor Δ induces “Strange Duality”-type isomorphisms between the Verlinde spaces at level one of the following pairs of groups $(\mathrm{SL}(5), \mathrm{SL}(5))$, $(\mathrm{Spin}(8), \mathrm{Spin}(8))$, $(\mathrm{SL}(3), E_6)$ and $(\mathrm{SL}(2), E_7)$.

1. INTRODUCTION

Let X be a smooth complex projective curve of genus g and let G be a simple and simply connected complex Lie group. We denote by $\mathcal{M}_X(G)$ the moduli stack parametrizing principal G -bundles over the curve X and by \mathcal{L}_G the ample line bundle over $\mathcal{M}_X(G)$ generating its Picard group. The starting point of our investigation is the observation (see e.g. [So], [F1], [F2]) that

$$\dim H^0(\mathcal{M}_X(E_8), \mathcal{L}_{E_8}) = 1.$$

for any genus g . In other words, the moduli stack $\mathcal{M}_X(E_8)$ carries a natural divisor Δ . Unfortunately a geometric interpretation of this divisor is not known.

In this paper we study the pull-back of this mysterious divisor Δ under the morphisms $\mathcal{M}_X(P) \rightarrow \mathcal{M}_X(E_8)$ induced by the group homomorphisms $\phi : P \rightarrow E_8$, where we assume that P is connected, simply connected and semi-simple, and that the differential $d\phi : \mathfrak{p} = \mathrm{Lie}(P) \rightarrow \mathfrak{e}_8 = \mathrm{Lie}(E_8)$ is a *conformal* embedding of Lie algebras (see Definition 3.1). We recall ([BB] p. 566) that any subalgebra of maximal rank 8 of \mathfrak{e}_8 (see [BD] Chapter 7 for a list) is actually a conformal subalgebra of \mathfrak{e}_8 with Dynkin (multi-)index one. Maximal conformal subalgebras of \mathfrak{e}_8 with Dynkin (multi-)indices one have been classified by [BB] and [SW], and the full list is as follows:

- (1) maximal rank : $\mathfrak{so}(16)$, $\mathfrak{sl}(9)$, $\mathfrak{sl}(5) \oplus \mathfrak{sl}(5)$, $\mathfrak{sl}(3) \oplus \mathfrak{e}_6$, $\mathfrak{sl}(2) \oplus \mathfrak{e}_7$
 non-maximal rank : $\mathfrak{g}_2 \oplus \mathfrak{f}_4$.

In Table (2) we list the corresponding simply connected Lie groups P and the finite kernel N of their natural maps to E_8 (see e.g. [CG] Lemma 3.3).

(2)

P	$\mathrm{Spin}(16)$	$\mathrm{SL}(9)$	$\mathrm{SL}(5) \times \mathrm{SL}(5)$	$\mathrm{SL}(3) \times E_6$	$\mathrm{SL}(2) \times E_7$	$G_2 \times F_4$
N	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/5\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1

Note that N is a subgroup of the center of P . We introduce the finite abelian group $\mathcal{M}_X(N)$ of principal N -bundles over X , which acts on $\mathcal{M}_X(P)$ by twisting P -bundles with N -bundles. Since N is the kernel of ϕ , the group $\mathcal{M}_X(N)$ acts on the fibers of the induced stack morphism

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$\tilde{\phi} : \mathcal{M}_X(P) \rightarrow \mathcal{M}_X(E_8)$. We emphasize that the $\mathcal{M}_X(N)$ -linearization of the line bundle \mathcal{L}_P over $\mathcal{M}_X(P)$ is not unique, but we obtain a *canonical* $\mathcal{M}_X(N)$ -linearization (hence a *canonical* linear action of $\mathcal{M}_X(N)$ on $H^0(\mathcal{M}_X(P), \mathcal{L}_P)$), namely, the one induced from the isomorphism $\tilde{\phi}^* \mathcal{L}_{E_8} = \mathcal{L}_P$.

If P has two simple factors, the line bundle \mathcal{L}_P denotes the tensor product of the ample generators on each factor (see section 2.1). With this notation our main result can be stated as follows.

Theorem 1. Let P be a Lie group of Table (2) and let

$$\phi_P : H^0(\mathcal{M}_X(E_8), \mathcal{L}_{E_8}) \longrightarrow H^0(\mathcal{M}_X(P), \mathcal{L}_P)$$

be the linear map on global sections induced by the homomorphism $\phi : P \rightarrow E_8$. Then

- (i) The linear map ϕ_P is nonzero.
- (ii) In all cases except $P = G_2 \times F_4$, its one-dimensional image coincides with the $\mathcal{M}_X(N)$ -invariant subspace of $H^0(\mathcal{M}_X(P), \mathcal{L}_P)$, where $\mathcal{M}_X(N)$ acts on $H^0(\mathcal{M}_X(P), \mathcal{L}_P)$ with the canonical action.

Unfortunately, even in the cases $\mathrm{SL}(9)$ and $\mathrm{Spin}(16)$, we are not able to give a geometric description of the zero-divisor of the $\mathcal{M}_X(N)$ -invariant section in the moduli stacks $\mathcal{M}_X(\mathrm{SL}(9))$ and $\mathcal{M}_X(\mathrm{Spin}(16))$ — see section 7.1 for further discussion.

A word about the proof of Theorem 1. We make use of the identification of the space of generalized G -theta functions $H^0(\mathcal{M}_X(G), \mathcal{L}_G)$ with the space of conformal blocks $\mathcal{V}_0^\dagger(X, \mathfrak{g})$ associated to the curve X with one marked point labelled with the zero weight. Here $\mathfrak{g} = \mathrm{Lie}(G)$. Under this identification the linear map ϕ_P becomes the map induced by the natural inclusion of the basic highest weight modules $\mathcal{H}_0(\mathfrak{p}) \hookrightarrow \mathcal{H}_0(\mathfrak{e}_8)$ of the affine Lie algebras $\hat{\mathfrak{p}}$ and $\hat{\mathfrak{e}}_8$. The proof has essentially two steps. First, we use a result by P. Belkale ([B] Proposition 5.8) saying that the linear map ϕ_P has constant rank when the curve X varies in a family of smooth curves. Here, the fact that the embedding $\mathfrak{p} \subset \mathfrak{e}_8$ is conformal, is crucial, since it ensures that ϕ_P is projectively flat with respect to the WZW connections on both sheaves of vacua over any family of smooth curves. Secondly, we study in section 4 the behaviour of the factorization rules of the spaces of conformal blocks associated to \mathfrak{g} under any conformal embedding $\mathfrak{p} \subset \mathfrak{g}$. This will follow once one has decomposed the “sewing-procedure” tensor $\tilde{\gamma}_\lambda \in \mathcal{H}_\lambda(\mathfrak{g}) \otimes \mathcal{H}_{\lambda^\dagger}(\mathfrak{g})[[q]]$ under the decomposition of the $\hat{\mathfrak{g}}$ -modules $\mathcal{H}_\lambda(\mathfrak{g})$ and $\mathcal{H}_{\lambda^\dagger}(\mathfrak{g})$ into irreducible $\hat{\mathfrak{p}}$ -modules. Finally, combining these two steps allows us to show by induction on the genus of X that ϕ_P is non-zero.

In the cases when P is not simple and N not trivial, an argument using the representation theory of Heisenberg groups allows us to show the following result, which can be seen as an instance of Strange Duality for exceptional groups at level one.

Theorem 2. Let (A, B) be one of the three pairs $(\mathrm{SL}(5), \mathrm{SL}(5))$, $(\mathrm{SL}(3), E_6)$, $(\mathrm{SL}(2), E_7)$. Consider *any* $\mathcal{M}_X(N)$ -linearization of the line bundle $\mathcal{L}_{A \times B}$ over $\mathcal{M}_X(A \times B)$. Let

$$\sigma \in H^0(\mathcal{M}_X(A \times B), \mathcal{L}_{A \times B}) = H^0(\mathcal{M}_X(A), \mathcal{L}_A) \otimes H^0(\mathcal{M}_X(B), \mathcal{L}_B)$$

be a non-zero element of the one-dimensional $\mathcal{M}_X(N)$ -invariant subspace. Then σ induces an isomorphism

$$\sigma : H^0(\mathcal{M}_X(A), \mathcal{L}_A)^* \longrightarrow H^0(\mathcal{M}_X(B), \mathcal{L}_B).$$

A similar isomorphism is obtained for the pair $(\text{Spin}(8), \text{Spin}(8))$ — see section 7.2.1.

We would like to mention that the proofs of Theorem 1 and Theorem 2 are independent, and that both results are related by the fact that the “Strange Duality” isomorphism of Theorem 2 corresponding to the *canonical* $\mathcal{M}_X(N)$ -linearization is obtained precisely by pulling-back the E_8 -theta divisor Δ to the moduli stack $\mathcal{M}_X(A \times B)$.

The last few years have seen important progress on “Strange Duality” or “rank-level” duality for Verlinde spaces. For a survey we refer, for example, to the papers [MO], [Po] or [Pa].

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2. NOTATION AND PRELIMINARIES

2.1. Moduli stacks and line bundles.

2.1.1. *Dynkin index.* Let \mathfrak{p} and \mathfrak{g} be two simple Lie algebras and let $\varphi : \mathfrak{p} \rightarrow \mathfrak{g}$ be a Lie algebra homomorphism. There exists [D] a unique integer d_φ , called the Dynkin index of the homomorphism φ , satisfying

$$(\varphi(x), \varphi(y))_{\mathfrak{g}} = d_\varphi(x, y)_{\mathfrak{p}}, \quad \text{for all } x, y \in \mathfrak{p},$$

where $(\ , \)_*$ denotes the Cartan-Killing form on \mathfrak{p} and \mathfrak{g} , normalized such that $(\theta, \theta) = 2$ for their respective highest roots θ . If \mathfrak{p} is semi-simple with two components $\mathfrak{p}_1 \oplus \mathfrak{p}_2$, then the Dynkin multi-index of $\varphi = \varphi_1 \oplus \varphi_2 : \mathfrak{p}_1 \oplus \mathfrak{p}_2 \rightarrow \mathfrak{g}$ is given by $d_\varphi = (d_{\varphi_1}, d_{\varphi_2})$, where each d_{φ_i} is defined using $\varphi_i : \mathfrak{p}_i \rightarrow \mathfrak{g}$.

2.1.2. *Line bundles over the moduli stack $\mathcal{M}_X(P)$.* If P is a simple and simply connected complex Lie group we refer to [LS] and [So] for the description of the ample generator \mathcal{L}_P of the Picard group of the moduli stack $\mathcal{M}_X(P)$. If $P = P_1 \times P_2$ with P_i simple and simply connected, we put $\mathcal{L}_P = \mathcal{L}_{P_1} \boxtimes \mathcal{L}_{P_2}$ and we note $\mathcal{L}_P^d = \mathcal{L}_{P_1}^{d_1} \boxtimes \mathcal{L}_{P_2}^{d_2}$ for a multi-index $d = (d_1, d_2)$. The following lemma follows easily from [LS] and [KNR].

