Introduction to semialgebraic, subanalytic and o-minimal sets and their application in analysis Trento, 2023

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CHAPTER I

Semialgebraic sets

The class of *semialgebraic subsets of* $\mathbb{R}^n$ is the smallest collection of subsets containing all $\{x \in \mathbb{R}^n : P(x) > 0\}$, where $P(x) = P(x_1, ..., x_n)$ is a polynomial, which is stable under finite intersection, finite union and complement.

Thus $X \subset \mathbb{R}^n$ is semialgebraic if and only if there exist polynomials $f_{ij}(x)$ and $g_{ij}(x)$, $i = 1, ..., p, j = 1, ..., q$, such that

$$X = \bigcup_{i=1}^{p} \{x \in \mathbb{R}^n : f_{ij}(x) = 0, g_{ij}(x) > 0, j = 1, ..., q\}.$$

A map $f : A \to \mathbb{R}^m$, where $A \subset \mathbb{R}^n$, is *semialgebraic* if the graph Γ_f of f is a semialgebraic subset of $\mathbb{R}^n \times \mathbb{R}^m$.

Example I.1. The semialgebraic subsets of $\mathbb{R}$ are precisely the finite unions of points and intervals (that includes the empty set $\emptyset$).

Example I.2. The (double) cone $X = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = z^2\} \subset \mathbb{R}^3$ is algebraic. Its projection on the x, z-plane,

$$\pi_{x,z}(X) = \{(x, z) \in \mathbb{R}^2; \exists_y \; x^2 + y^2 = z^2\} = \{(x, z) \in \mathbb{R}^2; z^2 - x^2 \geq 0\}$$

is not algebraic but it is semialgebraic.

Example I.3. The (simple) cone $X = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = z^2, z \geq 0\} \subset \mathbb{R}^3$ is semialgebraic but not algebraic (why ?). Find an algebraic subset Z of $\mathbb{R}^4$ such that $X = \pi(Z)$, where $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ is the standard projection on the first three coordinates.

1. Tarski-Seidenberg Theorem

Theorem I.4. (Tarski-Seidenberg Theorem, geometric form)
Let A be a semi-algebraic subset of $\mathbb{R}^{n+1}$ and let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ denote the projection on the first n coordinates. Then $\pi(A)$ is a semialgebraic subset of $\mathbb{R}^n$.

Theorem I.5. (Tarski-Seidenberg Theorem, quantifier elimination form)
Let Φ be a first order formula with parameters in $\mathbb{R}$. Then there exists a quantifier-free first order formula with parameters in $\mathbb{R}$, Ψ, with the same free variables $x_1, x_2, \ldots, x_n$ as Φ, such that for every $x \in \mathbb{R}^n$, $\Phi(x_1, x_2, \ldots, x_n) \iff \Psi(x_1, x_2, \ldots, x_n)$.

A first order formula (of the language of ordered fields with parameters in $\mathbb{R}$) $\mathcal{S} = (S_n))_{n\in\mathbb{N}}$ is constructed by the following rules

(1) If $P \in \mathbb{R}[X_1, \ldots, X_n]$, then

$$P(x_1, \ldots, x_n) = 0 \quad \text{and} \quad P(x_1, \ldots, x_n) > 0$$

are first order formulas.

(2) If $\Phi(x_1,\dots,x_n)$ and $\Psi(x_1,\dots,x_n)$ are first-order formulas, then

$$\Phi\wedge\Psi,\quad \Phi\vee\Psi,\quad \Phi\Rightarrow\Psi,\quad \neg\Phi$$

are first order formulas.

(3) If $\Phi(y,x)$ is a first order formula, where $y=(y_1,\dots,y_p)$ and $x=(x_1,\dots,x_n)$, then

$$\exists_{x\in A}\Phi(y,x) \text{ and } \forall_{x\in A}\Phi(y,x)$$

are first-order formulas.

1.1. Proof of Tarski-Seidenberg Theorem.

THEOREM I.6. (First structure theorem) *Let $\mathcal{F}=\{P_1(x,y),\dots,P_s(x,y)\}$ be a finite family of real polynomials in $(x,y)\in\mathbb{R}^n\times\mathbb{R}$. There exists a finite partition of $\mathbb{R}^n$ into finitely many connected semi-algebraic A_i, $i=1,\dots,k$, such that for each i there are l_i (l_i could be zero) continuous functions $\xi_{i,j}:A_i\to\mathbb{R}$ satisfying*

(i) *For each $x\in A_i$*

$$\xi_{i,1}(x)<\cdots<\xi_{i,l_i}(x)$$

and $\xi_{i,j}$ are all the roots of those polynomials of one variable y, $y\to P_r(x,y)$, that are not identically equal to zero.

(ii) *The graphs of $\xi_{i,j}$*

$$(\mathcal{G})\qquad \{(x,y)\in A_i\times\mathbb{R}; y=\xi_{i,j}(x)\},\qquad i=1,\dots,l_i,$$

and the bands between two such graphs

$$(\mathcal{B})\qquad \{(x,y)\in A_i\times\mathbb{R}; \xi_{i,j}(x)<y<\xi_{i,j+1}(x)\}\qquad i=0,1,\dots,l_i,$$

where $\xi_{i,0}\equiv-\infty,\xi_{i,l_i+1}\equiv+\infty$, are connected and semi-algebraic.

(iii) *For each r, the sign of $P_r(x,y)$ is constant on the above defined sets of type $\mathcal{G}$ or $\mathcal{B}$, in other words the sign of $P_r(x,y)$ depends only on the signs of $y-\xi_{i,j}(x)$.*

Let $X\subset\mathbb{R}^n\times\mathbb{R}$ be a semi-algebraic set defined by signs conditions in $P_1(x,y),\dots,P_s(x,y)$. Then X is a finite union of some sets of type $\mathcal{B}$ and $\mathcal{G}$. In particular, X has a finite number of connected components and each component is semi-algebraic.

The first structure theorem implies that the projection $\pi(X)$ of X by $\pi:\mathbb{R}^{n+1}\to\mathbb{R}^n$ is a semi-algebraic set, that is the Tarski-Seidenberg theorem. Indeed, X is a finite union of graphs and bands and the projection of each of these is one of the sets A_i.

EXERCISE I.7. Suppose $\mathcal{F}_1\subset\mathcal{F}$. Show that if the theorem holds for $\mathcal{F}$ then it holds for $\mathcal{F}_1$.

REMARK I.8. Suppose, moreover, that the family $\{P_1(x,y),\dots,P_s(x,y)\}$ is stable by $\partial/\partial y$. Then we may require additionally that

(iv) Each graph or band of (ii) is given by the sign conditions on $\{P_1(x,y),\dots,P_s(x,y)\}$, that is it coincides with one of the sets

$$\{(x,y)\in A_i\times\mathbb{R}; \operatorname{sgn}P_r(x,y)=\varepsilon(r), r=1,\dots,s\},$$

where $\varepsilon(r)=0$ or -1 or 1.

(v) each $\xi_{i,j}(x)$ is a simple root of one of $P_r(x,\cdot)$.

Given $\alpha = (\alpha_0, \dots, \alpha_m) \in \mathbb{C}^{m+1}$ and $k \in \mathbb{N}$. We denote by $\mathrm{B}_{m,k}$ the set of these α for which the polynomial

$$P_\alpha(z) = \alpha_0 z^m + \alpha_1 z^{m-1} + \dots + \alpha_m \in \mathbb{C}[z]$$

has exactly k distinct complex roots. We denote $\widetilde{\mathrm{B}}_{m,k} = \mathrm{B}_{m,k} \cap \{\alpha_0 \neq 0\}$.

Lemma I.9. *The set $\widetilde{\mathrm{B}}_{m,k}(\mathbb{R}) = \widetilde{\mathrm{B}}_{m,k} \cap \mathbb{R}^{m+1}$ is semi-algebraic.*

Lemma I.10. *Let $A \subset \widetilde{\mathrm{B}}_{m,k}(\mathbb{R}) = \widetilde{\mathrm{B}}_{m,k} \cap \mathbb{R}^{m+1}$ be connected. Then, there exists $r \le k$ and r continuos functions $\xi_i : A \to \mathbb{R}, i = 1, \dots, r$, such that for each $\alpha \in A$, $\xi_1(\alpha) < \dots < \xi_r(\alpha)$ are all real roots of P_α.*

Lemma I.11. Thom's Lemma
Let $\mathcal{F} = \{P_1(y), \dots, P_s(y)\}$ be a stable under differentiation finite family of one variable polynomials. Let $X \subset \mathbb{R}$ be given by sign conditions on $P_i(y)$

$$X = \{y \in \mathbb{R};\ \operatorname{sgn} P_r(y) = \varepsilon(r), r = 1, \dots, s\}, \tag{1}$$

$\varepsilon(r) = 0$ or -1 or 1. Then X is one of the following

1. *$X = \emptyset$,*
2. *$X =$ a single point. (possible only if $\exists P_i \not\equiv 0$ such that $\varepsilon_i = 0$),*
3. *X is an open (non-empty) interval.*

In particular X is connected.

Proof. Induction on the number of polynomials. We may suppose that $\max_{P\in\mathcal{F}} \deg P = \deg P_s$. Then $\mathcal{F}' = \mathcal{F} \backslash \{P_s\}$ is stable under differentiation and we apply the inductive assumption to it. Let X be a set given by sign conditions on $\mathcal{F}'$. The cases (1) and (2) being trivial, we may assume that X is a non-empty open interval $I = (a, b)$. Since P_s' is of constant sign on I, the polynomial P_s is either constant, or strictly monotone on I, and the lemma easily follows. □

Exercise I.12. For each sign condition $\operatorname{sgn} P(y) = \varepsilon$ define its closure $\operatorname{sgn} P(y) \in \overline{\varepsilon}$, where $\overline{1} = \{0, 1\}$, $\overline{-1} = \{0, -1\}$, $\overline{0} = \{0\}$. Show that if $X \neq \emptyset$ is given by (1) then

$$\overline{X} = \{y \in \mathbb{R};\ \operatorname{sgn} P_r(y) \in \overline{\varepsilon}(r), r = 1, \dots, s\}. \tag{2}$$

If $X = \emptyset$ then $\{y \in \mathbb{R};\ \operatorname{sgn} P_r(y) \in \overline{\varepsilon}(r), r = 1, \dots, s\}$ can be either a point or empty.

Proof of Theorem I.6. Induction on the number of variables.

We suppose first that the family $\mathcal{F} = \{P_1(x, y), \dots, P_s(x, y)\}$ is stable under $\frac{\partial}{\partial y}$. Let

$$P(x, y) = \prod P_i(x, y) = \alpha_0(x) y^m + \dots + \alpha_m(x), \qquad \alpha_0 \not\equiv 0, \tag{3}$$

where the product is taken over those $P_i \in \mathcal{F}$ that are not identically equal to zero. Consider α as a polynomial map

$$\alpha : \mathbb{R}^n \to \mathbb{R}^{m+1}.$$

Fix $k \in \mathbb{N}$ such that $\alpha^{-1}(\widetilde{\mathrm{B}}_{m,k}(\mathbb{R}))$ is non-empty. By Lemma I.9, $\alpha^{-1}(\widetilde{\mathrm{B}}_{m,k}(\mathbb{R}))$ is semi-algebraic. Let A be a connected component $\alpha^{-1}(\widetilde{\mathrm{B}}_{m,k}(\mathbb{R}))$. It is semi-algebraic by the inductive assumption on dimension. By Lemma I.10, there exist continuous functions defined on A, $\xi_1 < \dots < \xi_k$, such that $\xi_1(x), \dots, \xi_k(x)$ are all real roots of one variable polynomial $P(x, \cdot) = P_{\alpha(x)}$.

