Research Article

Bochra Méjri*

Shape sensitivity analysis for identification of voids under Navier's boundary conditions in linear elasticity

https://doi.org/10.1515/jiip-2018-0029
Received April 9, 2018; revised September 24, 2018; accepted November 10, 2018

Abstract: This work is devoted to the study of the void identification problem from partially overdetermined boundary data in the 2D-elastostatic case. In a first part, a shape identifiability result from a Cauchy data is presented, i.e. with traction field and boundary displacement as measurements. Then this geometric inverse problem is tackled by the minimization of two cost functionals, an energy gap functional and an L^2-gap functional, which enable the reconstruction of voids under Navier's boundary conditions. The shape derivatives of these cost functionals are computed for the purpose of sensitivity analysis.

Keywords: Voids identification, linear elasticity, Navier boundary conditions, identifiability, energy gap functional, L^2-gap functional, shape derivative

MSC 2010: 35R30, 74B05, 49Q12

1 Introduction

The reconstruction of flaws, buried in solids, is a classical geometric inverse problem that arises in several applications such as nondestructive material testing or underground object detection [15]. This shape problem can be tackled by transforming the inverse problem into an optimization one involving a cost functional that exploits mechanical boundary data only, that is, measurements of the traction and the boundary displacement. More precisely, the optimization problem revolves around the minimization of a cost function depending on a forward solution characterized by a trial defect (e.g., a cavity). Minimization-based approaches relevant to voids identification notably include methods based on the concepts of boundary elements [11], topological derivative [10, 19, 20] or shape derivative [1, 2, 16, 21], the latter being considered herein. The aim of the last concept revolves around the quantification of the first-order perturbation analysis with respect to small variations of the boundary of the domain of concern.

Voids identification using the shape gradient method under classical conditions have attracted the considerable interest of many authors [1, 2, 4, 16]. Moreover, the shape derivative is established for a large class of cost functionals, which includes the usual least-squares misfit functionals often used for identification, mentioning the work of Ben Ameur, Burger and Hackl [2] via the adjoint approach and the fictitious domain method. Recently, Ben Abda, Jaïem, Khalfallah and Zine [4] have investigated the misfit gap-cost functional, known as the Kohn–Vogelius functional [13, 14], for the identification of cavities under Neumann's boundary conditions. The energy-like error functional has been used with considerable success in wide applications [5, 6, 8, 16]. Besides, the shape derivative analysis has been extended to identification problems from par-

***Corresponding author: Bochra Méjri,** Ecole Nationale d'Ingénieurs de Tunis, LR99ES20 Modélisation Mathématique et Numérique dans les Sciences de l'Ingénieur, Université de Tunis El Manar, LAMSIN, B.P. 37, 1002 Tunis, Tunisia, e-mail: bochra.mejri@enit.utm.tn. http://orcid.org/0000-0003-0056-823X

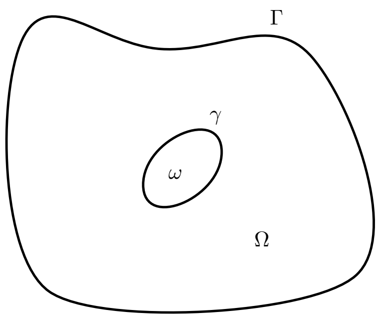

Figure 1: Domain configuration.

tially overdetermined boundary data such that the measurements are only the normal component of the normal stress and the displacement field [3, 12].

By following a recent work by Ben Abda, Jaïem, Khalfallah and Méjri [3] addressing cavities identification, this article treats the identification problem of voids under the Navier boundary conditions in the same context.

Voids identification model problem

Let $\mathcal{D}$ be a bounded, connected and open set of $\mathbb{R}^2$ with a Lipschitz external boundary Γ and let ω be a subdomain of $\mathcal{D}$ such that $\overline{\omega} \subset \mathcal{D}$, where $\overline{\omega}$ is the closure of ω. Consider the domain $\Omega = \mathcal{D} \setminus \overline{\omega}$ as a domain configuration of a homogeneous and isotropic elastic material and the subdomain ω as a void to be recovered. The present topological situation is depicted in Figure 1.

The elastic displacement field $\mathbf{u}$ satisfies the following direct problem:

$$\begin{cases} \mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{0} & \text{in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}_\Gamma = \mathbf{T}_g & \text{on } \Gamma, \\ \boldsymbol{\tau} \cdot \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = 0 & \text{on } \gamma = \partial\omega, \\ \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \gamma. \end{cases} \tag{1.1}$$

Here, $\boldsymbol{\sigma}(\mathbf{u})$ is the Cauchy stress tensor associated with the displacement field $\mathbf{u}$ and $\boldsymbol{\varepsilon}(\mathbf{u})$ is the linearized strain tensor given by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T).$$

Moreover, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are related by the Hooke constitutive law via

$$\boldsymbol{\sigma}(\mathbf{u}) = \lambda(\operatorname{tr} \boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}),$$

and conversely

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1+\nu}{E}\boldsymbol{\sigma}(\mathbf{u}) - \frac{\nu}{E}(\operatorname{tr} \boldsymbol{\sigma}(\mathbf{u}))\mathbf{I}.$$

Above, tr denotes the trace operator, $\mathbf{I}$ denotes the identity tensor and λ, μ are the Lamé coefficients related to Young's modulus E and Poisson's ratio ν via

$$\mu = \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}.$$

Furthermore, $\mathbf{n}_\Gamma$ denotes the outward unit normal to the boundary of Ω on Γ, and $\mathbf{n}$ and $\boldsymbol{\tau}$ denote the outward unit normal and the unit tangent, respectively, to ω on γ. By $\mathbf{v} \cdot \boldsymbol{\tau}$ we denote the tangential component of $\mathbf{v}$ on $\partial\Omega$ for any vector field $\mathbf{v}$ on $\partial\Omega$, i.e. $\mathbf{v} \cdot \boldsymbol{\tau} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n}$.

The direct problem (1.1) describes the displacement field $\mathbf{u}$ arising within Ω which is loaded by a given non-vanishing surface traction field $\mathbf{T}_g \in [H^{-1/2}(\Gamma)]^2$ such that $\int_\Gamma \mathbf{T}_g(\mathbf{x})\, \mathrm{d}s = \mathbf{0}$. The cavity ω obeys the Navier boundary conditions, more precisely, the elastic solid can slide in tangential direction, i.e. $\boldsymbol{\tau} \cdot \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = 0$, while in the normal direction a displacement is clamped, i.e. $\mathbf{u} \cdot \mathbf{n} = 0$.

The geometric inverse problem under consideration can be stated as follows: Given the traction field $\mathbf{T}_g$ on Γ and measuring the tangential displacement component u_τ on Γ (i.e. $\mathbf{u} \cdot \boldsymbol{\tau}_\Gamma = u_\tau$), recover the geometry of the void ω.

This paper addresses voids identification in elastic solids by means of partially overdetermined data on the external boundary, namely the traction field and the tangential component of displacement as measurements. A shape sensitivity analysis is developed for the reconstruction of voids under Navier's boundary conditions. To quantify the sensitivity of the so-called energy gap-misfit functional [13, 14], the existence of the shape derivative is proven via the material derivatives of the forward solutions and its expression is given. The difficulties affecting the study of the above-described problem stems from the structure of the boundary conditions prescribed on the boundary of voids, more precisely, from the slip boundary condition. These difficulties are reflected in the fact that each existing study for voids identification in elastic bodies [3, 4, 12] is formulated under classical conditions in terms of Neumann's conditions.

The outline of this paper is as follows: In Section 2 the identifiability issue for voids problems under Navier's boundary conditions from Cauchy data is discussed. Section 3 presents the forward and inverse problems and gives some insights on the solvability issues. The Eulerian derivative of the energy gap-misfit functional and the L^2-gap functional, in Hadamard's structure, are established in Section 4. Section 5 is devoted to some comments.

2 Voids identifiability issues

Classical Cauchy elasticity problems have been extensively studied by many authors in order to substantiate the uniqueness of the inverse problems from overspecified boundary data, which are the values of the traction and the displacement fields over an arbitrary portion (of non-vanishing length) of the boundary, using the Almansi lemma [17] for the plane elasticity (for more details, see [17, pp. 131–137]).

Shape identifiability

The main result of this section asserts that there is at most one solution in terms of void geometry in the elastic solid which yields the same surface measurements on an arbitrary small portion of the outer boundary.

Theorem 2.1. *Let $\omega^{(1)}$ and $\omega^{(2)}$ be two voids with smooth boundaries $\gamma^{(1,2)} = \partial\omega^{(1,2)}$ such that $\omega^{(1)} \cap \omega^{(2)} \neq \emptyset$. For $i = 1, 2$, suppose that $\mathbf{u}^{(i)}$ is the solution of the direct problem* (1.1) *defined in $\Omega^{(i)} = \mathcal{D} \setminus \overline{\omega^{(i)}}$ such that $\mathbf{u}^{(1)} \in [H^2(\Omega^{(1)})]^2$.*

Then, if $\omega^{(1)}$ and $\omega^{(2)}$ both lead to the same measured displacement field on an open portion Γ_0 of the outer boundary Γ, namely $\mathbf{u}^{(1)} = \mathbf{u}^{(2)} = \mathbf{u}_m$ on Γ_0, we have $\omega^{(1)} = \omega^{(2)}$.

Proof. Let $\Omega^{(e)} \subset \mathcal{D}$ be the external connected component of $\Omega^{(1)} \cap \Omega^{(2)}$ such that $\partial\Omega^{(e)} \subset (\gamma^{(1)} \cup \gamma^{(2)} \cup \Gamma)$.

