-) Representation of positive semidefinite global analytic functions on $\mathbb{R}^n$ (a manifold) as sums of squares

(j. w. with F. ACQUISTAPACE and J. FERNANDO)

-) Classic for
  Polynomials, Nash functions,
  Analytic germs.

-) Few results for this ring

$\cancel{\text{O-minimality}}$ $\cancel{\text{UFD}}$ $\cancel{\text{good compactification}}$

(... or "bad ring").

∘) Classic H17

$f \in \mathcal{O}(\mathbb{R}^n)$, $f \geq 0 \overset{?}{\Longrightarrow} f = \sum_{i=1}^{P} \frac{f_i^2}{f_0^2}$

∘) <u>Def</u> ① $f \in \mathcal{O}(\mathbb{R}^n)$

f is an <u>infinite sum of squares</u> if

$$f = \sum f_i^2$$

∘) $\exists\, \Omega$ open neighbourhood of $\mathbb{R}^n$ in $\mathbb{C}^n$ s.t. each f_i extends to a holomorphic function $F_i \in \mathcal{H}(\Omega)$

∘) $\sum F_i^2$ converges absolutely and uniformly on each compact K

$$\forall K \quad \sum \sup_K |F_i|^2 < +\infty .$$

∘) <u>Def</u> ② $f \in \mathcal{O}(\mathbb{R}^n)$

f is s.o.s. of meromorphic functions if $\exists\, g \neq 0$ s.t.

$$g^2 f = \sum_{i=1}^{\infty} f_i^2$$

(weak H17)

- Known results

  -) Low dimension

    dim 1   exercise
    dim 2   Risler

  -) Bochnak, Kucharz, Shiota (1981)

    if $f^{-1}(0)$ = discrete set

  -) Ruiz, Jaworski (1985)

    if $f^{-1}(0)$ = compact set

  -) Jaworski

    if $f^{-1}(0)$ = discrete set $\cup$ compact set

(Forster 1964)

$A = H^0(X, \mathcal{O})$ X stein space

$\underline{a}$ closed ideal in A.

$(\underline{a} = H^0(X, \underline{a}\mathcal{O}))$

$\Downarrow$

•) $\underline{a} = \bigcap_{i=1}^{\infty} \underline{q}_i$ q_i primary irredundant

$\{V(q_i)\}$ locally finite

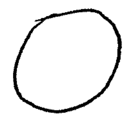
•) $\exists\, h_i \in \mathbb{N}$ $\left(\sqrt{\underline{q}_i}\right)^{h_i} \subset \underline{q}_i$

•) NSS holds for $\underline{a}$ $\Longleftrightarrow$

$\exists\, H \in \mathbb{N}$ s.t. $h_i \leq H \ \forall i$

BKS

R, J



Next case

i.e.

$f: \mathbb{R}^n \to \mathbb{R}$ $\quad f \geq 0$ and $f^{-1}(0) = \bigcup_{i=1}^{\infty} K_i$

K_i compact $\quad K_i \cap K_j = \emptyset$

$\Rightarrow (R, J)$ $\quad \forall j \ \exists g_j \neq 0$ s.t.

$$g_J^2 f_{K_J} = \sum_{i=1}^{p_J} f_{J i}^2$$

(f_{K_J} = germ of f at K_J)

<u>Then</u>

•) $\exists g \neq 0 \qquad g^2 f = \sum_{i=1}^{\infty} h_i^2$

•) if $p_j \leq p \ \forall J$
the sum is finite

A consequence

∘) Pythagoras number of a ring

$$p = \min \left\{ \begin{array}{l} \# \text{ squares needed to represent} \\ \text{any sum of squares} \end{array} \right\}$$

∘) If H17 finite holds for $\mathcal{O}(\mathbb{R}^4)$

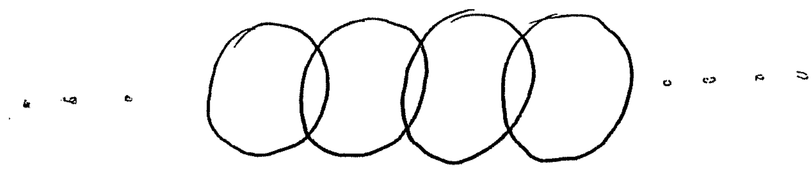
$\Rightarrow \quad p(\mathcal{M}(\mathbb{R}^4)) < +\infty$

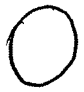
↳ (meromorphic functions field)

qualitative H17 : it is or not s.o.s.

quantitative H17 : how many squares

- in the algebraic case are completely independent
- it is not so in the global analytic case



Note  is not

-) an irreducible component of the zero set

but is

-) the zero set of an irreducible factor of f

Irreducible factors for f.

1) Extend f to a $F \in \mathcal{H}(\Omega)$ in a "suitable" neighborhood Ω of $\mathbb{R}^n$ in $\mathbb{C}^n$

2) $X = F^{-1}(0)$ $\quad X = \bar{X}$ $\quad$ codim $X = 1$.

Whitney Bruhat $\Rightarrow$ $X = \cup X_i$ irreducible

3) dim $X_i = n-1$ $\Rightarrow$ $\mathcal{J}_{X_i}$ is generated by $F_i \in \mathcal{H}(\Omega)$

4) if $\left.\begin{array}{ll} X_i = \bar{X}_i & \text{take } F_i \\ X_j = \bar{X}_i & \text{take } F_i F_j \end{array}\right\}$ restricts to $\mathbb{R}^n$

Call these factors f_J.