It is clear that the graphs and bands over A are connected and hence for each r, $\operatorname{sgn} P_r$ is constant on the band sets. In order to have it constant on the graph sets we may, using again

Lemma I.9, subdivide A so that the number of complex roots of each $P_r(x,\cdot)$ is constant on A, and use Lemma I.10. (Actually, this step is not necessary, see Exercise I.19 below).

We now show that the bands and the graphs are semi-algebraic (we need it for induction). By Thom's Lemma each band or graph coincides with a set of the form

$$\{(x,y) \in A \times \mathbb{R};\, \operatorname{sgn} P_r(x,y) = \varepsilon(r), r = 1,\dots,s\},$$

that is semi-algebraic. In this way we have partitioned $\mathbb{R}^n \setminus \alpha_0^{-1}(0)$.

It remains to consider the zero set of α_0. It can be partitioned by a similar type of argument. More precisely, for any $r = 1,\dots,s$, fix an integer $d_r \in \{0,1,\dots,\deg_y P_r, -\infty\}$. Here $\deg_y$ denotes the degree with repect to y. Denote $d = (d_1,\dots,d_s)$ and consider the set

$$A_d = \{x \in \mathbb{R}^n; \deg P_r(x,\cdot) = d_r \text{ for each } r = 1,\dots,s\},$$

where, by convention, the degree is equal to $-\infty$ means that the polynomial is identically equal to zero. If A_d is non-empty, then we consider the family of polynomials $\mathcal{F}_d\{P_{d,i}\}$ where we remove from the polynomials P_r of $\mathcal{F}$ all the terms with $y^k, k > d_r$, and proceed as before. Considering all such non-empty A_d we partition the complement of $A_{-\infty} = \{x \in \mathbb{R}^n; P_i(x,\cdot) \equiv 0, i = 1,\dots,s\}$. For $x \in A_{-\infty}$ the statement of theorem is obvious.

If $\mathcal{F} = \{P_1(x,y),\dots,P_s(x,y)\}$ is not stable under $\frac{\partial}{\partial y}$ we complete it by adding all the partial derivatives in y and use Exercise I.7. $\square$

1.2. Subresultants and continuity of roots. In this section we sketch the proofs of Lemma I.9 and Lemma I.10. The details are left to the reader as exercises. Lemma I.9 follows from the existence of subresultants.

Proposition I.13. *(Existence of subresultants, see e.g. [**1**])*
Let

$$P(z) = a_0 z^p + a_1 z^{p-1} + \cdots + a_p, \quad Q(z) = a_0 z^q + a_1 z^{p-1} + \cdots + a_q,$$

be one variable polynomials of $\mathbb{C}[z]$. Then there exists polynomials

$$r_i(P,Q) = r_i(a_0,\dots,a_p,b_0,\dots,b_q), \qquad i = 0,1,\dots \min\{p,q\},$$

with integer coefficients, such that the following conditions are equivalent

1. *P and Q have at least $k+1$ common roots (counted with multiplicities),*
2. *the degree of the gratest common divisor of P and Q is $\geq k+1$*
3. *$r_0(P,Q) = \cdots = r_k(P,Q) = 0$.*

By Proposition I.13 the set

$$\begin{aligned} W_k &= \{\alpha \in \mathbb{C}^{m+1};\, \alpha_0 \neq 0,\, P_\alpha \text{ has at most } k \text{ distinct roots}\} \\ &= \{\alpha \in \mathbb{C}^{m+1};\, \alpha_0 \neq 0,\, P_\alpha, P'_\alpha \text{ have at least } m-k \text{ common roots (with multiplicities)}\} \end{aligned}$$

is constructible. So is $\widetilde{\mathrm{B}}_{m,k} = W_k \setminus W_{k-1}$. Consequently, the sets $\widetilde{\mathrm{B}}_{m,k}(\mathbb{R}) = \widetilde{\mathrm{B}}_{m,k} \cap \mathbb{R}^{m+1}$ is semi-algebraic.

Now we show Lemma I.10. For simplicity we fix $\alpha_0 = 1$. We are interested in the map

$$\pi : \mathbb{C}^m \to \mathbb{C}^m, \quad \pi_i(\xi_1,\dots,\xi_m) = (-1)^i \sigma_i(\xi_1,\dots,\xi_m),$$

that is the map that associates to the roots of

$$P_\alpha(z) = z^m + \alpha_1 z^{m-1} + \cdots + \alpha_m$$

its coefficients.

EXERCISE I.14.
(a) Show that $|\xi_i| < 1 + \max_j\{|\pi_j(\xi_1, \dots, \xi_m)|\}$.
(b) Show that π is proper.

Let S_m be the symmetric group of m elements. The group S_m acts on $\mathbb{C}^m$ by permuting the coordinates and the map π is S_m invariant. This group action can be used in the following exercise.

EXERCISE I.15.
(a) Let $A \subset \mathbb{C}^n$. Show that if $\pi^{-1}(A)$ is closed so is A.
(b) Let $U \subset \mathbb{C}^n$. Show that if $\pi^{-1}(U)$ is open so is U.
(c) Show that π is open (the image of an open set is open).

REMARK I.16. The map π can be identified with the quotient map $q : \mathbb{C}^m \to \mathbb{C}^m/S_m$, that is we have a natural identification $\Phi : \mathbb{C}^m/S_m \to \mathbb{C}^m$. Exercise I.15 shows that Φ is a homeomorphism, for the quotient topology on $\mathbb{C}^m/S_m$. (The quotient topology on $\mathbb{C}^m/S_m$ is the finest topology for which the quotient map q is continuous. This means that a subset $U \subset \mathbb{C}^m/S_m$ is open (closed) if and only if $q^{-1}(U)$ is open (closed).)

Fix $\tilde{\alpha} \in \mathbb{C}^n$ and suppose that $P_{\tilde{\alpha}}$ has exactly k distinct complex roots $\rho_1, \dots, \rho_k$ of multiplicities, respectively, $m_1, \dots, m_k$, $m_1, + \cdots + m_k = m$. Let $\varepsilon > 0$ be small enough so that $U_i = \{z \in \mathbb{C};\ \|z - \rho_i\| < \varepsilon\}$ are disjoint. The set

$$U = \{\xi = (\xi_1, \dots, \xi_m); \text{such that exactly } m_i \text{ coordinates of } \xi \text{ are in } U_i, i = 1, \dots, k\}$$

is open in $\mathbb{C}^n$ and, by Exercise I.15, $\pi(U)$ contains a neighborhood V of $\tilde{\alpha}$. If $\alpha \in V \cap \widetilde{\mathrm{B}}_{m,k}$ then P_α has exactly one root in each of U_i, and this root is of multiplicity m_i. So these roots define the functions

$$\xi_i : V \cap \widetilde{\mathrm{B}}_{m,k} \to U_i \subset \mathbb{C},$$

that are clearly continuous.

For $\alpha \in V \cap \widetilde{\mathrm{B}}_{m,k}(\mathbb{R})$ the coefficients of the polynomials P_α are real so if $\xi_i(\alpha)$ is a root so is its complex conjugate $\overline{\xi_i(\alpha)}$.

EXERCISE I.17. Show that if $\xi_i(\tilde{\alpha})$ is real so is $\xi_i(\alpha)$ for each $\alpha \in V \cap \widetilde{\mathrm{B}}_{m,k}(\mathbb{R})$. Show that the number of real roots among $\xi(\alpha)$ is constant on $V \cap \widetilde{\mathrm{B}}_{m,k}(\mathbb{R})$.

EXERCISE I.18. Show that the set of these $\alpha \in \widetilde{\mathrm{B}}_{m,k}(\mathbb{R})$ for which the number of real distinct roots is r is open and closed in $\widetilde{\mathrm{B}}_{m,k}(\mathbb{R})$. Complete the proof of Lemma I.10.

EXERCISE I.19. We keep the assumption $\alpha_0 = 1$. Show that the subsets $W_k \subset \mathbb{C}^m$, defined above, are closed in $\mathbb{C}^m$. As a corollary show that the the number of distinct roots function $k(\alpha) : \mathbb{C}^m \to \mathbb{N}$, that associates to P_α the number of its distinct complex roots, is upper semi-continuous.
Suppose now

$$P(x, z) = \prod_i Q_i(x, z) = \prod_i (z^{d_i} + b_{i,1}(x) z^{d_i - 1} + \cdots + b_{i,d_i}(x))$$

where $x \in X$, where X is a connected topological space, and $b_{i,j}$ are continuous. Show that if the number of distinct complex roots of $P(x, \cdot)$ is constant on X so is the number of distinct complex roots of each factor $Q_i(x, \cdot)$

CHAPTER II

O-minimal structures

1. Definition

A structure expanding $\mathbb{R}$ as an ordered real closed field is a collection $\mathcal{S} = (S_n))_{n\in\mathbb{N}}$, where each S_n is a set of subsets of the affine space $\mathbb{R}^n$, satisfying the following axioms :

(1) All algebraic subsets of $\mathbb{R}^n$ are in S_n.
(2) For every n, S_n is a Boolean subalgebra of the powerset of $\mathbb{R}^n$ (that is S_n is stable by the set theoretic operations $\cup$, $\cap$, $\setminus$).
(3) If $A \in S_n$ and $B \in S_m$, then $A \times B \in S_{n+m}$.
(4) If $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates and $A \in S_{n+1}$, then $\pi(A) \in S_n$.

The structure $\mathcal{S}$ is called *o-minimal* (short for order-minimal) if, moreover, it satisfies:

(5) The elements of S_1 are precisely the finite unions of points and intervals.

The elements of S_n are called *the definable subsets of* $\mathbb{R}^n$. A map $f : A \to \mathbb{R}^m$, where $A \subset \mathbb{R}^n$, is called *definable* if the graph of f is a definable subset of $\mathbb{R}^n \times \mathbb{R}^m$.

Exercise II.1. If $f : A \to \mathbb{R}^m$ is definable then A is definable.

Proposition II.2. *Let $f : A \to \mathbb{R}^m$, $A \subset \mathbb{R}^n$, be definable. Let $A' \subset A$ and $B \subset \mathbb{R}^m$. Then the image $f(A')$ and the inverse image $f^{-1}(B)$ are definable.*

In particular if $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial then $\{x \in \mathbb{R}^n; f(x) > 0\}$ is defnable. As a corollary we obtain that every semi-algebraic set is definable in $\mathcal{S}$.

Exercise II.3. Show that the semialgebraic subsets of $\mathbb{R}^n$, $n \in \mathbb{N}$, form an o-minimal structure (use the Tarski-Seindeberg Theorem).

A first order formula in an o-minimal structure $\mathcal{S} = (S_n))_{n\in\mathbb{N}}$ is constructed by the following rules

(1) If $P \in \mathbb{R}[X_1, \dots, X_n]$, then

$$P(x_1, \dots, x_n) = 0 \quad \text{and} \quad P(x_1, \dots, x_n) > 0$$

are first order formulas.

(2) If $A \subset \mathbb{R}^n$ is definable then

$$x \in A,$$

where $x = (x_1, \dots, x_n)$, is a first order formula.

(3) If $\Phi(x_1, \dots, x_n)$ and $\Psi(x_1, \dots, x_n)$ are first-order formulas, then

$$\Phi \wedge \Psi, \quad \Phi \vee \Psi, \quad \Phi \Rightarrow \Psi, \quad \neg\Phi$$

are first order formulas.