Lemma 2.1. *Let $\phi : P \rightarrow G$ be a homomorphism between simply-connected complex Lie groups with G simple and P semi-simple. Let $\tilde{\phi} : \mathcal{M}_X(P) \rightarrow \mathcal{M}_X(G)$ be the induced stack morphism. Then we have the equality*

$$\tilde{\phi}^* \mathcal{L}_G = \mathcal{L}_P^{d_\varphi},$$

where d_φ is the Dynkin (multi-) index of the differential $\varphi = d\phi : \mathfrak{p} \rightarrow \mathfrak{g}$.

2.2. Spaces of conformal blocks.

2.2.1. *The case \mathfrak{g} simple.* Let \mathfrak{g} be a simple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. We denote as before by $(\ , \)$ the normalized Cartan-Killing form and we will use the same notation for the restricted form on \mathfrak{h} and for the induced form on \mathfrak{h}^* . We consider (see [K] Chapter 7) the non-twisted affine Lie algebra associated to \mathfrak{g} over $\mathbb{C}((z))$

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((z)) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with Lie bracket

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg + (x, y) \operatorname{Res}_{z=0}(gdf) \cdot c, \quad [\mathfrak{g}, c] = 0, \quad [d, c] = 0, \quad [d, x(n)] = nx(n),$$

for $x, y \in \mathfrak{g}$, $f, g \in \mathbb{C}((z))$ and $n \in \mathbb{Z}$. Here we put $x(n) = x \otimes z^n$. We identify \mathfrak{g} with the subalgebra $\mathfrak{g} \otimes 1$ of $\hat{\mathfrak{g}}$. The subalgebra

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

is the Cartan subalgebra of the affine Lie algebra $\hat{\mathfrak{g}}$. We extend $\lambda \in \mathfrak{h}^*$ to a linear form on $\hat{\mathfrak{h}}$ by putting $\langle \lambda, \mathbb{C}c \oplus \mathbb{C}d \rangle = 0$, where $\langle \ , \ \rangle$ is the standard pairing. We define the elements Λ_0 and δ in the dual $\hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ by $\langle \delta, d \rangle = \langle \Lambda_0, c \rangle = 1$ and $\langle \delta, \mathfrak{h} \oplus \mathbb{C}c \rangle = \langle \Lambda_0, \mathfrak{h} \oplus \mathbb{C}d \rangle = 0$. We extend the form $(\ , \)$ to $\hat{\mathfrak{h}}^*$ by putting

$$(\mathfrak{h}^*, \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta) = 0, \quad (\delta, \delta) = (\Lambda_0, \Lambda_0) = 0, \quad (\delta, \Lambda_0) = 1.$$

The Weyl group of \mathfrak{g} is denoted by $W(\mathfrak{g})$. Call $w_0^{\mathfrak{g}} \in W(\mathfrak{g})$ its longest element. Later we will need the following fact

Proposition 2.2. *Let $\mathfrak{p} \subset \mathfrak{g}$ be an embedding of semi-simple Lie algebras and choose Cartan subalgebras such that $\mathfrak{h}_{\mathfrak{p}} \subset \mathfrak{h}_{\mathfrak{g}}$. Then there exists an element $\tilde{w} \in W(\mathfrak{g})$ which preserves the subspace $\mathfrak{h}_{\mathfrak{p}} \subset \mathfrak{h}_{\mathfrak{g}}$ and such that the restriction $\tilde{w}|_{\mathfrak{h}_{\mathfrak{p}}}$ coincides with the longest element $w_0^{\mathfrak{p}}$.*

Proof. This can be deduced from a more general fact — see e.g. Theorem 2.1.4 [BS]. Moreover if \mathfrak{p} and \mathfrak{g} have the same rank (all our cases except $\mathfrak{g}_2 \oplus \mathfrak{f}_4$), i.e. $\mathfrak{h}_{\mathfrak{p}} = \mathfrak{h}_{\mathfrak{g}}$, we have a canonical inclusion $W(\mathfrak{p}) \subset W(\mathfrak{g})$. \square

Next, we recall some representation theory of the affine Lie algebra $\hat{\mathfrak{g}}$ from [K] Chapter 12.

We denote by $P(\mathfrak{g})$ the weight lattice of \mathfrak{g} and by $P_+(\mathfrak{g})$ the subset of $P(\mathfrak{g})$ consisting of dominant integral weights. Given a positive integer k , called level, we consider the finite set

$$P_k(\mathfrak{g}) := \{\lambda \in P_+(\mathfrak{g}) \mid (\lambda, \theta) \leq k\} \subset \mathfrak{h}^*.$$

Given $\lambda \in P_k(\mathfrak{g})$ we denote by $\mathcal{H}_{\lambda}(\mathfrak{g})$, or simply \mathcal{H}_{λ} if no confusion arises, the integrable $\hat{\mathfrak{g}}$ -module with highest weight $\lambda + k\Lambda_0$; in particular,

- (1) the center $c \in \hat{\mathfrak{h}}$ acts on \mathcal{H}_{λ} as $k \cdot \operatorname{Id}$.
- (2) the derivation $d \in \hat{\mathfrak{h}}$ acts trivially on the highest weight vector v_{λ} of \mathcal{H}_{λ} .

We introduce the set

$$\widehat{P_k(\mathfrak{g})} := \{\hat{\lambda} = \lambda + k\Lambda_0 + \zeta\delta \mid \lambda \in P(\mathfrak{g}), \zeta \in \mathbb{C}\} \subset \hat{\mathfrak{h}}^*.$$

Given $\hat{\lambda} = \lambda + k\Lambda_0 + \zeta\delta$ in $\widehat{P_k(\mathfrak{g})}$ we define $\hat{\lambda}^{\dagger} = -w_0^{\mathfrak{g}}(\lambda) + k\Lambda_0 + \zeta\delta$. This gives an involution $\hat{\lambda} \rightarrow \hat{\lambda}^{\dagger}$ on the set $\widehat{P_k(\mathfrak{g})}$.

Note that there is a projection map

$$\widehat{P_k(\mathfrak{g})} \rightarrow P(\mathfrak{g}), \quad \hat{\lambda} = \lambda + k\Lambda_0 + \zeta\delta \mapsto \lambda.$$

We will view $P_k(\mathfrak{g})$ as a subset of $\widehat{P_k(\mathfrak{g})}$ under the mapping $\lambda \mapsto \hat{\lambda} = \lambda + k\Lambda_0$ and we observe that the involution $\hat{\lambda} \rightarrow \hat{\lambda}^\dagger$ restricts to an involution $\lambda \mapsto \lambda^\dagger = -w_0^{\mathfrak{g}}(\lambda)$ on the finite set $P_k(\mathfrak{g})$. Note that $-\lambda^\dagger$ is the lowest weight of the irreducible right \mathfrak{g} -module V_λ^* , the dual of V_λ .

More generally, for any $\hat{\lambda} = \lambda + k\Lambda_0 + \zeta\delta \in \widehat{P_k(\mathfrak{g})}$ such that $\lambda \in P_k(\mathfrak{g})$ we denote by $\mathcal{H}_{\hat{\lambda}}(\mathfrak{g})$ the integrable $\hat{\mathfrak{g}}$ -module with highest weight $\hat{\lambda} = \lambda + k\Lambda_0 + \zeta\delta$; in particular,

- (1) the center $c \in \hat{\mathfrak{h}}$ acts on $\mathcal{H}_{\hat{\lambda}}$ as $k \cdot \text{Id}$.
- (2) the derivation $d \in \hat{\mathfrak{h}}$ acts on the highest weight vector v_λ of $\mathcal{H}_{\hat{\lambda}}$ as $d \cdot v_\lambda = \zeta v_\lambda$.

For $\hat{\lambda} = \lambda + k\Lambda_0 + \zeta\delta \in \widehat{P_k(\mathfrak{g})}$ with $\lambda \in P_k(\mathfrak{g})$, we note that the $\hat{\mathfrak{g}}$ -modules \mathcal{H}_λ and $\mathcal{H}_{\hat{\lambda}}$ become isomorphic as modules over the subalgebra $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}] = \mathfrak{g} \otimes \mathbb{C}((z)) \oplus \mathbb{C}c$. The $\hat{\mathfrak{g}}$ -module $\mathcal{H}_0(\mathfrak{g})$ with zero weight and level 1 (i.e., $\hat{\lambda} = \Lambda_0$) is called the *basic* $\hat{\mathfrak{g}}$ -module.

Given s points $\vec{p} = (p_1, \dots, p_s)$ on X we consider the open subset $U = X \setminus \{p_1, \dots, p_s\}$, and choose a local coordinate ξ_i at each point p_i . Following [U] Definition 3.1.1 we introduce the Lie algebra

$$\hat{\mathfrak{g}}_s := \bigoplus_{j=1}^s \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\xi_j)) \oplus \mathbb{C}c.$$

There is a natural embedding of the ring $H^0(U, \mathcal{O}_U)$ of regular functions on U into $\bigoplus_{j=1}^s \mathbb{C}((\xi_j))$ via Laurent expansions at the points p_i . By [U] Lemma 3.1.2, $\mathfrak{g}(U) := \mathfrak{g} \otimes H^0(U, \mathcal{O}_U)$ is a Lie subalgebra of $\hat{\mathfrak{g}}_s$.

For $\vec{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_s) \in P_k(\mathfrak{g})^s$ we introduce the left $\hat{\mathfrak{g}}_s$ -module

$$\mathcal{H}_{\vec{\lambda}} := \mathcal{H}_{\lambda_1} \otimes \mathcal{H}_{\lambda_2} \otimes \dots \otimes \mathcal{H}_{\lambda_s}.$$

The space of covacua associated to the data $(X, \vec{p}, \vec{\lambda}, \vec{\xi})$ is defined as

$$\mathcal{V}_{\vec{\lambda}}(X, \mathfrak{g}) := \mathcal{H}_{\vec{\lambda}} / \mathfrak{g}(U) \cdot \mathcal{H}_{\vec{\lambda}},$$

where $\mathfrak{g}(U)$ acts on $\mathcal{H}_{\vec{\lambda}}$ via the inclusion in $\hat{\mathfrak{g}}_s$. We refer to [U] for further details.

The space of vacua or the *space of conformal blocks* is defined as the dual of $\mathcal{V}_{\vec{\lambda}}(X, \mathfrak{g})$ and is denoted by $\mathcal{V}_{\vec{\lambda}}^\dagger(X, \mathfrak{g})$. We note that there is an inclusion

$$\mathcal{V}_{\vec{\lambda}}^\dagger(X, \mathfrak{g}) \hookrightarrow \mathcal{H}_{\vec{\lambda}}^\dagger.$$

We denote by $\langle | \rangle$ the natural pairing between $\mathcal{H}_{\vec{\lambda}}$ and its dual $\mathcal{H}_{\vec{\lambda}}^\dagger$.

The construction of the space of conformal blocks can be carried out for a family $\mathcal{X} \rightarrow S$ of pointed nodal curves and provides a sheaf over the base scheme S , called the sheaf of vacua and denoted by $\mathcal{V}_{\vec{\lambda}}^\dagger(\mathcal{X}, \mathfrak{g})$. A fundamental property is that the sheaf of vacua is locally free and that it commutes with any base change in S (see e.g. [U] Theorem 4.4.2).

