The exponential law for quasianalytic Denjoy-Carleman classes

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Structure

1. Introduction and preliminaries
2. Former approaches: testing along curves
3. Uniform approach: testing with Banach plots

Motivation

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Note that $C^\infty(\mathbb{R}, \mathbb{R})$ is not a Banach space. So this requires a smooth calculus beyond Banach spaces.

Moreover, classical smooth calculus works well up to Banach spaces. But Banach manifolds are not suitable for many questions of global analysis. For example, many Banach manifolds turn out to be just open subsets of the modeling space [Eells, Elworthy, 1970].

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Usually this comes hand in hand with (partly nonlinear) **uniform boundedness theorems** which are easy $\mathcal{C}$-detection principles:

- $f : E \to L(F, G)$ is $\mathcal{C}$ iff $\mathrm{ev}_x \circ f : E \to G$ is $\mathcal{C}$ for all $x \in F$.
- $f : E \to \mathcal{C}(F, G)$ linear is bounded iff $\mathrm{ev}_x \circ f : E \to G$ is bounded for all $x \in F$.

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These and some other natural properties determine the convenient setting.

Applications

Perturbation theory: Let $x \mapsto A(x)$ be a $\mathcal{C}$-family of unbounded normal (or self-adjoint) operators in a Hilbert space H with compact resolvents and common domain of definition. E.g., g_x are Riemann-metrics on a compact manifold M, $A(x) = \Delta(g_x)$, and $H = L^2(M)$. What is the regularity of the spectral decomposition? This is of particular interest if $\mathcal{C}$ is quasianalytic.

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Manifolds of Mappings: Generally, one can hope that the space $\mathcal{C}(A, B)$, where A, B are finite dimensional $\mathcal{C}$-manifolds (with A compact) is again a $\mathcal{C}$-manifold, that composition is $\mathcal{C}$, and that the group $\mathrm{Diff}_{\mathcal{C}}(A)$ of all $\mathcal{C}$-diffeomorphisms of A is a regular infinite dimensional $\mathcal{C}$-Lie group. Indeed, this is true for suitable Denjoy–Carleman classes.

Denjoy–Carleman classes

Let $M = (M_k)$ be a sequence of positive numbers, $U \subseteq \mathbb{R}^q$ open. The **Denjoy–Carleman class** $C^M(U)$ (of Roumieu type) is the set of all $f \in C^\infty(U)$ such that

$$\boxed{\forall K \subseteq U \text{ cp. } \exists \rho > 0 : \left\{ \frac{|\partial^\alpha f(x)|}{\rho^{|\alpha|}|\alpha|!M_{|\alpha|}} : \alpha \in \mathbb{N}^q, x \in K \right\} \text{ is bounded.}}$$

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If C^M is also **closed under derivation** ($\sup_k (M_{k+1}/M_k)^{1/k} < \infty$) and **quasianalytic** ($\sum_k (k!\, M_k)^{-1/k} = \infty$), then $\mathcal{C} = C^M$ admits resolution of singularities [Bierstone, Milman, 1997].

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E is **convenient** iff the following equivalent conditions hold:

- E is Mackey complete.
- If $\ell \circ c \in C^\infty$ for all $\ell \in E^*$, then $c \in C^\infty$.
- If $B \subseteq E$ is bounded closed absolutely convex, then the linear span $E_B := \langle B \rangle$ is a Banach space with $\|x\|_B := \inf\{\lambda > 0 : x \in \lambda B\}$.

Testing along curves: non-quasianalytic mappings

The convenient setting has been developed for:

- C^∞ [Frölicher, Kriegl, early 1980's]
- $\mathcal{H}$ [Kriegl, Nel, 1985]
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  Definition. A curve $c : \mathbb{R} \to E$ is called C^M if $\ell \circ c \in C^M(\mathbb{R}, \mathbb{R})$ for all $\ell \in E^*$. A mapping $f : E \supseteq U \to F$ is called C^M if $f \in C^\infty$ and $f \circ c \in C^M(\mathbb{R}, F)$ for every $c \in C^M(\mathbb{R}, U)$.
  Equip $C^M(U, F)$ with the initial locally convex structure w.r.t.

  $$C^M(U, F) \longrightarrow C^M(\mathbb{R}, \mathbb{R}), \quad f \mapsto \ell \circ f \circ c, \quad \ell \in E^*, c \in C^M(\mathbb{R}, U).$$

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  Theorem. On Banach spaces this notion coincides with the classical one. The above structure is weaker than the classical, but both have the same bounded sets. C^M can be tested along C^M-curves alone.

