Convergence of the Finite Volume MPFA O Scheme for Heterogeneous Anisotropic Diffusion Problems on General Meshes

L. Agelas — R. Masson

Institut Français du Pétrole
1 et 4 avenue Bois Préau
92852 Rueil Malmaison

leo.agelas@ifp.fr

ABSTRACT. *This paper proves the convergence of the finite volume MultiPoint Flux Approximation (MPFA) O scheme for anisotropic and heterogeneous diffusion problems. Our framework is based on a discrete variational formulation and a local coercivity condition. Its main originality is to hold for general polygonal and polyhedral meshes as well as L^∞ diffusion coefficients, which is essential in many practical applications.*

KEYWORDS: *Finite volume scheme, diffusion equation, general meshes, heterogeneities, anisotropy, convergence analysis*

1. Introduction

In this paper, we consider the second order elliptic equation

$$\begin{cases} \text{div}(-\Lambda \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad [1]$$

where Ω is an open bounded connected polygonal subset of $\mathbb{R}^d$, , $d \in \mathbb{N}^*$, and $f \in L^2(\Omega)$. It is assumed in the following that Λ is a measurable function from Ω to the set of square d dimensional matrices $\mathcal{M}_d(\mathbb{R})$ such that for all $x \in \Omega$, $\Lambda(x)$ is symmetric and its eigenvalues are in the interval $[\alpha(x), \beta(x)]$ with $\alpha, \beta \in L^\infty(\Omega)$, and $0 < \alpha_0 \le \alpha(x) \le \beta(x) \le \beta_0$.

The MultiPoint Flux Approximation (MPFA) O method is a cell centered finite volume discretization of such second order elliptic equations described for example in [Aav 02] and [Ed 02]. It is a widely used scheme in the oil industry for the discretization of diffusion fluxes in multiphase Darcy porous media flow models (see for example [GCHC 98], [JFG 00], and [LJT 02] for applications in reservoir simulations, and [EK 05], [AEK 06] for numerical experiments on 2D and 3D diffusion problems).

Let σ be any interior face of the mesh shared by the two cells K and L, and $\mathbf{n}_{K,\sigma}$ its normal vector outward K. Cell centered finite volume schemes use the cell unknowns u_M for each cell M of the mesh as degrees of freedom. They aim to build conservative approximations $F_{K,\sigma}$ of the fluxes $-\int_\sigma \Lambda\nabla u \cdot \mathbf{n}_{K,\sigma} d\sigma$ as linear combinations of the cell unknowns u_M using neighbouring cells M of the cells K or L. The fluxes are conservative in the sense that $F_{K,\sigma} + F_{L,\sigma} = 0$.

The main assets of the MPFA O scheme are to derive a consistent approximation of the fluxes on general meshes, and to be adapted to discontinuous anisotropic diffusion coefficients in the sense that it reproduces cellwise linear solutions for cellwise constant diffusion tensors.

Its construction uses in addition to the cell unknowns u_K for each cell K of the mesh, the intermediate unknowns u_σ^s for each face (edge in 2D) σ of the mesh and each vertex s of the face σ. Roughly speaking, assuming that each vertex s of any cell K is shared by exactly d faces σ of the cell K, subfluxes $F_{K,\sigma}^s$ are built using a cellwise constant diffusion coefficient and a linear approximation of u on the cell K using the cell unknown u_K and the d face unknowns u_σ^s. Then, the intermediate unknowns are eliminated by the flux continuity equations on each face around the vertex s, and the approximate flux $F_{K,\sigma}$ is the sum of the subfluxes over the vertices of the face σ. A generalization of this construction is proposed in [GCHC 98] for general polyhedral meshes.