(4) If $\Phi(y, x)$ is a first order formula, where $y = (y_1, \dots, y_p)$ and $x = (x_1, \dots, x_n)$, and $A \subset \mathbb{R}^n$ is definable, then

$$\exists_{x \in A} \Phi(y, x) \text{ and } \forall_{x \in A} \Phi(y, x)$$

are first-order formulas.

THEOREM II.4. *If $\Phi(x_1, \dots, x_n)$ is a first-order formula, then the set of $(x_1, \dots, x_n) \in \mathbb{R}^n$ for which the formula $\Phi(x_1, \dots, x_n)$ is true is definable.*

EXAMPLE II.5. If $A \subset \mathbb{R}^n$ is definable then the closure $\overline{A}$ of A is definable.

$$\overline{A} = \{x \in \mathbb{R}^n;\, \forall \varepsilon > 0\, \exists y \in A \text{ such that } \sum_{i=1}^n (x_i - y_i)^2 < \varepsilon^2\}.$$

EXERCISE II.6. Show that the interior of a definable subset of $\mathbb{R}^n$ is definable.

EXERCISE II.7. Show that the set $\{(x, y) \in \mathbb{R}^2;\, \exists n \in \mathbb{N}\, y = nx\}$ is not definable (whatever is the o-minimal structure).

1.0.1. *More exercises.*

(1) Show that the definable functions $A \to \mathbb{R}$ form an $\mathbb{R}$-algebra.
(2) Let $f = (f_1, \dots, f_p)$ be a map from $A \subset \mathbb{R}^n$ to $\mathbb{R}^p$. Show that f is definable if and only if each of its coordinate functions f_i is definable.
(3) Show that the composition of two definable maps is definable.
(4) Let $A \subset \mathbb{R}^n$ be definable non-empty. Show that the function $x \to \operatorname{dist}(x, A)$, defined on $\mathbb{R}^n$, is definable.
(5) Let $f : A \to \mathbb{R}^m$ be definable. Show that the set of points $x \in A$ such that f is continuous at x is definable.

1.0.2. *Monotonicity Theorem.*

THEOREM II.8. (Monotonicity Theorem, for a proof see e.g. [**5**])
Let $f : (a, b) \to \mathbb{R}$ be a definable function. Then there exists a finite subdivision $a = a_0 < a_1 < \cdots < a_k = b$ such that, on each interval (a_i, a_{i+1}), f is continuous and either constant or strictly monotone.

EXERCISE II.9. Let $f : (a, b) \to \mathbb{R}$ be definable. Then $\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^-} f(x)$ exist in $\mathbb{R} \cup \{-\infty, +\infty\}$.

COROLLARY II.10. (o-minimal Classical Łojasiewicz Inequality)
Let $f, g : A \to \mathbb{R}$ be two continuous proper definable function. Suppose that $f^{-1}(0) \subset g^{-1}(0)$. Then there is a continuous strictly increasing definable $\varphi : [0, \infty) \to [0, \infty)$, $\varphi(0) = 0$, such that $\varphi(0) = 0$

$$|g(x)| \le \varphi(|f(x)|), \qquad \textit{for all } x \in A.$$

Moreover if

PROOF. We may suppose that f and g are non-negative and that $f^{-1}(0) \neq \emptyset$. Consider

$$\psi(t) = \max\{g(x); x \text{ such that } f(x) \le t\}.$$

$\psi(t)$ is definable and increasing though not necessarily neither continuous nor strictly increasing. Then we replace ψ by a strictly increasing continuous definable function (complete the details).

$\square$

2. Cellular Decomposition.

DEFINITION II.11. *A cdcd (cylindrical definable cell decomposition) of* $\mathbb{R}^n$ is a finite partition of $\mathbb{R}^n$ into definable sets $\{C_i\}_{i\in I}$, called *cells*, constructed recursively as follows :

$n=1$ A cdcd of $\mathbb{R}^n$ is a finite subdivision $a_1 < \cdots < a_l$. The cells are the points $\{a_i\}$ and the intervals (a_i, a_{i+1}), $i = 0, \dots, l$, where $a_0 = -\infty$ and $a_{l+1} = +\infty$.

$n>1$ A cdcd of $\mathbb{R}^n$ is given by a cdcd of $\mathbb{R}^{n-1}$ and, for each cell D of $\mathbb{R}^{n-1}$, continuous definable functions $\zeta_{D,i} : D \to \mathbb{R}$ such that on D

$$\zeta_{D,1} < \cdots < \zeta_{D,l(D)}.$$

The cells of $\mathbb{R}^n$ are **the graphs**

$$\Gamma_{\zeta_{D,i}} := \{(x, \zeta_{D,i}(x));\, x \in D\}, \qquad 1 \le i \le l(D),$$

or **the bands**

$$\mathrm{B}_{\zeta_{D,i},\zeta_{D,i+1}} := \{(x,y);\, x \in D \text{ and } \zeta_{D,i}(x) < y < \zeta_{D,i+1}(x)\}, \qquad 0 \le i \le l(D),$$

where $\zeta_{D,0} = -\infty$ and $\zeta_{D,l(D)+1} = +\infty$.

For each cell we define its dimension by $\dim(\Gamma_{\zeta_{D,i}}) := \dim D$ and $\dim(\mathrm{B}_{\zeta_{D,i},\zeta_{D,i+1}}) := \dim D + 1$.

PROPOSITION II.12. *For each cell C of a cdcd of $\mathbb{R}^n$, there is a definable homeomorphism $\theta_C : C \to \mathbb{R}^{\dim C}$.*

THEOREM II.13. (Uniform Finiteness) *Let $A \subset \mathbb{R}^n$ be a definable set such that for every $x \in \mathbb{R}^{n-1}$, the set*

$$A_x = \{y \in \mathbb{R};\, (x,y) \in A\}$$

is finite. Then there exists $k \in \mathbb{N}$ such that for all $x \in \mathbb{R}^{n-1}$ the cardinality $|A_x| \le k$.

THEOREM II.14. (Cell Decomposition) *Let $A_1, \dots, A_k$ be definable subsets of $\mathbb{R}^n$. Then there is a cdcd of $\mathbb{R}^n$ such that each A_i is a union of cells.*

THEOREM II.15. (Piecewise Continuity) *Let $A \subset \mathbb{R}^n$ be definable and let $f : A \to \mathbb{R}$ be definable. Then there is a cdcd of $\mathbb{R}^n$ adapted to A such that for every cell C contained in A, $f|_C$ is continuous.*

REMARK II.16. Similarly, for $p \in \mathbb{N}$ or $p = \infty$ or $p = \omega$ we define a $\mathcal{C}^p$ cdcd decomposition. Such a decomposition always exists for p finite, but not always for $p = \infty$. For semialgebraic sets or subanalytic sets there exists always a $\mathcal{C}^\omega$ decomposition. Moreover for these two o-minimal structures a $\mathcal{C}^\infty$ definable function is automatically $\mathcal{C}^\omega$.

EXERCISE II.17. Prove (by induction on n) that a cell is open in $\mathbb{R}^n$ if and only if its dimension is n. Prove that the union of cells of dimension n is dense in $\mathbb{R}^n$.
Let $A \subset \mathbb{R}^n$ be definable. Show that if $\overline{A}$ has non-empty interior then A has non-empty interior as well.

2.0.1. *Definable choice and curve selection lemma.*

THEOREM II.18. (Definable choice)
Let A be a definable subset of $\mathbb{R}^m \times \mathbb{R}^n$. Denote by $p : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ the projection on the first m coordinates. Then there is a definable map $f : p(A) \to \mathbb{R}^n$, not necessarily continuous, such that, for every $x \in p(A)$, $(x, f(x)) \in A$.

THEOREM II.19. (Curve selection Lemma)
Let A be a definable subset of $\mathbb{R}^n$, $b \in \overline{A}$. Then there is a continuous definable map $\gamma : [0,1) \to \mathbb{R}^n$, such that $\gamma(0) = b$ and $\gamma((0,1)) \subset A$.

EXERCISE II.20. Show that a definable function $f : A \to \mathbb{R}$ is continuous if and only if, for every continuous definable $\gamma : [0,1) \to A$

$$\lim_{t \to 0_+} f(\gamma(t)) = f(\gamma(0)).$$

THEOREM II.21. (Compactness criterion)
Let A be a definable subset of $\mathbb{R}^n$. The following properties are equivalent:

(1) *A is compact.*
(2) *Every definable continuous map $(0,1) \to A$ extends by continuity to a map $[0,1) \to A$.*

EXERCISE II.22. (Properness criterion)
Let $f : A \to B$ be a definable continuous map. Show that the following properties are equivalent:

(1) f is proper.
(2) For every definable map $\gamma : (0,1) \to A$ if $\lim_{t \to 0^+} f \circ \gamma(t)$ exists in B then $\lim_{t \to 0^+} \gamma(t)$ exists in A.

2.0.2. *Connected Components.* Recall that a topological space Y is *connected* if $\emptyset$ and Y are the only open and closed subsets of Y. *A connected component* of a topological space X is a maximal (with respect to the inclusion) non-empty connected subset of X.

EXERCISE II.23. Show that the connected components of a topological space X form a partition of X (that is, they are disjoint and their union is the whole space). Every connected component of X is closed in X. Every nonempty connected open and closed subset of X is a connected component of X.
Give an example of a topological space X and a connected component Y of X such that Y is not open in X.

EXERCISE II.24. Let A be a definable subset of $\mathbb{R}^n$ and let $\{C_i\}_{i \in I}$ be a cdcd of $\mathbb{R}^n$ such that A is a union of cells. Show that :

(1) Each cell is connected.
(2) Each connected component of A is a union of cells.
(3) Each connected component of A is definable and open and closed in A.

EXERCISE II.25. Let $A \subset \mathbb{R}^n$ be definable and connected. Show that A is definably arc-wise connected : for all $x, y \in A$ there is a defiinable continuous $\gamma : [0,1] \to A$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

2.0.3. *Dimension.* The dimension of definable sets behaves in natural way and corresponds to the intuitive understanding of dimension. We follow the approach of [**5**] and quote main results.

Definition II.26. The dimension of a definable set A is the sup of d such that there exists an injective definable map from $\mathbb{R}^d$ to A. By convention, the dimension of the empty set is $-\infty$.

The main tool that allows to transform this definition to a reasonable notion is the following definable version of Brouwer's Invariance of Domain. In the definable set-up it can be shown by elementary arguments, see [**5**] .

Lemma II.27. *Let A be a definable open subset of $\mathbb{R}^n$. Let $F : U \to \mathbb{R}^n$ be definable, continuous and injective. Then $F(U)$ is open.*

Theorem II.28.

(1) $\dim \mathbb{R}^n = n$
(2) *if $f : A \to \mathbb{R}^n$ is definable then* $\dim f(A) \leq \dim A$.
(3) *if $f : A \to B$ is definable injective then* $\dim A \leq \dim B$.
(4) $\dim A \cup B = \max\{\dim A, \dim B\}$
(5) *Let $\mathcal{C}$ be a cdcd adapted to A. Then* $\dim A = \max_{C_1 \subset A} \dim C_i$.
(6) $\dim A \times B = \dim A + \dim B$.
(7) *Let $A \subset \mathbb{R}^n \times \mathbb{R}^m$ be definable. For $x \in \mathbb{R}^m$ we denote by $A_x = \{y \in \mathbb{R}^n ; (x, y) \in A\}$. Then, for each $d \in \mathbb{N}$*
$$X_d = \{x \in \mathbb{R}^m ; \dim A_x = d\}$$
is definable and $\dim(A \cap (X_d \times \mathbb{R}^n)) = d + \dim X_d$.
(8) *Let $A \neq \emptyset$ be definable. Then* $\dim(\overline{A}) = \dim A$ *and* $\dim(\overline{A} \setminus Int(A)) < \dim A$.

