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Wavelet adaptive method for second order elliptic problems: boundary conditions and domain decomposition

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Summary. Wavelet methods allow to combine high order accuracy, multilevel preconditioning techniques and adaptive approximation, in order to solve efficiently elliptic operator equations. One of the main difficulty in this context is the efficient treatment of non-homogeneous boundary conditions. In this paper, we propose a strategy that allows to append such conditions in the setting of space refinement (i.e. adaptive) discretizations of second order problems. Our method is based on the use of compatible multiscale decompositions for both the domain and its boundary, and on the possibility of characterizing various function spaces from the numerical properties of these decompositions. In particular, this allows the construction of a lifting operator which is stable for a certain range of smoothness classes, and preserves the compression of the solution in the wavelet basis. An explicit construction of the wavelet bases and the lifting is proposed on fairly general domains, based on C^0 conforming domain decomposition techniques.

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1 Introduction

Wavelet adaptive methods for operator equations have been mainly studied in the framework of elliptic problems where the unknown u_0 is the solution

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of the variational formulation: $u_0 \in V$,

$$a(u_0, v) = \langle f, v\rangle_{V',V} \text{ for all } v \in V, \tag{1}$$

and V denotes a Sobolev space possibly including homogeneous boundary conditions, a is a continuous bilinear form elliptic on $V \times V$, and V' is the dual space of V.

Let $\Psi^0_\Omega = \{\psi^0_\lambda,\ \lambda \in \nabla^0\}$ be a wavelet basis of V. These algorithms generate iteratively (through refinement and derefinement strategies) a series of subsets of ∇^0

$$\Lambda_0,\ \Lambda_1,\ \ldots,\ \Lambda_n\ \ldots$$

of cardinality $\#\Lambda_n$, and the Galerkin projections u_{Λ_n} (or a proper approximation $\tilde{u}_{\Lambda_n}$) of u_0 on $V_{\Lambda_n} := \text{ span}\{\psi^0_\lambda,\ \lambda \in \Lambda_n\}$. Throughout this paper, for two positive functions $A(v)$ and $B(v)$ on a set N, the notation $A(v) \lesssim B(v)$ will mean that there exits a constant C, independent of the various parameters, so that $A(v) \le C\,B(v)$ for all $v \in N$. Also $A(v) \sim B(v)$ denotes that $A(v) \lesssim B(v)$ and $B(v) \lesssim A(v)$. Then, the series $(u_{\Lambda_n}, \Lambda_n)$ is said to be *asymptotically optimal* if

$$\|u_0 - u_{\Lambda_n}\|_V \lesssim (\#\Lambda_n)^{-s}$$

whenever it is the case for the best approximation of the solution u_0 by linear combinations of N wavelets in Ψ^0_Ω. In particular, from the theory of approximation [17], this property is known to hold if (and almost only if) u_0 belongs to the Besov space $B^{sd+t}_{p,p}$ for $\frac{1}{p} = \frac{1}{2} + s$. In addition the best N-term approximation is asymptotically obtained by the truncated expansion using the N largest wavelet coefficients normalized in V. In this context, all the difficulty of adaptive strategies amounts to define an asymptotically optimal approximation without the a priori knowledge of the wavelet coefficients of the solution.

In [5] such an adaptive strategy has been developped for a large class of symmetric elliptic bilinear forms a. In addition the complexity of this algorithm is asymptotically optimal in the sense that all the computations can be carried out in $O(\#\Lambda_n)$ operations and storage. A simpler but similar strategy is tested in [9] where the optimality properties of the algorithm are illustrated on simple examples.

The objective of this paper is to propose a setting to extend these adaptive algorithms in the case of non homogeneous Dirichlet boundary conditions and second order elliptic problem which is not covered by the previous framework.

The only existing approach so far is based on a treatment of the boundary conditions by Lagrange multipliers [21], [22] which leads to a saddle point problem and raises serious difficulties related in particular to the proof of the inf-sup condition.