Recent papers have studied the convergence of the MPFA O scheme but there is yet no convergence result on general polygonal and polyhedral meshes, and none taking into account discontinuous diffusion coefficients which are essential in oil industry applications. In [KlWi 06], [AEKWY 07], [KW 06], the convergence of the scheme is obtained on quadrilateral meshes. The proofs are based on equivalences of the MPFA O scheme to mixed finite element methods using specific quadrature rules. The convergence of the scheme is obtained provided that a square d-dimensional matrix defined locally for each cell and each vertex of the cell, depending both on the distorsion of cell and on the cell diffusion tensor, is uniformly positive definite. This analysis confirms the numerical experiments showing that the coercivity and convergence of the scheme is lost in the cases of strong distorsion of the mesh and/or anisotropy of the diffusion tensor. For example, in [ADM 08], the MPFA O scheme is compared with symmetric finite volume schemes satisfying unconditional coercivity and convergence properties. These latter schemes are more robust for highly anisotropic test cases on distorted meshes but this is at the expense of a much larger stencil, and the MPFA O scheme offers a good compromise between robustness and compacity of the stencil.

In [LSY 05] a mimetic finite difference scheme is introduced which is equivalent to the MPFA O scheme for simplicial and parallepipedic cells and a proper choice of the continuity points. Their analysis provides a convergence result for such meshes with usual shape regularity assumptions and for smooth diffusion coefficients. In such specific cases, the MPFA O scheme is known to be symmetric and coercive whatever the diffusion tensor and the distorsion of the mesh.

In this paper, an hybrid discrete variational formulation is defined in section 3 using the framework introduced in [RGH 07], and [EH 07]. For usual meshes such that each vertex of any cell K is shared by exactly d faces of the cell K, our discrete variational formulation is equivalent to the usual MPFA O scheme, provided that the normal vectors to these d faces span $\mathbb{R}^d$. It will in addition provide a generalization of the O scheme on more general polyhedral cells, alternative to the one described in [GCHC 98]. A sufficient local condition for the coercivity of the scheme is derived in section 4 which will yield existence, and uniqueness of the solution. Under this coercivity condition, depending on the mesh and on the diffusion tensor anisotropy, the convergence of the scheme including the case of L^∞ diffusion coefficients can be proved.

Notations: in the following, the weak solution of (1) will be denoted by $\bar{u}$. For any vector $x, y \in \mathbb{R}^d$, let $x \cdot y$ denote the scalar product $\sum_{i=1}^d x_i\, y_i$, and $|x|$ be the norm $\sqrt{(x \cdot x)}$.

2. Discrete functional framework

Definition 2.1 *An admissible finite volume discretization of Ω, denoted by $\mathcal{D}$, is given by $\mathcal{D} = (\mathcal{T}, \mathcal{E}, \mathcal{P}, \mathcal{V})$, where:*

- $\mathcal{T}$ is a finite family of non-empty connected open disjoint subsets of Ω (the "cells") such that $\overline{\Omega} = \cup_{K\in\mathcal{T}} \overline{K}$. For any $K \in \mathcal{T}$, let $\partial K = \overline{K} \setminus K$ be the boundary of K and $\mathrm{m}_K > 0$ denote the measure of K.

- $\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "faces" of the mesh), such that, for all $\sigma \in \mathcal{E}$, σ is a non-empty closed subset of a hyperplane of $\mathbb{R}^d$, which has a $(d-1)$-dimensional measure $\mathrm{m}_\sigma > 0$. We assume that, for all $K \in \mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \cup_{\sigma\in\mathcal{E}_K} \sigma$. We then denote by $\mathcal{T}_\sigma$ the set $\{K \in \mathcal{T} \,|\, \sigma \in \mathcal{E}_K\}$. It is assumed that, for all $\sigma \in \mathcal{E}$, either $\mathcal{T}_\sigma$ has exactly one element and then $\sigma \subset \partial\Omega$ (boundary face) or $\mathcal{T}_\sigma$ has exactly two elements (interior face). For all $\sigma \in \mathcal{E}$, we denote by x_σ the center of gravity of σ.

- $\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{T}$ ("the centers of cells"), denoted by $\mathcal{P} = (x_K)_{K\in\mathcal{T}}$, such that $x_K \in K$ and K is star-shaped with respect to x_K.