We leave this theorem to the reader as exercise.

3. Derivabilty.

Exercise II.29. Let $f : I \to \mathbb{R}$ be a definable function defined on an open interval $I \subset \mathbb{R}$. Then f is differentiable outside a finite subset of I. (Hint: use monotonicity, Theorem II.8)

Exercise II.30. Let $f : I \to \mathbb{R}$ be a definable function defined on an open interval $I \subset \mathbb{R}$. Then for each $k \in \mathbb{N}$ there is a finite subset $M(k)$ of I such that f is of class C^k on $I \setminus M(k)$.

Exercise II.31.

(1) Let $A \subset \mathbb{R}^2$ be semialgebraic and nowhere dense in $\mathbb{R}^2$. Show that there exists a polynomial $P(x, y) \in \mathbb{R}[x, y]$, $P \not\equiv 0$, such that $A \subset P^{-1}(0)$.
(2) Let $f : I \to \mathbb{R}$ be a semi-algebraic function defined on an open interval $I \subset \mathbb{R}$. Then there is a finite set $M \subset I$ such that f is of class C^ω on $I \setminus M$. (Hint : use the Implicit Function Theorem).

Exercise II.32. Let $f : U \to \mathbb{R}$ be a definable function defined on an open definable $U \subset \mathbb{R}^n$. Then for each $k \in \mathbb{N}$ there is a definable subset $M(k) \subset U$, $\dim M(k) < n$, such that f is of class C^k on $U \setminus M(k)$.

Exercise II.33. Let $f : [0, \varepsilon) \to \mathbb{R}$, $\varepsilon > 0$, be definable and continuous. Suppose that $f(0) = 0$ and $f(t) > 0$ for $t > 0$. Show that

(1) If $\lim_{t\to 0^+} f'(t) = 0$ then f is differentiable at 0 and $f'(0) = 0$,
(2) Reciprocally, if $f'(0) = 0$ then there is $\varepsilon' > 0$ such that $f'(t)$ is continuous and strictly increasing on $[0, \varepsilon')$ and moreover
$$f(t) \le t f'(t), \qquad \text{for } t \in (0, \varepsilon'),$$

EXERCISE II.34. Let $f : [0, \varepsilon) \to \mathbb{R}$, $\varepsilon > 0$, be continuous definable. Suppose that $\lim_{t\to 0^+} f'(t)$ exists. Show that f is differentiable at 0 and $f'(0) = \lim_{t\to 0^+} f'(t)$,

EXERCISE II.35. (Definable L'Hôspital Rule)
Let $f, g : (0, \varepsilon) \to (0, +\infty)$, $\varepsilon > 0$, be definable. Suppose that
$$\lim_{t\to 0^+} f(t) = \lim_{t\to 0^+} g(t) = 0$$
Then
$$\lim_{t\to 0^+} \frac{f(t)}{g(t)} = \lim_{t\to 0^+} \frac{f'(t)}{g'(t)}.$$
(Hint: By a change of variable reduce to the case $g(t) \equiv t$.)

3.1. C^1-curve selection lemma. Consider continuous definable arcs $\gamma : [0, \varepsilon) \to \mathbb{R}^n$, where $\varepsilon > 0$. Replacing ε by a smaller positive number, if necessary, we may always assume that γ is C^1 on $(0, \varepsilon)$. By Monotonicity Theorem, replacing ε by an even smaller positive number, we can, moreover, assume that γ is either constant or injective.

EXERCISE II.36. Let $\gamma : [0, \varepsilon) \to \mathbb{R}^n$ be a continuous injective definable arc. Show that
$$\lim_{t\to 0^+} \frac{\gamma'(t)}{\|\gamma'(t)\|} \in S^{n-1} \qquad \text{exists .}$$

EXERCISE II.37. Suppose that a definable continuous $\gamma : [0, \varepsilon) \to \mathbb{R}^n$ is injective definable and that $\gamma(0) = 0$. Reparametrize γ by the distance to the origin $r = \|\gamma(t)\|$. Write for $r > 0$
$$\gamma(r) = r\theta(r), \qquad \theta(r) \in S^{n-1}.$$
Show that
$$\lim_{r\to 0^+} r\theta'(r) = 0$$
and then that $\lim_{r\to 0^+} \theta(r) = \lim_{r\to 0^+} (r\theta(r))'$. Therefore $\gamma(r)$ is of class $\mathcal{C}^1$.

As a corollary we get the following strengthening of the curve selection lemma.

THEOREM II.38. ($\mathcal{C}^1$-curve selection Lemma)
Let A be a definable subset of $\mathbb{R}^n$, $b \in \overline{A}$. Suppose that b is not an isolated point of A. Then there is a $\mathcal{C}^1$ definable map $\gamma : [0, 1) \to \mathbb{R}^n$, such that $\gamma((0, 1)) \subset A \setminus \{b\}$, $\gamma(0) = b$ and $\gamma'(0) \neq 0$.

EXERCISE II.39. Write an explicit γ satisfying the above theorem for $A = \{(x^2 = y^3\} \setminus \{(0, 0)\} \subset \mathbb{R}^2$, $b = (0, 0)$. Show that in this example we cannot require γ to be C^2.

EXERCISE II.40. Show that for an arbitrary continuous arc $\gamma(t) \to \gamma(0) = 0$, not identically equal to 0,
$$\lim_{t\to 0^+} \frac{\gamma(t)}{\|\gamma(t)\|} = \lim_{r\to 0^+} \theta(r) = \lim_{r\to 0^+} (r\theta(r))' = \lim_{t\to 0^+} \frac{\gamma'(t)}{\|\gamma'(t)\|} \in S^{n-1}.$$

3.2. Definable Sard Theorem. Let $f : U \to \mathbb{R}^m$ be a differentiable map defined on an open definable $U \subset \mathbb{R}^n$. A point $x \in U$ is called a *critical* (or *singular*) point of f if the rank of the Jacobian matrix of f at x satisfies

$$\operatorname{rank} J_f(x) < m.$$

Denote the set of critical points of f by $C(f)$. The map $x \to J_f(x)$ is definable and, hence, so is the set $C(f)$. If the Jacobian matrix $J_f(x)$ is of maximal rank then we say that x is a *regular* point of f.

Exercise II.41. Let $f : U \to \mathbb{R}$ be a differentiable definable function defined on an open definable $U \subset \mathbb{R}^n$. Then the set of critical values of f is finite.
Let A be a connected component of $C(f)$. Show that f is constant on A.

Exercise II.42. (Definable Sard Theorem)
Let $f : U \to \mathbb{R}^m$ be a differentiable definable map defined on an open definable $U \subset \mathbb{R}^n$. Then the set of critical values of f: $f(C(f))$, is definable and nowhere dense in $\mathbb{R}^m$ (i.e. $\dim f(C(f)) < m$).
(Hint: if $m \leq n$ then use definable choice)

4. Examples of o-minimal structures.

(1) $\mathbb{R}_{alg}$ - **Semialgebraic sets** form an o-minimal structure (check the axioms). This is the smallest o-minimal structure contained in any other o-minimal structure.
(2) $\mathbb{R}_{exp}$ is the smallest structure that contains the real exponential function $\exp(x) = e^x$. This structure contains also the logarithm function $\log : (0, \infty) \to \mathbb{R}$ and the functions $x^r : (0, \infty) \to \mathbb{R}$ for $r \in \mathbb{R}$.
(3) $\mathbb{R}_{an}$ - **Globally subanalytic sets** is the smallest structure that contains all restricted analytic functions, that is the restriction to $[-1, 1]^n$ of analytic functions defined in a neighbourhood of the cube $[-1, 1]^n$.
(4) $\mathbb{R}_{an}^{\mathbb{R}}$ - the smallest structure that contains $\mathbb{R}_{an}$ and the functions $x^r : (0, \infty) \to \mathbb{R}$ for $r \in \mathbb{R}$.
(5) $\mathbb{R}_{an,exp}$ - the smallest structure that contains $\mathbb{R}_{exp}$ and $\mathbb{R}_{an}$.

Exercise II.43. Show that there is no o-minimal structure that contains the graph of $x \to \sin x, x \in \mathbb{R}$.

Definition II.44. An o-minimal structure $\{S_n\}$ is called *polynomially bounded* if for every definable function $f : \mathbb{R} \to \mathbb{R}$ there exists $N \in \mathbb{N}$ such that

$$f(t) = O(t^N) \quad \text{as } t \to +\infty.$$

Exercise II.45. Show that the o-minimal structure of semialgebraic sets is polynomially bounded.

Exercise II.46. Show that an o-minimal structure is polynomially bounded if and only if for every $f : \mathbb{R} \to \mathbb{R}$ definable and continuous at 0, $f(0) = 0$, there is $N \in \mathbb{N}^*$ and $\varepsilon > 0$ such that for $|t| \leq \varepsilon$

$$|f(t)| \leq |t|^{\frac{1}{N}}.$$

The structures $\mathbb{R}_{an}$, and $\mathbb{R}_{an}^{\mathbb{R}}$ are polynomially bounded. An o-minimal structure that contains the real exponential function is not polynomially bounded (why ?).

Theorem II.47. (Miller, Growth Dychotomy)
An o-minimal structure is either polynomially bounded or it contains the real exponential function.

CHAPTER III

L-regular decomposition

A cdcd decomposition can be used to study several properties of definablity but not to study the topology since it is not clear from the construction how the cells are joined together. In particular an arbitrary cdcd is not a stratification. A cdcd constructed in a generic system of coordinates has better properties, see Theorem V.8 for instance, and is used to construct triangulations.

In this chapter we introduce an L-regular decomposition in cells, called L-regular cells, that allow to study various metric properties of definable sets. But its construction requires to work in many generic system of coordinates at the same time.

1. Triangulation

We just quote the results, see for instance [**5**]

THEOREM III.1. *Let A be a closed and bounded definable subset of $\mathbb{R}^n$ and let B_i, $i = 1, ..., k$, be definable subsets of A. Then there exist a finite simplicial complex K with vertices in $\mathbb{Q}^n$ and a definable homeomorphism $\Phi : |K| \to A$ such that each B_i is a union of images by Φ of open simplices of K.*

THEOREM III.2. *Let X be a closed and bounded definable subset of $\mathbb{R}^n$ and let $f : X \to \mathbb{R}$ be a continuous definable function. Then there exist a finite simplicial complex K in $\mathbb{R}^{n+1}$ and a definable homeomorphism $\rho : |K| \to X$ such that $f \circ \rho$ is an affine function on each simplex of K.*

Moreover, given finitely many definable subsets B_i, $i = 1, ..., k$, of X, we may choose the triangulation $\rho : |K| \to X$ so that each B_i is a union of images of open simplices of K.

As a corollary one obtains a local conic structure.

THEOREM III.3 ((Local Conic Structure)). *Let A be a closed definable subset of $\mathbb{R}^n$ and let $a \in A$. There are an $r > 0$ and a definable homeomorphism h from the cone with vertex a and base $S(a, r) \cap A$ onto $B(a, r) \cap A$, satisfying $h_{|S(a,r)\cap A} = Id$ and $\|h(x) - a\| = \|x - a\|$ for all x in the cone.*

The theorem of triangulation of continuous definable functions cannot be generalized to all continuous definable maps. The simplest example is the blowing-up map $f(x, y) = (x, xy)$. Indeed, consider f restricted to the unit square $f : [0, 1]^2 \to \mathbb{R}^2$. There is no way to choose triangulations $\Phi : |K| \to [0, 1]^2$ and $\Psi : |L| \to f([0, 1]^2)$ such that $\Psi^{-1} \circ f \circ \Phi$ is affine on every simplex of K.