Let $\boldsymbol{\psi} := \mathbf{u}^{(1)} - \mathbf{u}^{(2)}$ be the solution to the following problem:

$$\begin{cases} \mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{\psi}) = \mathbf{0} & \text{in } \Omega^{(e)}, \\ \boldsymbol{\sigma}(\boldsymbol{\psi})\mathbf{n}_\Gamma = \mathbf{0} & \text{on } \Gamma_0, \\ \boldsymbol{\psi} = \mathbf{0} & \text{on } \Gamma_0. \end{cases}$$

Since $\Omega^{(e)}$ is connected, $\boldsymbol{\psi}$ vanishes in the whole domain $\Omega^{(e)}$ by the Almansi lemma. Then

$$\mathbf{u}^{(1)} = \mathbf{u}^{(2)} \quad \text{and} \quad \boldsymbol{\sigma}(\mathbf{u}^{(1)}) = \boldsymbol{\sigma}(\mathbf{u}^{(2)}) \qquad \text{in } \Omega^{(e)}. \tag{2.1}$$

Now, we prove $\omega^{(1)} = \omega^{(2)}$ by contradiction. To this end, suppose $\omega^{(1)} \neq \omega^{(2)}$. Since numerous possibilities exist for the void (see Figure 2 for one particular situation), we assume that $\mathcal{O}$ is one of the connected compo-

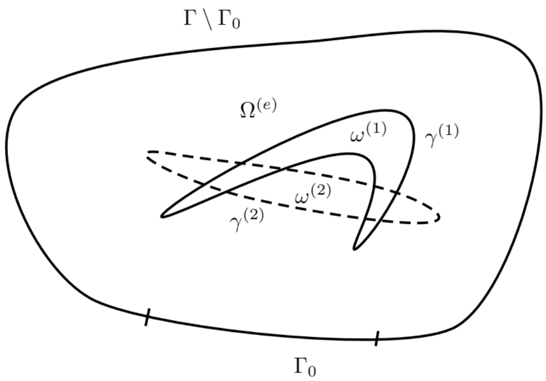

Figure 2: Possible voids configuration.

nents of $\Omega^{(1)} \setminus \Omega^{(2)}$. Subsequently, we have that $\partial\mathcal{O}$ is a union of a finite number of open smooth curves. So, for all $\mathbf{x} \in \partial\mathcal{O}$, there are only two cases to happen:

(i) $\mathbf{x} \in (\partial\mathcal{O} \cap \gamma^{(1)})$.

(ii) $\mathbf{x} \in (\partial\mathcal{O} \cap \gamma^{(2)})$.

In the first case, one has

$$\begin{cases} \boldsymbol{\tau} \cdot \boldsymbol{\sigma}(\mathbf{u}^{(1)})\mathbf{n} = 0 & \text{on } \gamma^{(1)}, \\ \mathbf{u}^{(1)} \cdot \mathbf{n} = 0 & \text{on } \gamma^{(1)}. \end{cases} \tag{2.2}$$

In the second case, by the result (2.1), we get

$$\mathbf{u}^{(1)}(\mathbf{x}) = \mathbf{u}^{(2)}(\mathbf{x}) \quad \text{and} \quad \boldsymbol{\sigma}(\mathbf{u}^{(1)}(\mathbf{x})) = \boldsymbol{\sigma}(\mathbf{u}^{(2)}(\mathbf{x})).$$

Since $\mathbf{x} \in (\partial\mathcal{O} \cap \gamma^{(2)}) \subset \gamma^{(2)}$, we have

$$\begin{cases} \boldsymbol{\tau} \cdot \boldsymbol{\sigma}(\mathbf{u}^{(1)})\mathbf{n} = 0 & \text{on } \gamma^{(2)}, \\ \mathbf{u}^{(1)} \cdot \mathbf{n} = 0 & \text{on } \gamma^{(2)}. \end{cases} \tag{2.3}$$

By (2.2) and (2.3), we get that $\mathbf{u}^{(1)}$ is the solution to the following problem in the open set $\mathcal{O}$:

$$\begin{cases} \mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}^{(1)}) = \mathbf{0} & \text{in } \mathcal{O}, \\ \boldsymbol{\tau} \cdot \boldsymbol{\sigma}(\mathbf{u}^{(1)})\mathbf{n} = 0 & \text{on } \partial\mathcal{O}, \\ \mathbf{u}^{(1)} \cdot \mathbf{n} = 0 & \text{on } \partial\mathcal{O}. \end{cases}$$

From the Green formula one obtains

$$\int_{\mathcal{O}} \boldsymbol{\sigma}(\mathbf{u}^{(1)}) : \boldsymbol{\varepsilon}(\mathbf{u}^{(1)})\, \mathrm{d}x = \int_{\partial\mathcal{O}} \boldsymbol{\sigma}(\mathbf{u}^{(1)})\mathbf{n} \cdot \mathbf{u}^{(1)}\, \mathrm{d}s = 0.$$

We conclude that $\mathbf{u}^{(1)}$ is a rigid displacement in $\mathcal{O}$. By Almansi's lemma, we deduce that $\mathbf{u}^{(1)}$ is a rigid displacement in the whole domain $\Omega^{(1)}$, which is a contradiction to the hypothesis of the load which is not identically equal to zero ($\mathbf{T}_g \neq \mathbf{0}$). Hence, $\omega^{(1)} = \omega^{(2)}$. □

Remark 2.2. For the case of monotonous and disjoint voids, the proof has the same spirit as the one presented above.

3 Voids identification problem

On the exterior boundary Γ, there are twice too many prescribed data. To address this issue, it is convenient to reformulate the model inverse problem by introducing two different well-posed problems, with a couple of solutions $(\boldsymbol{\sigma}_1, \mathbf{u}_1)$ and $(\boldsymbol{\sigma}_2, \mathbf{u}_2)$ defined in Ω, each of them satisfying the elasticity equations in Ω as well as the Navier boundary conditions on γ, and to attribute to the first problem Neumann conditions on the boundary Γ and to the second one Navier boundary conditions as follows:

$$\begin{cases} \mathbf{div}\,\boldsymbol{\sigma}_1 = \mathbf{0} & \text{in } \Omega, \\ \boldsymbol{\sigma}_1 \mathbf{n}_\Gamma = \mathbf{T}_g & \text{on } \Gamma, \\ \boldsymbol{\tau} \cdot \boldsymbol{\sigma}_1 \mathbf{n} = 0 & \text{on } \gamma, \\ \mathbf{u}_1 \cdot \mathbf{n} = 0 & \text{on } \gamma, \end{cases} \tag{3.1}$$

$$\begin{cases} \mathbf{div}\,\boldsymbol{\sigma}_2 = \mathbf{0} & \text{in } \Omega, \\ \mathbf{n}_\Gamma \cdot \boldsymbol{\sigma}_2 \mathbf{n}_\Gamma = \mathbf{T}_g \cdot \mathbf{n}_\Gamma & \text{on } \Gamma, \\ \mathbf{u}_2 \cdot \boldsymbol{\tau}_\Gamma = u_\tau & \text{on } \Gamma, \\ \boldsymbol{\tau} \cdot \boldsymbol{\sigma}_2 \mathbf{n} = 0 & \text{on } \gamma, \\ \mathbf{u}_2 \cdot \mathbf{n} = 0 & \text{on } \gamma. \end{cases} \tag{3.2}$$

To solve numerically the inverse problem presented in Section 1, we consider two shape functionals:

- The energy gap functional

$$\mathcal{J}_{\mathrm{KV}}(\omega) := \frac{1}{2} \int_\Omega (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1) : (\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1)\, dx, \tag{3.3}$$

  where “:” is the doubly contracted product.
- The L^2-gap functional

$$\mathcal{J}_{L^2}(\omega) := \frac{1}{2} \int_\Omega |\mathbf{u}_2 - \mathbf{u}_1|^2\, dx.$$

Remark 3.1.
- The L^2-gap functional measures the discrepancy between a Navier solution based on the measurements and a Neumann one based on the prescribed loads. However, the energy gap functional is based on an appropriate norm between both solutions, as the strain energy of their difference.
- For both functionals, $\mathbf{u}_1$ and $\mathbf{u}_2$ are obviously equal only when the geometry of the void reaches the real one.
- We remark that $\mathcal{J}_{\mathrm{KV}} = 0$ if and only if there is no misfit “up to the space $\mathcal{R}$” of rigid displacements between the solutions $(\boldsymbol{\sigma}_1, \mathbf{u}_1)$ and $(\boldsymbol{\sigma}_2, \mathbf{u}_2)$, that is, when $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2$ and $\mathbf{u}_1 = \mathbf{u}_2 + \mathbf{r}$, where $\mathbf{r}$ belongs to the linear space of rigid displacements defined by

$$\mathcal{R} = \{\mathbf{v}(\mathbf{x}) = \mathbf{a} + \mathbf{B} \cdot \mathbf{x} \mid \mathbf{a} \in \mathbb{R}^2 \text{ and } \mathbf{B} = -\mathbf{B}^T \text{ is an antisymmetric matrix}\}.$$

Thus, the inverse problem is formulated as a shape optimization one as follows:

$$\text{find } \Omega \text{ such that } \mathcal{J}(\Omega) = \min_{\tilde{\Omega} \subset \mathcal{D}} \mathcal{J}(\tilde{\Omega}).$$

Solvability issues

We recall briefly the weak formulation of problems (3.1) and (3.2), which are needed in the sequel and the associated solvability issues.

Define the spaces

$$\begin{aligned} L^2_s(\Omega) &= \{\boldsymbol{\alpha} = (\boldsymbol{\alpha}_{ij}) \in [L^2(\Omega)]^4 \mid \boldsymbol{\alpha}_{ij} = \boldsymbol{\alpha}_{ji}\}, \\ V &= \{\mathbf{v} \in [H^1(\Omega)]^2 \mid \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \gamma\}, \\ V_0 &= \{\mathbf{v} \in [H^1(\Omega)]^2 \mid \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \gamma \text{ and } \mathbf{v} \cdot \boldsymbol{\tau}_\Gamma = 0 \text{ on } \Gamma\}, \end{aligned}$$

and the bilinear symmetric form $a : L_s^2(\Omega) \times L_s^2(\Omega) \to \mathbb{R}$ and the bilinear form $b : L_s^2(\Omega) \times V \to \mathbb{R}$ by

$$a(\boldsymbol{\sigma}, \boldsymbol{\alpha}) = \int_\Omega \Big[\frac{1+\nu}{E} \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{\alpha}) - \frac{\nu}{E} \operatorname{tr}(\boldsymbol{\sigma}) \operatorname{tr}(\boldsymbol{\alpha}) \Big] \, dx, \quad b(\boldsymbol{\alpha}, \mathbf{v}) = -\int_\Omega \operatorname{tr}(\boldsymbol{\alpha} \nabla \mathbf{v}) \, dx.$$

The variational formulations of both problems (3.1) and (3.2) are respectively

$$\begin{cases} \text{find } (\boldsymbol{\sigma}_1, \mathbf{u}_1) \in L_s^2(\Omega) \times [H^1(\Omega)]^2, \mathbf{u}_1 \cdot \mathbf{n} = 0, \text{ on } \gamma \text{ such that} \\ a(\boldsymbol{\sigma}_1, \boldsymbol{\alpha}) + b(\boldsymbol{\alpha}, \mathbf{u}_1) = 0 \quad \text{for all } \boldsymbol{\alpha} \in L_s^2(\Omega), \\ b(\boldsymbol{\sigma}_1, \mathbf{v}) = -\int_\Gamma \mathbf{T}_g \cdot \mathbf{v} \, ds \quad \text{for all } \mathbf{v} \in V, \end{cases} \tag{3.4}$$

and

$$\begin{cases} \text{find } (\boldsymbol{\sigma}_2, \mathbf{u}_2) \in L_s^2(\Omega) \times [H^1(\Omega)]^2, \mathbf{u}_2 \cdot \mathbf{n} = 0 \text{ on } \gamma \\ \text{and } \mathbf{u}_2 \cdot \boldsymbol{\tau}_\Gamma = u_\tau \text{ on } \Gamma \text{ such that} \\ a(\boldsymbol{\sigma}_2, \boldsymbol{\alpha}) + b(\boldsymbol{\alpha}, \mathbf{u}_2) = 0 \quad \text{for all } \boldsymbol{\alpha} \in L_s^2(\Omega), \\ b(\boldsymbol{\sigma}_2, \mathbf{v}) = -\int_\Gamma (\mathbf{T}_g \cdot \mathbf{n}_\Gamma)(\mathbf{v} \cdot \mathbf{n}_\Gamma) \, ds \quad \text{for all } \mathbf{v} \in V_0. \end{cases} \tag{3.5}$$

We adopt the above formulation in two fields, namely the mixed Reissner formulation [9]. It consists of writing separately the equilibrium equation and the Hooke law in variational form.