- $\mathcal{V}$ is a family of points ("the vertices of the mesh"), such that for any $K \in \mathcal{T}$, for all subset H_K of $\mathcal{E}_K$ with $cardinal(H_K) \geq d$, then $\cap_{\sigma\in H_K} \sigma = \emptyset$ or $\cap_{\sigma\in H_K} \sigma = s$ where $s \in \mathcal{V}$. For all $s \in \mathcal{V}$, we denote by $\mathcal{E}_s$ the set $\{\sigma \in \mathcal{E} \,|\, s \in \sigma\}$ and by $\mathcal{T}_s$ the set $\{K \in \mathcal{T} \,|\, s \in \overline{K}\}$. For all $K \in \mathcal{T}$, the set $\mathcal{V}_K$ stands for $\{s \in \mathcal{V} \,|\, s \in \overline{K}\}$, and for all $\sigma \in \mathcal{E}$ the set $\{s \in \mathcal{V} \,|\, s \in \sigma\}$ is denoted by $\mathcal{V}_\sigma$.

The following notations are used. The size of the discretization is defined by: $h_\mathcal{D} = \sup\{\mathrm{diam}(K), K \in \mathcal{T}\}$. For all $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, we denote by $\mathbf{n}_{K,\sigma}$ the unit vector normal to σ outward to K, and by $d_{K,\sigma}$ the Euclidean distance between x_K and

σ. The set of interior (resp. boundary) faces is denoted by $\mathcal{E}_{\text{int}}$ (resp. $\mathcal{E}_{\text{ext}}$), defined by $\mathcal{E}_{\text{int}} = \{\sigma \in \mathcal{E} \,|\, \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E} \,|\, \sigma \subset \partial\Omega\}$). For any $K \in \mathcal{T}$ and $s \in \mathcal{V}_K$, q_K^s stands for the cardinal of $\mathcal{E}_K \cap \mathcal{E}_s$.

Shape regularity of the mesh. It will be measured by the parameters $\text{regul}(\mathcal{D}) = \min_{\sigma\in\mathcal{E}_K, K\in\mathcal{T}}\left\{\frac{d_{K,\sigma}}{\text{diam}(K)}\right\}$, and $\text{ratio}(\mathcal{D}) = \min_{\sigma\in\mathcal{E}_{\text{int}}, \mathcal{T}_\sigma=\{K,L\}} \left\{\frac{\min(d_{K,\sigma},d_{L,\sigma})}{\max(d_{K,\sigma},d_{L,\sigma})}\right\}$.

Parameters of the MPFA O finite volume scheme. In addition to the choice of the cell centers satisfying the above assumptions, the construction of the MPFA O scheme involves two families of parameters defined on the set $\{(\sigma, s) \,|\, s \in \mathcal{V}_\sigma, \sigma \in \mathcal{E}\}$. The first family of non-negative reals $(\mathrm{m}_\sigma^s)_{s\in\mathcal{V}_\sigma,\sigma\in\mathcal{E}}$ defines the distribution of the surface m_σ of each face σ to the face vertices $s \in \mathcal{V}_\sigma$ such that $\mathrm{m}_\sigma = \sum_{s\in\mathcal{V}_\sigma} \mathrm{m}_\sigma^s$. It results that the volume of each cell $K \in \mathcal{T}$ is also distributed to the vertices of the cell according to the subvolumes m_K^s, $s \in \mathcal{V}_K$ defined by $\mathrm{m}_K^s = \frac{1}{d}\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \mathrm{m}_\sigma^s d_{K,\sigma}$, and which satisfy $\mathrm{m}_K = \sum_{s\in\mathcal{V}_K} \mathrm{m}_K^s$ for all $K \in \mathcal{T}$. The second family is the set of the so called continuity points $(x_\sigma^s)_{\sigma\in\mathcal{E}_s,s\in\mathcal{V}}$ such that $x_\sigma^s \in \sigma$. On each continuity point x_σ^s, the intermediate unknown u_σ^s is defined which will be used together with the cell unknowns u_K, $K \in \mathcal{T}$ for the construction of the discrete gradients defined in the next section.