2.2.2. *The case \mathfrak{g} semi-simple.* We now adapt the previous constructions to semi-simple Lie algebras \mathfrak{g} . For our purposes it is enough to deal with the case when \mathfrak{g} is the direct sum of two simple Lie algebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. By [K] section 12.9 we define the affine Lie algebra associated to \mathfrak{g} by

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((z)) \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \mathbb{C}d,$$

with Cartan subalgebra $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \mathbb{C}d$. Similar to the case of simple algebras, one defines a Lie bracket and a non-degenerate bilinear form on $\hat{\mathfrak{h}}$ and on its dual $\hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0^{(1)} \oplus \mathbb{C}\Lambda_0^{(2)} \oplus \mathbb{C}\delta$.

Given a multi-index $k = (k_1, k_2)$, with k_i positive integers, we introduce the sets

$$\begin{aligned} P_k(\mathfrak{g}) &= P_{k_1}(\mathfrak{g}_1) \times P_{k_2}(\mathfrak{g}_2), \\ \widehat{P}_k(\mathfrak{g}) &= \{\hat{\lambda} = \lambda + k_1\Lambda_0^{(1)} + k_2\Lambda_0^{(2)} + \zeta\delta \mid \lambda \in P(\mathfrak{g}), \zeta \in \mathbb{C}\} \subset \hat{\mathfrak{h}}^*, \end{aligned}$$

and we associate to a weight $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in P_k(\mathfrak{g})$ the integrable $\hat{\mathfrak{g}}$ -module

$$\mathcal{H}_\lambda(\mathfrak{g}) = \mathcal{H}_{\lambda^{(1)}}(\mathfrak{g}_1) \otimes \mathcal{H}_{\lambda^{(2)}}(\mathfrak{g}_2).$$

Similarly, we introduce for any $\hat{\lambda} \in \widehat{P}_k(\mathfrak{g})$ the $\hat{\mathfrak{g}}$ -module $\mathcal{H}_{\hat{\lambda}}(\mathfrak{g})$.

With these definitions it is easy to deduce the following decomposition of the spaces of conformal blocks

$$\mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{g}) = \mathcal{V}_{\vec{\lambda}^{(1)}}^\dagger(\mathfrak{g}_1) \otimes \mathcal{V}_{\vec{\lambda}^{(2)}}^\dagger(\mathfrak{g}_2) \quad \text{with } \vec{\lambda} = (\vec{\lambda}^{(1)}, \vec{\lambda}^{(2)}) \in P_k(\mathfrak{g})^s.$$

2.3. **Generalized theta functions and the Verlinde formula.** For the convenience of the reader we recall (see [LS], [F3], [KNR]) that there is an isomorphism between the space $H^0(\mathcal{M}_X(G), \mathcal{L}_G^k)$ of generalized G -theta functions of level k and the space of conformal blocks $\mathcal{V}_0^\dagger(X, \mathfrak{g})$ associated to the curve X with one marked point labelled with the zero weight at level k , i.e.,

$$(3) \quad H^0(\mathcal{M}_X(G), \mathcal{L}_G^k) \xrightarrow{\sim} \mathcal{V}_0^\dagger(X, \mathfrak{g}).$$

The dimension of this space is given by the Verlinde formula (see [F3], [T]). Its value for the groups G of type ADE and at level one is $|Z|^g$, where $|Z|$ denotes the order of the center Z of G (see [F1]). The list is as follows:

$$(4) \quad \begin{array}{c|ccccc} G & \text{SL}(n) & \text{Spin}(2n) & E_6 & E_7 & E_8 \\ \hline \dim H^0(\mathcal{M}_X(G), \mathcal{L}_G) & n^g & 4^g & 3^g & 2^g & 1 \end{array}$$

3. CONFORMAL PAIRS: PROPERTIES OF THEIR REPRESENTATIONS

3.1. **The Virasoro algebra and its representation on $\mathcal{H}_{\hat{\lambda}}(\mathfrak{g})$.** The Virasoro algebra

$$\text{Vir} := \bigoplus_{n \in \mathbb{Z}} \mathbb{C}d_n \oplus \mathbb{C}\tilde{c},$$

is defined by the relations $[d_i, d_j] = (i - j)d_{i+j} + \frac{1}{12}(i^3 - i)\delta_{i,-j}\tilde{c}$ and $[d_i, \tilde{c}] = 0$.

If \mathfrak{g} is simple, we define for any level k and any $\hat{\lambda} \in \widehat{P}_k(\mathfrak{g})$, the conformal anomaly $c(\mathfrak{g}, k)$ and the trace anomaly $\Delta_{\hat{\lambda}}(\mathfrak{g})$ as

$$c(\mathfrak{g}, k) = \frac{k \dim \mathfrak{g}}{\check{h}(\mathfrak{g}) + k}, \quad \text{and} \quad \Delta_{\hat{\lambda}}(\mathfrak{g}) = \frac{(\hat{\lambda}, \hat{\lambda} + 2\hat{\rho})}{2(\check{h}(\mathfrak{g}) + k)}.$$

Here $\check{h}(\mathfrak{g})$ is the dual Coxeter number of \mathfrak{g} and $\hat{\rho} = \rho + \check{h}(\mathfrak{g})\Lambda_0$, where ρ denotes the half-sum of the positive roots of \mathfrak{g} . We choose dual bases $\{u_i\}$ and $\{u^i\}$ of the simple algebra \mathfrak{g} and introduce for any $n \in \mathbb{Z}$ the Sugawara operator (see [KW] section 3.2)

$$(5) \quad L_n^{\mathfrak{g}} = \frac{1}{2(k + \check{h}(\mathfrak{g}))} \sum_{j \in \mathbb{Z}} \sum_i : u_i(-j) u^i(j + n) :,$$

where the notation $: :$ stands for the normal ordering. Then $L_n^{\mathfrak{g}}$ acts linearly on $\mathcal{H}_{\hat{\lambda}}(\mathfrak{g})$ and, by putting $d_n = L_n^{\mathfrak{g}}$ and $\tilde{c} = c(\mathfrak{g}, k)\text{Id}$, we obtain a representation of Vir on $\mathcal{H}_{\hat{\lambda}}(\mathfrak{g})$.

If \mathfrak{g} is semi-simple with $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, we define for $k = (k_1, k_2)$, see [KW] formula (1.4.7),

$$c(\mathfrak{g}, k) = c(\mathfrak{g}_1, k_1) + c(\mathfrak{g}_2, k_2), \quad \text{and} \quad \Delta_{\hat{\lambda}}(\mathfrak{g}) = \Delta_{\hat{\lambda}^{(1)}}(\mathfrak{g}_1) + \Delta_{\hat{\lambda}^{(2)}}(\mathfrak{g}_2),$$

and we put $L_n^{\mathfrak{g}} = L_n^{\mathfrak{g}_1} + L_n^{\mathfrak{g}_2}$. As in the simple case, we obtain a representation of Vir on $\mathcal{H}_{\hat{\lambda}}(\mathfrak{g})$.

For later use we recall the following relation ([KW] formula (1.4.5))

$$(6) \quad \Delta_{\hat{\lambda} + n\delta}(\mathfrak{g}) = \Delta_{\hat{\lambda}}(\mathfrak{g}) + n.$$

The endomorphism $L_0^{\mathfrak{g}}$ of the $\hat{\mathfrak{g}}$ -module $\mathcal{H}_{\hat{\lambda}}(\mathfrak{g})$ can be diagonalized

$$\mathcal{H}_{\hat{\lambda}} = \bigoplus_{m=0}^{\infty} \mathcal{H}_{\hat{\lambda}}(m) \quad \text{with} \quad \mathcal{H}_{\hat{\lambda}}(m) := \{u \in \mathcal{H}_{\hat{\lambda}} \mid L_0^{\mathfrak{g}}(u) = (\Delta_{\hat{\lambda}}(\mathfrak{g}) + m)u\}.$$

We recall that the endomorphism $L_0^{\mathfrak{g}}$ of $\mathcal{H}_{\hat{\lambda}}(\mathfrak{g})$ defined by (5) can also be written as ([KW] (3.2.6) or [K] Corollary 12.8 (b))

$$(7) \quad L_0^{\mathfrak{g}} = \Delta_{\hat{\lambda}}(\mathfrak{g})\text{Id} - d.$$

Note that $\mathcal{H}_{\hat{\lambda}}(0)$ equals the irreducible \mathfrak{g} -module V_{λ} with highest weight λ .

3.2. Definition of conformal pair.

Definition 3.1 (e. g. [K] Chapter 13). *Let \mathfrak{p} be a semisimple subalgebra of a simple Lie algebra \mathfrak{g} , and let ℓ denote the Dynkin (multi-)index of the inclusion homomorphism $\mathfrak{p} \subset \mathfrak{g}$. We say that \mathfrak{p} is a conformal subalgebra of \mathfrak{g} at level k if $c(\mathfrak{p}, k\ell) = c(\mathfrak{g}, k)$.*

The equality $c(\mathfrak{p}, k\ell) = c(\mathfrak{g}, k)$ in the above definition in fact holds only if $k = 1$. Classification of conformal pairs are given in [BB] and [SW]. We recall (see [KW]) that, since \mathfrak{p} is semisimple, $\mathfrak{p} \subset \mathfrak{g}$ is a conformal subalgebra is equivalent to the statement that any irreducible $\hat{\mathfrak{g}}$ -module $\mathcal{H}_{\hat{\lambda}}(\mathfrak{g})$ of level 1 decomposes into a finite sum of irreducible $\hat{\mathfrak{p}}$ -modules of level ℓ .

A fundamental property of a conformal embedding $\mathfrak{p} \subset \mathfrak{g}$ is the following.

Proposition 3.2 ([KW] Proposition 3.2 (c)). *If $\mathfrak{p} \subset \mathfrak{g}$ is a conformal embedding, then*

$$L_n^{\mathfrak{g}} = L_n^{\mathfrak{p}} \in \text{End}(\mathcal{H}_{\hat{\lambda}}(\mathfrak{g})) \quad \text{for all } n \in \mathbb{Z}.$$

In the above statement, we warn that (though implicit in the notation), the operator $L_n^{\mathfrak{g}}$ is acting at level $k = 1$ and $L_n^{\mathfrak{p}}$ at level ℓ . In all the cases we will consider the multi-integer ℓ is equal to 1 (see Table (11)).

3.3. The pairing on $\mathcal{H}_\lambda \times \mathcal{H}_{\lambda^\dagger}$. First we recall the following lemma

Lemma 3.3 ([U] Lemma 2.2.12). *There exists a bilinear pairing*

$$(\cdot|\cdot)_\lambda : \mathcal{H}_\lambda \times \mathcal{H}_{\lambda^\dagger} \longrightarrow \mathbb{C},$$

unique up to a multiplicative constant such that

$$(8) \quad (X(n)u|v)_\lambda + (u|X(-n)v)_\lambda = 0,$$

for any $X \in \mathfrak{g}$, $n \in \mathbb{Z}$, $u \in \mathcal{H}_\lambda$ and $v \in \mathcal{H}_{\lambda^\dagger}$. Moreover the restriction of this pairing to $\mathcal{H}_\lambda(m) \times \mathcal{H}_{\lambda^\dagger}(m')$ is zero if $m \neq m'$, and non-degenerate if $m = m'$.