Lemma 3.2 (Korn-type inequality). *Let Ω be a domain with a Lipschitz boundary and let $V_0 = \mathcal{R}_{V_0} \oplus \mathcal{Q}_{V_0}$, where $\mathcal{R}_{V_0} = \mathcal{R} \cap V_0$ and $\mathcal{Q}_{V_0}$ is the orthogonal complement of $\mathcal{R}_{V_0}$ in V_0. Then the following Korn inequality holds:*

$$\int_\Omega \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx \geq c \|\mathbf{v}\|_{H^1(\Omega)}^2 \quad \textit{for all } \mathbf{v} \in V_0.$$

Proof. One can follow the same argument as in the proof of the Korn inequality without boundary conditions (see [18, Theorem 3.5]) to obtain a Korn inequality for this case. □

Theorem 3.3. *There is a unique solution $(\boldsymbol{\sigma}_1, \mathbf{u}_1) \in L_s^2(\Omega) \times [H^1(\Omega)]^2$ (resp. $(\boldsymbol{\sigma}_2, \mathbf{u}_2) \in L_s^2(\Omega) \times [H^1(\Omega)]^2$) satisfying the variational problem* (3.4) *(resp.* (3.5)*) which depends linearly and continuously on the data $\mathbf{T}_g$ (resp. $(\mathbf{T}_g, u_\tau)$).*

Proof. By using Lemma 3.2, one can prove the existence and the uniqueness of the weak solutions of the direct problems (3.1) and (3.2) by the Brezzi–Babuška theorem [7, 9] as follows:
- The bilinear symmetric form $a(\boldsymbol{\alpha}, \boldsymbol{\alpha})$ defines an equivalent norm to $L_s^2(\Omega)$.
- The linear form $b(\boldsymbol{\alpha}, \mathbf{v})$ satisfies the inf-sup condition by choosing $\boldsymbol{\alpha}$ as the Cauchy stress tensor associated with the vector $\mathbf{v}$ and by using an algebraic inequality and the Korn-type inequality

$$\sup_{\boldsymbol{\alpha} \in L_s^2(\Omega)} \frac{b(\boldsymbol{\alpha}, \mathbf{v})}{\|\boldsymbol{\alpha}\|_{L_s^2(\Omega)}} \geq C(\mu, \lambda) \|\mathbf{v}\|_{V_0} \quad \text{for all } \mathbf{v} \in V_0.$$ □

4 Shape sensitivity analysis

In this section, we prove that the cost functionals are differentiable with respect to the variations of the boundary of the void. Below a rigorous explanation of the calculus is provided.

4.1 Transformations of domains

To evaluate the sensitivity analysis of the shape functional $\mathcal{J}$, a family of perturbations is acquired for which one considers transformations of domains $\{\Omega_t\}$ of a fixed domain Ω for $t \in \mathbb{R}$. It is assumed that every member of the family $\{\Omega_t\}$ has the same topological properties. Let $\mathbf{F}_t$ be a perturbation of the identity operator defined by $\mathbf{F}_t = \mathbf{id} + t\mathbf{h}$ for a deformation field $\mathbf{h}$ belonging to the space

$$\mathcal{S} = \{\mathbf{h} \in C^{1,1}(\overline{\mathcal{U}}, \mathbb{R}^2) \mid \mathbf{h} = \mathbf{0} \text{ on } \partial\mathcal{U} \cup \Gamma \text{ and } \mathbf{h}\cdot\boldsymbol{\tau} = 0 \text{ on } \gamma\},$$

where $\mathcal{U}$ is an open and bounded domain containing $\overline{\Omega}$. The condition $\mathbf{h}_{|\Gamma} = \mathbf{0}$ means that the boundary Γ is a part of Ω_t for all t and the condition $\mathbf{h}\cdot\boldsymbol{\tau}_{|\gamma} = 0$ means that we perturb the geometry of the void ω in the normal direction. For sufficiently small t, the perturbation $\mathbf{F}_t$ is a $C^{1,1}$-diffeomorphism from Ω onto its image. One defines the families of domains

$$\Omega_t = \mathbf{F}_t(\Omega) \quad \text{and} \quad \gamma_t = \mathbf{F}_t(\gamma).$$

For each t sufficiently small, one considers $(\boldsymbol{\sigma}_{1,t}, \mathbf{u}_{1,t}) \in L^2_s(\Omega_t) \times V_t$ and $(\boldsymbol{\sigma}_{2,t}, \mathbf{u}_{2,t}) \in L^2_s(\Omega_t) \times V_{0,t}$ the solutions of problems (3.1) and (3.2), respectively, defined on the perturbed domain $(\Omega_t, \Gamma, \gamma_t)$, where

$$V_t = \{\mathbf{v}_t \in [H^1(\Omega_t)]^2 \mid \mathbf{v}_t\cdot\mathbf{n}_t = 0 \text{ on } \gamma_t\},$$
$$V_{0,t} = \{\mathbf{v}_t \in [H^1(\Omega_t)]^2 \mid \mathbf{v}_t\cdot\mathbf{n}_t = 0 \text{ on } \gamma_t \text{ and } \mathbf{v}_t\cdot\boldsymbol{\tau}_\Gamma = 0 \text{ on } \Gamma\},$$

where $\mathbf{n}_t$ denotes the unit normal vector to γ_t.

Set the misfit cost functional $\mathcal{J}_{KV}$ depending on the domain Ω_t,

$$\mathcal{J}_{KV}(\Omega_t) := \frac{1}{2}\int_{\Omega_t} (\boldsymbol{\sigma}_{2,t} - \boldsymbol{\sigma}_{1,t}) : (\boldsymbol{\varepsilon}_{2,t} - \boldsymbol{\varepsilon}_{1,t})\,dx, \tag{4.1}$$

and the L^2-gap functional $\mathcal{J}_{L^2}$ depending on the domain Ω_t,

$$\mathcal{J}_{L^2}(\Omega_t) := \frac{1}{2}\int_{\Omega_t} |\mathbf{u}_{2,t} - \mathbf{u}_{1,t}|^2\,dx.$$

Then the Eulerian derivative of the functional $\mathcal{J}$ at Ω in the direction $\mathbf{h}$ is defined by

$$\mathcal{J}'(\Omega, \mathbf{h}) = \lim_{t\to 0} \frac{\mathcal{J}(\Omega_t) - \mathcal{J}(\Omega)}{t}. \tag{4.2}$$

The functional $\mathcal{J}$ is called shape derivative if $\mathcal{J}'(\Omega, \mathbf{h})$ exists for all $\mathbf{h} \in \mathcal{S}$ and defines a continuous linear functional in $\mathcal{S}$.

For a function $\mathbf{v}_t$ defined in the perturbed domain Ω_t, we denote the transported function defined in the fixed domain Ω by $\mathbf{v}^t = \mathbf{v}_t \circ \mathbf{F}_t$. Thus, one considers the following limits:

- Shape derivative:
$$\mathbf{v}' = \frac{d\mathbf{v}_t}{dt}_{|t=0}.$$
- Material derivative:
$$\dot{\mathbf{v}} = \frac{d\mathbf{v}^t}{dt}_{|t=0} = \frac{d(\mathbf{v}_t \circ \mathbf{F}_t)}{dt}_{|t=0}.$$

Herein, we use the material derivatives for the calculation of the shape derivative of $\mathcal{J}_{KV}$. For its determination, one recalls the following two lemmas.

Lemma 4.1 ([21]). (i) *If $\mathbf{v}_t \in L^1(\Omega_t)$, then $\mathbf{v}^t \in L^1(\Omega)$ and we have*

$$\int_{\Omega_t} \mathbf{v}_t\,dx = \int_{\Omega} \delta_t \mathbf{v}^t\,dx,$$

where $\delta_t = \det(D\mathbf{F}_t) = \det(\mathbf{id} + t\nabla\mathbf{h}^T)$ (here $D\mathbf{F}_t$ is the Jacobian matrix of $\mathbf{F}_t$).

(ii) *If $\mathbf{v}_t \in H^1(\Omega_t)$, then $\mathbf{v}^t \in H^1(\Omega)$ and we have*

$$(\nabla\mathbf{v}_t)\circ\mathbf{F}_t = M_t \nabla\mathbf{v}^t,$$

with $M_t = D\mathbf{F}_t^{-T} = \nabla\mathbf{F}_t^{-1}$.

Lemma 4.2 ([21]). *The mappings $t \mapsto \delta_t$ and $t \mapsto M_t$ with values in $C(\Omega)$ and $C(\Omega)^{2\times 2}$, respectively, are C^1 in a neighborhood of 0, and we have*

$$\frac{d\delta_t}{dt}_{|t=0} = \operatorname{div}\mathbf{h}, \quad \frac{dM_t}{dt}_{|t=0} = -\nabla\mathbf{h}.$$

4.2 Asymptotic expansions

In order to perform the material derivatives of the solutions to the variational forms (3.4) and (3.5), one has to direct attention to the differentiability of the transported solutions defined in the fixed domain. At this stage of calculus, there is a significant difference to the classical cases at the level of the slip boundary condition on the boundary γ of the void ω for the transported solutions.

