Finite Volume Schemes for multi-phase flow simulation on near well grids

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Objective

Study of Finite Volume Schemes on
3D near well multi-phase flow simulations.

Outline

- Applications and Difficulties
- 3D Near Well Grids
- Finite Volume Schemes
- Numerical Experiments
- Conclusion

Multi-phase Flow Simulation on Near Well Grids

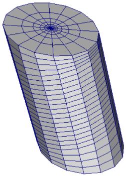
Applications

- Reservoir Simulation
- CO_2 geological storage

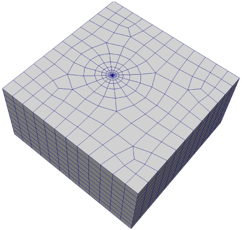
Difficulties

- Singular Pressure Distribution
- Well-Radius $<<$ Reservoir Dimension
- Deviated Well
- Anisotropy

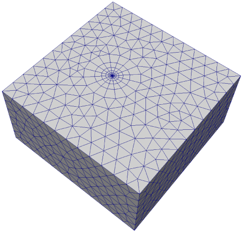
3D Near-well model



Exponentially refined radial mesh



Unstructured mesh with only hexahedra



Hybrid mesh with hexahedra, tetrahedra and pyramids

Discretization on complex 3D general meshes

$\Longrightarrow$ MultiPoint Flux Approximation (MPFA) Finite Volume Schemes

Finite Volume Scheme
Model Problem

Model Equation

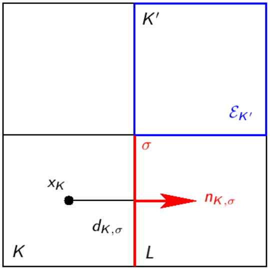
Find u (*potential*) in $H_0^1(\Omega)$ | $\begin{cases} -\nabla \cdot (\Lambda \nabla u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$

Ω	:	bounded polygonal domain of $\mathbb{R}^d$
Λ	:	symmetric positive definite tensor field
f	:	function of $L^2(\Omega)$

Finite Volume Scheme
Flux Formulation

$\mathcal{T}_h$: set of cells K

V_h : space of piecewise constant functions on $\mathcal{T}_h$



- $F_{K,\sigma}(u_h) \approx \int_\sigma \Lambda \nabla u \cdot n_{K,\sigma}$ linearly
- Conservativity : $F_{K,\sigma}(u_h) + F_{L,\sigma}(u_h) = 0$, $\sigma = K|L$

Find $u_h \in V_h \mid -\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u_h) = \int_K f \quad \forall K \in \mathcal{T}_h$

Finite Volume Scheme
Discrete Variational Formulation

For all $u_h,\ v_h \in V_h$, let

$$a_h(u_h, v_h) = \sum_{\sigma=\mathcal{E}_K\cap\mathcal{E}_L\in\mathcal{E}_h} F_{K,\sigma}(u_h)(v_L - v_K) - \sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_h^b} F_{K,\sigma}(u_h)v_K$$

The finite volume scheme is equivalent to

$$\text{Find } u_h \in V_h \mid a_h(u_h, v_h) = \int_\Omega f v_h \quad \forall v_h \in V_h$$

Sample of MPFA Finite Volume Schemes

O scheme *[Aavatsmark et al., 1996, Edwards and Rogers, 1998]*

L scheme *[Aavatsmark, 2007]*

G scheme inspired by the L scheme *[Agélas et al., 2010a]*

$\longrightarrow$ Subcell gradients satisfying continuity conditions

$\longrightarrow$ Subfluxes $F^G_{L,\sigma}$

$\longrightarrow$ Convex linear combination $F_{K,\sigma} = \sum \theta^G_\sigma \, F^G_{K,\sigma}$

$\longrightarrow$ Choose the θ^G_σ to enhance the coercivity and remove singularities

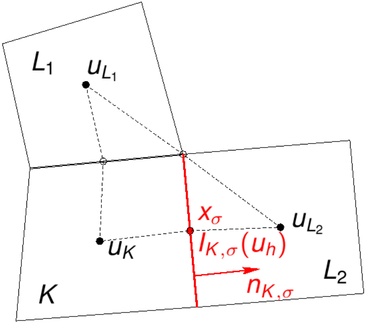
The GradCell scheme uses a discrete variational formulation