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  It is unknown if such an equivalence is true for quasianalytic $C^M \supsetneq C^\omega$ (even in $\mathbb{R}^2$).

  Moreover, there is a subtlety in the proof of the C^ω-exponential law which was resolved by using that on Banach spaces real analytic mappings extend locally as holomorphic mappings on the complexification.

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Definition. Let M be an $\mathcal{L}$-**intersection**, i.e., for $U \subseteq \mathbb{R}^n$,

$$C^M(U) = \bigcap C^L(U), \text{ where}$$
$$L \in \mathcal{L}(M) := \{L \geq M, \text{ non-quasianalytic, log-convex}\}.$$

A mapping $f : E \supseteq U \to F$ is called C^M if for each $L \in \mathcal{L}(M)$ the composite $f \circ c$ is C^L for each C^L-curve c. Consider the convenient structure induced by $C^M(U,F) \to C^L(U,F)$ for $L \in \mathcal{L}(M)$.

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Theorem. On Banach spaces this notion coincides with the classical one. There exist (at least) countably many $\mathcal{L}$-intersections M such that C^M admits a convenient setting.

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Remark. We cannot get all quasianalytic C^M in that way:

$$\bigcap\{C^L : L \text{ non-quasianalytic, log-convex}\} = C^M \supsetneq C^\omega$$
$$\text{where } M_k = (k \log(k+e))^k / k! \quad \text{[cf. Rudin, 1962]}$$

Definition of function spaces: on Banach spaces

Let $M = (M_k)$, E, F Banach spaces, $U \subseteq E$ open, $K \subseteq U$ compact, $\rho > 0$. Consider the non-Hausdorff Banach space

$$C^M_{K,\rho}(U,F) := \{f \in C^\infty(U,F) : \|f\|_{K,\rho} < \infty\}, \text{ where}$$
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$$C^M(U, F) := \varprojlim_{K \subseteq U} \varinjlim_{\rho > 0} C^M_{K,\rho}(U, F).$$

In the inductive limit it suffices to take $\rho \in \mathbb{N}$. It is a Silva space, i.e., an inductive limit of a sequence of Banach spaces with compact connecting mappings.

Definition of function spaces: on convenient spaces

Let E, F be convenient, $U \subseteq E$ c^∞-open, and $B \subseteq E$ bounded closed absolutely convex. Consider

$$C_b^M(U,F) := \Big\{ f \in C^\infty(U,F) : \forall B\, \forall K \subseteq U \cap E_B\, \exists \rho > 0 : \\ \Big\{ \tfrac{f^{(k)}(x)(v_1,\dots,v_k)}{k!\, \rho^k\, M_k} : k \in \mathbb{N}, x \in K, \|v_i\|_B \le 1 \Big\} \text{ is bounded in } F \Big\}$$

$$\boxed{C^M(U,F) := \Big\{ f \in C^\infty(U,F) : \forall \ell \in F^*\, \forall B : \ell \circ f \circ i_B \in C^M(U_B, \mathbb{R}) \Big\}}$$

where $i_B : E_B \to E$ is the inclusion of E_B in E, and $U_B := i_B^{-1}(U)$.
$C^M(U,F) = C_b^M(U,F)$ if F^* has a Baire topology s.t. all ev_x are continuous.