We propose here a new strategy that is “fully adaptative” in the sense that it will be adaptative for both the “homogeneous part” of the solution and the boundary conditions. It is based on the construction of wavelet bases Ψ_Ω^* on the domain and $\Psi_{\Gamma_{\rm D}}$ on the boundary satisfying some compatibility conditions in order to define a natural stable lifting operator.

Let a domain $\Omega \subset \mathbb{R}^d$ of globally Lipschitz, piecewise regular boundary Γ, and $\Gamma_{\rm D} \subset \Gamma$ a manifold of dimension $d-1$ of same regularity as Γ. The trace operator $\gamma_{\rm D}$ classically defines a continuous mapping from $H^1(\Omega)$ onto $H^{1/2}(\Gamma_{\rm D})$ and we denote by $V := H^1_{\rm D}(\Omega)$ its kernel. We consider the following second order elliptic problem:

$$\text{(2)} \qquad \begin{cases} a(u,v) = \langle f, v\rangle_{V',V}, \text{ for all } v \in V \\ u \in H^1(\Omega),\ \gamma_{\rm D} u = g \end{cases}$$

where a is a continuous bilinear form on $H^1(\Omega) \times V$ elliptic on V, $f \in V'$ is the right-hand side and $g \in H^{1/2}(\Gamma_{\rm D})$ is the Dirichlet boundary condition.

The basic idea is to apply the wavelet adaptive strategy displayed in [5] to the homogeneous problem obtained by correction of the right-hand side through a lifting of the boundary condition g. Let $\mathcal{R}(g) \in H^1(\Omega)$ be a stable lifting of the trace g so that

$$\gamma_{\rm D}\mathcal{R}(g) = g \text{ and } \|\mathcal{R}(g)\|_{H^1(\Omega)} \lesssim \|g\|_{H^{1/2}(\Gamma_{\rm D})}.$$

The function

$$u_0 := u - \mathcal{R}(g)$$

is the unique solution (by Lax Milgram theorem) of the classical variational formulation

$$\text{(3)} \qquad \begin{cases} a(u_0,v) = \langle f, v\rangle_{V',V} - a(\mathcal{R}(g), v), \text{ for all } v \in V \\ u_0 \in V \end{cases}$$

that fits into the framework (1).

In order to preserve the potential efficiency of adaptive algorithms applied to (3), we need to build explicitely a wavelet basis Ψ_Ω^* of $H^1(\Omega)$ and a lifting operator $\mathcal{R}$ so that u_0 and $\mathcal{R}(g)$ have the same compression properties as the solution u in Ψ_Ω^*.

To achieve this objective we shall build a wavelet basis

$$\Psi_\Omega^* = \{\psi_\lambda^*,\ \lambda \in \nabla\}$$

of $H^1(\Omega)$ and satisfying the following properties.

Property 1.1 *The wavelet basis Ψ^*_Ω admits the partition*

$$\Psi^*_\Omega = \Psi^0_\Omega \cup \Psi^b_\Omega$$

with the following properties.
(i) $\Psi^0_\Omega = \{\psi^*_\lambda,\ \lambda \in \nabla^0\}$ is a wavelet basis of $H^1_{\rm D}(\Omega)$,
(ii) there exist scaling factors c_λ so that the trace operator $c_\lambda \gamma_{\rm D}$ defines a one to one mapping from Ψ^b_Ω to a wavelet basis

$$\Psi_{\Gamma_{\rm D}} = \{\psi_\sigma,\ \sigma \in \Sigma_b\}$$

of $H^{1/2}(\Gamma_{\rm D})$, i.e. there exists a one to one mapping λ^r from Σ_b to the set of subscripts ∇^b of Ψ^b_Ω, so that $c_{\lambda^r(\sigma)}\gamma_{\rm D}\psi^*_{\lambda^r(\sigma)} = \psi_\sigma$ for all $\sigma \in \Sigma_b$.