Consider the restriction of the pairing $(\cdot|\cdot)_\lambda$ to $\mathcal{H}_\lambda(0) \times \mathcal{H}_{\lambda^\dagger}(0) = V_\lambda \times V_{\lambda^\dagger}$. By definition $V_{\lambda^\dagger} = V_\lambda^*$, so that $V_\lambda \otimes V_{\lambda^\dagger} = \text{End}(V_\lambda)$. The pairing on $V_\lambda \times V_{\lambda^\dagger}$ is given by a multiple of the natural evaluation map. More generally, the pairing $(\cdot|\cdot)_\lambda$ induces for any integer m an isomorphism $\phi_m^\lambda : \mathcal{H}_{\lambda^\dagger}(m) \xrightarrow{\sim} \mathcal{H}_\lambda(m)^*$ and therefore a distinguished element, which we denote by $\gamma_\lambda(m) \in \mathcal{H}_\lambda(m) \otimes \mathcal{H}_{\lambda^\dagger}(m)$ and which maps to the identity element in $\text{End}(\mathcal{H}_\lambda(m))$ under the isomorphism

$$(9) \quad id \otimes \phi_m^\lambda : \mathcal{H}_\lambda(m) \otimes \mathcal{H}_{\lambda^\dagger}(m) \xrightarrow{\sim} \text{End}(\mathcal{H}_\lambda(m)), \quad \gamma_\lambda(m) \mapsto \text{Id}_{\mathcal{H}_\lambda(m)}.$$

Note that the family $\{\gamma_\lambda(m)\}_{m \in \mathbb{Z}_+}$ is uniquely defined up to a multiplicative constant. More precisely, if we multiply $(\cdot|\cdot)_\lambda$ with $\alpha \in \mathbb{C}^*$ the family $\{\gamma_\lambda(m)\}_{m \in \mathbb{Z}_+}$ is transformed into the family $\{\frac{1}{\alpha}\gamma_\lambda(m)\}_{m \in \mathbb{Z}_+}$.

Consider a conformal embedding $\mathfrak{p} \subset \mathfrak{g}$ and an integrable $\hat{\mathfrak{g}}$ -module $\mathcal{H}_\lambda(\mathfrak{g})$ of level one. Then we have a decomposition as $\hat{\mathfrak{p}}$ -module

$$\mathcal{H}_\lambda(\mathfrak{g}) = \bigoplus_{\hat{\mu} \in B(\lambda)} M(\hat{\mu}, \lambda) \otimes \mathcal{H}_{\hat{\mu}}(\mathfrak{p}),$$

where $B(\lambda)$ is a finite subset of $\widehat{P_\ell(\mathfrak{p})}$ (see [K] equation (13.14.6)) and the $M(\hat{\mu}, \lambda)$ are finite-dimensional vector spaces ([KW] section 1.6). The integer $\dim M(\hat{\mu}, \lambda)$ is the multiplicity of the representation $\mathcal{H}_{\hat{\mu}}(\mathfrak{p})$ in $\mathcal{H}_\lambda(\mathfrak{g})$. Note that the weights $\hat{\mu} \in B(\lambda)$ do not necessarily lie in $P_\ell(\mathfrak{p})$. We can write $\hat{\mu} = \mu + \sum_i \ell_i \Lambda_0^{(i)} - n_\mu \delta$.

Using Proposition 3.2 we deduce an equality between the trace anomalies

$$(10) \quad \Delta_\lambda(\mathfrak{g}) = \Delta_{\hat{\mu}}(\mathfrak{p}) \quad \text{for any } \hat{\mu} \in B(\lambda).$$

Moreover by (6) we also have $\Delta_{\hat{\mu}}(\mathfrak{p}) = \Delta_\mu(\mathfrak{p}) - n_\mu$. Since $-n_\mu$ is the d -eigenvalue of the highest weight vector $v_\mu \in \mathcal{H}_{\hat{\mu}}(\mathfrak{p}) \subset \mathcal{H}_\lambda(\mathfrak{g})$ and since all d -eigenvalues of $\mathcal{H}_\lambda(\mathfrak{g})$ are negative integers, we conclude that $n_\mu \in \mathbb{Z}_+$.

Remark 3.4. We immediately deduce from the above that $n_\mu = 0$ if and only if the \mathfrak{p} -module $V_\mu = \mathcal{H}_\mu(\mathfrak{p})(0)$ appears in the decomposition into irreducible \mathfrak{p} -modules of the \mathfrak{g} -module $V_\lambda = \mathcal{H}_\lambda(\mathfrak{g})(0)$.

Thus we conclude that given $\mu \in P_\ell(\mathfrak{p})$ there exists at most one $\hat{\mu} \in B(\lambda)$ — since n_μ is given by the difference of the trace anomalies. So we will write

$$\text{mult}_\lambda(\mu, \mathfrak{p}) := \dim M(\hat{\mu}, \lambda)$$

for the multiplicity of occurrence of $\mathcal{H}_\mu(\mathfrak{p})$ in $\mathcal{H}_\lambda(\mathfrak{g})$ as $[\hat{\mathfrak{p}}, \hat{\mathfrak{p}}]$ -modules.

Proposition 3.5. *We have the equality*

$$B(\lambda^\dagger) = B(\lambda)^\dagger := \{\hat{\mu} \in \widehat{P_\ell(\mathfrak{p})} \mid \hat{\mu}^\dagger \in B(\lambda)\}.$$

Moreover $\text{mult}_\lambda(\mu, \mathfrak{p}) = \text{mult}_{\lambda^\dagger}(\mu^\dagger, \mathfrak{p})$.

Proof. For $\hat{\nu} \in \widehat{P_k(\mathfrak{g})}$ and $\lambda \in P_k(\mathfrak{g})$ we will denote by $V_{\hat{\nu}}^\lambda \subset \mathcal{H}_\lambda(\mathfrak{g})$ the weight space of the $\hat{\mathfrak{g}}$ -module $\mathcal{H}_\lambda(\mathfrak{g})$ associated to the weight $\hat{\nu}$. It follows from relation (7) that $V_{\hat{\nu}}^\lambda \subset \mathcal{H}_\lambda(\mathfrak{g})(m)$ if and only if $\hat{\nu}$ is of the form $\hat{\nu} = \nu + k\Lambda_0 - m\delta$. By Lemma 3.3 we know that $\mathcal{H}_\lambda(\mathfrak{g})(m)$ and $\mathcal{H}_{\lambda^\dagger}(\mathfrak{g})(m)$ are dual spaces, and it follows from relation (8) that the weight spaces

$$V_{\nu+k\Lambda_0-m\delta}^\lambda \subset \mathcal{H}_\lambda(\mathfrak{g})(m) \quad \text{and} \quad V_{-\nu+k\Lambda_0-m\delta}^{\lambda^\dagger} \subset \mathcal{H}_{\lambda^\dagger}(\mathfrak{g})(m)$$

are dual to each other. Hence their dimensions coincide

$$\text{mult}_{\mathcal{H}_\lambda}(\nu + k\Lambda_0 - m\delta) = \text{mult}_{\mathcal{H}_{\lambda^\dagger}}(-\nu + k\Lambda_0 - m\delta).$$

Consider $\hat{\mu} \in B(\lambda)$. Then by (10) we have $\Delta_\lambda(\mathfrak{g}) = \Delta_{\hat{\mu}}(\mathfrak{p})$ and since the bilinear form (\cdot, \cdot) is invariant under the finite Weyl group and $w_0(\rho) = -\rho$ we also have $\Delta_{\lambda^\dagger}(\mathfrak{g}) = \Delta_{\hat{\mu}^\dagger}(\mathfrak{p})$. By [K] Proposition 12.11, there exists a weight $\hat{\nu}$ of $\mathcal{H}_\lambda(\mathfrak{g})$ such that $\hat{\nu}|_{\hat{\mathfrak{p}}} = \hat{\mu}$. By Proposition 2.2 there exists an element $\tilde{w} \in W(\mathfrak{g})$ which restricts to $w_0^\mathfrak{p} \in W(\mathfrak{p})$ and by [K] Proposition 10.1,

$$\text{mult}_{\mathcal{H}_{\lambda^\dagger}}(-\nu + k\Lambda_0 - m\delta) = \text{mult}_{\mathcal{H}_{\lambda^\dagger}}(-\tilde{w}(\nu) + k\Lambda_0 - m\delta).$$

But $-\tilde{w}(\nu) + k\Lambda_0 - m\delta|_{\hat{\mathfrak{p}}} = \hat{\mu}^\dagger$, therefore, using [K] Proposition 12.11 once more, we obtain that $\hat{\mu}^\dagger \in B(\lambda^\dagger)$. The same reasoning combined with [K] formula (12.11.1) shows that $\text{mult}_\lambda(\mu, \mathfrak{p}) = \text{mult}_{\lambda^\dagger}(\mu^\dagger, \mathfrak{p})$. \square

3.4. Conformal pairs and the sewing procedure. Let q be a formal variable. Given $\lambda \in P_k(\mathfrak{g})$ we define the element (cf. section 3.3)

$$\tilde{\gamma}_\lambda := \sum_{m=0}^{\infty} \gamma_\lambda(m) q^m \in \mathcal{H}_\lambda(\mathfrak{g}) \otimes \mathcal{H}_{\lambda^\dagger}(\mathfrak{g})[[q]].$$

Note that $\tilde{\gamma}_\lambda$ is well-defined up to a multiplicative constant, and that the choice of a bilinear pairing $(\cdot|\cdot)_\lambda$ on $\mathcal{H}_\lambda \times \mathcal{H}_{\lambda^\dagger}$ introduced in Lemma 3.3 uniquely determines $\tilde{\gamma}_\lambda$.

Let $\mathfrak{p} \subset \mathfrak{g}$ be a conformal subalgebra. It follows from Proposition 3.5 that the decompositions of $\mathcal{H}_\lambda(\mathfrak{g})$ and $\mathcal{H}_{\lambda^\dagger}(\mathfrak{g})$ into irreducible $\hat{\mathfrak{p}}$ -modules are of the form

$$(11) \quad \mathcal{H}_\lambda(\mathfrak{g}) = \bigoplus_{\hat{\mu} \in B(\lambda)} \mathcal{H}_{\hat{\mu}}(\mathfrak{p}), \quad \mathcal{H}_{\lambda^\dagger}(\mathfrak{g}) = \bigoplus_{\hat{\mu} \in B(\lambda)} \mathcal{H}_{\hat{\mu}^\dagger}(\mathfrak{p}).$$

Here we suppose for simplicity that $\text{mult}_\lambda(\mu, \mathfrak{p}) = 1$ for all $\hat{\mu} \in B(\lambda)$, which will be the case of all our examples (see Table (13) below). We start with decomposing the bilinear form $(\cdot|\cdot)_\lambda$ introduced in Lemma 3.3 with respect to the direct sums (11).