For $\mathbf{n}$ being a unit outward norm vector to γ, the transported vector of $\mathbf{n}_{\mathrm{t}}$ denoted by $\mathbf{n}^{\mathrm{t}}$ satisfies (for details see [6])

$$\mathbf{n}^{\mathrm{t}} = \frac{M_{\mathrm{t}}\mathbf{n}}{|M_{\mathrm{t}}\mathbf{n}|}.$$

By that fact, the slip boundary condition $\mathbf{u}_{\mathrm{t}} \cdot \mathbf{n}_{\mathrm{t}} = 0$ on γ_{t}, for the perturbed solution, is equivalent to

$$(\mathbf{u}_{\mathrm{t}} \cdot \mathbf{n}_{\mathrm{t}}) \circ \mathbf{F}_{\mathrm{t}} = \mathbf{u}^{\mathrm{t}} \cdot \mathbf{n}^{\mathrm{t}} = \mathbf{u}^{\mathrm{t}} \cdot \frac{M_{\mathrm{t}}\mathbf{n}}{|M_{\mathrm{t}}\mathbf{n}|} = 0 \quad \text{on } \gamma.$$

This consideration led to the definition of the following functional framework where the transported solutions exist:

$$\begin{aligned} V^{\mathrm{t}} &= \{\mathbf{v}^{\mathrm{t}} \in [H^1(\Omega)]^2 \mid M_{\mathrm{t}}^T\mathbf{v}^{\mathrm{t}} \cdot \mathbf{n} = 0 \text{ on } \gamma\}, \\ V_0^{\mathrm{t}} &= \{\mathbf{v}^{\mathrm{t}} \in [H^1(\Omega)]^2 \mid M_{\mathrm{t}}^T\mathbf{v}^{\mathrm{t}} \cdot \mathbf{n} = 0 \text{ on } \gamma \text{ and } \mathbf{v}^{\mathrm{t}} \cdot \boldsymbol{\tau}_\Gamma = 0 \text{ on } \Gamma\}. \end{aligned}$$

One defines the following bilinear forms:

$$\begin{aligned} a^{\mathrm{t}}(\boldsymbol{\sigma}, \boldsymbol{\alpha}) &= \int_\Omega \delta_{\mathrm{t}}\Big[\frac{1+\nu}{E}\,\mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{\alpha}) - \frac{\nu}{E}\,\mathrm{tr}(\boldsymbol{\sigma})\,\mathrm{tr}(\boldsymbol{\alpha})\Big]\,\mathrm{d}x, \\ b^{\mathrm{t}}(\boldsymbol{\alpha}, \mathbf{v}) &= -\int_\Omega \delta_{\mathrm{t}}\,\mathrm{tr}(\boldsymbol{\alpha}(M_{\mathrm{t}}\nabla\mathbf{v}))\,\mathrm{d}x, \end{aligned}$$

and

$$\begin{aligned} \dot{a}(\boldsymbol{\sigma}, \boldsymbol{\alpha}) &= \int_\Omega \operatorname{div}\mathbf{h}\Big[\frac{1+\nu}{E}\,\mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{\alpha}) - \frac{\nu}{E}\,\mathrm{tr}(\boldsymbol{\sigma})\,\mathrm{tr}(\boldsymbol{\alpha})\Big]\,\mathrm{d}x, \\ \dot{b}(\boldsymbol{\alpha}, \mathbf{v}) &= -\int_\Omega \operatorname{div}\mathbf{h}\,\mathrm{tr}(\boldsymbol{\alpha}\nabla\mathbf{v})\,\mathrm{d}x + \int_\Omega \mathrm{tr}(\boldsymbol{\alpha}(\nabla\mathbf{h}^T\nabla\mathbf{v}))\,\mathrm{d}x. \end{aligned}$$

Neumann's problem

Here, we will detail the Neumann case; the treatment of Navier's case is analogue.

Lemma 4.3. *For each* t *sufficiently small, one associates the pair* $(\boldsymbol{\sigma}_1^{\mathrm{t}}, \mathbf{u}_1^{\mathrm{t}})$*, the solution transported to the fixed domain* Ω*, by*

$$\mathbf{u}_1^{\mathrm{t}} = \mathbf{u}_{1,\mathrm{t}} \circ \mathbf{F}_{\mathrm{t}} \quad \text{and} \quad \boldsymbol{\sigma}_1^{\mathrm{t}} = \boldsymbol{\sigma}_{1,\mathrm{t}} \circ \mathbf{F}_{\mathrm{t}},$$

which is the unique solution of the following formulation in two fields:

$$\begin{cases} \text{find } (\boldsymbol{\sigma}_1^{\mathrm{t}}, \mathbf{u}_1^{\mathrm{t}}) \in L_s^2(\Omega) \times [H^1(\Omega)]^2,\ M_{\mathrm{t}}^T\mathbf{u}_1^{\mathrm{t}} \cdot \mathbf{n} = 0 \text{ on } \gamma \text{ such that} \\ a^{\mathrm{t}}(\boldsymbol{\sigma}_1^{\mathrm{t}}, \boldsymbol{\alpha}) + b^{\mathrm{t}}(\boldsymbol{\alpha}, \mathbf{u}_1^{\mathrm{t}}) = 0 \quad \text{for all } \boldsymbol{\alpha} \in L_s^2(\Omega), \\ b^{\mathrm{t}}(\boldsymbol{\sigma}_1^{\mathrm{t}}, \mathbf{v}^{\mathrm{t}}) = -\int_\Gamma \mathbf{T}_g \cdot \mathbf{v}^{\mathrm{t}}\,\mathrm{d}s \quad \text{for all } \mathbf{v}^{\mathrm{t}} \in V^{\mathrm{t}}. \end{cases}$$

Proof. In the first step, we use Lemma 4.1 and Lemma 4.2 to write the transformation of the volume integrals in the weak formulation satisfied by $(\boldsymbol{\sigma}_{1,\mathrm{t}}, \mathbf{u}_{1,\mathrm{t}})$, the perturbed solution of problem (3.1) defined on the perturbed domain $(\Omega_{\mathrm{t}}, \Gamma, \gamma_{\mathrm{t}})$. In the second one, we take into account the transformation of the slip boundary condition on the boundary γ. □

The perturbed solution $(\boldsymbol{\sigma}_1^{\rm t}, \mathbf{u}_1^{\rm t})$ is well-posed only in the variable space $L_s^2(\Omega) \times V^{\rm t}$. By this fact, we can not prove its differentiability. To avoid this difficulty, one has to consider the function

$$\mathbf{w}_1^{\rm t} = M_{\rm t}^T(\mathbf{u}_1^{\rm t} - \mathbf{u}_1^*), \tag{4.3}$$

where $(\boldsymbol{\sigma}_1^*, \mathbf{u}_1^*)$ is the solution to the following value-boundary problem:

$$\begin{cases} \mathbf{div}\,\boldsymbol{\sigma}_1^* = \mathbf{0} & \text{in } \Omega, \\ \boldsymbol{\varepsilon}_1^* = \dfrac{1+\nu}{E}\boldsymbol{\sigma}_1^* - \dfrac{\nu}{E}(\operatorname{tr}\boldsymbol{\sigma}_1^*)\mathbf{I} & \text{in } \Omega, \\ \boldsymbol{\sigma}_1^*\mathbf{n}_\Gamma = \mathbf{T}_g & \text{on } \Gamma, \\ \mathbf{u}_1^* = \mathbf{0} & \text{on } \gamma. \end{cases}$$

For each t sufficiently small, one observes that $\mathbf{w}_1^{\rm t} \in V$ satisfies the following problem:

$$\begin{cases} \text{find } (\boldsymbol{\sigma}_1^{\rm t}, \mathbf{w}_1^{\rm t}) \in L_s^2(\Omega) \times V \text{ such that} \\ a^{\rm t}(\boldsymbol{\sigma}_1^{\rm t}, \boldsymbol{\alpha}) + b^{\rm t}(\boldsymbol{\alpha}, M_{\rm t}^{-T}\mathbf{w}_1^{\rm t} + \mathbf{u}_1^*) = 0 \quad \text{for all } \boldsymbol{\alpha} \in L_s^2(\Omega), \\ b^{\rm t}(\boldsymbol{\sigma}_1^{\rm t}, M_{\rm t}^{-T}\mathbf{v}) = -\displaystyle\int_\Gamma \mathbf{T}_g \cdot \mathbf{v}\,ds \quad \text{for all } \mathbf{v} \in V. \end{cases}$$

Lemma 4.4. *The map* $\mathrm{t} \mapsto (\boldsymbol{\sigma}_1^{\rm t}, \mathbf{w}_1^{\rm t})$ *is continuously differentiable in a neighborhood of* 0 *and one has that* $(\dot{\boldsymbol{\sigma}}_1, \dot{\mathbf{w}}_1) = \frac{\rm d}{\rm dt}(\boldsymbol{\sigma}_1^{\rm t}, \mathbf{w}_1^{\rm t})_{|\mathrm{t}=0}$ *satisfies the following variational problem:*

$$\begin{cases} \text{find } (\dot{\boldsymbol{\sigma}}_1, \dot{\mathbf{w}}_1) \in L_s^2(\Omega) \times V \text{ such that} \\ a(\dot{\boldsymbol{\sigma}}_1, \boldsymbol{\alpha}) + b(\boldsymbol{\alpha}, \dot{\mathbf{w}}_1) = \tilde{\ell}_1(\boldsymbol{\alpha}) \quad \text{for all } \boldsymbol{\alpha} \in L_s^2(\Omega), \\ b(\dot{\boldsymbol{\sigma}}_1, \mathbf{v}) = -b(\boldsymbol{\sigma}_1, \nabla\mathbf{h}^T\mathbf{v}) - \dot{b}(\boldsymbol{\sigma}_1, \mathbf{v}) \quad \text{for all } \mathbf{v} \in V, \end{cases} \tag{4.4}$$

where

$$\tilde{\ell}_1(\boldsymbol{\alpha}) = -\dot{a}(\boldsymbol{\sigma}_1, \boldsymbol{\alpha}) - \dot{b}(\boldsymbol{\alpha}, \mathbf{u}_1) - b(\boldsymbol{\alpha}, \nabla\mathbf{h}^T(\mathbf{u}_1 - \mathbf{u}_1^*)).$$

Proof. Consider the linear and continuous mapping

$$\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2) : \mathbb{R} \times L_s^2(\Omega) \times [H^1(\Omega)]^2 \to L_s^2(\Omega) \times ([H^1(\Omega)]^2)'$$

given by

$$\begin{aligned} \langle \mathcal{M}_1(\mathrm{t}, \boldsymbol{\sigma}, \mathbf{w}), \boldsymbol{\alpha}\rangle &= a^{\rm t}(\boldsymbol{\sigma}, \boldsymbol{\alpha}) + b^{\rm t}(\boldsymbol{\alpha}, M_t^{-T}\mathbf{w} + \mathbf{u}_1^*) \quad && \text{for all } \boldsymbol{\alpha} \in L_s^2(\Omega), \\ \langle \mathcal{M}_2(\mathrm{t}, \boldsymbol{\sigma}, \mathbf{w}), \mathbf{v}\rangle &= b^{\rm t}(\boldsymbol{\sigma}, M_t^{-T}\mathbf{v}) + \int_\Gamma \mathbf{T}_g \cdot \mathbf{v}\,ds && \text{for all } \mathbf{v} \in V. \end{aligned}$$