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Equip $C^M(U,F)$ with the initial convenient structure w.r.t.

$$C^M(U,F) \xrightarrow{C^M(i_B,\ell)} C^M(U_B,\mathbb{R}), \quad f \mapsto \ell \circ f \circ i_B.$$

The exponential law fails for C_b^M

Example. Let $M = (M_k)$ be weakly log-convex. There exists $f \in C^M(\mathbb{R}^2, \mathbb{R})$, whereas there is no reasonable topology on $C^M(\mathbb{R}, \mathbb{R})$ (i.e. all $\mathrm{ev}_t : C^M(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ are bounded linear functionals) such that $f^\vee : \mathbb{R} \to C^M(\mathbb{R}, \mathbb{R})$ is C_b^M.

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Sketch of Proof. There is $g \in C^M(\mathbb{R}, \mathbb{R})$ such that $|g^{(k)}(0)| \ge k!\, M_k$. Then $f(s,t) := g(st)$ is C^M. Suppose that $f^\vee$ is C_b^M. Then for $s = 0$ there is ρ such that

$$\left\{ \frac{(f^\vee)^{(k)}(0)}{k!\, \rho^k\, M_k} : k \in \mathbb{N} \right\}$$

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is bounded in $C^M(\mathbb{R}, \mathbb{R})$. Apply the bounded linear functional ev_t for $t = 2\rho$ and then get

$$\frac{(f^\vee)^{(k)}(0)(2\rho)}{k!\, \rho^k\, M_k} = \frac{(2\rho)^k g^{(k)}(0)}{k!\, \rho^k\, M_k} \ge 2^k,$$

a contradiction. □

C^M is a category

Let $M = (M_k)$ be log-convex. A C^M **Banach plot** is a C^M-mapping $c : D \to E$, where D is open in a Banach space.

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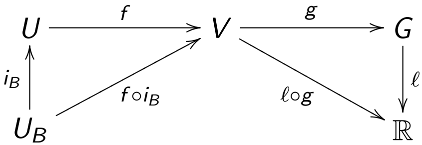

□

C^M is cartesian closed, I

Let $M = (M_k)$ be log-convex and of **moderate growth**, i.e., there is $\sigma > 0$ s.t. $M_{k+l} \le \sigma^{k+l} M_k M_l$ for all k, l.

Theorem. The category C^M is cartesian closed, i.e.,

$$\boxed{f \in C^M(U_1 \times U_2, F) \Longleftrightarrow f^\vee \in C^M(U_1, C^M(U_2, F))}$$

The direction $(\Leftarrow)$ holds without moderate growth.

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Sketch of Proof.

$$\begin{aligned}
& f \in C^M(U_1 \times U_2, F) \\
& \Longleftrightarrow \forall \ell \in F^* \ \forall B : \ell \circ f \circ i_B \in C^M((U_1 \times U_2)_B, \mathbb{R}) \\
& \Longleftrightarrow \forall \ell \in F^* \ \forall B_1, B_2 : \ell \circ f \circ (i_{B_1} \times i_{B_2}) \in C^M((U_1)_{B_1} \times (U_2)_{B_2}, \mathbb{R})
\end{aligned}$$

C^M is cartesian closed, II

Sketch of Proof (continued).

$$f^\vee \in C^M(U_1, C^M(U_2, F))$$
$$\iff \forall B_1 : f^\vee \circ i_{B_1} \in C^M((U_1)_{B_1}, C^M(U_2, F))$$
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So it suffices to consider $U_i \subseteq E_i$ open in Banach spaces E_i and $F = \mathbb{R}$.

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$$f^\vee \in C^M(U_1, C^M(U_2, F))$$
$$\iff \forall B_1 : f^\vee \circ i_{B_1} \in C^M((U_1)_{B_1}, C^M(U_2, F))$$
$$\iff \forall \ell \in F^* \ \forall B_1, B_2 : C^M(i_{B_2}, \ell) \circ f^\vee \circ i_{B_1} \in C^M((U_1)_{B_1}, C^M((U_2)_{B_2}, \mathbb{R}))$$

So it suffices to consider $U_i \subseteq E_i$ open in Banach spaces E_i and $F = \mathbb{R}$.