Lemma 3.6. *Given $(\hat{\mu}, \hat{\nu}) \in B(\lambda) \times B(\lambda^\dagger)$ the restriction of the bilinear form $(\cdot|\cdot)_\lambda$ to the direct summand $\mathcal{H}_{\hat{\mu}}(\mathfrak{p}) \times \mathcal{H}_{\hat{\nu}}(\mathfrak{p})$ is*

- zero, if $\hat{\nu} \neq \hat{\mu}^\dagger$.
- a non-zero multiple of the bilinear form $(\cdot|\cdot)_\mu$ on $\mathcal{H}_\mu(\mathfrak{p}) \times \mathcal{H}_{\mu^\dagger}(\mathfrak{p})$, if $\hat{\nu} = \hat{\mu}^\dagger$.

Proof. We consider the decomposition into d -eigenspaces

$$\mathcal{H}_\lambda(\mathfrak{g})(m) = \bigoplus_{\hat{\mu} \in B(\lambda)} \mathcal{H}_{\hat{\mu}}(\mathfrak{p})(m).$$

Note that $\mathcal{H}_{\hat{\mu}}(\mathfrak{p})(m) = \mathcal{H}_\mu(\mathfrak{p})(m - n_\mu)$ for all $m \in \mathbb{Z}_+$ and that $\mathcal{H}_\mu(\mathfrak{p})(l) = \{0\}$ for $l < 0$. First we observe that the restriction of $(\cdot|\cdot)_\lambda$ to $\mathcal{H}_{\hat{\mu}}(\mathfrak{p}) \times \mathcal{H}_{\hat{\nu}}(\mathfrak{p})$ is determined by values on the finite-dimensional subspace

$$V_\mu \times V_\nu = \mathcal{H}_\mu(\mathfrak{p})(0) \times \mathcal{H}_\nu(\mathfrak{p})(0) = \mathcal{H}_{\hat{\mu}}(\mathfrak{p})(n_\mu) \times \mathcal{H}_{\hat{\nu}}(\mathfrak{p})(n_\nu).$$

This follows from the fact that $(\cdot|\cdot)_\lambda$ satisfies relation (8) for $X \in \mathfrak{p}$, $u \in \mathcal{H}_{\hat{\mu}}(\mathfrak{p})$ and $v \in \mathcal{H}_{\hat{\nu}}(\mathfrak{p})$, which enables one to reconstruct the bilinear form on $\mathcal{H}_{\hat{\mu}}(\mathfrak{p}) \times \mathcal{H}_{\hat{\nu}}(\mathfrak{p})$ from its values on $V_\mu \times V_\nu$ — see e.g. [U] Lemma 2.2.12.

The restriction of $(\cdot|\cdot)_\lambda$ to $V_\mu \times V_\nu$ induces a linear map $\varphi : V_\mu \rightarrow (V_\nu)^*$ and relation (8) written for $X \in \mathfrak{p}$, $n = 0$, $u \in V_\mu$ and $v \in V_\nu$ implies that φ is a \mathfrak{p} -module homomorphism (here we endow the vector space $(V_\nu)^*$ with the structure of a left \mathfrak{p} -module). Hence in the case $\hat{\nu} \neq \hat{\mu}^\dagger$, we deduce that $\varphi = 0$ and therefore that the restriction of $(\cdot|\cdot)_\lambda$ to $\mathcal{H}_{\hat{\mu}}(\mathfrak{p}) \times \mathcal{H}_{\hat{\nu}}(\mathfrak{p})$ is zero. In the case $\hat{\nu} = \hat{\mu}^\dagger$, we obtain by Schur's lemma that $\varphi : V_\mu \rightarrow (V_{\mu^\dagger})^* = V_\mu$ is a homothety and that the restriction of $(\cdot|\cdot)_\lambda$ to $\mathcal{H}_{\hat{\mu}}(\mathfrak{p}) \times \mathcal{H}_{\hat{\mu}^\dagger}(\mathfrak{p})$ is a multiple of $(\cdot|\cdot)_\mu$. It now remains to check that this multiple is non-zero. In order to show that, we consider for m large the isomorphism ϕ_m^λ induced by $(\cdot|\cdot)_\lambda$

$$\mathcal{H}_{\lambda^\dagger}(m) = \bigoplus_{\hat{\mu} \in B(\lambda)} \mathcal{H}_{\hat{\mu}^\dagger}(m) \longrightarrow \mathcal{H}_\lambda(m)^* = \bigoplus_{\hat{\mu} \in B(\lambda)} \mathcal{H}_{\hat{\mu}}(m)^*.$$

By the preceding arguments, ϕ_m^λ is block-diagonal and decomposes as $\phi_m^\lambda = \sum_{\hat{\mu} \in B(\lambda)} \phi_m^\mu$. Hence all restrictions $\phi_m^\mu : \mathcal{H}_{\hat{\mu}^\dagger}(m) \rightarrow \mathcal{H}_{\hat{\mu}}(m)^*$ for $\hat{\mu} \in B(\lambda)$ are isomorphisms and we are done. \square

Remark 3.7. If we fix a pairing $(\cdot|\cdot)_\lambda$ on $\mathcal{H}_\lambda \times \mathcal{H}_{\lambda^\dagger}$, we also obtain by restriction a pairing $(\cdot|\cdot)_\mu$ on $\mathcal{H}_\mu \times \mathcal{H}_{\mu^\dagger}$ for any $\hat{\mu} \in B(\lambda)$. Thus we obtain uniquely defined elements $\tilde{\gamma}_\lambda$ and $\tilde{\gamma}_\mu$ for $\hat{\mu} \in B(\lambda)$.

Proposition 3.8. *With the choices made in Remark 3.7 we have a decomposition in $\mathcal{H}_\lambda(\mathfrak{g}) \otimes \mathcal{H}_{\lambda^\dagger}(\mathfrak{g})[[q]]$*

$$\tilde{\gamma}_\lambda = \sum_{\hat{\mu} \in B(\lambda)} q^{n_\mu} \tilde{\gamma}_\mu,$$

where the positive integer n_μ equals $\Delta_\mu(\mathfrak{p}) - \Delta_\lambda(\mathfrak{g})$.

Proof. As in the proof of Lemma 3.6, we consider the decomposition into d -eigenspaces

$$\mathcal{H}_\lambda(\mathfrak{g})(m) = \bigoplus_{\hat{\mu} \in B(\lambda)} \mathcal{H}_{\hat{\mu}}(\mathfrak{p})(m)$$

and note that $\mathcal{H}_{\hat{\mu}}(\mathfrak{p})(m) = \mathcal{H}_\mu(\mathfrak{p})(m - n_\mu)$ for all $m \in \mathbb{Z}_+$. The identity transformation of $\mathcal{H}_\lambda(\mathfrak{g})(m)$ obviously decomposes as

$$(12) \quad \text{Id}_{\mathcal{H}_\lambda(\mathfrak{g})(m)} = \sum_{\hat{\mu} \in B(\lambda)} \text{Id}_{\mathcal{H}_\mu(\mathfrak{p})(m - n_\mu)}$$

for any $m \in \mathbb{Z}_+$.

Since by Lemma 3.6 the pairing $(\cdot|\cdot)_\lambda$ on $\mathcal{H}_\lambda \times \mathcal{H}_{\lambda^\dagger}$ restricts to the pairing $(\cdot|\cdot)_\mu$ on $\mathcal{H}_\mu \times \mathcal{H}_{\mu^\dagger}$ for any $\hat{\mu} \in B(\lambda)$, we obtain after applying the inverse of the isomorphism $id \otimes \phi_m^\lambda = \sum_{\hat{\mu} \in B(\lambda)} id \otimes \phi_m^\mu$ (see (9)) to the equality (12),

$$\gamma_\lambda(m) = \sum_{\hat{\mu} \in B(\lambda)} \gamma_\mu(m - n_\mu) \quad \text{for any } m \in \mathbb{Z}_+.$$

Here we put $\gamma_\mu(l) = 0$ if $l < 0$. Multiplying with q^m and summing over \mathbb{Z}_+ gives the relation

$$\begin{aligned} \tilde{\gamma}_\lambda &= \sum_{m=0}^{\infty} \gamma_\lambda(m) q^m = \sum_{\hat{\mu} \in B(\lambda)} \sum_{m \geq n_\mu} \gamma_\mu(m - n_\mu) q^{m - n_\mu} q^{n_\mu} \\ &= \sum_{\hat{\mu} \in B(\lambda)} q^{n_\mu} \tilde{\gamma}_\mu. \end{aligned}$$

□

The following table is extracted from [KS] page 2235 and gives for any conformal subalgebra \mathfrak{p} of Table (1) the decomposition of the basic representation $\mathcal{H}_0(\mathfrak{e}_8)$ as $[\hat{\mathfrak{p}}, \hat{\mathfrak{p}}]$ -module, in particular its Dynkin (multi-)index ℓ , its subset $B(0)$ and the action of the involution $\mu \mapsto \mu^\dagger$. We use Bourbaki's notation for the fundamental weights of a simple Lie algebra.

(13)

\mathfrak{p}	ℓ	$B(0)$	$\varpi \mapsto \varpi^\dagger$
$\mathfrak{so}(16)$	1	$\{0, \varpi_7\}$	$\varpi_7^\dagger = \varpi_7$
$\mathfrak{sl}(9)$	1	$\{0, \varpi_3, \varpi_6\}$	$\varpi_3^\dagger = \varpi_6$
$\mathfrak{sl}(5) \oplus \mathfrak{sl}(5)$	(1,1)	$\{(0,0), (\varpi_1, \varpi_2), (\varpi_2, \varpi_4), (\varpi_3, \varpi_1), (\varpi_4, \varpi_3)\}$	$(\varpi_i, \varpi_j)^\dagger = (\varpi_{5-i}, \varpi_{5-j})$
$\mathfrak{sl}(3) \oplus \mathfrak{e}_6$	(1,1)	$\{(0,0), (\varpi_1, \varpi_1), (\varpi_2, \varpi_6)\}$	$(\varpi_1, \varpi_1)^\dagger = (\varpi_2, \varpi_6)$
$\mathfrak{sl}(2) \oplus \mathfrak{e}_7$	(1,1)	$\{(0,0), (\varpi_1, \varpi_7)\}$	$(\varpi_1, \varpi_7)^\dagger = (\varpi_1, \varpi_7)$
$\mathfrak{g}_2 \oplus \mathfrak{f}_4$	(1,1)	$\{(0,0), (\varpi_1, \varpi_4)\}$	$(\varpi_1, \varpi_4)^\dagger = (\varpi_1, \varpi_4)$

We recall that if $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{b}$, then we have the decomposition as $[\hat{\mathfrak{p}}, \hat{\mathfrak{p}}]$ -module

$$\mathcal{H}_0(\mathfrak{e}_8) = \bigoplus_{(\lambda_1, \lambda_2) \in B(0)} \mathcal{H}_{\lambda_1}(\mathfrak{a}) \otimes \mathcal{H}_{\lambda_2}(\mathfrak{b}).$$

We also note that, for any \mathfrak{p} in Table (13), the trace anomalies $\Delta_\mu(\mathfrak{p}) = n_\mu$ are equal to 1 for all non-zero $\mu \in B(0)$.