$\{f_J\}$ good in this sense

$$f = u \prod f_J$$

u = analytic unity

$\prod f_J$ = sheaf$^{(*)}$ product

(*) Take the ideal sheaf

$$\mathcal{J}_x = \begin{cases} x \in X \cap \mathbb{R}^n & (\prod f_{J,x})\,\mathcal{O}^n_x \\ & f_{J,x} \text{ vanishing at } x \\ x \notin X \cap \mathbb{R}^n & \mathcal{O}^n_x \end{cases}$$

$\mathcal{J}$ is locally principal

$\Rightarrow$ ($\mathbb{R}^n$ is contractible) $\mathcal{J}$ is principal

call a generator $\prod f_J$

- Result

let $\{f_k\}$ a sequence of analytic functions

1) $\forall_k \quad g_k^2 f_k = \sum_{j=1}^{2^{r_k}} f_{k,j}^2$

2) $\{Z(f_k)\}$ locally finite

Then

∘) $\exists g \neq 0$

$$g^2 \prod f_k = \sum_{i=1}^{\infty} h_i^2$$

∘) if $r_k = r \ \forall k$

$$g^2 \prod f_k = \sum_{i=1}^{2^{r+1}} h_j^2$$

Idea of the proof

(case $r_k = r \ \forall k$).

Notation

$$f = \boxed{p} \rightsquigarrow f = \sum_{i=1}^{p} f_i^2$$

Pfister result: in a field

$$\boxed{2^r} \cdot \boxed{2^r} = \boxed{2^r}$$

We adapt the proof of the Pfister result and we prove

<u>Lemma</u> $f, a \in \mathcal{O}(\mathbb{R}^4)$ s.t.

$$f = a^2 + \boxed{2^r} \qquad \{a=0\} = \{f=0\}$$

$\Rightarrow \exists g \quad \{g=0\} \subset \{f=0\}$

$\exists M \in \mathcal{M}_{2^{r+1}}(\mathcal{O}(\mathbb{R}^4))$

$${}^tM \cdot M = g^2 f \, I_{2^{r+1}}$$

Pfister bundle for a function f

Def $f \in O(\mathbb{R}^n)$
a Pfister bundle for f is
- a fiber bundle E on $\mathbb{R}^n$ (or a manifold) endowed with a Riemanian metric
- a section s s.t. $\langle s, s \rangle = f$

This lemma justifies the name

Lemma f is a sum of 2^r squares

$$\Updownarrow$$

$\exists\, g \neq 0$ s.t. $g^2 f$ admits a Pfister bundle.

(Remark that any fiber bundle is trivial on $\mathbb{R}^n$)

Proof

1) Suppose $g_0^2 f = \boxed{2^2}$

$\exists\, g_1$, M s.t $\quad {}^tM . M = g_1^2 g_0^2 f = h^2 f$

$E = \mathbb{R}^4 \times \mathbb{R}^p \quad (p = 2^2)$

$s_1, \dots s_p$ orthonormal basis in $\mathbb{R}^p$ (°)

u a unitary vector

Define $s : x \longrightarrow (x,\ M(x) \cdot u)$

$$<s, s> = h^2 f$$

2) Viceversa

s_0 s.t $\quad <s, s> = f$

$s = \sum a_i s_i \qquad <s, s> = \sum a_i^2$

(°) $\exists$ by triviality of the fiber bundle.

Construction of a Pfister bundle for πf_k

•) $g_k^2 f_k = h_k = \boxed{2^r}$ (Case $r_k = r \ \forall k$)

$\exists M_k \ : \ {}^tM_k M_k = h_k I_{2^r}$

•) Covering of $\mathbb{R}^n$

$X = \bigcup X_k \qquad X_k = f_k^{-1}(0)$

$U_0 = \mathbb{R}^n \setminus X$

$U_1 = \mathbb{R}^n \setminus \bigcup_{k>1} X_k$

$U_2 = \mathbb{R}^n \setminus \bigcup_{k>2} X_k$

. . . .

$U_0 \subset U_1 \subset U_2 \subset \dots$

-) Transition functions

$$g_{ii} = id$$

$$i < j \quad g_{ij} : U_i \cap U_j = U_i \longrightarrow GL(\mathbb{R}^{2^t})$$

$$g_{ij}(x) = \frac{1}{\sqrt{h_j' \circ \cdots \circ h_{i+1}'}} M_j \circ \cdots \circ M_{i+1}$$

$$h_\alpha' = h_\alpha|_{U_i}$$

$\{g_{ij}\}$ is a cocycle. ~~wit~~ with values in $O(\mathbb{R}^{2^t})$

-) Riemann metric

in U_0 the usual metric

on U_i extended by $\{g_{0j}\}$.

The section

u unitary vector.

$$s = \{ s_i : U_i \longrightarrow U_i \times \mathbb{R}^p \}$$

$$\{ s_i : x \longrightarrow (x, \sigma_i(x)) \}$$

$$\sigma_0(x) = \sqrt{h|_{U_0}}\, u$$

$$\sigma_1(x) = \sigma_0(x)\, g_{01} = \sqrt{\frac{h|_{W_0}}{h_1|_{W_0}}}\, M_1(x)\, u$$

$$\sigma_i(x) = g_{0i}\, \sigma_0(x) = \sqrt{\frac{h|_{W_0}}{h_i' \cdots h_1'}}\, M_i \cdots M_1\, u$$

<u>Note</u> $\dfrac{h}{h_1' \cdots h_i'}$ is analytic in U_i and $\neq 0$.

$$\langle s, s \rangle|_{U_0} = h|_{W_0}(x)$$