Lemma. Let E be a Banach space, $U \subseteq E$ open, F convenient, and $\mathcal{S} \subseteq F'$ s.t. $B \subseteq F$ is bounded iff $\ell(B)$ is bounded for all $\ell \in \mathcal{S}$. Then:

$$f \in C^M(U, F) \iff \ell \circ f \in C^M(U, \mathbb{R}) \text{ for all } \ell \in \mathcal{S}$$

C^M is cartesian closed, III

Sketch of Proof (continued). $(\Rightarrow)$

$$
\begin{array}{ccccc}
U_1 & \xrightarrow{f^\vee} & C^M(U_2,\mathbb{R}) = \varprojlim_{K_2} \varinjlim_{\rho_2} C^M_{K_2,\rho_2}(U_2,\mathbb{R}) & \xrightarrow{\ell} & \mathbb{R} \\
\uparrow & & \downarrow & \nearrow & \\
 & & \varinjlim_{\rho_2} C^M_{K_2,\rho_2}(U_2,\mathbb{R}) & & \\
 & & \uparrow & & \\
K_1 & \dashrightarrow & C^M_{K_2,\rho_2}(U_2,\mathbb{R}) & &
\end{array}
$$

C^M is cartesian closed, III

Sketch of Proof (continued). ($\Rightarrow$)

$$\begin{array}{ccccccc}
U_1 & \xrightarrow{f^\vee} & C^M(U_2,\mathbb{R}) & = & \varprojlim_{K_2} \varinjlim_{\rho_2} C^M_{K_2,\rho_2}(U_2,\mathbb{R}) & \xrightarrow{\ell} & \mathbb{R} \\
\uparrow & & & & \downarrow & \nearrow & \\
 & & & & \varinjlim_{\rho_2} C^M_{K_2,\rho_2}(U_2,\mathbb{R}) & & \\
 & & & & \uparrow & & \\
K_1 & \cdots\cdots & \cdots\cdots & \cdots\cdots\to & C^M_{K_2,\rho_2}(U_2,\mathbb{R}) & &
\end{array}$$

We have to show that:

$$\forall K_1, K_2\ \exists \rho_1 > 0 : \ \left\{ \frac{d^{k_1} f^\vee(x_1)(v_1^1,\ldots,v_{k_1}^1)}{k_1!\, \rho_1^{k_1}\, M_{k_1}} : x_1 \in K_1, k_1 \in \mathbb{N}, \|v_j^1\|_{E_1} \le 1 \right\}$$
$$\text{is bounded in } \varinjlim_{\rho_2} C^M_{K_2,\rho_2}(U_2,\mathbb{R}).$$

C^M is cartesian closed, IV

Sketch of Proof (continued). As $M_{k_1+k_2} \le \sigma^{k_1+k_2} M_{k_1} M_{k_2}$ for some $\sigma > 0$, we obtain, for $x_1 \in K_1$, $k_1 \in \mathbb{N}$, and $\|v_j^1\|_{E_1}$,

$$\Big\| \frac{d^{k_1} f^\vee(x_1)(v_1^1,\dots,v_{k_1}^1)}{k_1!\, \rho_1^{k_1}\, M_{k_1}} \Big\|_{K_2,\rho_2} =$$

$$= \sup \Big\{ \frac{|\partial_2^{k_2} \partial_1^{k_1} f(x_1,x_2)(v_1^1,\dots;v_1^2,\dots)|}{k_2!\, k_1!\, \rho_2^{k_2}\, \rho_1^{k_1}\, M_{k_2}\, M_{k_1}} : x_2 \in K_2, k_2 \in \mathbb{N}, \|v_j^2\|_{E_2} \le 1 \Big\}$$

$$\le \sup \Big\{ (2\sigma)^{k_1+k_2} \frac{|\partial_2^{k_2} \partial_1^{k_1} f(x_1,x_2)(v_1^1,\dots;v_1^2,\dots)|}{(k_1+k_2)!\, \rho_1^{k_1}\, \rho_2^{k_2}\, M_{k_1+k_2}} : x_2, k_2, v_j^2 \Big\}.$$