The lifting operator $\mathcal{R}$ is formaly defined by this mapping

$$\mathcal{R}(g) = \sum_{\sigma \in \Sigma_b} c_{\lambda^r(\sigma)} \langle g, \tilde{\psi}_\sigma \rangle \psi^*_{\lambda^r(\sigma)}.$$

where $\tilde{\Psi}_{\Gamma_{\rm D}} = \{\tilde{\psi}_\sigma,\ \sigma \in \Sigma_b\}$ denotes the dual basis of $\Psi_{\Gamma_{\rm D}}$. Its stability properties in the class of Besov spaces $B^s_{p,q}$ (with $B^s_{2,2} = H^s$) is related to the stability of the wavelet bases on the domain and the boundary i.e. the possibility to characterize these function spaces with discrete weighted norms of their wavelet coefficients.

Such wavelet bases and lifting operator built, the adaptive algorithm [5] applies directly to the variational formulation (3) and the optimality properties of this algorithm carry over to our setting. Note that the compression of the right-hand side of (3) in the dual wavelet basis $\tilde{\Psi}^0_\Omega$ of Ψ^0_Ω requires as usual compressing the data f, but also $a(\mathcal{R}(g), .)$ as element of V'. For this last term, it leads to compress the trace g in the wavelet basis $\Psi_{\Gamma_{\rm D}}$ as given by the estimate: for $\Sigma \subset \Sigma_b$ and $P_{\Sigma,\Gamma_{\rm D}} := \sum_{\sigma\in\Sigma} \langle ., \tilde{\psi}_\sigma \rangle \psi_\sigma$,

$$\begin{aligned} \|a(\mathcal{R}(g - P_{\Sigma,\Gamma_{\rm D}} g), .)\|_{V'} &\leq \|a\| \, \|\mathcal{R}(g - P_{\Sigma,\Gamma_{\rm D}} g)\|_{H^1(\Omega)} \\ &\lesssim \|g - P_{\Sigma,\Gamma_{\rm D}} g\|_{H^{1/2}(\Gamma_{\rm D})}. \end{aligned}$$

We introduce in Sect. 2 a theoretical setting in order to build wavelet bases Ψ^*_Ω and $\Psi_{\Gamma_{\rm D}}$ satisfying the Properties 1.1 and to analyse their stability in the class of Besov spaces. This construction starts from the data of wavelet bases Ψ^0_Ω and $\Psi_{\Gamma_{\rm D}}$ built on sequences of nested spaces $V^0_j(\Omega) \subset V^0_{j+1}(\Omega)$ and $V_j(\Gamma_{\rm D}) \subset V_{j+1}(\Gamma_{\rm D})$, $j \in \mathbb{N}$ obtained by restriction to V and trace $\gamma_{\rm D}$ from the same sequence $V_j(\Omega) \subset V_{j+1}(\Omega)$.

Section 2.1 recalls basic definitions and properties of wavelet bases that will be used in the analysis carried out in Sect. 2.2.

In Sect. 3 we propose, for a fairly general domain Ω, explicit constructions of the wavelet bases Ψ_Ω^*, Ψ_Ω^0, Ψ_{Γ_D} and the lifting operator $\mathcal{R}$ based on domain decomposition techniques with C^0 matching.

In these techniques, the domain Ω (or similarly the manifold Γ_D) is decomposed into disjoint subdomains Ω_i, $i = 1, \dots, N$, so that $\overline{\Omega} = \bigcup_{i=1}^N \overline{\Omega}_i$. Each subdomain Ω_i is parametrized by a regular one to one mapping κ_i from the reference domain $\hat{\Omega} =]0,1[^d$ to Ω_i. The specific interest of this setting is to benefit from well known constructions of wavelet bases on tensor product domains that will be briefly recalled in Sect. 3.1. Then, the wavelet bases on $\hat{\Omega}$ are pushed-forwards onto the subdomains Ω_i and the difficulty of the construction reduces to match these wavelets at the interfaces $\bigcup_{i=1}^N \partial\Omega_i$ of the decomposition.