Discrete function spaces. Let us define the discrete function space $\mathcal{H}_\mathcal{D}$ as the subspace of $\{(u_K)_{K\in\mathcal{T}}, (u_\sigma^s)_{\sigma\in\mathcal{E}_s,s\in\mathcal{V}}, u_K, u_\sigma^s \in \mathbb{R}\}$ such that $u_\sigma^s = 0$ for all $s \in \mathcal{V}_\sigma$, $\sigma \in \mathcal{E}_{\text{ext}}$. The space $\mathcal{H}_\mathcal{D}$ is equipped with the following Euclidean structure defined by the inner product:

$$[v, w]_\mathcal{D} = \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \sum_{s\in\mathcal{V}_\sigma} \frac{\mathrm{m}_\sigma^s}{d_{K,\sigma}}(v_\sigma^s - v_K)(w_\sigma^s - w_K),$$

for $(v, w) \in (\mathcal{H}_\mathcal{D})^2$, and the associated norm: $\|u\|_\mathcal{D} = ([u, u]_\mathcal{D})^{1/2}$. Let $H_\mathcal{T}(\Omega) \subset L^2(\Omega)$ be the space of piecewise constant functions on each cell of the mesh $\mathcal{T}$, equipped with the following norm: $\|u\|_\mathcal{T} = \inf\{\|v\|_\mathcal{D}, v \in \mathcal{H}_\mathcal{D}, P_\mathcal{T} v = u\}$ where for all $v \in \mathcal{H}_\mathcal{D}$, $P_\mathcal{T} v \in H_\mathcal{T}(\Omega)$ denotes the vector of $H_\mathcal{T}(\Omega)$ defined by $(v_K)_{K\in\mathcal{T}}$.

3. The MPFA scheme and its discrete variational formulation

The definition of the finite volume scheme is based on an hybrid variational formulation on the space $\mathcal{H}_\mathcal{D}$ using the construction of two discrete gradients for each $s \in \mathcal{V}_K$ and $K \in \mathcal{T}$. The first gradient defined by

$$(\widetilde{\nabla}_\mathcal{D} u)_K^s \;=\; \frac{1}{\mathrm{m}_K^s} \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \mathrm{m}_\sigma^s (u_\sigma^s - u_K)\mathbf{n}_{K,\sigma}, \qquad [2]$$

is built to have a weak convergence property once averaged for each cell K over its vertices $s \in \mathcal{V}_K$ with the weights m_K^s. The second gradient is defined by

$$(\overline{\nabla}_\mathcal{D} u)_K^s \;=\; (B_K^s)^{-1}(\widetilde{\nabla}_\mathcal{D} u)_K^s, \qquad [3]$$

using the square d dimensional matrix

$$B_K^s = \frac{1}{m_K^s} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} \mathrm{m}_\sigma^s \mathbf{n}_{K,\sigma} (x_\sigma^s - x_K)^t. \qquad [4]$$

The matrices B_K^s are always assumed in the following to be non-singular for each cell K of the mesh and each vertex s of the K. In the next section, a stronger assumption (8) is made ensuring the coercivity of the scheme which in particular guarantees the non-singularity of the matrices B_K^s for all $s \in \mathcal{V}_K$, $K \in \mathcal{T}$. The gradient $(\overline{\nabla}_{\mathcal{D}} u)_K^s$ is built to be consistent in the sense that it is exact for linear functions. More precisely, we can check from definition (4) of B_K^s that $\frac{1}{\mathrm{m}_K^s}(B_K^s)^{-1} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} \mathrm{m}_\sigma^s (\varphi(x_\sigma^s) - \varphi(x_K))\mathbf{n}_{K,\sigma} = \nabla\varphi$ for all linear functions φ provided that B_K^s is non-singular.

Let $a_{\mathcal{D}}$ be the bilinear form defined for all $(u,v) \in \mathcal{H}_{\mathcal{D}} \times \mathcal{H}_{\mathcal{D}}$ by

$$a_{\mathcal{D}}(u,v) = \sum_{K \in \mathcal{T}} \sum_{s \in \mathcal{V}_K} \left(\mathrm{m}_K^s (\overline{\nabla}_{\mathcal{D}} u)_K^s \cdot \Lambda_K (\widetilde{\nabla}_{\mathcal{D}} v)_K^s + \alpha_K^s \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} \frac{\mathrm{m}_\sigma^s}{d_{K,\sigma}} R_{K,\sigma}^s(u) R_{K,\sigma}^s(v) \right),$$