C^M is cartesian closed, IV

Sketch of Proof (continued). As $M_{k_1+k_2} \le \sigma^{k_1+k_2} M_{k_1} M_{k_2}$ for some $\sigma > 0$, we obtain, for $x_1 \in K_1$, $k_1 \in \mathbb{N}$, and $\|v_j^1\|_{E_1}$,

$$\Big\| \frac{d^{k_1} f^\vee(x_1)(v_1^1,\dots,v_{k_1}^1)}{k_1!\,\rho_1^{k_1}\,M_{k_1}} \Big\|_{K_2,\rho_2} =$$

$$= \sup \Big\{ \frac{|\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v_1^1,\dots;v_1^2,\dots)|}{k_2!\,k_1!\,\rho_2^{k_2}\,\rho_1^{k_1}\,M_{k_2}\,M_{k_1}} : x_2 \in K_2, k_2 \in \mathbb{N}, \|v_j^2\|_{E_2} \le 1 \Big\}$$

$$\le \sup \Big\{ (2\sigma)^{k_1+k_2} \frac{|\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v_1^1,\dots;v_1^2,\dots)|}{(k_1+k_2)!\,\rho_1^{k_1}\,\rho_2^{k_2}\,M_{k_1+k_2}} : x_2, k_2, v_j^2 \Big\}.$$

Since f is C^M, there exists $\rho > 0$ such that

$$\sup \Big\{ \frac{|\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v_1^1,\dots;v_1^2,\dots)|}{(k_1+k_2)!\,\rho^{k_1+k_2}\,M_{k_1+k_2}} : x_2 \in K_2, k_2 \in \mathbb{N}, \|v_j^2\|_{E_2} \le 1 \Big\} < \infty$$

Set $\rho_i := 2\sigma\rho$. Thus, $f^\vee$ is C^M.

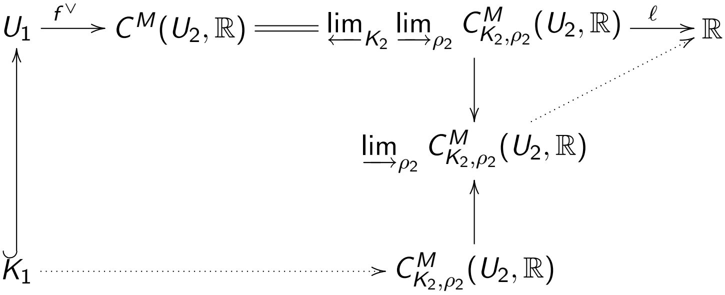
C^M is cartesian closed, V

Sketch of Proof (continued). $(\Leftarrow)$

$$
\begin{array}{ccccccc}
U_1 & \xrightarrow{f^\vee} & C^M(U_2,\mathbb{R}) & = & \varprojlim_{K_2} \varinjlim_{\rho_2} C^M_{K_2,\rho_2}(U_2,\mathbb{R}) & \xrightarrow{\ell} & \mathbb{R} \\
\uparrow & & & & \downarrow & \nearrow & \\
 & & & & \varinjlim_{\rho_2} C^M_{K_2,\rho_2}(U_2,\mathbb{R}) & & \\
 & & & & \uparrow & & \\
K_1 & \cdots\cdots & \cdots\cdots & \cdots\cdots\to & C^M_{K_2,\rho_2}(U_2,\mathbb{R}) & &
\end{array}
$$

C^M is cartesian closed, V

Sketch of Proof (continued). $(\Leftarrow)$



$f^\vee : U_1 \to \varinjlim_{\rho_2} C^M_{K_2,\rho_2}(U_2,\mathbb{R})$ is C^M and thus C^M_b, since the dual $(\varinjlim_{\rho_2} C^M_{K_2,\rho_2}(U_2,\mathbb{R}))^*$ can be equipped with the Baire topology of the countable limit $\varprojlim_{\rho_2} C^M_{K_2,\rho_2}(U_2,\mathbb{R})^*$ of Banach spaces.