4. CONFORMAL PAIRS: FACTORIZATION RULES

In this section we describe how the factorization rules behave under conformal embeddings. First we recall the factorization rules. Let X_0 be a nodal (not necessarily irreducible) curve with one node x_0 . We call \tilde{X} the normalization of X_0 with $\pi : \tilde{X} \rightarrow X_0$ and $\pi^{-1}(x_0) = \{a, b\}$.

Proposition 4.1 (Factorization rules, [U] Theorem 4.4.9). *Let \mathfrak{g} be a semi-simple Lie algebra and X_0 a nodal curve with s marked points with labels $\vec{\lambda} \in P_k(\mathfrak{g})^s$. There is an isomorphism*

$$\bigoplus_{\lambda \in P_k(\mathfrak{g})} \mathcal{V}_{\vec{\lambda}, \lambda, \lambda^\dagger}^\dagger(\tilde{X}, \mathfrak{g}) \xrightarrow{\oplus \iota_\lambda} \mathcal{V}_{\vec{\lambda}}^\dagger(X_0, \mathfrak{g}).$$

Remark 4.2. For any $\lambda \in P_k(\mathfrak{g})$ the linear map ι_λ is only defined up to homothety. More precisely, ι_λ depends on the choice of an element $\tilde{\gamma}_\lambda$ (or, equivalently, of its degree zero piece $\gamma_\lambda(0)$), which are only defined up to a multiplicative constant — see equation (14) below.

We will denote by \mathcal{O} the ring of formal power series $\mathbb{C}[[q]]$ and by $K = \mathbb{C}((q))$ its field of fraction. We consider a family of curves \mathcal{X} over $\text{Spec } \mathcal{O}$ such that its special fiber \mathcal{X}_0 is a nodal curve X_0 over \mathbb{C} and its generic fiber \mathcal{X}_K a smooth curve over the field K . Consider the sheaf of conformal blocks $\mathcal{V}_\lambda^\dagger(\mathcal{X}, \mathfrak{g})$ for the family \mathcal{X} , which is an \mathcal{O} -module of finite type. Moreover since the formation of the sheaf of conformal blocks commutes with base change, the fibre $\mathcal{V}_\lambda^\dagger(\mathcal{X}, \mathfrak{g})_0$ over the closed point $0 \in \text{Spec } \mathcal{O}$ of $\mathcal{V}_\lambda^\dagger(\mathcal{X}, \mathfrak{g})$ coincides with $\mathcal{V}_\lambda^\dagger(X_0, \mathfrak{g})$. We thus obtain a restriction map

$$r_0 : \mathcal{V}_\lambda^\dagger(\mathcal{X}, \mathfrak{g}) \longrightarrow \mathcal{V}_\lambda^\dagger(X_0, \mathfrak{g}).$$

On the other hand, there exists for every $\lambda \in P_k(\mathfrak{g})$ a \mathbb{C} -linear map — the so-called sewing procedure, see [U] formula (4.4.3) and Lemma 4.4.5

$$\mathfrak{s}_\lambda : \mathcal{V}_{\tilde{\lambda}, \lambda, \lambda^\dagger}^\dagger(\tilde{X}, \mathfrak{g}) \longrightarrow \mathcal{V}_\lambda^\dagger(\mathcal{X}, \mathfrak{g}), \quad \psi_\lambda \mapsto \tilde{\psi}_\lambda := \mathfrak{s}_\lambda(\psi_\lambda).$$

The linear maps ι_λ and \mathfrak{s}_λ are defined as follows: for $\psi_\lambda \in \mathcal{V}_{\tilde{\lambda}, \lambda, \lambda^\dagger}^\dagger(\tilde{X}, \mathfrak{g})$

$$(14) \quad \langle \iota_\lambda(\psi_\lambda) | u \rangle := \langle \psi_\lambda | u \otimes \gamma_\lambda(0) \rangle \in \mathbb{C} \quad \text{and} \quad \langle \tilde{\psi}_\lambda | \tilde{u} \rangle := \langle \psi_\lambda | \tilde{u} \otimes \tilde{\gamma}_\lambda \rangle \in \mathbb{C}$$

for any vectors $u \in \mathcal{H}_{\tilde{\lambda}}$ and $\tilde{u} \in \mathcal{H}_{\tilde{\lambda}}[[q]]$. We recall (see [U] Lemma 4.4.6) that $\mathcal{V}_\lambda^\dagger(\mathcal{X}, \mathfrak{g})$ identifies with the subset of linear forms in $\mathcal{H}_{\tilde{\lambda}}^\dagger[[q]]$ satisfying the formal gauge condition. It is clear from these definitions that the map \mathfrak{s}_λ is an extension of ι_λ , i.e., $\iota_\lambda = r_0 \circ \mathfrak{s}_\lambda$.

We consider now a conformal embedding $\mathfrak{p} \subset \mathfrak{g}$. We assume that all level one representations $\mathcal{H}_\lambda(\mathfrak{g})$ decompose with multiplicities one, i.e. $\text{mult}_\lambda(\mu, \mathfrak{p}) = 1$ for all $\hat{\mu} \in B(\lambda)$. For $\vec{\lambda} = (\lambda_1, \dots, \lambda_s)$ define $B(\vec{\lambda}) = B(\lambda_1) \times \dots \times B(\lambda_s)$. Consider $\vec{\mu} \in B(\vec{\lambda})$ and the corresponding inclusion

$$\mathcal{H}_{\vec{\mu}}(\mathfrak{p}) \hookrightarrow \mathcal{H}_{\vec{\lambda}}(\mathfrak{g}).$$

The gauge condition is preserved under restriction of linear forms to $\mathcal{H}_{\vec{\mu}}(\mathfrak{p})$ — see [NT] formula (2.9), so that we obtain an \mathcal{O} -linear map

$$(15) \quad \alpha : \mathcal{V}_\lambda^\dagger(\mathcal{X}, \mathfrak{g}) \longrightarrow \mathcal{V}_{\vec{\mu}}^\dagger(\mathcal{X}, \mathfrak{p}).$$

The restriction of α to $0 \in \text{Spec } \mathcal{O}$ gives a \mathbb{C} -linear map

$$\alpha_0 : \mathcal{V}_\lambda^\dagger(X_0, \mathfrak{g}) \longrightarrow \mathcal{V}_{\vec{\mu}}^\dagger(X_0, \mathfrak{p}).$$

The next result will describe how α_0 decomposes in the direct sums given by the factorization rules (Proposition 4.1) on both sides. More precisely, we consider the composite map

$$\tilde{\alpha} : \bigoplus_{\lambda \in P_1(\mathfrak{g})} \mathcal{V}_{\tilde{\lambda}, \lambda, \lambda^\dagger}^\dagger(\tilde{X}, \mathfrak{g}) \xrightarrow{\oplus \iota_\lambda} \mathcal{V}_\lambda^\dagger(X_0, \mathfrak{g}) \xrightarrow{\alpha_0} \mathcal{V}_{\vec{\mu}}^\dagger(X_0, \mathfrak{p}) \xrightarrow{(\oplus \iota_\mu)^{-1}} \bigoplus_{\mu \in P_\ell(\mathfrak{p})} \mathcal{V}_{\vec{\mu}, \mu, \mu^\dagger}^\dagger(\tilde{X}, \mathfrak{p})$$

and define for any pair $(\lambda, \mu) \in P_1(\mathfrak{g}) \times P_\ell(\mathfrak{p})$ the linear map $\alpha_0^{\lambda, \mu} : \mathcal{V}_{\tilde{\lambda}, \lambda, \lambda^\dagger}^\dagger(\tilde{X}, \mathfrak{g}) \rightarrow \mathcal{V}_{\vec{\mu}, \mu, \mu^\dagger}^\dagger(\tilde{X}, \mathfrak{p})$ to be the (λ, μ) -component of $\tilde{\alpha}$. In particular, we have for any $\lambda \in P_1(\mathfrak{g})$, $\sum_{\mu \in P_\ell(\mathfrak{p})} \iota_\mu \circ \alpha_0^{\lambda, \mu} = \alpha_0 \circ \iota_\lambda$.

Remark 4.3. Since ι_λ and ι_μ are only defined up to homothety, the linear maps $\alpha_0^{\lambda,\mu}$ are only defined up to homothety.

Proposition 4.4. *Given a pair $(\lambda, \mu) \in P_1(\mathfrak{g}) \times P_\ell(\mathfrak{p})$, the linear map $\alpha_0^{\lambda,\mu}$*

- *is identically zero, if $\hat{\mu} \notin B(\lambda)$ or $\Delta_\mu(\mathfrak{p}) \neq \Delta_\lambda(\mathfrak{g})$.*
- *is, up to non-zero homothety, induced by the natural inclusion $\mathcal{H}_{\tilde{\mu}}(\mathfrak{p}) \otimes \mathcal{H}_{\hat{\mu}}(\mathfrak{p}) \otimes \mathcal{H}_{\hat{\mu}^\dagger}(\mathfrak{p}) \hookrightarrow \mathcal{H}_{\tilde{\lambda}}(\mathfrak{g}) \otimes \mathcal{H}_\lambda(\mathfrak{g}) \otimes \mathcal{H}_{\lambda^\dagger}(\mathfrak{g})$, if $\hat{\mu} \in B(\lambda)$ and $\Delta_\mu(\mathfrak{p}) = \Delta_\lambda(\mathfrak{g})$.*

Proof. We fix a weight $\lambda \in P_1(\mathfrak{g})$. In order to lift the \mathbb{C}^* -indeterminacy in the definition of $\alpha_0^{\lambda,\mu}$, we fix a bilinear pairing $(\cdot|\cdot)_\lambda$ on $\mathcal{H}_\lambda \times \mathcal{H}_{\lambda^\dagger}$, which by Remarks 3.7 and 4.2 determines the elements $\tilde{\gamma}_\lambda$ and $\tilde{\gamma}_\mu$, hence the linear maps ι_λ and ι_μ for any $\hat{\mu} \in B(\lambda)$; we choose any ι_μ for $\hat{\mu} \notin B(\lambda)$.