We follow here the strategy introduced in [18], [19] and in more generality in [14] (see also [2] for a slightly different analysis) which is based on a C^0 matching of the wavelets at the interfaces recalled in Sect. 3.2.1. Under conforming assumptions on the boundary Γ_D, the decomposition $(\Omega_i, \kappa_i)_{i=1,\dots,N}$ will induce a natural “trace” decomposition $(\Gamma_i, \xi_i)_{i=1,\dots,N_\mathrm{D}}$ of Γ_D which enables to define wavelet bases Ψ_Ω^0 and Ψ_{Γ_D} that fit into the framework of Sect. 2.2.

One specific difficulty of these domain decomposition techniques is the explicit construction of the wavelet bases and in particular the matching wavelets at the interfaces of the decomposition. In [2] an explicit construction is proposed starting from the definition of a linear system of matching conditions at the interfaces involving the wavelets on the subdomains. We introduce in Sect. 3.2.2 a new construction which differs from the previous one and is based on the definition of a natural hierarchy of the interfaces of the decomposition. One specific interest of this new approach is that the same ideas will readily apply in Sect. 3.3 for the construction of the lifting wavelets of $\Psi_{j,\Omega}^b$.

For the sake of a concise presentation, numerical experiments illustrating these constructions and the optimality of our adaptive strategy in the framework of non homogeneous Dirichlet boundary conditions will not be displayed here but can be found in [26].

2 Abstract setting

2.1 Wavelet bases

We briefly recall in this section the general framework of wavelet bases on a domain or a manifold and refer to the two surveys [4] and [11] for a detailed presentation.

Let L^2 be the space of square integrable functions on a domain of $\mathbb{R}^d$ or a d dimensional manifold Ω. We consider a Multiresolution Analysis (MRA) of L^2 that is a sequence of nested subspaces whose union is dense in L^2

$$V_j \subset V_{j+1} \subset L^2,\ j \in \mathbb{N}$$

where in practise, j will denote the dyadic scale of the approximation spaces V_j at the resolution level 2^{-j}. The spaces V_j are assumed to be endowed with local Riesz bases

$$\Phi_j := \{\varphi_{j,k},\ k \in \Delta_j\}$$

so that $\|\sum_{k\in\Delta_j} u_{j,k}\varphi_{j,k}\|_{L^2} \sim \|(u_{j,k})_{k\in\Delta_j}\|_{l^2}$.

The data of complement spaces (also called wavelet spaces) W_j so that

$$V_{j+1} = V_j \oplus W_j$$

formally gives rise to a multiscale decomposition

$$v = v_0 + \sum_{j\geq 0} w_j,\ v_0 \in V_0,\ w_j \in W_j$$

of a function v into a coarse approximation v_0 and the successive details w_j at the resolution 2^{-j}. In order to analyse the convergence and stability properties of this decomposition in classical function spaces, the complement spaces are usually defined through a family of projectors P_j onto V_j satisfying the property of biorthogonality

(4) $$P_j P_{j+1} = P_j,$$

stating equivalently that $Q_j := P_{j+1} - P_j$ are also projectors. Then, the multiscale decomposition is rewritten in terms of projections

(5) $$v = P_0 v + \sum_{j\geq 0} Q_j v,$$

corresponding to the choice $W_j := Q_j(V_{j+1})$.

The theory of approximation enables to relate, through the theory of interpolation, the stability properties of the decomposition (5) in the class of Besov spaces $B^s_{p,q}$, to the fulfillment of direct and inverse estimates in the spaces V_j. We refer to [10] or [4] for a detailed presentation of such a mechanism. We first state a theorem taken from [4] Chap. III that covers the cases $p, q \in]0, \infty]$ (i.e. includes the quasi-Banach spaces obtained for $p < 1$) and will be constantly used in the following analysis.