C^M is cartesian closed, VI

Sketch of Proof (continued). The inductive limit $\varinjlim_{\rho_2} C^M_{K_2,\rho_2}(U_2,\mathbb{R})$ is countable and regular, since it is a Silva space. Hence,

$$\forall K_1, K_2\ \exists \rho_1, \rho_2 : \ \Big\{ \frac{d^{k_1} f^\vee(x_1)(v^1_1,\dots,v^1_{k_1})}{k_1!\, \rho_1^{k_1}\, M_{k_1}} : x_1 \in K_1, k_1 \in \mathbb{N}, \|v^1_j\|_{E_1} \le 1 \Big\}$$

is bounded in $C^M_{K_2,\rho_2}(U_2,\mathbb{R})$.

C^M is cartesian closed, VI

Sketch of Proof (continued). The inductive limit $\varinjlim_{\rho_2} C^M_{K_2,\rho_2}(U_2,\mathbb{R})$ is countable and regular, since it is a Silva space. Hence,

$$\forall K_1, K_2\ \exists \rho_1, \rho_2 :\ \Big\{ \frac{d^{k_1} f^\vee(x_1)(v^1_1,\dots,v^1_{k_1})}{k_1!\,\rho_1^{k_1}\,M_{k_1}} : x_1 \in K_1, k_1 \in \mathbb{N}, \|v^1_j\|_{E_1} \le 1 \Big\}$$

$$\text{is bounded in } C^M_{K_2,\rho_2}(U_2,\mathbb{R}).$$

As $k_1!k_2!M_{k_1}M_{k_2} \le (k_1+k_2)!M_{k_1+k_2}$, for $x_1 \in K_1$, $k_1 \in \mathbb{N}$, and $\|v^1_j\|_{E_1}$,

$$\Big\| \frac{d^{k_1} f^\vee(x_1)(v^1_1,\dots,v^1_{k_1})}{k_1!\,\rho_1^{k_1}\,M_{k_1}} \Big\|_{K_2,\rho_2} =$$

$$= \sup \Big\{ \frac{|\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v^1_1,\dots,v^1_{k_1};v^2_1,\dots,v^2_{k_2})|}{k_2!\,k_1!\,\rho_2^{k_2}\,\rho_1^{k_1}\,M_{k_2}\,M_{k_1}} : x_2 \in K_2, k_2 \in \mathbb{N}, \|v^2_j\|_{E_2} \le 1 \Big\}$$

$$\ge \sup \Big\{ \frac{|\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v^1_1,\dots;v^2_1,\dots)|}{(k_1+k_2)!\,\rho_1^{k_1}\,\rho_2^{k_2}\,M_{k_1+k_2}} : x_2 \in K_2, k_2 \in \mathbb{N}, \|v^2_j\|_{E_2} \le 1 \Big\}.$$

So f is C^M. □

The exponential law, I

Corollary. Let $M = (M_k)$ be log-convex and of moderate growth. Then: The exponential law holds:

$$\boxed{C^M(U \times V, G) \cong C^M(U, C^M(V, G)),\ C^M\text{-diffeomorphically}}$$

The following canonical mappings are C^M.

$$\mathrm{ev} : C^M(U, F) \times U \to F, \quad \mathrm{ev}(f, x) = f(x) \tag{1}$$

$$\mathrm{ins} : E \to C^M(F, E \times F), \quad \mathrm{ins}(x)(y) = (x, y) \tag{2}$$

$$(\ )^\wedge : C^M(U, C^M(V, G)) \to C^M(U \times V, G) \tag{3}$$

$$(\ )^\vee : C^M(U \times V, G) \to C^M(U, C^M(V, G)) \tag{4}$$

$$\mathrm{comp} : C^M(F, G) \times C^M(U, F) \to C^M(U, G) \tag{5}$$

$$C^M(\ ,\ ) : C^M(F, F_1) \times C^M(E_1, E) \to C^M\Big(C^M(E, F), C^M(E_1, F_1)\Big) \tag{6}$$

$$\prod : \prod C^M(E_i, F_i) \to C^M\Big(\prod E_i, \prod F_i\Big) \tag{7}$$

The exponential law, II

Sketch of Proof.
The exponential law follows from (3) and (4).