In order to compute the decomposition of $\alpha_0(\iota_\lambda(\psi_\lambda))$ for an element $\psi_\lambda \in \mathcal{V}_{\tilde{\lambda}, \lambda, \lambda^\dagger}^\dagger(\tilde{X}, \mathfrak{g})$ in the direct sum $\bigoplus_{\mu \in P_\ell(\mathfrak{p})} \mathcal{V}_{\tilde{\mu}, \mu, \mu^\dagger}^\dagger(\tilde{X}, \mathfrak{p})$, we will first decompose the extension $\alpha(\widetilde{\psi}_\lambda)$ and then restrict to the special fiber. In fact, we have $\alpha_0(\iota_\lambda(\psi_\lambda)) = r_0(\alpha(\widetilde{\psi}_\lambda))$. On the other hand, using Proposition 3.8 and the definition (14) of $\widetilde{\psi}_\lambda$, we easily obtain the following equalities in \mathcal{O} , which hold for any $\tilde{u} \in \mathcal{H}_{\tilde{\mu}}(\mathfrak{p})[[q]] \hookrightarrow \mathcal{H}_{\tilde{\lambda}}(\mathfrak{g})[[q]]$

$$(16) \quad \langle \alpha(\widetilde{\psi}_\lambda) | \tilde{u} \rangle = \langle \widetilde{\psi}_\lambda | \tilde{u} \rangle = \langle \psi_\lambda | \tilde{u} \otimes \tilde{\gamma}_\lambda \rangle = \sum_{\hat{\mu} \in B(\lambda)} q^{n_\mu} \langle \psi_\lambda | \tilde{u} \otimes \tilde{\gamma}_\mu \rangle.$$

Now we restrict this expression to the special fiber, i.e., we put $q = 0$: we write $\tilde{u} = \sum_{m \geq 0} u(m)q^m$ and $\tilde{\gamma}_\mu = \sum_{m \geq 0} \gamma_\mu(m)q^m$, so that the evaluation at $q = 0$ of the first term of (16) gives

$$(17) \quad \langle \alpha(\widetilde{\psi}_\lambda) | \tilde{u} \rangle(0) = \langle r_0(\alpha(\widetilde{\psi}_\lambda)) | u(0) \rangle = \langle \alpha_0(\iota_\lambda(\psi_\lambda)) | u(0) \rangle.$$

On the other hand the evaluation at $q = 0$ of the last term of (16) annihilates all terms corresponding to $\hat{\mu} \in B(\lambda)$ with $n_\mu \neq 0$ and, since $[\tilde{u} \otimes \tilde{\gamma}_\mu](0) = u(0) \otimes \gamma_\mu(0)$, the last term becomes

$$(18) \quad \sum_{\substack{\hat{\mu} \in B(\lambda) \\ n_\mu = 0}} \langle \psi_\lambda | u(0) \otimes \gamma_\mu(0) \rangle = \sum_{\substack{\hat{\mu} \in B(\lambda) \\ n_\mu = 0}} \langle \text{res}_\mu(\psi_\lambda) | u(0) \otimes \gamma_\mu(0) \rangle = \sum_{\substack{\hat{\mu} \in B(\lambda) \\ n_\mu = 0}} \langle \iota_\mu(\text{res}_\mu(\psi_\lambda)) | u(0) \rangle,$$

where $\text{res}_\mu(\psi_\lambda)$ denotes the restriction of the linear form ψ_λ to the subspace $\mathcal{H}_{\tilde{\mu}}(\mathfrak{p}) \otimes \mathcal{H}_{\hat{\mu}}(\mathfrak{p}) \otimes \mathcal{H}_{\hat{\mu}^\dagger}(\mathfrak{p})$. Since the equality between (17) and (18) holds for any vector $u(0) \in \mathcal{H}_{\tilde{\lambda}}(\mathfrak{p})$, we obtain the following equality in $\mathcal{V}_{\tilde{\mu}}^\dagger(X_0, \mathfrak{p})$

$$\alpha_0(\iota_\lambda(\psi_\lambda)) = \sum_{\substack{\hat{\mu} \in B(\lambda) \\ n_\mu = 0}} \iota_\mu(\text{res}_\mu(\psi_\lambda)).$$

Projecting onto the subspace $\text{im}(\iota_\mu) \subset \mathcal{V}_{\tilde{\mu}}^\dagger(X_0, \mathfrak{p})$ leads to $\alpha_0^{\lambda,\mu} = 0$ if $\hat{\mu} \notin B(\lambda)$ or $n_\mu \neq 0$, and $\alpha_0^{\lambda,\mu}(\psi_\lambda) = \text{res}_\mu(\psi_\lambda)$ if $\hat{\mu} \in B(\lambda)$ and $n_\mu = 0$. \square

5. PROOF OF THEOREM 1

5.1. Proof of part (i). First of all, we will restate Theorem 1 in terms of spaces of conformal blocks. By [B] Proposition 5.2 the linear map ϕ_P identifies under the ‘‘Verlinde’’ isomorphism (3) with the linear map

$$\phi_{X,\mathfrak{p}} : \mathcal{V}_0^\dagger(X, \mathfrak{e}_8) \longrightarrow \mathcal{V}_0^\dagger(X, \mathfrak{p}),$$

induced by the inclusion of the basic $\hat{\mathfrak{p}}$ -module $\mathcal{H}_0(\mathfrak{p})$ into the basic $\hat{\mathfrak{e}}_8$ -module $\mathcal{H}_0(\mathfrak{e}_8)$. We recall that we choose a point $p \in X$ and we denote $U = X \setminus \{p\}$. It is therefore equivalent to show that $\phi_{X,\mathfrak{p}}$ is non-zero for any smooth curve X .

We then observe that the rank of the linear map $\phi_{X,\mathfrak{p}}$ is constant when the smooth curve X varies by [B] Proposition 5.8 and Lemma A.1. It is therefore sufficient to show that there exists a smooth curve X for which the map $\phi_{X,\mathfrak{p}}$ is non-zero. We will prove that by induction on the genus g of X .

The case $g = 0$ is easily seen as follows. Over the projective line we have $\mathfrak{g}(U) \cdot \mathcal{H}_0(\mathfrak{g}) = \bigoplus_{m>0} \mathcal{H}_0(\mathfrak{g})(m)$ for any semi-simple Lie algebra \mathfrak{g} . Hence the one-dimensional space of covacua $\mathcal{V}_0(\mathbb{P}^1, \mathfrak{g})$ is generated by the image under the projection

$$\mathcal{H}_0(\mathfrak{g}) \longrightarrow \mathcal{V}_0(\mathbb{P}^1, \mathfrak{g}) = \mathcal{H}_0(\mathfrak{g}) / \mathfrak{g}(U) \cdot \mathcal{H}_0(\mathfrak{g})$$

of the highest weight vector $v_0(\mathfrak{g}) \in \mathcal{H}_0(\mathfrak{g})(0) = V_0 = \mathbb{C}$. Since the trivial \mathfrak{e}_8 -module $V_0 = \mathbb{C}$ restricts to the trivial \mathfrak{p} -module, the highest weight vector $v_0(\mathfrak{p}) \in \mathcal{H}_0(\mathfrak{p})(0) = V_0$ coincides with the highest weight vector $v_0(\mathfrak{g})$ under the inclusion $\mathcal{H}_0(\mathfrak{p}) \hookrightarrow \mathcal{H}_0(\mathfrak{g})$, which implies that $\phi_{\mathbb{P}^1,\mathfrak{p}}$ is non-zero.

Next, we consider as in section 4 a family \mathcal{X} of genus g curves parametrized by $\text{Spec } \mathcal{O}$ such that $\mathcal{X}_0 = X_0$ is a nodal curve and \mathcal{X}_K a smooth curve defined over $K = \mathbb{C}((q))$. We also consider (15) the \mathcal{O} -linear map α associated to the conformal embedding $\mathfrak{p} \subset \mathfrak{e}_8$ and the trivial weights $\lambda = 0$ and $\mu = 0$

$$\alpha : \mathcal{V}_0^\dagger(\mathcal{X}, \mathfrak{e}_8) \longrightarrow \mathcal{V}_0^\dagger(\mathcal{X}, \mathfrak{p}).$$

By Proposition 4.4 and by the induction hypothesis, the restriction α_0 of the map α to the special fiber is non-zero: in fact, the genus of the normalization \tilde{X} equals $g - 1$ and α_0 decomposes, up to non-zero homothety in each direct summand, as follows

$$\alpha_0 = (\alpha_0^{0,\mu}) : \mathcal{V}_0^\dagger(\tilde{X}, \mathfrak{e}_8) \xleftarrow{\sim} \mathcal{V}_{0,0,0}^\dagger(\tilde{X}, \mathfrak{e}_8) \longrightarrow \bigoplus_{\mu \in P_\ell(\mathfrak{p})} \mathcal{V}_{0,\mu,\mu^\dagger}^\dagger(\tilde{X}, \mathfrak{p}),$$

where the first isomorphism is the so-called *propagation of vacua* isomorphism (see e.g. [U] Theorem 3.3.1). We note that $P_1(\mathfrak{e}_8) = \{0\}$. By Remark 3.4 and Proposition 4.4 we have $\alpha_0^{0,\mu} = 0$ if $\mu \neq 0$ and, again due to the isomorphism $\mathcal{V}_{0,0,0}^\dagger(\tilde{X}, \mathfrak{p}) \xrightarrow{\sim} \mathcal{V}_0^\dagger(\tilde{X}, \mathfrak{p})$, the map $\alpha_0^{0,0}$ is identified with the map $\phi_{\tilde{X},\mathfrak{p}} : \mathcal{V}_0^\dagger(\tilde{X}, \mathfrak{e}_8) \longrightarrow \mathcal{V}_0^\dagger(\tilde{X}, \mathfrak{p})$, which is non-zero by the induction hypothesis. Hence α_0 is also non-zero.

By semi-continuity we conclude that the restriction $\alpha_K = \phi_{\mathcal{X}_K,\mathfrak{p}} : \mathcal{V}_0^\dagger(\mathcal{X}_K, \mathfrak{e}_8) \longrightarrow \mathcal{V}_0^\dagger(\mathcal{X}_K, \mathfrak{p})$ of α to the generic fiber is non-zero. Hence, again by [B] Proposition 5.8 and Lemma A.1, the K -linear map $\phi_{X_K,\mathfrak{p}}$ is non-zero for any genus g curve X_K defined over the field $K = \mathbb{C}((q))$. Given a genus g curve X defined over \mathbb{C} , the result then follows from the equality $\phi_{X,\mathfrak{p}} \otimes_{\mathbb{C}} K = \phi_{X \otimes_{\mathbb{C}} K,\mathfrak{p}}$.

5.2. Irreducible representations of Heisenberg groups. Before pursuing the proof of Theorem 1, we need to recall some known facts on Heisenberg groups and their irreducible representations. We consider a semi-simple simply connected group P with center Z of the following table

(19)

P	$\text{Spin}(4n)$	$\text{SL}(n)$	E_6	E_7	$\text{SL}(n) \times \text{SL}(n)$	$\text{SL}(3) \times E_6$	$\text{SL}(2) \times E_7$
Z	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

The finite abelian group $\mathcal{M}_X(Z)$ of principal Z -bundles acts on $\mathcal{M}_X(P)$ by twisting P -bundles with Z -bundles. Note that $|\mathcal{M}_X(Z)| = |Z|^{2g}$. We denote by t_ζ the automorphism of $\mathcal{M}_X(P)$ induced by the twist with $\zeta \in \mathcal{M}_X(Z)$. We introduce the Mumford group associated to the line bundle \mathcal{L}_P

$$\mathcal{G}(\mathcal{L}_P) := \{(\zeta, \psi) \mid \zeta \in \mathcal{M}_X(Z) \text{ and } \psi : t_\zeta^* \mathcal{L}_P \xrightarrow{\sim} \mathcal{L}_P\}.$$

The Mumford group is a central extension of the group $\mathcal{M}_X(Z)$ by \mathbb{C}^* and acts via $s \mapsto \psi(t_\zeta^*(s))$ on the space of global sections $H^0(\mathcal{M}_X(P), \mathcal{L}_P)$. Note that the center \mathbb{C}^* of $\mathcal{G}(\mathcal{L}_P)$ acts by scalar multiplication.