where $\Lambda_K = \frac{1}{\mathrm{m}_K} \int_K \Lambda(x)dx$, and the residual functions $R_{K,\sigma}^s$ are defined by $R_{K,\sigma}^s(u) = u_\sigma^s - u_K - (\overline{\nabla}_{\mathcal{D}} u)_K^s \cdot (x_\sigma^s - x_K)$ for all $u \in \mathcal{H}_{\mathcal{D}}$. Our scheme is defined by the following discrete hybrid variational formulation: find $u \in \mathcal{H}_{\mathcal{D}}$ such that $a_{\mathcal{D}}(u,v) = \int_\Omega f(P_{\mathcal{T}} v)$ for all $v \in \mathcal{H}_{\mathcal{D}}$. Checking that $a_{\mathcal{D}}(u,v) = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \sum_{s \in \mathcal{V}_\sigma} F_{K,\sigma}^s(u)(v_K - v_\sigma^s)$, for all $u, v \in \mathcal{H}_{\mathcal{D}}$ with the following definition of the subfluxes

$$\begin{aligned} F_{K,\sigma}^s(u) &= -\mathrm{m}_\sigma^s \Lambda_K (\overline{\nabla}_{\mathcal{D}} u)_K^s \cdot \mathbf{n}_{K,\sigma} \\ &\quad - \alpha_K^s \, \mathrm{m}_\sigma^s \left(\frac{R_{K,\sigma}^s(u)}{d_{K,\sigma}} - \frac{(B_K^s)^{-t}}{m_K^s} \sum_{\sigma' \in \mathcal{E}_K \cap \mathcal{E}_s} \frac{\mathrm{m}_{\sigma'}^s}{d_{K,\sigma'}} R_{K,\sigma'}^s(u)(x_{\sigma'}^s - x_K) \cdot \mathbf{n}_{K,\sigma} \right), \end{aligned}$$

for all $s \in \mathcal{V}_\sigma$, $\sigma \in \mathcal{E}_K$, $K \in \mathcal{T}$, it is easily shown that the hybrid variational formulation is equivalent to the following hybrid finite volume scheme: find $u \in \mathcal{H}_{\mathcal{D}}$ such that

$$\left\{ \begin{array}{rll} -\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u) &= \int_K f(x)dx & \text{for all } K \in \mathcal{T}, \\ F_{K,\sigma}(u) &= \sum_{s \in \mathcal{V}_\sigma} F_{K,\sigma}^s(u) & \text{for all } \sigma \in \mathcal{E}_K, K \in \mathcal{T}, \\ F_{K,\sigma}^s(u) + F_{L,\sigma}^s(u) &= 0 & \text{for all } s \in \mathcal{V}_\sigma, \mathcal{T}_\sigma = \{K,L\}, \sigma \in \mathcal{E}_{\text{int}}. \end{array} \right. \qquad [5]$$

Note that around each vertex $s \in \mathcal{V}$, the face unknowns $(u_\sigma^s)_{\sigma \in \mathcal{E}_s}$ can be eliminated in terms of the $(u_K)_{K \in \mathcal{T}_s}$ solving the local linear system

$$\left\{ \begin{array}{rll} F_{K,\sigma}^s(u) + F_{L,\sigma}^s(u) &= 0 & \text{for all } \sigma \in \mathcal{E}_s \cap \mathcal{E}_{\text{int}} \text{ with } \mathcal{T}_\sigma = \{K,L\}, \\ u_\sigma^s &= 0 & \text{for all } \sigma \in \mathcal{E}_s \cap \mathcal{E}_{\text{ext}}. \end{array} \right. \qquad [6]$$

The well-posedness of this system derives from coercivity condition (8) stated below in section 4. It results that the hybrid finite volume scheme reduces to a cell centered finite volume scheme

$$\left\{\begin{array}{l} -\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{int}},\mathcal{T}_\sigma=\{K,L\}} F_{K,L} + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{ext}}} F_\sigma = \int_K f(x)dx \quad \text{for all } K\in\mathcal{T}, \\ u_\sigma^s = 0 \quad \text{for all } s\in\mathcal{V}_\sigma, \sigma\in\mathcal{E}_{\text{ext}}, \end{array}\right. \tag{7}$$

where the inner fluxes $F_{K,L}$, $\mathcal{T}_\sigma = \{K,L\}$, $\sigma\in\mathcal{E}_{\text{int}}$, and the boundary fluxes F_σ, $\sigma\in\mathcal{E}_{\text{ext}}$, are linear combinations of the cell unknowns u_M with $M\in\bigcup_{s\in\mathcal{V}_\sigma}\mathcal{T}_s$.