Then $(\boldsymbol{\sigma}_1^{\rm t}, \mathbf{w}_1^{\rm t})$ is the unique element of $L_s^2(\Omega) \times [H^1(\Omega)]^2$ satisfying for all $t \in \mathbb{R}$,

$$\langle \mathcal{M}(\mathrm{t}, \boldsymbol{\sigma}_1^{\rm t}, \mathbf{w}_1^{\rm t}), (\boldsymbol{\alpha}, \mathbf{v})\rangle = \mathbf{0}, \quad (\boldsymbol{\alpha}, \mathbf{v}) \in L_s^2(\Omega) \times V.$$

For the state solution $(\boldsymbol{\sigma}_1, \mathbf{w}_1)$, we remark that $\mathcal{M}(0, \boldsymbol{\sigma}_1, \mathbf{w}_1) = 0$, and also that $\mathcal{M}$ is a linear mapping on $(\boldsymbol{\sigma}, \mathbf{w})$ and differentiable on t. So, the derivative of $\mathcal{M}$ at t = 0 is of the form

$$\langle \mathrm{D}_{(\boldsymbol{\sigma},\mathbf{w})}\mathcal{M}(0, \boldsymbol{\sigma}_1, \mathbf{w}_1)(\partial\boldsymbol{\sigma}, \partial\mathbf{w}), (\boldsymbol{\alpha}, \mathbf{v})\rangle = \langle a(\partial\boldsymbol{\sigma}, \boldsymbol{\alpha}) + b(\boldsymbol{\alpha}, \partial\mathbf{w}), b(\partial\boldsymbol{\sigma}, \mathbf{v})\rangle.$$

By the implicit function theorem, one has that the mapping $\mathrm{t} \to (\boldsymbol{\sigma}_1^{\rm t}, \mathbf{w}_1^{\rm t})$ is strongly differentiable in a neighborhood of 0. Its derivative at t = 0, i.e. $(\dot{\boldsymbol{\sigma}}_1, \dot{\mathbf{w}}_1)$, is the unique solution to the following system

$$\begin{aligned} \frac{\partial}{\partial \mathrm{t}}\langle \mathcal{M}_1(\mathrm{t}, \boldsymbol{\sigma}, \mathbf{w}), \boldsymbol{\alpha}\rangle &= \dot{a}(\boldsymbol{\sigma}, \boldsymbol{\alpha}) + \dot{b}(\boldsymbol{\alpha}, \mathbf{w} + \mathbf{u}_1^*) + b(\boldsymbol{\alpha}, \nabla\mathbf{h}^T\mathbf{w}), \\ \frac{\partial}{\partial \mathrm{t}}\langle \mathcal{M}_2(\mathrm{t}, \boldsymbol{\sigma}, \mathbf{w}), \mathbf{v}\rangle &= \dot{b}(\boldsymbol{\sigma}, \mathbf{v}) + b(\boldsymbol{\sigma}, \nabla\mathbf{h}^T\mathbf{v}), \end{aligned}$$

which proves the lemma. □

Finally, one gets the main result which proves the existence of the strong material derivative.

Theorem 4.5. *The map* $\mathrm{t} \mapsto (\boldsymbol{\sigma}_1^{\mathrm{t}}, \mathbf{u}_1^{\mathrm{t}})$ *is continuously differentiable in a neighborhood of* 0 *and one has that* $(\dot{\boldsymbol{\sigma}}_1, \dot{\mathbf{u}}_1)$ *is the solution to the following variational problem:*

$$\begin{cases} \textit{find } (\dot{\boldsymbol{\sigma}}_1, \dot{\mathbf{u}}_1) \in L_s^2(\Omega) \times [H^1(\Omega)]^2,\ \dot{\mathbf{u}}_1 \cdot \mathbf{n} = \nabla \mathbf{h}^T \mathbf{u}_1 \cdot \mathbf{n} \textit{ on } \gamma \textit{ such that} \\ a(\dot{\boldsymbol{\sigma}}_1, \boldsymbol{\alpha}) + b(\boldsymbol{\alpha}, \dot{\mathbf{u}}_1) = \ell_1(\boldsymbol{\alpha}) \quad \textit{for all } \boldsymbol{\alpha} \in L_s^2(\Omega), \\ b(\dot{\boldsymbol{\sigma}}_1, \mathbf{v}) = -b(\boldsymbol{\sigma}_1, \nabla \mathbf{h}^T \mathbf{v}) - \dot{b}(\boldsymbol{\sigma}_1, \mathbf{v}) \quad \textit{for all } \mathbf{v} \in V, \end{cases}$$

where

$$\ell_1(\boldsymbol{\alpha}) = -\dot{a}(\boldsymbol{\sigma}_1, \boldsymbol{\alpha}) - \dot{b}(\boldsymbol{\alpha}, \mathbf{u}_1).$$

Proof. By (4.3), one has $(\boldsymbol{\sigma}_1^{\mathrm{t}}, \mathbf{u}_1^{\mathrm{t}}) = (\boldsymbol{\sigma}_1^{\mathrm{t}}, M_{\mathrm{t}}^{-T} \mathbf{w}_1^{\mathrm{t}} + \mathbf{u}_1^*)$, so the differentiability of $\mathrm{t} \mapsto (\boldsymbol{\sigma}_1^{\mathrm{t}}, \mathbf{u}_1^{\mathrm{t}})$ follows from Lemma 4.4. In particular, one obtains $(\dot{\boldsymbol{\sigma}}_1, \dot{\mathbf{u}}_1) = (\dot{\boldsymbol{\sigma}}_1, \nabla \mathbf{h}^T(\mathbf{u}_1 - \mathbf{u}_1^*) + \dot{\mathbf{w}}_1)$. Inserting it in (4.4) results in

$$a(\dot{\boldsymbol{\sigma}}_1, \boldsymbol{\alpha}) + b(\boldsymbol{\alpha}, \dot{\mathbf{u}}_1) = \tilde{\ell}_1(\boldsymbol{\alpha}) + b(\boldsymbol{\alpha}, \nabla \mathbf{h}^T(\mathbf{u}_1 - \mathbf{u}_1^*)),$$

which yields

$$a(\dot{\boldsymbol{\sigma}}_1, \boldsymbol{\alpha}) + b(\boldsymbol{\alpha}, \dot{\mathbf{u}}_1) = \ell_1(\boldsymbol{\alpha}) = -\dot{a}(\boldsymbol{\sigma}_1, \boldsymbol{\alpha}) - \dot{b}(\boldsymbol{\alpha}, \mathbf{u}_1).$$

In order to derive the condition $\dot{\mathbf{u}}_1 \cdot \mathbf{n} = \nabla \mathbf{h}^T \mathbf{u}_1 \cdot \mathbf{n}$ on γ, the following equation is used:

$$\mathbf{u}_1^{\mathrm{t}} \cdot \mathbf{n}^{\mathrm{t}} = 0 \quad \text{on } \gamma.$$

It follows that

$$\dot{\mathbf{u}}_1 \cdot \mathbf{n} = -\mathbf{u}_1 \cdot \dot{\mathbf{n}} \quad \text{on } \gamma.$$

On the other hand,

$$\dot{\mathbf{n}} = -\nabla \mathbf{h} \cdot \mathbf{n} \quad \text{on } \gamma.$$

Then

$$\dot{\mathbf{u}}_1 \cdot \mathbf{n} = \nabla \mathbf{h}^T \mathbf{u}_1 \cdot \mathbf{n} \quad \text{on } \gamma.$$

□

Navier's problem

Similarly to the previous case, one gets the following lemma.

Lemma 4.6. *For each* t *sufficiently small, one associates the pair* $(\boldsymbol{\sigma}_2^{\mathrm{t}}, \mathbf{u}_2^{\mathrm{t}})$*, the solution transported to the fixed domain* Ω*, by*

$$\mathbf{u}_2^{\mathrm{t}} = \mathbf{u}_{2,\mathrm{t}} \circ \mathbf{F}_{\mathrm{t}} \quad \textit{and} \quad \boldsymbol{\sigma}_2^{\mathrm{t}} = \boldsymbol{\sigma}_{2,\mathrm{t}} \circ \mathbf{F}_{\mathrm{t}},$$

which is the unique solution of the following formulation in two fields:

$$\begin{cases} \textit{find } (\boldsymbol{\sigma}_2^{\mathrm{t}}, \mathbf{u}_2^{\mathrm{t}}) \in L_s^2(\Omega) \times [H^1(\Omega)]^2,\ M_{\mathrm{t}}^T \mathbf{u}_2^{\mathrm{t}} \cdot \mathbf{n} = 0 \textit{ on } \gamma \\ \textit{and } \mathbf{u}_2^{\mathrm{t}} \cdot \boldsymbol{\tau}_\Gamma = u_\tau \ \textit{ on } \Gamma \textit{ such that} \\ a^{\mathrm{t}}(\boldsymbol{\sigma}_2^{\mathrm{t}}, \boldsymbol{\alpha}) + b^{\mathrm{t}}(\boldsymbol{\alpha}, \mathbf{u}_2^{\mathrm{t}}) = 0 \quad \textit{for all } \boldsymbol{\alpha} \in L_s^2(\Omega), \\ b^{\mathrm{t}}(\boldsymbol{\sigma}_2^{\mathrm{t}}, \mathbf{v}^{\mathrm{t}}) = -\displaystyle\int_\Gamma (\mathbf{T}_g \cdot \mathbf{n}_\Gamma)(\mathbf{v}^{\mathrm{t}} \cdot \mathbf{n}_\Gamma)\, ds \quad \textit{for all } \mathbf{v}^{\mathrm{t}} \in V_0^{\mathrm{t}}. \end{cases}$$

Proof. In the first step, we use Lemma 4.1 and Lemma 4.2 to write the transformation of the volume integrals in the weak formulation satisfied by $(\boldsymbol{\sigma}_{2,\mathrm{t}}, \mathbf{u}_{2,\mathrm{t}})$, the perturbed solution of problem (3.2) defined on the perturbed domain $(\Omega_{\mathrm{t}}, \Gamma, \gamma_{\mathrm{t}})$. In the second one, we take into account the transformation of the slip boundary condition on the boundary γ. □