2. L-regular cells

We define, by induction on n, a class of subsets of $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ let us write $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. We say that $A \subset \mathbb{R}^n$ is a *standard L-regular cell in $\mathbb{R}^n$ with constant*

C, if $A = \{0\}$ for $n = 0$, and for $n > 0$ the set A is of one of the following forms:

(graph) $$A = \Gamma_h = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\ x_n = h(x'),\ x' \in A'\}$$

or

(band) $$A = B_{f,g} = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\ f(x') < x_n < g(x'),\ x' \in A'\}$$

where A' is a standard L-regular cell in $\mathbb{R}^{n-1}$ with constant C, $f, g, h : A' \to \mathbb{R}$ are C^1 functions (or functions of the regularity C^p, $p = 1, 2, ..., \infty, \omega$, we fix at the beginning) such that $f(x') < g(x')$ for $x' \in A'$, and

$$\|df(x')\| \leq C,\ \|dg(x')\| \leq C,\ \|dh(x')\| \leq C \tag{4}$$

for all $x' \in A'$. We call A' *the base* of the cell A.

We say that $B \subset \mathbb{R}^n$ is an *L-regular cell in* $\mathbb{R}^n$ *with constant* C, if there exists an orthogonal change of variables $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(B)$ is a standard L-regular cell (with constant C) in $\mathbb{R}^n$. By convention the empty set is an L-regular set (with any constant).

We say that $A \subset \mathbb{R}^n$ verifies *the Whitney property with constant* $M > 0$, or is *M-quasi-convex*, if any two points $x, y \in A$ can be joined in A by a piecewise smooth arc of length $\leq M|x - y|$. It is easily seen by induction on dimension that

Lemma III.4. *Any L-regular cell in $\mathbb{R}^n$ with constant C is M-quasi-convex, where $M = (C+1)^{n-1}$. Moreover $\overline{A}$ is also M-quasi-convex.*

3. L-regular decomposition with parameter

The following result was proven in [**9**, Proposition 1.4].

Theorem III.5. *Let $A^k \subset \mathbb{R}^n \times \mathbb{R}^p$, $k \in K$, be a finite collection of definable sets in an o-minimal structure. Then there exist finitely many disjoint definable sets $B^i \subset \mathbb{R}^n \times \mathbb{R}^p$, $i \in I$, and linear orthogonal mappings $\varphi^i : \mathbb{R}^n \to \mathbb{R}^n$, $i \in I$, such that:*

a) *for every $t \in \mathbb{R}^p$, each $\varphi^i(B^i_t)$ is a standard L-regular cell in $\mathbb{R}^n$ with constant C. The constant $C = C_n$ depends only on n.*
b) *For every $t \in \mathbb{R}^p$, the family $B^i_t \subset \mathbb{R}^n$, $i \in I$, is a stratification of $\mathbb{R}^n$.*
c) *For any $k \in K$ there exists $I_k \subset I$ such that $A^k_t = \bigcup_{i \in I_k} B^i_t$, for every $t \in \mathbb{R}^p$.*

Remark III.6. Let $\varepsilon > 0$, we say that Γ, a d-dimensional C^1 submanifold of $\mathbb{R}^n$, is *ε-flat* if for every pair $x, y \in \Gamma$ we have $\delta(T_x, T_y) \leq \varepsilon$, where δ is a fixed metric on the grasmannian $\mathcal{G}_{d,n}$.

Given $\varepsilon > 0$, it follows from the proof of [**9**, Proposition 1.4], that we can additionally require that

d) for each $i \in I$, B^i_t is a definable family of ε-flat cells.

More precisely, let us, for each $d < n$, fix a finite covering $\mathcal{G}_{d,n} = \bigcup \Theta^\varepsilon_{d,\nu}$. Then we may require that for each $i \in I$, $d_i = \dim B^i_t$, there exists ν_i such that for eavery $(x, t) \in B^i$

$$T_x B^i_t \in \Theta^\varepsilon_{d_i, \nu_i}.$$

CHAPTER IV

Łojasiewicz Inequalities

1. Classical Łojasiewicz Inequality

Let us recall first

THEOREM IV.1. (o-minimal classical Łojasiewicz Inequality)
Let $f, g : A \to \mathbb{R}$ be two continuous proper definable functions. Suppose that $f^{-1}(0) \subset g^{-1}(0)$. Then there is a continuous strictly increasing definable $\varphi : [0, \infty) \to [0, \infty)$, $\varphi(0) = 0$, such that $\varphi(0) = 0$

$$|g(x)| \leq \varphi(|f(x)|), \qquad \textit{for all } x \in A.$$

If the structure is polynomially bounded we obtain the classical Łojasiewicz Inequality.

THEOREM IV.2 (classical Łojasiewicz Inequality). *Let A be compact definable and let $f, g : A \to \mathbb{R}$ be two continuous definable functions such that $f^{-1}(0) \subset g^{-1}(0)$. Then there are an exponent $N \in \mathbb{N}$ and a constant $C > 0$ such that*

$$|g(x)|^N \leq C|f(x)|, \qquad \textit{for all } x \in A.$$

EXERCISE IV.3. Let $f : X \to \mathbb{R}$ be a continuous definable function defined on a compact $X \subset \mathbb{R}^n$ and let $Y = f^{-1}(0)$. Show that there are an exponent $N \in \mathbb{N}$ and constants $c, C > 0$ such that

$$c \operatorname{dist}(x, Y)^N \leq |f(x)| \leq C \operatorname{dist}(x, Y)^{1/N}, \qquad \text{for all } x \in X.$$

EXERCISE IV.4. Let $f : U \to \mathbb{R}$ be a real analytic function defined on a open $U \subset \mathbb{R}^n$ and let $Y = f^{-1}(0)$. Let $x_0 \in U$. Show that there are an exponent $N \in \mathbb{N}_{\geq 1}$, a constant $C > 0$ and a neigbourhood U' of x_0 such that

$$|f(x)| \leq C \operatorname{dist}(x, Y)^N, \qquad \text{for all } x \in X.$$

EXERCISE IV.5 (Hölder Continuity). Let $f : X \to \mathbb{R}$ be a continuous definable function defined on a closed $X \subset \mathbb{R}^n$. Show that f is locally Hölder continuous.

EXERCISE IV.6 (Regular separation). Let X, Y be two closed definable subsets of $\mathbb{R}^n$. Let $x_0 \in X \cap Y$. Then there are a neighbourhood U of x_0, an exponent $N \in \mathbb{N}$, and a positive constant $c > 0$ such that

$$c \operatorname{dist}(x, X \cap Y)^N \leq \operatorname{dist}(x, X) + \operatorname{dist}(x, Y), \qquad \text{for all } x \in X \in U.$$

2. Łojasiewicz Gradient Inequality

Theorem IV.7. *Let $f : \Omega \to \mathbb{R}$ be a differentiable definable function defined on an open bounded definable $\Omega \subset \mathbb{R}^n$. Suppose $f(x) > 0$ for all $x \in \Omega$. Then there exist $c > 0$, $\rho > 0$ and a continuous definable function $\Psi : [0, \infty) \to [0, \infty)$, such that*

$$\| \operatorname{grad}(\Psi \circ f)(x) \| \geq c,$$

for every $x \in \Omega$, $f(x) \in (0, \rho)$.

The above result is due to K. Kurdyka [**8**]. It implies the original Łojasiewicz Gradient Inequality.

Theorem IV.8. (Łojasiewicz Gradient Inequality)
Let $f : U \to \mathbb{R}$ be a real analytic function defined on an open neighbourhood $U \subset \mathbb{R}^n$ of the origin. Suppose $f(0) = 0$. Then there exist an exponent $\alpha < 1$, a constant $C > 0$ and a neigbourhood U' of 0, such that

$$|f(x)|^\alpha \leq C \| \operatorname{grad} f(x) \| \quad \textit{for } x \in U'.$$

The strength and the usufulness of this result is the fact the exponent α is strictly bigger than 1. The above inequality holds (locally) for any differentiable function definable in a polynomially bounded o-minimal structure as also follows from Kurdyka's result.

The following inequality is valid in an arbitrary o-minimal structure.

Exercise IV.9. (o-minimal version of Bochnak-Łojasiewicz Inequality)
Let $f : U \to \mathbb{R}$ be a differentiable definable function defined on an open neighbourhood $U \subset \mathbb{R}^n$ of the origin. Suppose $f(0) = 0$. Then there exist $C > 0$ and a neigbourhood U' of 0 such that

$$|f(x)| \leq C \, \|x\| \, \| \operatorname{grad} f(x) \| \quad \text{for } x \in U'.$$

(Hint: by the C^1-curve selection lemma it suffices to check it any C^1 definable arc $\gamma : [0, \varepsilon) \to U$, $\gamma(0) = 0$ and $\gamma'(0) \neq 0$.)

CHAPTER V

Regular points. Stratifications

In this chapter we suppose that k is either an integer ≥ 1, or in the semi-algebraic case we consider additionally $k = \omega$. In the latter case, C^ω means real analytic.

Let $k \in \mathbb{N} \cup \{\omega\}, k \geq 1$. We say that a definable $X \subset \mathbb{R}^n$ is *k-regular (of dimension d) at $p \in X$* if there is a neighbourhood U of p in $\mathbb{R}^n$ such that $X \cap U$ is C^k-submanifold of U of dimension d. In this case we also say that p is *a k-regular point of X*.

We denote by $Reg_k(X)$ the set of all k-regular $p \in X$ of dimension $\dim X$. We say that X is *k-regular* if $Reg_k(X) = X$, that is if X is a pure-dimensional locally closed C^k-submanifold of $\mathbb{R}^n$.

1. Good coordinates

THEOREM V.1. *Let $X \subset \mathbb{R}^n, \dim X < n$, be closed and definable. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be a generic affine projection. Then $\pi|_X$ is proper with finite fibres.*

To be more precise, in the above statement we mean that there exists a definable subset $A \subset \mathbb{RP}(n-1)$, $\dim A < n-1$, such that every affine projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ such that $[\ker \pi] \notin A$ satisfies the statement.

PROOF. Define the limit set of X at infinity as

$$X_\infty = \{p \in \mathbb{RP}(n-1); \exists \gamma : (0, \varepsilon) \to X, \lim_{t \to 0^+} \|\gamma(t)\| = +\infty, \lim_{t \to 0^+} [\gamma(t)] = p\},$$

where γ are definable. It can be checked easily (exercise) that X_∞ is closed and definable, $\dim X_\infty \leq \dim X - 1$. Note that in the definition of X_∞ we may replace $\lim_{t \to 0^+}[\gamma(t)]$ by $\lim_{t \to 0^+}[\gamma'(t)]$, since the identity of exercise II.40 holds also for the arcs going to infinity. It is easy to check (exercise), that $\pi|_X$ is proper if and only if $[\ker \pi] \notin X_\infty$.