Non symmetric discrete variational formulation based on two cellwise constant gradients and residuals for stabilization

$$a_h(u_h, v_h) = \sum_{K\in\mathcal{T}_h} m_K \Lambda_K (\nabla_h u_h)_K \cdot (\widetilde{\nabla}_h v_h)_K + \sum_{K\in\mathcal{T}_h} \sum_{\sigma\in\mathcal{E}_K} \frac{m_\sigma}{d_{K,\sigma}} R_{K,\sigma}(u_h) R_{K,\sigma}(v_h)$$

$$(\nabla_h v_h)_K = \frac{1}{m_K} \sum_{\sigma\in\mathcal{E}_K} m_\sigma (I_{K,\sigma}(v_h) - v_K) n_{K,\sigma}$$

$$(\widetilde{\nabla}_h v_h)_K = \frac{1}{m_K} \sum_{\sigma\in\mathcal{E}_K} m_\sigma (\gamma_\sigma(v_h) - v_K) n_{K,\sigma}$$



The GradCell scheme has a compact stencil

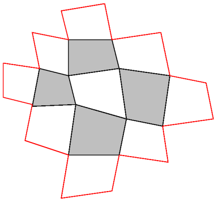
Fluxes are derived from the bilinear form.

$$a_h(u_h, v_h) = \sum_{\sigma=\mathcal{E}_K\cap\mathcal{E}_L\in\mathcal{E}_h} F_{K,\sigma}(u_h)(v_L - v_K) - \sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_h^b} F_{K,\sigma}(u_h)v_K$$

Fluxes $F_{K,\sigma}(u_h)$ only between cells sharing a face

The stencil is compact: neighbours of the neighbours

For topologically cartesian grids :
13 cells in 2D, 21 cells in 3D



[Agélas et al., 2010b]

Outcome on Symmetry and Sparsity properties

Fact

- Previous schemes : compact but non symmetric $\Rightarrow$ conditionnal coercivity
- If GradCell symmetric $\Rightarrow$ large stencil (81 cells in 3D)

Difficult to combine both properties

Wish

- Symmetric unconditionally coercive scheme
- Sparse stencil: 9 points in 2D and 27 points in 3D on topologically Cartesian meshes

SUSHI
combining smart ideas...

Scheme **U**sing **S**tabilization and **H**armonic **I**nterfaces

combines...

- O scheme ideas : subcell gradients $(\nabla_h u)^s_K$ and subfaces unknowns u^s_σ
- A symmetric variational bilinear form for coercivity
- Weak and consistent subcell gradient for convergence
- Two point harmonic interpolation at the faces

SUSHI
Nice harmonic interpolation formula

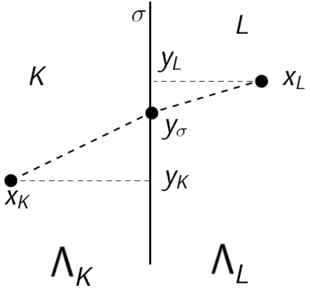
Find a point y_σ and a coefficient α with...

- a linear two point interpolation...
- ... exact on piecewise linear functions,
- normal flux and potential continuity.

Harmonic point y_σ

Harmonic interpolation
$u(y_\sigma) = \alpha\ u(x_K) + (1-\alpha)\ u(x_L)$

[Agélas et al., 2009]

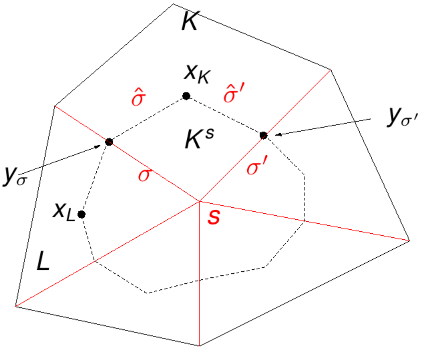


How SUSHI uses the interpolation formula ?