Theorem 2.1 *Let $p \in]0, +\infty]$. Assuming that, for $0 \leq s_0(p) < s_1(p)$, the projectors P_j satisfy the direct estimate*

(6) $$\|f - P_j f\|_{L^p} \lesssim 2^{-sj}\|f\|_{B^s_{p,p}},\ \textit{for all } s_0(p) < s < s_1(p),$$

and the MRA spaces V_j the inverse estimate

(7) $\|f_j\|_{B^s_{p,p}} \lesssim 2^{sj}\|f_j\|_{L^p}$, *for all* $s_0(p) < s < s_1(p)$, *and* $f_j \in V_j$,

then, for all $s_0(p) < s < s_1(p)$ *and* $q \in]0, +\infty]$

$$\|f\|_{B^s_{p,q}} \sim \|P_0 f\|_{L^p} + \|(2^{js}\|Q_j f\|_{L^p})_{j\in\mathbb{N}}\|_{l^q}. \tag{8}$$

Remark 2.1 If in addition the wavelet spaces are endowed with (local) bases

$$\Psi_j := \{\psi_{j,k},\ k \in \nabla_j\}$$

uniformly stable in L^p in the sense that

$$\|\sum_{k\in\nabla_j} w_{j,k}\psi_{j,k}\|_{L^p} \sim 2^{d(\frac{1}{2}-\frac{1}{p})j}\|(w_{j,k})_{k\in\nabla_j}\|_{l^p}, \tag{9}$$

then, the multiscale basis $\Psi := \Phi_0 \cup \bigcup_{j\in\mathbb{N}} \Psi_j$ satisfies the norm equivalence

$$\|f\|_{B^s_{p,q}} \sim \|\left(2^{js}\|(f_{j,k})_{k\in\nabla_j}\|_{l^p}\right)_{j\geq -1}\|_{l^q}, \tag{10}$$

where we have used the notations $\nabla_{-1} := \Delta_0$ and $\Psi_{-1} := \Phi_0$ (also assumed to be stable in L^p). Such stability properties at the level j will be derived from the local linear independence of the basis functions of Ψ_j.

The multiscale basis Ψ will shortly be denoted by

$$\Psi = \{\psi_\lambda,\ \lambda \in \nabla\}$$

where the subscript λ stands for the scale $j := |\lambda|$ and the localization in space k of the wavelets.

If the projectors P_j are uniformly stable in L^2 i.e. satisfy $\|P_j f\|_{L^2} \lesssim \|f\|_{L^2}$, then the ranges $\widetilde{V}_j$ of their adjoint projectors $\widetilde{P}_j$ define a "dual" MRA of L^2. The sequence $(V_j)_{j\in\mathbb{N}}$ is then referred to as the primal MRA. Conversely, the data of a pair of primal and dual MRA $(V_j, \widetilde{V}_j)_{j\in\mathbb{N}}$ *biorthogonal* in the sense that they can be endowed with Riesz bases (with some constants independent of j) Φ_j and $\tilde{\Phi}_j$ satisfying the biorthogonality relations

$$\langle\varphi_{j,k}, \tilde{\varphi}_{j,k'}\rangle = \delta_{k,k'}, k, k' \in \Delta_j, \tag{11}$$

defines a family of biorthogonal projectors

$$P_j := \sum_{k\in\Delta_j} \langle f, \tilde{\varphi}_{j,k}\rangle\varphi_{j,k}$$

uniformly stable in L^2. The corresponding wavelet spaces are also defined as $W_j = V_{j+1} \cap (\widetilde{V}_j)^\perp$ for the primal decomposition and $\widetilde{W}_j = \widetilde{V}_{j+1} \cap V_j^\perp$ for the dual decomposition.

In that case, combining the above mentioned mechanism with duality arguments, one can obtain the following theorem (see e.g. Chap. III of [4]).