Let $\Pi : \mathbb{R}^n \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ be given by $\Pi(x, u) = (\pi_u(x), u)$, where

$$\pi_u(x) = (x_1, \ldots, x_{n-1}) - x_n u$$

is a linear projection with kernel spanned by $(u, 1)$. We show that for generic $u \in \mathbb{R}^{n-1}$, the restriction of π_u to X is finite-to-one. Suppose that this is not the case. Then, by Definable Choice, there is an open non-empty definable $U \subset \mathbb{R}^{n-1}$, a definable C^1 map $\varphi : U \to \mathbb{R}^n$, and $\varepsilon > 0$ such that

$$\psi(u, \lambda) = \varphi(u) + \lambda(u, 1) \in X \qquad \text{for } u \in U,\ \lambda \in [0, \varepsilon).$$

But the Jacobian determinant of ψ, is of the form

$$\det D\psi(u, \lambda) = \lambda^{n-1} + \sum_{i=0}^{n-2} a_i(u)\lambda^i$$

and hence cannot be identically equal to zero. Therefore the image of ψ is of dimension n and cannot be a subset of X. $\square$

Proposition V.2. *Let $X \subset \mathbb{R}^n, \dim X < n$, be closed and definable and suppose such that the restriction $\pi|_X$ of the standard projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is proper with finite fibres. Let $Y \subset \pi(X)$ be a definable set such that every $y \in \overline{Y}$ satisfies :*

(5) *there is a basis of neighbourhoods V of y such that $V \cap Y$ is connected.*

Then every continuous $\zeta : Y \to \mathbb{R}$ whose graph is contained in X extends continuously to $\overline{Y}$.

Proof. Consider the closure of the graph of ζ, $\overline{\Gamma}_\zeta \subset X$. It suffices to show that the projection $\overline{\Gamma}_\zeta \to \overline{Y}$ is a bijection. For this suffices, by the assumption on X, to show that the fibres of this projection are connected.

Let $y \in \overline{Y}$ and let $U_i, i \in I$, be a basis of neighbourhoods of y as in the assumption. Then the closures A_i of $\zeta(U_i \cap Y)$ are connected, so they are closed intervals. The intersection of A_i equals $\pi^{-1}(y) \cap \overline{\Gamma}_\zeta$ so the latter set is connected as claimed. □

Exercise V.3. Let $Y \subset \mathbb{R}^n$ be an open cell of a cdcd decomposition. Show that there is a definable $A \subset Fr(Y)$, $\dim A \le n-2$, such that every $y \in Fr(Y) \setminus A$ satisfies condition (5).

Exercise V.4. Let $f : Y \to \mathbb{R}$ be a continuous bounded definable function defined on an open $Y \subset \mathbb{R}^n$. Suppose moreover that there is a definable $A \subset Fr(Y)$, $\dim A \le n-2$, such that every $y \in Fr(Y) \setminus A$ satisfies condition (5). Show that there $B \subset \overline{Y} \setminus Y$, $\dim B \le n-2$, such that f extends by continuity on $\overline{Y} \setminus B$.

2. Stratifications.

We call a cdcd (cylindrical definable cell decomposition) of $\mathbb{R}^n$, see definition II.11, *k-regular* if all the functions $\zeta_{D,i}$ of the definition II.11, are of class C^k. Then, the cells are k-regular. The first part of the next theorem is an easy application of exercise II.30.. The second part is more delicate and follows from the IFT (Implicit function theorem) similarly to exercice II.31.

Theorem V.5. *Let $A_1, \dots, A_m$ be definable subsets of $\mathbb{R}^n$ and $k \in \mathbb{N}$ Then there is a k-regular cdcd of $\mathbb{R}^n$ such that each A_i is a union of cells. In the semi-algebraic case the theorem also holds for $k = \omega$.*

Exercise V.6. Let $X \subset \mathbb{R}^n$ be definable and $d = \dim X$. Let X_d denote the set of these points $x \in X$ that X is of dimension d at x (that is X intersected with any sufficiently small neighbourhood of x is of dimension d). Show that X_d is definable of pure dimension d and $\dim(X \setminus X_d) < d$.

Moreover, show that $Reg_k(X)$ of X contains a definable open dense subset of X_d. (We again assume $k \in \mathbb{N}, k \ge 1$ in general and additionally $k = \omega$ in the semi-algebraic case.)

Definition V.7. By a *(definable) C^k-stratification* of $X \subset \mathbb{R}^n$ we mean a finite decomposition (disjoint union)

$$X = \bigsqcup_{S_i \in \mathcal{S}} S_i,$$

where the sets S_i, called *strata*, are definable and k-regular. Unless otherwise stated we always assume that the strata S_i are connected and that the stratification satisfies the following condition:

(*Frontier condition*) For each stratum $S \in \mathcal{S}$, $(\bar{S} \setminus S) \cap X$ is a union of strata.

Theorem V.8. *Let $A_1, \dots, A_m$ be definable subsets of $\mathbb{R}^n$ and $k \in \mathbb{N}$, $k \ge 1$. Then, after a generic linear change of coordinates in $\mathbb{R}^n$, there is a k-regular cdcd of $\mathbb{R}^n$, that is also a*

C^k-stratification of $\mathbb{R}^n$, such that each A_i is a union of cells. In the semi-algebraic case the theorem holds also for $k = \omega$.

Proof. The construction of a cdcd adapted to A_i is algorithmic. Before the first step we choose a system of coordinates in $\mathbb{R}^n$ so that the projection $\pi_{n-1} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ restricted to $X = \bigcup Fr(A_i)$ is proper and finite. Then all the cells of the graph type are contained in X.

Suppose that a cdcd $\mathcal{C}$ of $\mathbb{R}^n$ is defined by a cdcd $\mathcal{D}$ of $\mathbb{R}^{n-1}$ and C^k functions $\zeta_{D,i}$, $D \in \mathcal{D}$ with graphs contained in X. We also assume that the cdcd $\mathcal{D}$ satisfies two conditions :

- $\mathcal{D}$ satisfies frontier condition.
- each $D \in \mathcal{D}$ satisfies condition (5) at all its points

Then it suffices to show that $\mathcal{C}$ also satisfies these two conditions. If the cell $C \in \mathcal{C}$ is the graph $\zeta_{D,i} : D \to \mathbb{R}$, then it follows from Proposition V.2 since $\zeta_{D,i}$ extends continuously onto $\overline{D}$. A similar argument works for the cells of the band type. The details are left as exercise. $\square$

A similar argument gives the following result (exercise).

Proposition V.9. *Let $U \subset \mathbb{R}^n$ be a open definable. Then there exists a closed definable subset $Y \subset \overline{U} \setminus U$, $\dim Y \leq n-2$, such that for every $p \in \overline{U} \setminus (U \cup Y)$, either $p \in Int(\overline{U})$ or the pair $(\overline{U}, \overline{U} \setminus U)$ near p is a C^k-manifold with boundary.*

Proposition V.10. *Let $X \subset \mathbb{R}^n$ be definable. Then for each $d \leq \dim X$ and for each $k \in \mathbb{N}, k \geq 1$, the set of k-regular of dimension d points of X is definable.*

Proof. We show it for $d = \dim X$. Fix the projection $\pi_d : \mathbb{R}^n \to \mathbb{R}^d$, for instance the standard one $\pi_d(x_1, \ldots, x_n) = (x_1, \ldots, x_d)$. We claim that the set of points of $p \in X$ such that in a neighbourhood of p, X coincides the graph a C^k definable function $\varphi : U \to \mathbb{R}^{n-d}$, defined on an open definable neighbourhood of $\pi_d(p)$ in $\mathbb{R}^d$, is definable. Indeed, firstly if we skip the k-regularity requirement for φ then the set of these points can be expressed by a first order formula. Then the regularity condition can be added by exercise II.32. We denote this set by $Top_{\pi_d,k}(X)$, the set of topographic points with respect to the projection π_d.

Now consider the finite set of orthogonal projection $\pi_I : \mathbb{R}^n \to \mathbb{R}^d$, onto the coordinate d-subspaces of $\mathbb{R}^n$, thus indexed by $I \subset \{1, \ldots, n\}$, $|I| = d$. Then $Reg_k(X)$ is the union of all $Top_{\pi_I,k}(X)$. $\square$

The analogous statement holds also in the semi-algebraic case and $k = \omega$, but its proof is much more difficult.

3. Singularities in codimension 1

Proposition V.11. *Fix $k \in \mathbb{N}$, $k \geq 1$. Let $X \subset \mathbb{R}^n$ be a definable closed of dimension d. Then there exists a closed definable subset $Y \subset X$, $\dim Y \leq d-2$ such that any $p \in X \setminus (Reg_k(X) \cup Y$ satisfies the following property:*

(6) *there are $m \in \mathbb{N}$ and a neighbourhood V of p such that $Reg_k(X) \cap V = X_1 \sqcup \ldots \sqcup X_m$, with X_i connected, and such that for every $i = 1, \ldots, m$, the closure of each X_i in V is a C^1manifold with boundary $V \cap (X \setminus Reg_k(X))$.*

Proof. The case $n = d$ is given in Proposition V.9. Thus we may suppose that $n > d$.

Suppose first that $Reg_k(X)$ is L-topographic, the graph of $\varphi : U \to \mathbb{R}^{n-d}$, and that the projection $\pi_d : \mathbb{R}^n \to \mathbb{R}^d$ restricted to X is proper and finite. By Proposition V.9 we may suppose that $(\overline{U}, \overline{U} \setminus U)$ is a manifold with boundary. Moreover, by Exercise V.4, we may suppose that φ and the first order partial derivatives of φ extend to the boundary. It is a nice exercise to show that these properites imply that the graph of φ is a C^1 manifold. (One can easily reduce to the case $\overline{U}$ being the half space $x_d \geq 0$.)

Now the proposition in the general case follows from Proposition VI.5. $\square$

The next lemma is a corollary of Proposition V.11 and is left as exercise.

LEMMA V.12. Wing lemma.
Let X, Y be definable subsets of $\mathbb{R}^n$. Suppose that $Y \subset \overline{X} \setminus X$. Then there is a definable subset $V \subset X$ and a definable subset $S \subset Y$, $\dim S < \dim Y$ such that every $p \in Y \setminus S$ has a neighbourhood $U_p \subset \mathbb{R}^n$ such that $(\overline{V} \cap U_p, Y \cap U_p)$ is a C^1 manifold with boundary.

Note that the C^1 curve selection lemma is a special case of this theorem.

CHAPTER VI

Integration on definable sets

Let $X \in \mathbb{R}^n$ be definable $\dim X = d$. In this section introduce the integration along X. Roughly speaking the idea is the following. Firstly we show that d-dimensional volume of X is locally finite (that is finite on any relatively compact subset of X). This allows us to introduce the d-dimensional volume on X which we denote by $dvol_d$ or $\mathcal{H}^d$. In the process of integration we may ignore any definable subset of X of dimension smaller than d since its d-dimensional volume is zero. Then we show that the bounded definable functions (or forms) are locally integrable on X. Our main tool will the decomposition into L-topographic sets.

1. L-topographic sets.

We call a definable subset $X \subset \mathbb{R}^n$ *topographic of class* $\mathcal{C}^k$, $k \in \mathbb{N} \cup \{\omega\}, k \geq 1$, if, after a linear change of coordinates in $\mathbb{R}^n$, X is the graph of a C^k definable map $\varphi : U \to \mathbb{R}^{n-d}$, defined on an open definable $U \subset \mathbb{R}^d$. Such X is called *L-topographic of class* $\mathcal{C}^k$ if, moreover, the partial derivatives $\partial\varphi/\partial u_i$ of φ are bounded on U.

Exercise VI.1. Show that a cell of a k-regular cdcd is topographic of class $\mathcal{C}^k$.

Exercise VI.2. Let $X \subset \mathbb{R}^n$ be L-topographic, the graph of a C^k definable function $\varphi : U \to \mathbb{R}^{n-d}$. Show that the projection $\pi_d : \mathbb{R}^n \to \mathbb{R}^d$ restricted to $\overline{X}$ is proper and finite.
(Hint: Consider the case $n = d + 1$.)

The importance of L-topographic sets follows from the following two propositions.