For all $K\in\mathcal{T}$ and all $s\in\mathcal{V}$, let us assume that $q_K^s = d$, and that both sets $(x_\sigma^s - x_K)_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s}$ and $(\mathbf{n}_{K,\sigma})_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s}$ span $\mathbb{R}^d$. Then, it can be shown that our finite volume scheme (5) is equivalent to the usual MPFA O scheme, since in that case each discrete gradient $(\overline{\nabla}_{\mathcal{D}}u)_K^s$ matches with the gradient of the linear function uniquely defined by the $d+1$ points $(x_\sigma^s, u_\sigma^s)_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s}$, (x_K, u_K), and each residual $R_{K,\sigma}^s(u)$ vanishes for all $u\in\mathcal{H}_{\mathcal{D}}$. For more general polyhedral meshes, the above formulation of the scheme provides a generalization of the MPFA O scheme described in [Aav 02], [Ed 02].

4. Coercivity and Convergence of the MPFA O scheme

In order to obtain existence, uniqueness of the solution and stability estimates, a coercivity property is needed in the sense that there exists a real $\beta > 0$ such that, for all $u\in\mathcal{H}_{\mathcal{D}}$, $a_{\mathcal{D}}(u,u)\geq\beta\|u\|_{\mathcal{D}}^2$.
This is achieved imposing the following sufficient condition: there exists a real $\theta > 0$ such that

$$\text{coer}(\mathcal{D},\Lambda)\geq\theta. \tag{8}$$

where $\text{coer}(\mathcal{D},\Lambda)$ is defined by

$$\text{coer}(\mathcal{D},\Lambda) = \min_{K\in\mathcal{T},s\in\mathcal{V}_K}\lambda_{min}\left(\Lambda_K B_K^s + (\Lambda_K B_K^s)^t\right), \tag{9}$$

$\lambda_{\min}(M)$ denoting the smallest eigenvalue of a symmetric square matrix M. This condition can be easily computed for any given finite volume discretization $\mathcal{D}$ and diffusion tensor Λ. Assuming that this condition holds uniformly we can prove the following theorem.

Theorem 4.1 *[Convergence of the scheme] Let $(\mathcal{D}^{(n)})_{n\in\mathbb{N}}$ be a family of finite volume discretizations, and let* $\text{regul}(\mathcal{D}^{(n)})\geq\beta$ *for some $\beta>0$, $ratio(\mathcal{D}^{(n)})\geq\zeta$ for some $\zeta>0$ and $coer(\mathcal{D}^{(n)},\Lambda)\geq\theta$ for some $\theta>0$. Then, for each $n\in\mathbb{N}$, there exists a unique solution $u_{\mathcal{D}^{(n)}}\in\mathcal{H}_{\mathcal{D}^{(n)}}$ to (5), and the sequence $P_{\mathcal{T}}u_{\mathcal{D}^{(n)}}$ converges to $\bar{u}$ in $L^q(\Omega)$, for all $q\in[1,+\infty)$ if $d=2$ and all $q\in[1,2d/(d-2))$ if $d>2$, as $h_{\mathcal{D}^{(n)}}\to 0$. Moreover, the cellwise constant gradient function $\widehat{\nabla}_{\mathcal{D}^{(n)}}u_{\mathcal{D}^{(n)}}\in H_{\mathcal{T}}(\Omega)^d$ defined by*

$\mathrm{m}_K(\widehat{\nabla}_{\mathcal{D}^{(n)}} u_{\mathcal{D}^{(n)}})_K = \sum_{s\in\mathcal{V}_K} \mathrm{m}_K^s(\overline{\nabla}_{\mathcal{D}^{(n)}} u_{\mathcal{D}^{(n)}})_K^s$ *for all* $K \in \mathcal{T}$*, converges to* $\nabla\bar{u}$ *in* $L^2(\Omega)^d$.