Theorem 2.2 *Let $p \in [1, +\infty]$ and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. We assume that the projectors P_j satisfy the direct estimate*

$$\|f - P_j f\|_{L^p} \lesssim 2^{-sj} \|f\|_{W^{s,p}}, \text{ for all } 0 \le s \le n \tag{12}$$

and the MRA spaces V_j the inverse estimate

$$\|f_j\|_{W^{s,p}} \lesssim 2^{sj} \|f_j\|_{L^p}, \text{ for all } 0 \le s < \tau(p) \text{ and } f_j \in V_j, \tag{13}$$

and similar estimates for the adjoint projectors $\widetilde{P}_j$ and the dual MRA spaces $\widetilde{V}_j$ in $W^{s,p'}$ norms and for some parameters $\tilde{n}$ and $\tilde{\tau}(p')$. Then, for all $-\min(\tilde{n}, \tilde{\tau}(p')) < s < \min(n, \tau(p))$ and $q \in [1, +\infty]$ we have the norm equivalences

$$\|f\|_{B^s_{p,q}} \sim \|P_0 f\|_{L^p} + \|(2^{js} \|Q_j f\|_{L^p})_{j \in \mathbb{N}}\|_{l^q}. \tag{14}$$

where here for $s < 0$, $B^s_{p,q} := (B^{-s}_{p,q})'$.

Let us finally remark that this analysis carries over for spaces including boundary conditions in their definition with the usual attention to $s - 1/p$ integer in the use of interpolation results (see [4] for a detailed discussion).

2.2 Stable liftings

Let Ω a Lipschitz domain of $\mathbb{R}^d$ and

$$V_j(\Omega) \subset V_{j+1}(\Omega), \; j \in \mathbb{N}$$

a sequence of nested subspaces (MRA) of $W^{s,p}(\Omega)$ with $s > 1/p$ in order to define the trace operator γ_{D} from $W^{s,p}(\Omega)$ onto $W^{s-1/p,p}(\Gamma_{\mathrm{D}})$. We denote by $W^{s,p}_{\mathrm{D}}(\Omega)$ the kernel of γ_{D}.

From this MRA, we derive by restriction and trace the following MRA of respectively $W^{s,p}_{\mathrm{D}}(\Omega)$ and $W^{s-1/p,p}(\Gamma_{\mathrm{D}})$:

$$\begin{cases} V^0_j(\Omega) := V_j(\Omega) \cap W^{s,p}_{\mathrm{D}}(\Omega) \subset V^0_{j+1}(\Omega) \\ V_j(\Gamma_{\mathrm{D}}) := \gamma_{\mathrm{D}} V_j(\Omega) \subset V_{j+1}(\Gamma_{\mathrm{D}}). \end{cases}$$

Each of these MRA is endowed with a multiscale decomposition defined by a family of biorthogonal projectors denoted by $P^0_{j,\Omega}$ for the MRA $V^0_j(\Omega)$ and by $P_{j,\Gamma_{\mathrm{D}}}$ for the MRA $V_j(\Gamma_{\mathrm{D}})$. We shall denote by

$$W^0_j(\Omega) := Q^0_{j,\Omega} V^0_{j+1}(\Omega) \text{ with } Q^0_{j,\Omega} = P^0_{j+1,\Omega} - P^0_{j,\Omega}$$

and

$$W_j(\Gamma_{\mathrm{D}}) := Q_{j,\Gamma_{\mathrm{D}}} V_{j+1}(\Gamma_{\mathrm{D}}) \text{ with } Q_{j,\Gamma_{\mathrm{D}}} = P_{j+1,\Gamma_{\mathrm{D}}} - P_{j,\Gamma_{\mathrm{D}}}$$

the corresponding wavelet spaces and by Ψ^0_Ω, $\Psi_{\Gamma_{\mathrm{D}}}$ a choice of multiscale wavelet bases whose stability properties will be specified later.

We then have the following definition-proposition.