Theorem 4.7. *The mapping* $t \mapsto (\boldsymbol{\sigma}_2^t, \mathbf{u}_2^t)$ *is continuously differentiable in a neighborhood of* 0 *and one has that* $(\dot{\boldsymbol{\sigma}}_2, \dot{\mathbf{u}}_2)$ *is the solution to the following variational problem:*

$$\begin{cases} \text{find } (\dot{\boldsymbol{\sigma}}_2, \dot{\mathbf{u}}_2) \in L_s^2(\Omega) \times [H^1(\Omega)]^2,\ \dot{\mathbf{u}}_2 \cdot \mathbf{n} = \nabla \mathbf{h}^T \mathbf{u}_2 \cdot \mathbf{n} \text{ on } \gamma \\ \text{and } \dot{\mathbf{u}}_2 \cdot \boldsymbol{\tau}_\Gamma = 0 \text{ on } \Gamma \text{ such that} \\ a(\dot{\boldsymbol{\sigma}}_2, \boldsymbol{\alpha}) + b(\boldsymbol{\alpha}, \dot{\mathbf{u}}_2) = \ell_2(\boldsymbol{\alpha}) \quad \text{for all } \boldsymbol{\alpha} \in L_s^2(\Omega), \\ b(\dot{\boldsymbol{\sigma}}_2, \mathbf{v}) = -b(\boldsymbol{\sigma}_2, \nabla \mathbf{h}^T \mathbf{v}) - \dot{b}(\boldsymbol{\sigma}_2, \mathbf{v}) \quad \text{for all } \mathbf{v} \in V_0, \end{cases}$$

where

$$\ell_2(\boldsymbol{\alpha}) = -\dot{a}(\boldsymbol{\sigma}_2, \boldsymbol{\alpha}) - \dot{b}(\boldsymbol{\alpha}, \mathbf{u}_2).$$

Proof. The proof has the same spirit as the one of Theorem 4.5, with the consideration of the function

$$\mathbf{w}_2^t = M_t^T(\mathbf{u}_2^t - \mathbf{u}_2^*), \tag{4.5}$$

where $(\boldsymbol{\sigma}_2^*, \mathbf{u}_2^*)$ is the solution to the following value-boundary problem:

$$\begin{cases} \operatorname{\mathbf{div}} \boldsymbol{\sigma}_2^* = \mathbf{0} & \text{in } \Omega, \\ \boldsymbol{\varepsilon}_2^* = \dfrac{1+\nu}{E} \boldsymbol{\sigma}_2^* - \dfrac{\nu}{E} (\operatorname{tr} \boldsymbol{\sigma}_2^*) \mathbf{I} & \text{in } \Omega, \\ \mathbf{n}_\Gamma \cdot \boldsymbol{\sigma}_2^* \mathbf{n}_\Gamma = \mathbf{T}_g \cdot \mathbf{n}_\Gamma & \text{on } \Gamma, \\ \mathbf{u}_2^* \cdot \boldsymbol{\tau}_\Gamma = u_\tau & \text{on } \Gamma, \\ \mathbf{u}_2^* = \mathbf{0} & \text{on } \gamma, \end{cases}$$

which proves the theorem. □

4.3 Shape derivatives

Shape derivative of the energy gap functional

Before proceeding further with the calculus of the shape gradient $\mathcal{J}_{KV}$, we present the following lemma.

Lemma 4.8 ([9]). *One has*

$$[\operatorname{div}(\boldsymbol{\sigma} \nabla \mathbf{u})]^T = \frac{1}{2} \nabla [\operatorname{tr}(\boldsymbol{\sigma} \nabla \mathbf{u})].$$

We present the main result of this section.

Theorem 4.9. *The mapping* $t \mapsto \mathcal{J}_{KV}(\Omega_t)$ *is* C^1 *in a neighborhood of* 0 *and its derivative at* 0 *is given by*

$$\mathcal{J}'_{KV}(\Omega, \mathbf{h}) = \frac{1}{2} \int_\gamma G_{KV} (\mathbf{h} \cdot \mathbf{n})\, ds,$$

with

$$\begin{aligned} G_{KV} = (\boldsymbol{\sigma}_2 : \boldsymbol{\varepsilon}_2) - (\boldsymbol{\sigma}_1 : \boldsymbol{\varepsilon}_1) - 2((\mathbf{n} \cdot \boldsymbol{\sigma}_2 \nabla \mathbf{u}_2 \mathbf{n}) - (\mathbf{n} \cdot \boldsymbol{\sigma}_1 \nabla \mathbf{u}_1 \mathbf{n})) \\ - 2(\nabla((\mathbf{n} \cdot \boldsymbol{\sigma}_2 \mathbf{n})(\mathbf{u}_2 \cdot \boldsymbol{\tau})) \cdot \boldsymbol{\tau} - \nabla((\mathbf{n} \cdot \boldsymbol{\sigma}_1 \mathbf{n})(\mathbf{u}_1 \cdot \boldsymbol{\tau})) \cdot \boldsymbol{\tau}). \end{aligned}$$

Proof. To carry out the shape derivative of the cost functional $\mathcal{J}_{KV}$ given by (3.3), one has to calculate its Eulerian derivative (4.2) by transforming the cost functional (4.1) to the fixed domain Ω:

$$\mathcal{J}_{KV}(\Omega_t, \mathbf{h}) = a^t(\boldsymbol{\sigma}_2^t - \boldsymbol{\sigma}_1^t, \boldsymbol{\sigma}_2^t - \boldsymbol{\sigma}_1^t).$$

By Theorem 4.5 and Theorem 4.7, one gets

$$\mathcal{J}'_{KV}(\Omega, \mathbf{h}) = a(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1, \dot{\boldsymbol{\sigma}}_2 - \dot{\boldsymbol{\sigma}}_1) + \frac{1}{2} \dot{a}(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1).$$

From the variational forms verified by the state solutions $(\boldsymbol{\sigma}_1, \mathbf{u}_1)$ and $(\boldsymbol{\sigma}_2, \mathbf{u}_2)$ and by the material derivative $(\dot{\boldsymbol{\sigma}}_1, \dot{\mathbf{u}}_1)$, one has

$$a(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1, \dot{\boldsymbol{\sigma}}_1) = -b(\dot{\boldsymbol{\sigma}}_1, \mathbf{u}_2 - \mathbf{u}_1) = b(\boldsymbol{\sigma}_1, \nabla \mathbf{h}^T(\mathbf{u}_2 - \mathbf{u}_1)) + \dot{b}(\boldsymbol{\sigma}_1, \mathbf{u}_2 - \mathbf{u}_1).$$

From the variational form verified by the material derivative $(\dot{\boldsymbol{\sigma}}_2, \dot{\mathbf{u}}_2)$, one has

$$a(\dot{\boldsymbol{\sigma}}_2, \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1) = -b(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1, \dot{\mathbf{u}}_2) - \dot{a}(\boldsymbol{\sigma}_2, \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1) - \dot{b}(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1, \mathbf{u}_2).$$

One summarizes the above equalities to obtain the following form of the Eulerian derivative $\mathcal{J}'_{KV}(\Omega, \mathbf{h})$:

$$\begin{aligned}\mathcal{J}'_{KV}(\Omega, \mathbf{h}) = &-b(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1, \dot{\mathbf{u}}_2) - \dot{a}(\boldsymbol{\sigma}_2, \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1) - \dot{b}(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1, \mathbf{u}_2) - b(\boldsymbol{\sigma}_1, \nabla \mathbf{h}^T(\mathbf{u}_2 - \mathbf{u}_1))\\ &- \dot{b}(\boldsymbol{\sigma}_1, \mathbf{u}_2 - \mathbf{u}_1) + \frac{1}{2}\dot{a}(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1).\end{aligned}$$

By simplifying the last form, one gets

$$\begin{aligned}\mathcal{J}'_{KV}(\Omega, \mathbf{h}) = &-b(\boldsymbol{\sigma}_2, \dot{\mathbf{u}}_2) + b(\boldsymbol{\sigma}_1, \dot{\mathbf{u}}_2) - b(\boldsymbol{\sigma}_1, \nabla \mathbf{h}^T \mathbf{u}_2) + b(\boldsymbol{\sigma}_1, \nabla \mathbf{h}^T \mathbf{u}_1)\\ &- \dot{b}(\boldsymbol{\sigma}_2, \mathbf{u}_2) + \dot{b}(\boldsymbol{\sigma}_1, \mathbf{u}_1) - \frac{1}{2}\dot{a}(\boldsymbol{\sigma}_2, \boldsymbol{\sigma}_2) + \frac{1}{2}\dot{a}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_1).\end{aligned}$$

In other words, one gets

$$\begin{aligned}\mathcal{J}'_{KV}(\Omega, \mathbf{h}) &= \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_2 \nabla \dot{\mathbf{u}}_2)\, dx - \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_1 \nabla \dot{\mathbf{u}}_2)\, dx + \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_1 \nabla(\nabla \mathbf{h}^T \mathbf{u}_2))\, dx\\ &\quad - \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_1 \nabla(\nabla \mathbf{h}^T \mathbf{u}_1))\, dx - \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_2 (\nabla \mathbf{h}^T \nabla \mathbf{u}_2))\, dx + \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_1 (\nabla \mathbf{h}^T \nabla \mathbf{u}_1))\, dx\\ &\quad + \frac{1}{2}\int_\Omega \operatorname{div} \mathbf{h} (\boldsymbol{\sigma}_2 : \boldsymbol{\varepsilon}_2)\, dx - \frac{1}{2}\int_\Omega \operatorname{div} \mathbf{h} (\boldsymbol{\sigma}_1 : \boldsymbol{\varepsilon}_1)\, dx\\ &= \sum_{i=1}^{8} \mathfrak{I}_i.\end{aligned}$$

One treats the eight integrals by using Green's formula and the boundary conditions verified by the solutions:

$$\mathfrak{I}_1 = \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_2 \nabla \dot{\mathbf{u}}_2)\, dx = \int_\Gamma (\mathbf{T}_g \cdot \mathbf{n}_\Gamma)(\dot{\mathbf{u}}_2 \cdot \mathbf{n}_\Gamma)\, ds + \int_\gamma (\mathbf{n} \cdot \boldsymbol{\sigma}_2 \mathbf{n})(\nabla \mathbf{h}^T \mathbf{u}_2 \cdot \mathbf{n})\, ds.$$