Proposition VI.3. *A bounded L-topographic set of dimension d is of finite d-volume.*

Proof. For a C^1 injective map $\Phi : U \to \mathbb{R}^n$ defined on an open bounded $U \subset \mathbb{R}^d$, the d-volume of its image is given in terms of the Gramm determinant of $\det(\langle\partial\Phi/\partial u_i, \partial\Phi/\partial u_j\rangle), i, j = 1, \dots, d$ as follows

$$\mathrm{vol}_d(\Phi(U)) = \int_U \sqrt{\det(\langle\partial\Phi/\partial u_i, \partial\Phi/\partial u_j\rangle)}\, du_1 \dots du_d. \tag{7}$$

Thus the proposition follows from the following lemma, left as exercise. □

Lemma VI.4. *Let $U \subset \mathbb{R}^d$ be definable and bounded and let the function $f : U \to \mathbb{R}$ be a bounded definable function. Then f is integrable with respect to the standard Lebesgue measure on $\mathbb{R}^d$. Moreover if $Y \in U$ be definable of dimension smaller than d then*

$$\int_U f\, du_1 \dots du_d. = \int_{U\setminus Y} f\, du_1 \dots du_d..$$

Proposition VI.5. *Let X be definable of dimension d, $k \in \mathbb{N}$, $k \geq 1$. Then X can be decomposed as a finite disjoint union of L-topographic sets of class $\mathcal{C}^k$ and a definable set of dimension strictly smaller of d.*

Moreover, given $\varepsilon > 0$ we may choose choose these topographic sets to be ε-flat (in the sense of Remark III.6).

We will show Proposition VI.5 in subsection 1.

Corollary VI.6. *Let $X \subset \mathbb{R}^n$ be a bounded definable of dimension d. Then the d-dimensional volume of X is finite.*

2. Integration on definable sets. Volume form

Let $X \subset \mathbb{R}^n$ be definable of dimension d. The formula (7) and allows us to define the integration with respect to the d-density of X. Thus for given $f : X \to \mathbb{R}$, the integral of f, if it exists, can be expressed in terms of local parametrisation, as follows. Let $\Phi : U \to X$ be an injective C^1 map $\Phi : U \to X$ defined on an open $U \subset \mathbb{R}^d$. Then

$$\int_{\varphi(U)} f \; dvol_d = \int_U f(\Phi(u)) \sqrt{\det(\langle \partial\Phi/\partial u_i, \partial\Phi/\partial u_j\rangle)} \; du_1 \dots du_d. \tag{8}$$

The integration along X can be reduced to the subvariety case since, if $X' \subset X$ is a definable C^1 regular set such that $\dim X \setminus X' < d$, for instance $X' = Reg_1(X)$, then

$$\int_X f \; dvol_d = \int_{X'} f \, dvol_d.$$

Similarly to Lemma VI.4 and Corollary VI.6 we have the following result.

Corollary VI.7. *Let $X \subset \mathbb{R}^n$ be a bounded definable of dimension d and let $f : X \to \mathbb{R}$ be definable and bounded. Then f is integrable.*

Let M be a smooth oriented d-dimensional submanifold of $\mathbb{R}^n$. The volume form ω_M of M is a d form on M that is characterized in terms of local parametrization as follows. Let $\Phi : U \to V$ be an orientation preserving C^1 diffeomorphism from an open $U \subset \mathbb{R}^d$ onto an open $V \subset M$. Then

$$\Phi^*\omega_M = \sqrt{\det(\langle \partial\Phi/\partial u_i, \partial\Phi/\partial u_j\rangle)} \; du_1 \wedge \dots \wedge du_d.$$

Exercise VI.8. Let M be a d-dimensional oriented submanifold of $\mathbb{R}^n$ and let ω_M be the volume form of M. Let $p \in M$ and let $\vec{v}_1, \dots, \vec{v}_d$ be a system of vectors in T_pM. Show that $\omega_M(p)(\vec{v}_1, \dots, \vec{v}_d)$ equals, up to a sign, the d-volume of the parallelepiped spanned by $\vec{v}_1, \dots, \vec{v}_d$.

Exercise VI.9. Let $M \subset \mathbb{R}^n$ be a smooth hypersurface defined by $f(x) = 0$ and oriented by $\frac{\operatorname{grad} f}{\|\operatorname{grad} f\|}$. Show that the volume form of M is

$$\omega = \sum_{i=1}^n (-1)^{i-1} \frac{\partial f/\partial x_i}{\|\operatorname{grad} f\|} \, dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n.$$

In particular the volume form of the sphere S^{n-1} oriented by the outward unit normal vector field is

$$\omega = \sum_{i=1}^n (-1)^{i-1} x_i \, dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n.$$

Exercise VI.10. Let $X \subset \mathbb{R}^n$ be a 1-regular definable set of dimension d. Suppose that X is bounded and oriented. Let η be a differential form of degree d on $\mathbb{R}^n$ with definable coefficients that is

$$\eta = \sum \eta_I \, dx_I,$$

where $I = \{i_1 < \ldots < i_d\}$, $dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_d}$, and η_I are definable functions defined on X. Show that, if η_I are bounded on X, then η is integrable on X.

CHAPTER VII

Further properties

1. Tangent and normal bundles

Throughout this section we suppose $X \subset \mathbb{R}^n$ to be k-regular definable of dimension d. We identify the underlying total spaces of the tangent and the normal bundles to X with the following subsets of $\mathbb{R}^n \times \mathbb{R}^n$.

$$TX = \{(x, v) \in X \times \mathbb{R}^n; v \in T_x X\},$$
$$NX = \{(x, v) \in X \times \mathbb{R}^n; v \perp T_x X\}$$

Exercise VII.1.

(1) Suppose that V is topographic given by the graph of $\varphi : U \subset \mathbb{R}^d \to \mathbb{R}^{n-d}$. Write explicit parmetrisations of TV and NV. Show that TV and NV can be trivialised by definable trivialisations.
(2) Prove that TX and NX are definable and $k-1$-regular. (In the semialgebraic case they are C^ω if so is X.)
(3) Denote $SNX = NX \cap \mathbb{R}^n \times S^{n-1}$ the unit bundle of NX. Show that SNX is a $(k-1)$-regular definable set of dimension $n-1$. Show that NX is orientable.

Proof of Proposition VI.5. Suppose first that $X \subset \mathbb{R}^n$ is an oriented hypersurface. The Gauss map of X

$$\nu : X \to S^{n-1}$$

associates to $p \in X$ the unit oriented normal vector to $T_p X$.

Exercise VII.2. Suppose that X is the graph of $\varphi : U \to \mathbb{R}$, where $U \subset \mathbb{R}^{n-1}$. Write a formula for the Gauss map in terms of the partial derivatives of φ. Show that X is L-topographic if and only if there is $C > 0$ such that for all $x \in X$

$$\max_{j<n} |\nu_j(x)| < C|\nu_n(x)|.$$

We continue the proof. The graph of ν is a definable subset of SNX. By dimensional reasons the generic fibres of ν are finite. For $i = 1, \dots n$ consider the set

$$S_i = \{v \in S^{n-1}, |v_i| > \max_{j \neq i} |v_j| \ \}$$

and its inverse image $X_i = \nu^{-1}(S_i)$. Thus we have

$$X = X_1 \cup \dots \cup X_n \cup Y,$$

where $\dim Y < n-1$ provided the system of coordinates of $\mathbb{R}^n$ is chosen sufficiently generic (it follows from the next subsection). Then the projection $\pi_i : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $\pi_i(x_1, \dots x_n) = (x_1, \dots, \widehat{x_i}, \dots, x_n)$, restricted to X_i is open and has finite fibres. Let $A_i = \pi_i(\overline{X_i}) \subset \mathbb{R}_{n-1}$.

Choose a cdcd of $\mathbb{R}^{n-1}$ adapted to A_i and let C be an open cell. Denote $X_C = \pi_i^{-1}(C) \cap X_i$. Then

$$X_C \to C$$

is a finite covering of C. Since the cell C is homeomorphic to $(0,1)^n$ this covering has to be trivial. Hence X_C is a disjoint union of topographic sets. They are L-topographic by Exercise VII.2. This ends the proof in the oriented hypersurface case. The same argument applies to the non-oriented case since the definition of S_i does not depend on the choice of orientation.

A similar argument can be used in the general d-dimensional case by replacing the Gauss map by the classifying map of the normal bundle NX of X (or, alternatively, of TX)

$$\gamma : X \to G(n, n-d),$$

where $G(n, n-d)$ is the Grassmanian of non-oriented $(n-d)$-vector subspaces of $\mathbb{R}^n$. □

2. The transversality of a general translate

We give an o-minimal version of a theorem of Kleiman [The transversality of a general translate, *Compositio Math.*, tome 28, no. 3 (1974), p. 287–297].

A group G is will be called *a definable group* if G is definable as a set and the group operations: multiplication and taking the inverse, are definable. We say that G is a $\mathcal{C}^k$ group, if G and the group operations are $\mathcal{C}^k$ regular. We are mostly interested in the algebraic subgroups of the linear group $GL(n,\mathbb{R})$ or the affine group $Aff(n,\mathbb{R}) = \mathbb{R}^n \ltimes GL(n,\mathbb{R})$, such as the orthogonal group $O(n)$ or the group of isometries $Isom(n)$. These groups are real algerbaic, in particular C^ω.

Theorem VII.3. *Let G be a definable group. Suppose that G acts transitively on a definable set X and the action $G \times X \to X$ is definable. Let $f : Y \to X$ and $g : Z \to X$ be definable. For $s \in G$ denote by $sf : Y \to X$ the composition of f and the multiplication by s, and by $(sY) \times_X Z$ the fibred product of sf and g.*

(1) *there is an open dense definable $U \subset G$ such that for $s \in U$*

$$\dim (sY) \times_X Z \le \dim Y + \dim Z - \dim X$$

(2) *Suppose, moreover, that G, X, Y, Z are k-regular for $k \ge 1$, and that the action of G on X and the maps $f : Y \to X$ and $g : Z \to X$ are k-regular. Then, there is an open dense definable $U \subset G$ such that for $s \in U$, $(sY) \times_X Z$ is either empty or k-regular of dimension* $\dim Y + \dim Z - \dim X$.

Proof of (1). Recall first the definition of fiber product

$$(sY) \times_X Z = \{(y,z) \in Y \times Z;\ sf(y) = g(z)\} \subset Y \times Z.$$

Consider

$$V = \{(s,y,z) \in G \times Y \times Z;\ sf(y) = g(z)\} \subset G \times Y \times Z.$$

and the induced projections $\pi_1 : V \to G$ and $\pi_2 :\to Y \times Z$. Then $(sY) \times_X Z = \pi_1^{-1}(s)$. The dimension of V can be computed using the other projection since $\pi_2^{-1}(y,z)$ equals $\{s; sf(y) = g(z)\}$ and, hence, is of dimension $\dim G - \dim X$. Thus

$$\dim V = \dim Y + \dim Z + \dim G - \dim X$$

and the generic fibres of π_1 are of dimension not greater than $\dim Y + \dim Z - \dim X$, as claimed. □

Suppose that the action $G \times X \to X$ is transitive definable and C^k, $k \geq 1$. Consider

$$V_0 = \{(s, x_1, x_2) \in G \times X \times X;\ sx_1 = x_2\} \subset G \times X \times X \tag{9}$$
$$\pi : V_0 \to X \times X, \quad \pi(s, x_1, x_2) = (x_1, x_2).$$

By transitivity of the action, π is surjective. The group $G \times G$ acts on V_0 by $(s_1, s_2)(s, x_1, x_2) = (s_2 s s_1, s_1^{-1} x_1, s_2 x_2)$.