Proposition 2.1 *For $u \in \mathcal{D}(\overline{\Omega})$ and r_j any lifting from $V_j(\Gamma_{\rm D})$ into $V_j(\Omega)$, we set*

$$(15)\quad P^*_{j,\Omega}u := P^0_{j,\Omega}\Big(u - r_j \circ P_{j,\Gamma_{\rm D}} \circ \gamma_{\rm D}(u)\Big) + r_j \circ P_{j,\Gamma_{\rm D}} \circ \gamma_{\rm D}(u).$$

This definition is independent of r_j and satisfies the following properties:

(i) $P^*_{j,\Omega}$ *is a biorthogonal projector onto $V_j(\Omega)$ i.e.*

$$\begin{cases} \text{range } P^*_{j,\Omega} = V_j(\Omega) \\ (P^*_{j,\Omega})^2 = P^*_{j,\Omega} \\ P^*_{j,\Omega}P^*_{j+1,\Omega} = P^*_{j,\Omega}. \end{cases}$$

(ii) The triplet $(P^*_{j,\Omega}, P^0_{j,\Omega}, P_{j,\Gamma_{\rm D}})$ verifies the relations

$$(16)\quad \begin{cases} \gamma_{\rm D} \circ P^*_{j,\Omega} = P_{j,\Gamma_{\rm D}} \circ \gamma_{\rm D} \ \textit{(trace commuting property)}, \\ P^*_{j,\Omega}|_{V^0_{j+1}(\Omega)} = P^0_{j,\Omega}|_{V^0_{j+1}(\Omega)}. \end{cases}$$

Proof. The proof is achieved by elementary computations from the definition (15) and the properties of biorthogonality of the projectors $P^0_{j,\Omega}$ and $P_{j,\Gamma_{\rm D}}$.
□

The biorthogonal projector $P^*_{j,\Omega}$ defines a multiscale decomposition associated to the MRA $V_j(\Omega)$. Let

$$W^*_j(\Omega) := Q^*_{j,\Omega}V_{j+1}(\Omega) \text{ with } Q^*_{j,\Omega} = P^*_{j+1,\Omega} - P^*_{j,\Omega}$$

the corresponding wavelet spaces. The properties (16) imply the relations

$$(17)\quad \begin{cases} W^0_j(\Omega) = W^*_j(\Omega) \cap V^0_{j+1}(\Omega) \\ \gamma_{\rm D}W^*_j(\Omega) = W_j(\Gamma_{\rm D}) \end{cases}$$

that formally show the existence of a wavelet basis Ψ^*_Ω (associated to the complement spaces $W^*_j(\Omega)$) that admits the partition

$$(18)\quad \Psi^*_\Omega = \Psi^0_\Omega \cup \Psi^b_\Omega$$

so that the trace operator $2^{-\frac{|\lambda|}{2}}\gamma_{\rm D}$ defines a one to one mapping from the collection Ψ^b_Ω to $\Psi_{\Gamma_{\rm D}}$.

Let

$$\Psi_{\Gamma_{\rm D}} = \{\psi_\sigma,\ \sigma \in \Sigma_b\},$$

$$\Psi^b_\Omega = \{\psi^*_\lambda,\ \lambda \in \nabla_b\},$$

then the mapping $2^{-\frac{|\lambda|}{2}}\gamma_{\rm D}$ on these collections defines a one to one mapping on the set of subscripts $\lambda^r : \Sigma_b \to \nabla_b$ of inverse σ^r and such that $|\lambda^r(\sigma)| = |\sigma|$. This leads us to define formally a lifting operator $\mathcal{R}$

$$\mathcal{R}(g) = \sum_{\sigma\in\Sigma_b} 2^{-\frac{|\sigma|}{2}} \langle g, \tilde{\psi}_\sigma\rangle \psi^*_{\lambda^r(\sigma)}. \tag{19}$$

where $\tilde{\Psi}_{\Gamma_{\rm D}} = \{\tilde{\psi}_\sigma,\ \sigma \in \Sigma_b\}$ is the dual basis of $\Psi_{\Gamma_{\rm D}}$ and $< .,. >$ the duality product.