Sketch of the proof: The existence, uniqueness, and a stability estimate of $u_{\mathcal{D}^{(n)}}$ in $\mathcal{H}_{\mathcal{D}^{(n)}}$ are readily obtained from the uniform coercivity of the bilinear forms $a_{\mathcal{D}^{(n)}}$. Then, it results from the discrete Rellich theorem already proved in [RGH 07] that there exist a function $\widetilde{u} \in H_0^1(\Omega)$ and a subsequence of $n \in \mathbb{N}$, still denoted by $n \in \mathbb{N}$ for simplicity, such that $P_{\mathcal{T}} u_{\mathcal{D}^{(n)}}$, $n \in \mathbb{N}$ converges to $\widetilde{u} \in H_0^1(\Omega)$ in $L^q(\Omega)$ for all $q \in [1, +\infty)$ if $d = 2$ and all $q \in [1, 2d/(d-2))$ if $d > 2$, and such that the cellwise gradient function $\nabla_{\mathcal{D}} u_{\mathcal{D}^{(n)}} = \sum_{s\in\mathcal{V}_K} \mathrm{m}_K^s/\mathrm{m}_K(\widetilde{\nabla}_{\mathcal{D}^{(n)}} u_{\mathcal{D}^{(n)}})_K^s$ on each cell $K \in \mathcal{T}$, $n \in \mathbb{N}$ weakly converges to $\nabla\widetilde{u}$ in $L^2(\Omega)^d$. For all $\varphi \in C_c^\infty(\Omega)$, let $P_{\mathcal{D}}\varphi$ be the function of $\mathcal{H}_{\mathcal{D}}$ defined by the values $\varphi(x_K)$, $\varphi(x_\sigma^s)$, $K \in \mathcal{T}$, $\sigma \in \mathcal{E}_s$, $s \in \mathcal{V}$. Using these properties, the consistency of the discrete gradients $(\overline{\nabla}_{\mathcal{D}}(P_{\mathcal{D}}\varphi))_K^s$, and of the residual functions $R_{K,\sigma}^s(P_{\mathcal{D}}\varphi)$ for $\varphi \in C_c^\infty(\Omega)$, the stability of the gradient function $\widehat{\nabla}_{\mathcal{D}} u$ in $\mathcal{H}_{\mathcal{D}}$, and the coercivity of the bilinear form $a_{\mathcal{D}}$, we can then prove the convergence in $L^2(\Omega)^d$ up to a subsequence of the gradient function $\widehat{\nabla}_{\mathcal{D}_n} u_{\mathcal{D}^{(n)}}$ $n \in \mathbb{N}$ to $\nabla\widetilde{u}$. To complete the proof of Theorem 4.1 it is then shown that $\widetilde{u}$ is the unique weak solution $\bar{u}$ of (1) by passing to the limit in the discrete hybrid variational formulation with $v = P_{\mathcal{D}}\varphi$, $\varphi \in C_c^\infty(\Omega)$.

Examples: let us set $m_\sigma^s = \frac{m_\sigma}{\text{cardinal}(\mathcal{V}_\sigma)}$, and x_σ^s be the center of gravity of the face σ for all $s \in \mathcal{V}_\sigma$, $\sigma \in \mathcal{E}$. Let x_K be the isobarycenter of the vertices of the cell K for all $K \in \mathcal{T}$. Then, for parallelogram and parallelepiped cells, the matrix B_K^s is equal to I. In such a case, the MPFA O scheme is symmetric and our sufficient condition of coercivity (8) is always satisfied. The same result holds for triangles with x_σ^s the barycenter with weights $2/3$ at point s and $1/3$ at the second end point of the edge σ. It holds again for tetrahedrons with x_σ^s the barycenter with weights $1/2$ at point s and $1/4$ at the two remaining end points of the face σ.

Let us now consider the case $d = 2$ with $\Lambda = I$, and let σ_1 and σ_2 be the two edges shared by a given vertex s of a given cell K. For $\sigma = \sigma_1, \sigma_2$, we assume that the continuity point x_σ^s is the center of gravity x_σ of the edge σ and that $m_\sigma^s = |x_\sigma - s|$. Then, the condition $\lambda_{min}(B_K^s + (B_K^s)^t) \geq \theta$ is equivalent to $|x_{\sigma_1} - x_{\sigma_2}|\ |\overrightarrow{sx_{\sigma_1}} - \overrightarrow{x_{\sigma_2}x_K}| \leq 2(1-\frac{\theta}{2})m_K^s$.