At each face σ, choose the harmonic point y_σ

Subcell K_s around a vertex s

$K_s = (x_K, y_\sigma, s, y_{\sigma'}, x_K)$



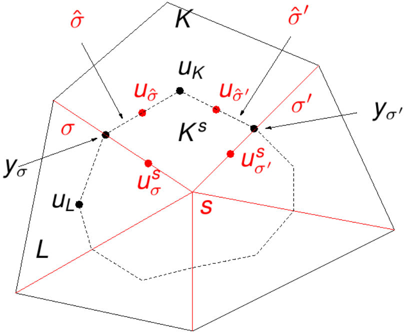
SUSHI
Discrete subcell gradient

$$(\nabla_h u)_{K_s} = \frac{1}{m_{K_s}}\Big(m_\sigma^s(u_\sigma^s - u_K)n_{K,\sigma}$$

$$+ m_{\sigma'}^s(u_{\sigma'}^s - u_K)n_{K,\sigma'}$$

$$+ m_{\hat\sigma}(u_{\hat\sigma} - u_K)n_{K_s,\hat\sigma}$$

$$+ m_{\hat\sigma'}(u_{\hat\sigma'} - u_K)n_{K_s,\hat\sigma'}\Big)$$



SUSHI
A symmetric formulation

Symmetric discrete variational formulation

$$a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \sum_{s \in \mathcal{V}_K} \left(m_{K_s} \Lambda_K (\nabla_h u_h)_K^s \cdot (\nabla_h v_h)_K^s + \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} \frac{m_{K_s}}{(d_{K,\sigma})^2} R_{K,\sigma}^s(u_h) R_{K,\sigma}^s(v_h) \right)$$

Subfluxes

$$a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \sum_{s \in \mathcal{V}_K} \sum_{\sigma \in \mathcal{E}_s \cap \mathcal{E}_K} F_{K,\sigma}^s(u_h)(v_\sigma^s - v_K)$$

$$F_{K,\sigma}^s(u_h) + F_{L,\sigma}^s(u_h) = 0$$

Schemes comparison on single-phase analytical solution

Problem studied

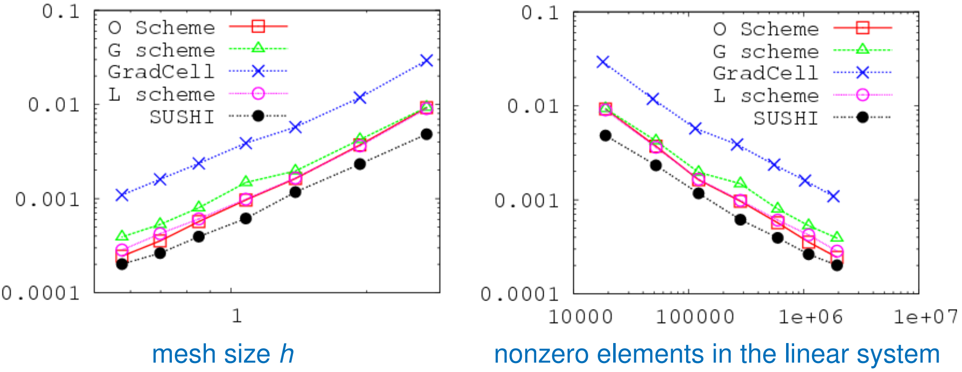
- Single phase flow
- Anisotropy of the tensor permeability Λ
- Slanted well

$\Rightarrow$ Analytical solution [Aavatsmark and Klausen, 2003]

Numerical study

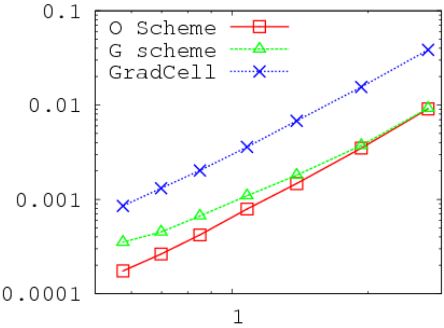
- O, L, G, GradCell and SUSHI schemes
- Hexahedra and Hybrid mesh families
- $\Lambda = \text{diag}(1, 1, \frac{1}{20})$