With the same arguments, one proves

$$\mathfrak{I}_2 = -\int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_1 \nabla \dot{\mathbf{u}}_2)\, dx = -\int_\Gamma (\mathbf{T}_g \cdot \mathbf{n}_\Gamma)(\dot{\mathbf{u}}_2 \cdot \mathbf{n}_\Gamma)\, ds - \int_\gamma (\mathbf{n} \cdot \boldsymbol{\sigma}_1 \mathbf{n})(\nabla \mathbf{h}^T \mathbf{u}_2 \cdot \mathbf{n})\, ds.$$

With the fact that $\mathbf{h}_{|\Gamma} = \mathbf{0}$, the third term is as follows:

$$\mathfrak{I}_3 = \int_\Omega \operatorname{tr}[\boldsymbol{\sigma}_1 \nabla(\nabla \mathbf{h}^T \mathbf{u}_2)]\, dx = \int_\gamma (\mathbf{n} \cdot \boldsymbol{\sigma}_1 \mathbf{n})(\nabla \mathbf{h}^T \mathbf{u}_2 \cdot \mathbf{n})\, ds.$$

Similarly, one gets

$$\mathfrak{I}_4 = -\int_\Omega \operatorname{tr}[\boldsymbol{\sigma}_1 \nabla(\nabla \mathbf{h}^T \mathbf{u}_1)]\, dx = -\int_\gamma (\mathbf{n} \cdot \boldsymbol{\sigma}_1 \mathbf{n})(\nabla \mathbf{h}^T \mathbf{u}_1 \cdot \mathbf{n})\, ds.$$

Using the fact that $\mathbf{h}_{|\Gamma} = \mathbf{0}$ and also $\mathbf{h} \cdot \boldsymbol{\tau}_{|\gamma} = 0$, one has

$$\mathfrak{I}_5 = -\int_\Omega \operatorname{tr}[\boldsymbol{\sigma}_2 (\nabla \mathbf{u}_2 \nabla \mathbf{h})]\, dx = -\int_\gamma (\mathbf{n} \cdot \boldsymbol{\sigma}_2 \nabla \mathbf{u}_2 \mathbf{n}) \mathbf{h} \cdot \mathbf{n}\, ds + \int_\Omega \operatorname{div}(\boldsymbol{\sigma}_2 \nabla \mathbf{u}_2) \cdot \mathbf{h}\, dx.$$

Similarly to the last one, one obtains

$$\mathcal{I}_6 = \int_\Omega \operatorname{tr}[\boldsymbol{\sigma}_1(\nabla\mathbf{u}_1\nabla\mathbf{h})]\,dx = \int_\gamma (\mathbf{n}\cdot\boldsymbol{\sigma}_1\nabla\mathbf{u}_1\mathbf{n})\mathbf{h}\cdot\mathbf{n}\,ds - \int_\Omega \operatorname{div}(\boldsymbol{\sigma}_1\nabla\mathbf{u}_1)\cdot\mathbf{h}\,dx.$$

By Green's formula, one gets the last terms

$$\mathcal{I}_7 = \frac{1}{2}\int_\Omega \operatorname{div}\mathbf{h}(\boldsymbol{\sigma}_2 : \boldsymbol{\varepsilon}_2)\,dx = \frac{1}{2}\int_\gamma (\boldsymbol{\sigma}_2 : \boldsymbol{\varepsilon}_2)\mathbf{h}\cdot\mathbf{n}\,ds - \frac{1}{2}\int_\Omega \nabla[\operatorname{tr}(\boldsymbol{\sigma}_2\nabla\mathbf{u}_2)]\cdot\mathbf{h}\,dx$$

and

$$\mathcal{I}_8 = -\frac{1}{2}\int_\Omega \operatorname{div}\mathbf{h}(\boldsymbol{\sigma}_1 : \boldsymbol{\varepsilon}_1)\,dx = -\frac{1}{2}\int_\gamma (\boldsymbol{\sigma}_1 : \boldsymbol{\varepsilon}_1)\mathbf{h}\cdot\mathbf{n}\,ds + \frac{1}{2}\int_\Omega \nabla[\operatorname{tr}(\boldsymbol{\sigma}_1\nabla\mathbf{u}_1)]\cdot\mathbf{h}\,dx.$$

By summarizing the above terms and using Lemma 4.8, one gets the following form of the cost functional $\mathcal{J}'_{\mathrm{KV}}$:

$$\begin{aligned}\mathcal{J}'_{\mathrm{KV}}(\Omega,\mathbf{h}) &= \frac{1}{2}\int_\gamma (\boldsymbol{\sigma}_2 : \boldsymbol{\varepsilon}_2)\mathbf{h}\cdot\mathbf{n}\,ds - \frac{1}{2}\int_\gamma (\boldsymbol{\sigma}_1 : \boldsymbol{\varepsilon}_1)\mathbf{h}\cdot\mathbf{n}\,ds - \int_\gamma (\mathbf{n}\cdot\boldsymbol{\sigma}_2\nabla\mathbf{u}_2\mathbf{n})\mathbf{h}\cdot\mathbf{n}\,ds \\ &\quad + \int_\gamma (\mathbf{n}\cdot\boldsymbol{\sigma}_1\nabla\mathbf{u}_1\mathbf{n})\mathbf{h}\cdot\mathbf{n}\,ds + \int_\gamma (\mathbf{n}\cdot\boldsymbol{\sigma}_2\mathbf{n})(\nabla\mathbf{h}^T\mathbf{u}_2\cdot\mathbf{n})\,ds - \int_\gamma (\mathbf{n}\cdot\boldsymbol{\sigma}_1\mathbf{n})(\nabla\mathbf{h}^T\mathbf{u}_1\cdot\mathbf{n})\,ds.\end{aligned}$$

To write the Eulerian derivative of the energy gap-misfit functional $\mathcal{J}_{\mathrm{KV}}$ in Hadamard's structure, we reformulate the last two terms of the above formula by using integration by parts as follows:

$$\int_\gamma (\mathbf{n}\cdot\boldsymbol{\sigma}_2\mathbf{n})(\nabla\mathbf{h}^T\mathbf{u}_2\cdot\mathbf{n})\,ds = -\int_\gamma \nabla((\mathbf{n}\cdot\boldsymbol{\sigma}_2\mathbf{n})(\mathbf{u}_2\cdot\boldsymbol{\tau}))\cdot\boldsymbol{\tau}\mathbf{h}\cdot\mathbf{n}\,ds.$$

Finally, one gets

$$\begin{aligned}\mathcal{J}'_{\mathrm{KV}}(\Omega,\mathbf{h}) &= \frac{1}{2}\int_\gamma (\boldsymbol{\sigma}_2 : \boldsymbol{\varepsilon}_2)\mathbf{h}\cdot\mathbf{n}\,ds - \frac{1}{2}\int_\gamma (\boldsymbol{\sigma}_1 : \boldsymbol{\varepsilon}_1)\mathbf{h}\cdot\mathbf{n}\,ds - \int_\gamma (\mathbf{n}\cdot\boldsymbol{\sigma}_2\nabla\mathbf{u}_2\mathbf{n})\mathbf{h}\cdot\mathbf{n}\,ds + \int_\gamma (\mathbf{n}\cdot\boldsymbol{\sigma}_1\nabla\mathbf{u}_1\mathbf{n})\mathbf{h}\cdot\mathbf{n}\,ds \\ &\quad - \int_\gamma \nabla((\mathbf{n}\cdot\boldsymbol{\sigma}_2\mathbf{n})(\mathbf{u}_2\cdot\boldsymbol{\tau}))\cdot\boldsymbol{\tau}\mathbf{h}\cdot\mathbf{n}\,ds + \int_\gamma \nabla((\mathbf{n}\cdot\boldsymbol{\sigma}_1\mathbf{n})(\mathbf{u}_1\cdot\boldsymbol{\tau}))\cdot\boldsymbol{\tau}\mathbf{h}\cdot\mathbf{n}\,ds.\end{aligned}$$ □

Remark 4.10. The proof of Theorem 4.9 reveals that the essential ingredients for establishing the expression for the shape derivative of the cost functional $\mathcal{J}_{\mathrm{KV}}$ are the properties of Theorem 4.5 and Theorem 4.7 which are based on the implicit function theorem applied to the proposed functions (4.3) and (4.5). This point is one of the difficulties stemming from the imposed Navier boundary conditions.

Remark 4.11. In our case, the rigorous analysis of the calculation of the shape derivative is a non-trial task, and for a clear idea one refers to the work by Jaïem [12] for the case of cavities under Neumann's conditions.

Shape derivative of the L^2-gap functional

Theorem 4.12. *The mapping* $t \mapsto \mathcal{J}_{L^2}(\Omega_t)$ *is* C^1 *in a neighborhood of* 0 *and its derivative at* 0 *is given by*

$$\mathcal{J}'_{L^2}(\Omega,\mathbf{h}) = \frac{1}{2}\int_\gamma G_{L^2}(\mathbf{h}\cdot\mathbf{n})\,ds,$$

with

$$G_{L^2} = |\mathbf{u}_2 - \mathbf{u}_1|^2 + \nabla((\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}_2\mathbf{n})(\mathbf{u}_2\cdot\boldsymbol{\tau}))\cdot\boldsymbol{\tau} - \nabla((\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}_1\mathbf{n})(\mathbf{u}_1\cdot\boldsymbol{\tau}))\cdot\boldsymbol{\tau}.$$

Proof. According to [21], the shape derivative of $\mathcal{J}_{L^2}$ is given by

$$\mathcal{J}'_{L^2}(\Omega,\mathbf{h}) = \int_\Omega 2(\mathbf{u}_2 - \mathbf{u}_1)(\mathbf{u}'_2 - \mathbf{u}'_1)\,dx + \int_{\partial\Omega} |\mathbf{u}_2 - \mathbf{u}_1|^2\mathbf{h}\cdot\mathbf{n}\,ds, \tag{4.6}$$

where $\mathbf{u}_1'$ and $\mathbf{u}_2'$ are the so-called shape derivatives of $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively, and are defined by the boundary-initial value problems (for more details, we refer to [21])

$$
\begin{cases}
\operatorname{\mathbf{div}} \boldsymbol{\sigma}_1' = \mathbf{0} & \text{in } \Omega,\\
\boldsymbol{\varepsilon}_1' = \dfrac{1+\nu}{E}\boldsymbol{\sigma}_1' - \dfrac{\nu}{E}(\operatorname{tr}\boldsymbol{\sigma}_1')\mathbf{I} & \text{in } \Omega,\\
\boldsymbol{\sigma}_1'\mathbf{n}_\Gamma = \mathbf{0} & \text{on } \Gamma,\\
\boldsymbol{\tau}\cdot\boldsymbol{\sigma}_1'\mathbf{n} = 0 & \text{on } \gamma,\\
\mathbf{u}_1'\cdot\mathbf{n} = \nabla\mathbf{h}^T\mathbf{u}_1\cdot\mathbf{n} & \text{on } \gamma,
\end{cases}
$$

$$
\begin{cases}
\operatorname{\mathbf{div}} \boldsymbol{\sigma}_2' = \mathbf{0} & \text{in } \Omega,\\
\boldsymbol{\varepsilon}_2' = \dfrac{1+\nu}{E}\boldsymbol{\sigma}_2' - \dfrac{\nu}{E}(\operatorname{tr}\boldsymbol{\sigma}_2')\mathbf{I} & \text{in } \Omega,\\
\mathbf{n}_\Gamma\cdot\boldsymbol{\sigma}_2'\mathbf{n}_\Gamma = 0 & \text{on } \Gamma,\\
\mathbf{u}_2'\cdot\boldsymbol{\tau}_\Gamma = 0 & \text{on } \Gamma,\\
\boldsymbol{\tau}\cdot\boldsymbol{\sigma}_2'\mathbf{n} = 0 & \text{on } \gamma,\\
\mathbf{u}_2'\cdot\mathbf{n} = \nabla\mathbf{h}^T\mathbf{u}_2\cdot\mathbf{n} & \text{on } \gamma.
\end{cases}
\tag{4.7}
$$