We will need the following results.

Lemma 5.1 ([F1] Lemma 16). *The Mumford group $\mathcal{G}(\mathcal{L}_P)$ is isomorphic as an extension to a finite Heisenberg group $\mathcal{G}(\delta)$. The type δ depends only on the center Z and on the genus g of X .*

We refer, for example, to [M] page 294 for the definition of the Heisenberg group $\mathcal{G}(\delta)$. Under the above identification, for all groups P of Table (19) the space of global sections $H^0(\mathcal{M}_X(P), \mathcal{L}_P)$ is an irreducible representation of $\mathcal{G}(\mathcal{L}_P)$. This is an immediate consequence of the fact that there exists a unique irreducible representation of $\mathcal{G}(\delta)$ of dimension $|Z|^g$ on which \mathbb{C}^* acts as scalar multiplication (see e.g. [M] Proposition 3), and the numerical identity $\dim H^0(\mathcal{M}_X(P), \mathcal{L}_P) = |Z|^g$ provided by the Verlinde formula (see Table (4)).

We remark that for all pairs (P, N) of Table (2), $\mathcal{M}_X(N)$ is a maximal isotropic subgroup of $\mathcal{M}_X(Z)$. Here isotropic means with respect to the standard symplectic form on $\mathcal{M}_X(Z)$ induced by the commutators in the Mumford group (see e.g. [M] page 293). The isotropic subgroup $\mathcal{M}_X(N)$ is maximal since $|N| = |Z|^{1/2}$, hence $|\mathcal{M}_X(N)| = |\mathcal{M}_X(Z)|^{1/2}$. Therefore, there exists a lift $\mathcal{M}_X(N) \hookrightarrow \mathcal{G}(\mathcal{L}_P)$; in other words, there exists an $\mathcal{M}_X(N)$ -linearization of the line bundle \mathcal{L}_P .

Since $H^0(\mathcal{M}_X(P), \mathcal{L}_P)$ is the unique irreducible representation of $\mathcal{G}(\mathcal{L}_P)$ of level one, i.e., $\mathbb{C}^* \subset \mathcal{G}(\mathcal{L}_P)$ acts as scalar multiplication, we deduce from Mumford's theory of theta groups (see [M] Proposition 3) the following:

Lemma 5.2. *For any $\mathcal{M}_X(N)$ -linearization of \mathcal{L}_P , the subspace of $\mathcal{M}_X(N)$ -invariant sections of $H^0(\mathcal{M}_X(P), \mathcal{L}_P)$ is one-dimensional.*

5.3. Proof of part (ii). It is clear that the image of ϕ is contained in the $\mathcal{M}_X(N)$ -invariant subspace. Hence it suffices to show that the $\mathcal{M}_X(N)$ -invariant subspace of $H^0(\mathcal{M}_X(P), \mathcal{L}_P)$ is one-dimensional, which is precisely the result of Lemma 5.2.

6. PROOF OF THEOREM 2

First of all, we recall that by Lemma 5.2 the subspace of $\mathcal{M}_X(N)$ -invariant sections is one-dimensional. The main observation is that, in all three cases, the finite group $N \subset P = A \times B$ projects isomorphically to the centers $Z(A)$ and $Z(B)$ of the groups A and B respectively. Hence we obtain canonical isomorphisms $Z(B) \xrightarrow{\sim} N \xrightarrow{\sim} Z(A)$, which induce an isomorphism $\iota : \mathcal{M}_X(Z(B)) \xrightarrow{\sim} \mathcal{M}_X(N) \xrightarrow{\sim} \mathcal{M}_X(Z(A))$. Now we observe that the isomorphism ι lifts to an isomorphism

$$\tilde{\iota} : \mathcal{G}(\mathcal{L}_B) \longrightarrow \mathcal{G}(\mathcal{L}_A), \quad x \mapsto \tilde{\iota}(x) := \Lambda(x) \cdot x^{-1}.$$

Here Λ denotes the composite map $\mathcal{G}(\mathcal{L}_B) \rightarrow \mathcal{M}_X(Z(B)) \xrightarrow{\sim} \mathcal{M}_X(N) \hookrightarrow \mathcal{G}(\mathcal{L}_P)$, where the first arrow is the canonical projection and the last arrow is the chosen lift of $\mathcal{M}_X(N)$ to the Mumford group. In order to show that $\tilde{\iota}$ is a group homomorphism, we need the fact that the two Mumford groups $\mathcal{G}(\mathcal{L}_A)$ and $\mathcal{G}(\mathcal{L}_B)$ are commuting subgroups of $\mathcal{G}(\mathcal{L}_{A \times B})$. Note that the restriction of $\tilde{\iota}$ to the center \mathbb{C}^* is $t \mapsto t^{-1}$.

Under the identification of the Mumford groups $\tilde{\iota} : \mathcal{G}(\mathcal{L}_B) \xrightarrow{\sim} \mathcal{G}(\mathcal{L}_A)$ the vector space $H^0(\mathcal{M}_X(A), \mathcal{L}_A)^*$ can be viewed as an irreducible representation of $\mathcal{G}(\mathcal{L}_B)$ of level one. Moreover the $\mathcal{M}_X(N)$ -invariance of the section $\sigma \in H^0(\mathcal{M}_X(P), \mathcal{L}_P)$ translates into the equivariance of the linear map $\sigma : H^0(\mathcal{M}_X(A), \mathcal{L}_A)^* \rightarrow H^0(\mathcal{M}_X(B), \mathcal{L}_B)$ under the Mumford group $\mathcal{G}(\mathcal{L}_B)$. Since both spaces are irreducible representations of $\mathcal{G}(\mathcal{L}_B)$ by section 5.2, the non-zero map σ is an isomorphism by Schur's lemma.

7. FURTHER REMARKS

7.1. Invariant sections.

7.1.1. $G = \mathrm{SL}(9)$. In this case the restriction of the E_8 -theta divisor is the unique (up to a scalar) $\mathrm{Jac}(X)[3]$ -invariant section in $H^0(\mathcal{M}_X(\mathrm{SL}(9)), \mathcal{L}_{\mathrm{SL}(9)})$. Here the group $\mathrm{Jac}(X)[3]$ of 3-torsion line bundles over X acts by tensor product on the moduli stack of rank-9 vector bundles with trivial determinant. Note that we take here the linear action of $\mathrm{Jac}(X)[3]$ on $H^0(\mathcal{M}_X(\mathrm{SL}(9)), \mathcal{L}_{\mathrm{SL}(9)})$ induced by the isomorphism $\tilde{\phi}^* \mathcal{L}_{E_8} = \mathcal{L}_{\mathrm{SL}(9)}$. This leads to the natural question (see also [F2] section 6): does there exist a geometrical description of the zero-divisor of this invariant section?

7.1.2. $G = \mathrm{Spin}(16)$. The $\mathcal{M}_X(N)$ -invariant section $\sigma \in H^0(\mathcal{M}_X(\mathrm{Spin}(16)), \mathcal{L}_{\mathrm{Spin}(16)})$ can be described as follows. The center Z of $\mathrm{Spin}(16)$ equals $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{\pm 1, \pm \gamma\}$, where $\{\pm 1\}$ is the kernel K of the homomorphism $\mathrm{Spin}(16) \rightarrow \mathrm{SO}(16)$ and $\pm \gamma$ covers the element $-\mathrm{Id} \in \mathrm{SO}(16)$. Note that $N = \{1, \gamma\}$. By [PR] Proposition 8.2, the space $H^0(\mathcal{M}_X(\mathrm{Spin}(16)), \mathcal{L}_{\mathrm{Spin}(16)})$ admits a basis $\{s_\kappa\}$ indexed by the 4^g theta-characteristics κ of the curve X and such that the set-theoretical support of the zero-divisor of s_κ equals

$$D_\kappa = \{E \in \mathcal{M}_X(\mathrm{Spin}(16)) \mid \dim H^0(X, E(\mathbb{C}^{16}) \otimes \kappa) > 0\}.$$

Here $E(\mathbb{C}^{16})$ denotes the orthogonal rank-16 vector bundle associated to E . The group $\mathcal{M}_X(N)$ identifies with the group $\mathrm{Jac}(X)[2]$ of 2-torsion line bundles over X . We can use the canonical lift $\mathrm{Jac}(X)[2] \hookrightarrow \mathcal{G}(\mathcal{L}_{\mathrm{Spin}(16)})$ and a fixed theta-characteristic κ_0 to normalize the basis $\{s_\kappa\}$ by putting $s_{\alpha\kappa} := \alpha \cdot s_{\kappa_0}$, since any theta-characteristic κ can be written as $\alpha\kappa_0$ for a unique $\alpha \in \mathrm{Jac}(X)[2]$. It is then clear that $\sigma = \sum_\kappa s_\kappa = \sum_{\alpha \in \mathrm{Jac}(X)[2]} \alpha \cdot s_{\kappa_0}$ and that σ does not depend on the choice of κ_0 . As in the previous case, a geometrical description of its zero-divisor is still missing.

7.2. Other conformal subalgebras of \mathfrak{e}_8 .

7.2.1. $(\text{Spin}(8), \text{Spin}(8))$. We observe that $\mathfrak{p} = \mathfrak{so}(8) \oplus \mathfrak{so}(8)$ is a *non-maximal* conformal subalgebra of \mathfrak{e}_8 with Dynkin multi-index $(1, 1)$. In fact, the inclusion $\mathfrak{p} \subset \mathfrak{e}_8$ factorizes through $\mathfrak{so}(16)$. We therefore obtain a group homomorphism $\phi : P = \text{Spin}(8) \times \text{Spin}(8) \rightarrow \text{Spin}(16) \rightarrow E_8$ with kernel $N = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We note that N sits diagonally in $Z(\text{Spin}(8)) \times Z(\text{Spin}(8))$. Theorem 1 and Theorem 2 also hold in this case.

7.2.2. (G_2, F_4) . Using the Verlinde formula one computes that the spaces $H^0(\mathcal{M}_X(G_2), \mathcal{L}_{G_2})$ and $H^0(\mathcal{M}_X(F_4), \mathcal{L}_{F_4})$ have the same dimension, which is $\left(\frac{5+\sqrt{5}}{2}\right)^{g-1} + \left(\frac{5-\sqrt{5}}{2}\right)^{g-1}$. Theorem 1 says that the E_8 -theta divisor Δ induces a non-zero linear map $\sigma : H^0(\mathcal{M}_X(G_2), \mathcal{L}_{G_2})^* \rightarrow H^0(\mathcal{M}_X(F_4), \mathcal{L}_{F_4})$. Since G_2 and F_4 have no center, the argument used in the proof of Theorem 2 breaks down for this case.

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