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(1) The mapping associated to ev via cartesian closedness is the identity on $C^M(U,F)$, which is C^M, thus ev is also C^M.

The exponential law, II

Sketch of Proof.

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(3) The mapping associated to $(\ )^\wedge$ via cartesian closedness is the composite of evaluations $\mathrm{ev} \circ (\mathrm{ev} \times \mathrm{Id}) : (f; x, y) \mapsto f(x)(y)$.

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The exponential law follows from (3) and (4).

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(3) The mapping associated to $(\ )^\wedge$ via cartesian closedness is the composite of evaluations $\mathrm{ev} \circ (\mathrm{ev} \times \mathrm{Id}) : (f; x, y) \mapsto f(x)(y)$.

(4) Apply cartesian closedness twice to get the associated mapping $(f; x; y) \mapsto f(x, y)$, which is just an evaluation mapping. □

Cartesian closedness fails without moderate growth

Example. Let $M = (M_k)$ be weakly log-convex, but not of moderate growth. Then there exists $f \in C^M(\mathbb{R}^2, \mathbb{R})$ such that $f^\vee : \mathbb{R} \to C^M(\mathbb{R}, \mathbb{R})$ is not C^M.

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Sketch of Proof. There exists $f \in C^M(\mathbb{R}^2, \mathbb{R})$ with $\partial^\alpha f(0,0) \ge |\alpha|! M_{|\alpha|}$.

There exist $j_n \nearrow \infty$ and $k_n > 0$ such that $\left(\frac{M_{k_n+j_n}}{M_{k_n} M_{j_n}}\right)^{1/(k_n+j_n)} \ge n$.

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Consider the linear functional $\ell : C^M(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$, $\ell(g) = \sum_n \frac{g^{(j_n)}(0)}{j_n!\, M_{j_n}\, n^{j_n}}$. It is continuous, since, for suitable ρ,

$$\Big| \sum_n \frac{g^{(j_n)}(0)}{j_n!\, M_{j_n}\, n^{j_n}} \Big| \le \sum_n \frac{|g^{(j_n)}(0)|}{j_n!\, \rho^{j_n}\, M_{j_n}} \frac{\rho^{j_n}}{n^{j_n}} \le \|g\|_{[-1,1],\rho} \sum_n \rho^{j_n} \frac{1}{n^{j_n}} < \infty.$$

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$$\Big| \sum_n \frac{g^{(j_n)}(0)}{j_n!\, M_{j_n}\, n^{j_n}} \Big| \le \sum_n \frac{|g^{(j_n)}(0)|}{j_n!\, \rho^{j_n}\, M_{j_n}} \frac{\rho^{j_n}}{n^{j_n}} \le \|g\|_{[-1,1],\rho} \sum_n \rho^{j_n} \frac{1}{n^{j_n}} < \infty.$$

But $\ell \circ f^\vee$ is not C^M, since, for all $\rho_1 > 0$,

$$\|\ell \circ f^\vee\|_{[-1,1],\rho_1} \ge \sup_k \frac{1}{\rho_1^k\, k!\, M_k} \sum_n \frac{f^{(j_n,k)}(0,0)}{j_n!\, M_{j_n} n^{j_n}} \ge \sup_n \frac{1}{\rho_1^{k_n}\, k_n!\, M_{k_n}} \frac{f^{(j_n,k_n)}(0,0)}{j_n!\, M_{j_n}\, n^{j_n}}$$

$$\ge \sup_n \frac{(j_n+k_n)!\, M_{j_n+k_n}}{\rho_1^{k_n}\, k_n!\, j_n!\, M_{k_n}\, M_{j_n}\, n^{j_n}} \ge \sup_n \frac{n^{j_n+k_n}}{\rho_1^{k_n}\, n^{j_n}} = \infty. \qquad \square$$