For example, the trapezoidal mesh shown in Figure 1 satisfies the coercivity condition (8) if and only if $\frac{b-a}{h} \leq (1-\frac{\theta}{2})\frac{3a+b}{(b^2+h^2)^{1/2}}$ which exhibits the lack of robustness of the MPFA O scheme for very distorted quadrangular meshes.

5. References

[ADM 08] AGELAS L. , DI PIETRO D. , MASSON R. “ A symmetric finite volume scheme for multiphase porous media flow problems with applications in the oil industry ”, accepted in the Proceedings of FVCA5 2008.

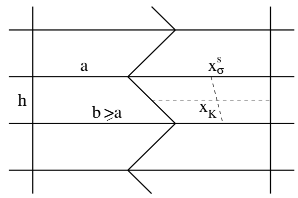

Figure 1. *Example of a trapezoidal mesh.*

[AEK 06] AAVATSMARK I. , EIGESTAD G.T. , KLAUSEN R.A. " Numerical convergence of MPFA for general quadrilateral grids in two and three dimensions", *Compatible Spacial Discretizations for Partial Differential Equations*, IMA vol. Ser. 142, p. 1-22, 2006.

[Aav 02] AAVATSMARK I. "An introduction to multipoint flux approximations for quadrilateral grids", *Computational Geosciences* 6, 2002, p. 405-432.

[AEKWY 07] AAVATSMARK I. , EIGESTAD G.T. , KLAUSEN R.A. , WHEELER M.F. , YOTOV I. "Convergence of a symmetric MPFA method on quadrilateral grids", *Computational Geosciences*, (2007).

[Ed 02] EDWARDS M.G. "Unstructured control-volume distributed full tensor finite volume schemes with flow based grids", *Computational Geosciences*, 6, 2002 p. 433-452.

[EK 05] EIGESTAD G.T., KLAUSEN R.A. "On the convergence of the Multi-point flux approximation O-method: Numerical experiments for discontinuous permeability", *Num. Meth. for PDE*, No. 6, Vol. 21, 2005, p. 1079-1098.

[EH 07] EYMARD R., HERBIN R.. "A new colocated finite volume scheme for the incompressible Navier-Stokes equations on general non matching grids", *Comptes rendus Mathématiques de l'Académie des Sciences*, 344(10) p. 659-662, 2007.

[RGH 07] EYMARD R. , GALLOUËT T. , HERBIN R. " A new finite volume scheme for anisotropic diffusion problems on general grids: convergence analysis", *Comptes rendus Mathématiques de l'Académie des Sciences*, 344,6, 2007, p. 403-406.

[GCHC 98] GUNASEKERA D. , CHILDS P. , HERRING J. , COX J. " A multi-point flux discretization scheme for general polyhedral grids", *Proc. SPE 6th International Oil and Gas Conference and Exhibition, China*, SPE 48855, nov. 1998.

[JFG 00] JEANNIN L. , FAILLE I. , GALLOUËT T. " How to model compressible two phase flows on hybrid grids", *Oil and Gas Science and Technology - Rev. IFP*, vol. 55, 2000, 3, p. 269-279.

[KlWi 06] KLAUSEN R.A. , WINTHER R. "Convergence of Multi-Point Flux Approximations on Quadrilateral Grids", *Num. Met. for PDE's*, 22, 6, 2006, p. 1438-1454.

[LSY 05] LIPNIKOV K. , SHASHKOV M., YOTOV I. "Local flux mimetic finite difference methods", *Technical Report LA-UR-05-8364, Los Alamos National Laboratory*, 2005.

[LJT 02] LEE S.H. , JENNY P. , TCHELEPI H. "A finite-volume method with hexahedral multiblock grids for modeling flow in porous media", *Computational Geosciences*, 6,3, p. 269-277, 2002.

[KW 06] KLAUSEN R.A. , WINTHER R. "Robust convergence of multi point flux approximation on rough grids", *Numer. Math.*, 104, 3, 2006, p. 317–337.