We introduce the adjoint states $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$:

$$
\begin{cases}
-\operatorname{\mathbf{div}} \hat{\boldsymbol{\sigma}}_1 = 2(\mathbf{u}_2 - \mathbf{u}_1) & \text{in } \Omega,\\
\hat{\boldsymbol{\varepsilon}}_1 = \dfrac{1+\nu}{E}\hat{\boldsymbol{\sigma}}_1 - \dfrac{\nu}{E}(\operatorname{tr}\hat{\boldsymbol{\sigma}}_1)\mathbf{I} & \text{in } \Omega,\\
\hat{\boldsymbol{\sigma}}_1\mathbf{n}_\Gamma = \mathbf{0} & \text{on } \Gamma,\\
\boldsymbol{\tau}\cdot\hat{\boldsymbol{\sigma}}_1\mathbf{n} = 0 & \text{on } \gamma,\\
\hat{\mathbf{u}}_1\cdot\mathbf{n} = 0 & \text{on } \gamma,
\end{cases}
$$

$$
\begin{cases}
-\operatorname{\mathbf{div}} \hat{\boldsymbol{\sigma}}_2 = 2(\mathbf{u}_2 - \mathbf{u}_1) & \text{in } \Omega,\\
\hat{\boldsymbol{\varepsilon}}_2 = \dfrac{1+\nu}{E}\hat{\boldsymbol{\sigma}}_2 - \dfrac{\nu}{E}(\operatorname{tr}\hat{\boldsymbol{\sigma}}_2)\mathbf{I} & \text{in } \Omega,\\
\mathbf{n}_\Gamma\cdot\hat{\boldsymbol{\sigma}}_2\mathbf{n}_\Gamma = 0 & \text{on } \Gamma,\\
\hat{\mathbf{u}}_2\cdot\boldsymbol{\tau}_\Gamma = 0 & \text{on } \Gamma,\\
\boldsymbol{\tau}\cdot\hat{\boldsymbol{\sigma}}_2\mathbf{n} = 0 & \text{on } \gamma,\\
\hat{\mathbf{u}}_2\cdot\mathbf{n} = 0 & \text{on } \gamma.
\end{cases}
\tag{4.8}
$$

Using the value boundary problems of the shape derivative (4.7), the adjoint state (4.8) and the reciprocity identity, we are able to compute

$$
\begin{aligned}
\int_\Omega 2(\mathbf{u}_2-\mathbf{u}_1)\mathbf{u}_2'\,dx &= \int_\Omega -\operatorname{\mathbf{div}}\hat{\boldsymbol{\sigma}}_2\mathbf{u}_2'\,dx\\
&= \int_\Omega -\operatorname{\mathbf{div}}\boldsymbol{\sigma}_2'\hat{\mathbf{u}}_2\,dx - \int_{\partial\Omega}(\hat{\boldsymbol{\sigma}}_2\mathbf{n}\mathbf{u}_2' - \hat{\mathbf{u}}_2\boldsymbol{\sigma}_2'\mathbf{n})\,ds\\
&= -\int_\gamma (\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}_2\mathbf{n})(\nabla\mathbf{h}^T\mathbf{u}_2\cdot\mathbf{n})\,ds.
\end{aligned}
$$

To write the Eulerian derivative of the L^2-gap functional $\mathcal{J}_{L^2}$ in Hadamard's structure, we reformulate the last term of the above formula by using integration by parts as follows:

$$
\int_\gamma (\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}_2\mathbf{n})(\nabla\mathbf{h}^T\mathbf{u}_2\cdot\mathbf{n})\,ds = -\int_\gamma \nabla((\mathbf{n}\cdot\hat{\boldsymbol{\sigma}}_2\mathbf{n})(\mathbf{u}_2\cdot\boldsymbol{\tau}))\cdot\boldsymbol{\tau}\mathbf{h}\cdot\mathbf{n}\,ds.
$$

By the same steps, one gets for the second term of (4.6),

$$\int_{\Omega} 2(\mathbf{u}_2 - \mathbf{u}_1)\mathbf{u}_1' \,\mathrm{d}x = \int_{\gamma} \nabla((\mathbf{n} \cdot \hat{\boldsymbol{\sigma}}_1 \mathbf{n})(\mathbf{u}_1 \cdot \boldsymbol{\tau})) \cdot \boldsymbol{\tau}\mathbf{h} \cdot \mathbf{n} \,\mathrm{d}s.$$

By summarizing the above terms, one gets the final expression of the shape derivative of the L^2-gap cost functional. □

Remark 4.13. It is demonstrated that, for a constitutive law error functional, the first-order of the shape derivative can be obtained without recourse to the adjoint-based form, while the shape derivative of the L^2-gap functional is numerically expensive to evaluate such that one needs to solve four PDES, namely two state equations and two adjoint equations.

5 Conclusion

A uniqueness result for the void identification problem from a single pair of a Cauchy data is obtained. A shape sensitivity analysis for identification of voids under Navier's boundary conditions is formulated for two cost functionals. The existence of the shape derivative of a constitutive law error functional is proven via the material derivatives of the forward solutions, and its expression is given such that it is expressed in terms of a boundary integral. This information can be combined with a level set technique to construct an efficient numerical iterative scheme to solve the shape problem. Also, the topological derivative approach is a relevant method to treat the considered problem in view of its advantages. Both works will be pursued in the future.

Acknowledgment: The author wishes to thank Pr. Amel Ben Abda for numerous discussions on the subject of this article.

Funding: This work was supported by the Tunisian Ministry of Higher Education and Scientific Research.

References

[1] G. Allaire, F. de Gournay, F. Jouve and A.-M. Toader, Structural optimization using topological and shape sensitivity via a level set method, *Control Cybernet.* **34** (2005), no. 1, 59–80.
[2] H. B. Ameur, M. Burger and B. Hackl, Cavity identification in linear elasticity and thermoelasticity, *Math. Methods Appl. Sci.* **30** (2007), no. 6, 625–647.
[3] A. Ben Abda, E. Jaïem, S. Khalfallah and B. Méjri, Voids identification from partially overspecified boundary data, *New Trends Math. Sci.*, to appear.
[4] A. Ben Abda, E. Jaïem, S. Khalfallah and A. Zine, An energy gap functional: Cavity identification in linear elasticity, *J. Inverse Ill-Posed Probl.* **25** (2017), no. 5, 573–595.
[5] M. Bonnet, Topological sensitivity of energy cost functional for wave-based defect identification, *C. R. Mecanique* **338** (2010), 377–389.
[6] F. Bouchon, G. H. Peichl, M. Sayeh and R. Touzani, A free boundary problem for the Stokes equations, *ESAIM Control Optim. Calc. Var.* **23** (2017), no. 1, 195–215.
[7] F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers, *Rev. Franc. Automat. Inform. Rech. Operat. R* **8** (1974), no. R-2, 129–151.
[8] F. Caubet, M. Dambrine, D. Kateb and C. Z. Timimoun, A Kohn–Vogelius formulation to detect an obstacle immersed in a fluid, preprint (2012), https://hal.archives-ouvertes.fr/hal-00678036.
[9] P. Destuynder and M. Djaoua, Sur une interprétation mathématique de l'intégrale de Rice en théorie de la rupture fragile, *Math. Methods Appl. Sci.* **3** (1981), no. 1, 70–87.
[10] S. Garreau, P. Guillaume and M. Masmoudi, The topological asymptotic for PDE systems: The elasticity case, *SIAM J. Control Optim.* **39** (2001), no. 6, 1756–1778.
[11] S.-C. Hsieh and T. Mura, Nondestructive cavity identification in structures, *Internat. J. Solids Structures* **30** (1993), no. 12, 1579–1587.

[12] E. Jaïem, Shape derivative of an energy error functional for voids detection from sub-Cauchy data, *Electron. J. Differential Equations* **2016** (2016), Paper No. 265.

[13] R. Kohn and M. Vogelius, Determining conductivity by boundary measurements, *Comm. Pure Appl. Math.* **37** (1984), no. 3, 289–298.

[14] R. V. Kohn and M. Vogelius, Relaxation of a variational method for impedance computed tomography, *Comm. Pure Appl. Math.* **40** (1987), no. 6, 745–777.

[15] S. Kubo, Inverse problems related to the mechanics and fracture of solids and structures, *JSME Int. J.* **31** (1988), no. 2, 157–166.

[16] H. Meftahi and J.-P. Zolésio, Sensitivity analysis for some inverse problems in linear elasticity via minimax differentiability, *Appl. Math. Model.* **39** (2015), no. 5–6, 1554–1576.

[17] N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff International, Leiden, 1977.

[18] J. Nečas and I. Hlaváček, *Mathematical Theory of Elastic and Elasto-plastic Bodies: An Introduction*, Elsevier Scientific, Amsterdam, 1980.

[19] A. A. Novotny, R. A. Feijóo, E. Taroco and C. Padra, Topological sensitivity analysis for three-dimensional linear elasticity problem, *Comput. Methods Appl. Mech. Engrg.* **196** (2007), no. 41–44, 4354–4364.

[20] J. Sokoł owski and A. Żochowski, On the topological derivative in shape optimization, *SIAM J. Control Optim.* **37** (1999), no. 4, 1251–1272.

[21] J. Sokoł owski and J.-P. Zolésio, *Introduction to Shape Optimization. Shape Sensitivity Analysis*, Springer Ser. Comput. Math. 16, Springer, Berlin, 1992.