Exercise VII.4.

(1) Using the definable choice and the group action show that π admits locally, in a neighbourhood of any $(x_1, x_2) \in X \times X$, a definable C^k section.

(2) Fix $x_0 \in X$. Show that for any $x \in X$ there is a neighbourhood U in X, and a C^k definable map $\rho_{x,x_0} : U \to G$ such that for every $y \in U$

$$\rho_{x,x_0}(y)\, y = x_0.$$

(3) Fix $x_0 \in X$ and let G_{x_0} denotes the isotropy subgroup of x_0. Show that π is locally trivial in the following sense : for any $(x_1, x_2) \in X$ there is a neighbourhood $U = U_1 \times U_2 \subset X \times X$ and definable C^k maps $\rho_i : U_i \to G$, $i = 1, 2$, such that the following diagram is commutative

$$\begin{tikzcd} G_{x_0} \times U \arrow[r, "\mu"] \arrow[d, "pr_2"'] & \pi^{-1}(U) \arrow[d, "\pi"] \\ U \arrow[r, "id"] & U \end{tikzcd}$$

where $\mu(s, x_1, x_2) = ((\rho_2(x_2))^{-1} s\, \rho_1(x_1), x_1, x_2)$ is a definable C^k diffeomorphism and pr_2 is the projection on the second factor.

Proof of (2) of Theorem VII.3. We keep the notation the proof of first part of theorem. The map $\pi_2 : V \to Y \times Z$ is the pull-back of π of (9)

$$\begin{tikzcd} V \arrow[r, "(f,g)^*"] \arrow[d, "\pi_2"'] & G \times X \arrow[d, "\pi"] \\ Y \times Z \arrow[r, "(f,g)"] & X \times X \end{tikzcd}$$

and therefore, by Exercise VII.4, is a definable C^k locally trivial fibration. As a consequence we obtain that V is k-regular. Now (2) of Theorem VII.3 follows from Sard Theorem, Exercise II.42 that can be easily extended to maps with values in regular sets. □

Exercise VII.5. Let $g : Z \to X$ be definable, $Y \subset X$. Show that for $S \in G$ generic

$$\dim g^{-1}(sY) \leq \dim Z - \dim X + \dim Y.$$

3. Invariant forms

Consider $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ and denote its elements as $(x, v) \in T\mathbb{R}^n$. In order to distinguish the factors we sometimes write $x \in \mathbb{R}^n_x, v \in \mathbb{R}^n_v$. Denote by $\alpha = vdx = \sum_i v_i dx_i$ the canonical 1-form on $T\mathbb{R}^n$ and let $\beta = vdv = \sum_i v_i dv_i$.

Consider the map

$$h : T\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad hi(x, v, \rho) = x + \rho v.$$

PROPOSITION VII.6. *There are $\kappa_i \in (\bigwedge^i \mathbb{R}^n_x) \wedge (\bigwedge^{n-i-1} \mathbb{R}^n_v)$ such that*

$$\|v\|^2 h^*(dx_1 \wedge \ldots \wedge dx_n) = (\alpha + \rho\beta + \|v\|^2 d\rho) \wedge \sum_{i=0}^{n-1} \rho^{n-i-1}\kappa_i. \tag{10}$$

PROOF.

$$hi^*(dx_1 \wedge \ldots \wedge dx_n) = (dx_1 + \rho dv_1) \wedge \ldots \wedge (dx_n + \rho dv_n) + \\ d\rho \wedge \sum_{i=1}^{n} (-1)^{i-1} v_i \,(dx_1 + \rho dv_1) \wedge \cdots \wedge (\widehat{dx_i + \rho dv_i}) \wedge \cdots \wedge (dx_n + \rho dv_n).$$

Thus the forms κ_i are defined be developing the second summand

$$\sum_{i=1}^{n} (-1)^{i-1} v_i \,(dx_1 + \rho dv_1) \wedge \cdots \wedge (\widehat{dx_i + \rho dv_i}) \wedge \cdots \wedge (dx_n + \rho dv_n) = \sum_{i=0}^{n-1} \rho^{n-i-1}\kappa_i.$$

As for the first summand we have

$$\|v\|^2 (dx_1 + \rho dv_1) \wedge \ldots \wedge (dx_n + \rho dv_n) \\ = \sum_i v_i(dx_i + \rho dv_i) \wedge \sum_{i=1}^{n} (-1)^{i-1} v_i \,(dx_1 + \rho dv_1) \wedge \cdots \wedge (\widehat{dx_i + \rho dv_i}) \wedge \cdots \wedge (dx_n + \rho dv_n).$$

□

The forms α, β, and κ_i are G-invariant, for the group $G = Isom^+(\mathbb{R}^n)$ of oriented euclidean motions of $\mathbb{R}^n$.

REMARK VII.7. Replace $T\mathbb{R}^n$ by the unit tangent bundle $ST\mathbb{R}^n = \mathbb{R}^n \times S^{n-1}$. Since $\|v\| = 1$ and $\beta = \frac{1}{2} d\|v\|^2 = 0$ on S^{n-1}, the formula (10) takes on $ST\mathbb{R}^n \times \mathbb{R}$ the form

$$h^*(dx_1 \wedge \ldots \wedge dx_n) = (\alpha + d\rho) \wedge \sum_{i=0}^{n-1} \rho^{n-i-1}\kappa_i. \tag{11}$$

It is shown in [Joe Fu, *Kinematic formulas in integral geometry*, Indiana Univ. Math J, **39** (1990), 1115–1154] that the exterior algebra of $Isom^+(\mathbb{R}^n)$-invariant differential forms on $ST\mathbb{R}^n$ is generated by the canonical 1-form α, its differential $\omega = d\alpha$, and the forms $\kappa_0, \ldots, \kappa_{n-1}$.

EXERCISE VII.8. Show that κ_0 depends only on v and κ_0 restricted to S^{n-1} coincides with the volume form on S^{n-1}.

PROPOSITION VII.9. *Let M be a m-dimensional submanifold of $\mathbb{R}^n$. Then κ_m restricted to the unit normal bundle SNM to M is nowhere zero. In particular, SNM is canonically oriented by κ_m.*

PROOF. Suppose that the tangent space T_pM at $p \in M$ is given by $x_{m+1} = \cdots = x_n = 0$. Then the fibre SN_pM over p is the $(n-m-1)$-sphere given by $v_1 = \cdots = v_m = 0$. Let $v \in SN_pM$. Let $(p, v) \in SNM$. Since $SNM \subset \mathbb{R}^n \times \mathbb{R}^n$ we may identify the tangent space $T_{(p,v)}SNM$ with a linear subspace of $\mathbb{R}^n \times \mathbb{R}^n$. Then κ_m as an exterior form on $T_{(p,v)}SNM$ equals

$$dx_1 \wedge \ldots \wedge dx_m \wedge \sum_{i=m+1}^{n} (-1)^{i-1} v_i \, dv_{m+1} \wedge \cdots \wedge \widehat{dv_i} \wedge \cdots \wedge dv_n.$$

This form does not vanish on $T_{(p,v)}SNM$. Moreover $\sum_{i=m+1}^{n} (-1)^{i-1} v_i \, dv_{m+1} \wedge \cdots \wedge \widehat{dv_i} \wedge \cdots \wedge dv_n$ is, up to a sign, the volume form of SN_pM. □

Exercise VII.10. Let $M \subset \mathbb{R}^n$ be a compact m dimensional submanifold. Show that

$$\sigma_{n-m-1} Vol_m(M) = \int_{SNM} \kappa_m,$$

where σ_{n-m-1} denotes the volume of S^{n-m-1}.

Exercise VII.11. Let M be an m-dimensional submanifold of $\mathbb{R}^n$. Then

(1) $d\kappa_m$ restricted to the normal bundle NM to M is nowhere zero.
(2) Let $\alpha = \sum_i v_i dx_i$ be the canonical 1-form. Show that α restricted to on NM is identically zero.

4. Normal (characteistic) cycle

4.1. Lagrangian and Legandrian cycles. We say that a connected submanifold $L \subset ST\mathbb{R}^n$ is *Legandrian* if there is a submanifold $M \subset \mathbb{R}^n$ such that L is an open subset of SNM. By Proposition VII.9 every Legandrian submanifold of $ST\mathbb{R}^n$ is canonically oriented.

Remark VII.12. The classical definition of Legandrian manifold says that L is Legandrian if it is of dimension $n-1$ and the canonical 1-form α vanishes on L. It can be shown that both definitions are equivalent.

We say that a definable set $L \subset ST\mathbb{R}^n$ is *Legandrian* if $\dim L = n-1$ and $Reg_k(L)$ is Legandrian. *A Legandrian chain* is a formal finite integral combination of Legandrian definable sets $\sum n_i L_i$, with the regular part of each L_i oriented. We identify $-L$ and L with the opposite orientation, as well as $L_1 + L_2$ with $L_1 \cup L_2$, if $\dim L_1 \cap L_2 < n-1$ and the orientation on the regular part of $L_1 \cup L_2$ is given by those of L_1 and L_2.

A Legandrian cycle is a Legendrian chain whose boundary, in the sense we explain below, is zero.

Let $\sum m_j L_j$ be a Legandrian chain of $ST\mathbb{R}^n$. We may suppose that L_j are disjoint and $X = \bigcup L_j$ is k-regular. Then *the boundary of* L is a formal integral combination of $(n-2)$-dimensional oriented definable C^1 manifolds, included in $\overline{X} \setminus X$ defined as follows. Fix a generic point of $p \in \overline{X} \setminus X$ that satisfies (6) of Proposition V.11. Fix an orientation of $V \cap (X \setminus Reg_k(X))$, and the compatible with it orientations of X_i. The chain structure on L gives the integral coefficient n_i to each of X_i. Now the coefficient assigned to $V \cap (X \setminus Reg_k(X))$, with the given orientation, is defined to be $\sum n_i$.

We say that a connected submanifold $L \subset T\mathbb{R}^n$ is *Lagrangian* if it is of dimension n and the canonical 2-form $\omega = d\alpha$ vanishes on L. There are two typical examples to Lagrangian manifolds : the normal bundle NM to a submanifold M of $\mathbb{R}^n$ and the graph of the gradient of a C^2 function defined on an open subset of $\mathbb{R}^n$.

We say that a Lagrangian submanifold L is *conical* (or $\mathbb{R}_+$-*homogeneous*) if it is stable by the action of R_+, $t(p,v) = (p, tv)$. Then its intersection with $ST\mathbb{R}^n$ is Legandrian. Conversely, the semi-cone over a Legandrian submanifold M is Lagrangian. This way we establish the correspondence between the conical Lagrangian submanifolds of $T\mathbb{R}^n$ and the Legandrian submanifolds of $ST\mathbb{R}^n$, that is bijective if we do not take into account the zero section of $T\mathbb{R}^n$, that is conical and Lagrangian. Similarly to the Legandrian case we define the Lagrangian cycles.

Proposition VII.13. *Let L be a conical Lagrangian cycle of $T\mathbb{R}^n$. Then $L \cap ST\mathbb{R}^n$ is a Legandrian cycle.*

Conversely, let L' be a Legandrian cycle of $ST\mathbb{R}^n$ and denote by $\mathbb{R}_+ L'$ the semi-cone over L'. Then there exists a Lagrangian cycle $\tilde{L}$ supported on the zero section of $T\mathbb{R}^n$ such that $\tilde{L} + \mathbb{R}_+ L'$ is a conical Lagrangian cycle. Moreover, $\tilde{L}$ is unique up to a multiple of the zero section of $T\mathbb{R}^n$.

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