Hexahedral mesh family
l^2 pressure error

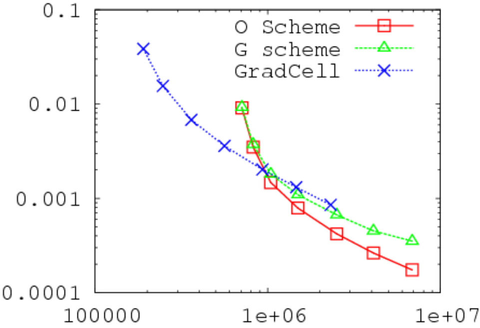


- O and L schemes have the same behavior

Hybrid mesh family
l^2 pressure error



mesh size h



nonzero elements in the linear system

- GradCell stencil ≈ 4 times smaller than O scheme
- L scheme fails but not the more flexible G scheme

Two-phase flow (w-g) near well simulation

Injection of gaseous CO_2 miscible in a reservoir full of water

- Two-component

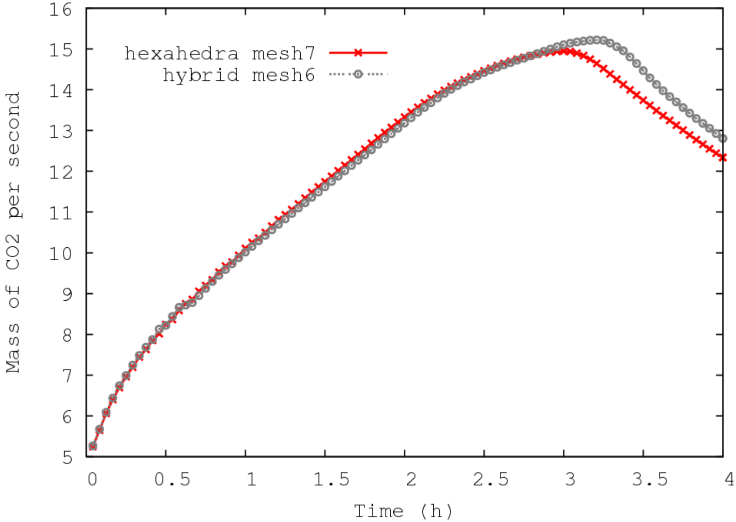
$$\left| \begin{array}{ll} H_2O & (w) \\ CO_2 & (w\text{-}g) \end{array} \right.$$

- Thermodynamic equilibrium defined by the solubility $\bar{C}$

$$\left\{ \begin{array}{ccccc} (w-g) & : & C^w_{CO_2} < \bar{C} & S_g = 0 \\ (g) & : & C^w_{CO_2} = \bar{C} & S_g > 0 \end{array} \right.$$

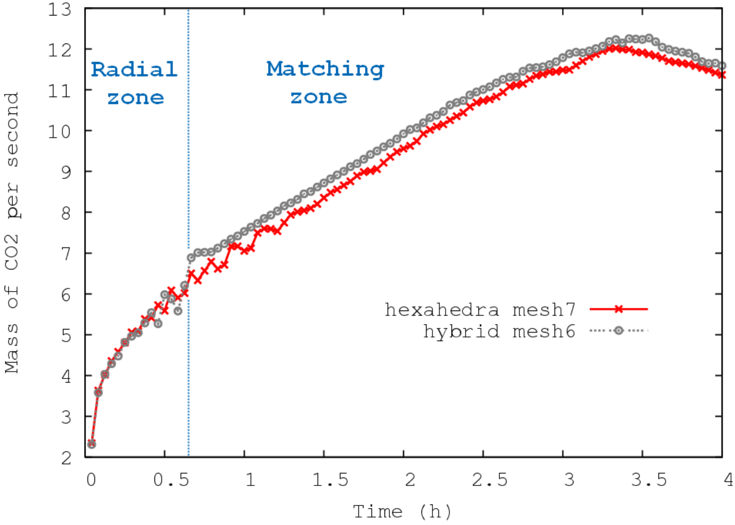
Test the O scheme on both types of meshes

Total Mass of CO_2 function of time



GOE = **G**rid **O**rientation **E**ffect

Mass of CO_2 in phase gas function of time



Cell size affect the oscillations

Conclusion

- SUSHI scheme exhibits very promising results thanks to its unconditional coercivity
- Hybrid meshes show drawbacks of the schemes :
  - O scheme has a stencil $\approx$ 4 times bigger than GradCell
  - L scheme fails but not the more flexible G scheme
- Two-phase flow numerical solution is sensitive to GOE and size of the cells

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