The stability properties of this lifting operator $\mathcal{R}$ in Besov spaces will be determined by the stability of the wavelet bases Ψ^*_Ω and $\Psi_{\Gamma_{\rm D}}$. From Sect. 2.1, this analysis is achieved through the derivation of direct and inverse estimates on the MRA spaces $V_j(\Omega)$, $V_j(\Gamma_{\rm D})$ that are studied in the next subsection.

2.2.1 Direct and inverse estimates on $V_j(\Omega)$ and $V_j(\Gamma_{\rm D})$. The following localness hypothesis introduced in [22] (see also [21]) will play a major role in the subsequent analysis.

Hypothesis 2.1 *For all $g_j \in V_j(\Gamma_{\rm D})$ we have*

$$\inf_{u_j\in V_j(\Omega):\gamma_{\rm D} u_j = g_j} \|u_j\|_{L^p(\Omega)} \sim 2^{-j/p}\|g_j\|_{L^p(\Gamma_{\rm D})}. \tag{20}$$

In practise, such a property will be a direct consequence of the following elementary lemma.

Lemma 2.1 *The localness hypothesis is satisfied as soon as there exists a family of (local) bases of $V_j(\Omega)$ stable in $L^p(\Omega)$ (in the sense of (9))*

$$\Phi_{j,\Omega} = \{\varphi_{j,x},\ x \in \mathcal{G}_{j,\Omega}\},$$

(with $\mathcal{G}_{j,\Omega}$ being typically a set of grid points), and so that

$$\Phi_{j,\Gamma_{\rm D}} = \{2^{-\frac{j}{2}}\gamma_{\rm D}\varphi_{j,x},\ x \in \mathcal{G}^b_{j,\Omega} \subset \mathcal{G}_{j,\Omega}\}$$

defines a family of bases of $V_j(\Gamma_{\rm D})$ stable in $L^p(\Gamma_{\rm D})$, while the remaining basis functions for $x \in \mathcal{G}_{j,\Omega}/\mathcal{G}^b_{j,\Omega}$ have a vanishing trace on $\Gamma_{\rm D}$.

We shall use in addition the following assumption of approximation on the MRA spaces $V_j(\Omega)$.

Hypothesis 2.2 *Let $p \in [1,\infty]$, there exist biorthogonal projectors $P_{j,\Omega}$ onto $V_j(\Omega)$ satisfying the direct estimate*

$$\|u - P_{j,\Omega}u\|_{L^p(\Omega)} \lesssim 2^{-js}\|u\|_{W^{s,p}(\Omega)} \tag{21}$$

for all $0 \le s \le n$ and $u \in W^{s,p}(\Omega)$.

Note that the same estimate in Besov norms $B^s_{p,q}(\Omega)$ is deduced by interpolation for $0 < s < n$.

We also assume the stability in L^p of the biorthogonal projectors $P^0_{j,\Omega}$ and $P_{j,\Gamma_{\rm D}}$.

Hypothesis 2.3 *Let* $p \in [1,\infty]$, *then* $\|P^0_{j,\Omega}u\|_{L^p(\Omega)} \lesssim \|u\|_{L^p(\Omega)}$.

Hypothesis 2.4 *Let* $p \in [1,\infty]$, *then* $\|P_{j,\Gamma_{\rm D}}g\|_{L^p(\Gamma_{\rm D})} \lesssim \|g\|_{L^p(\Gamma_{\rm D})}$.

Finally the MRA spaces $V_j(\Omega)$ will also be assumed to satisfy inverse estimates.

Hypothesis 2.5 *Let* $\tau(p) := \sup\{s : V_j(\Omega) \subset W^{s,p}(\Omega)\}$. *Then for all* $u_j \in V_j(\Omega)$ *and* $0 < s < \tau(p)$

(22) $$\|u_j\|_{B^s_{p,p}(\Omega)} \lesssim 2^{js}\|u_j\|_{L^p(\Omega)}.$$

We need to specify the definition of the function spaces on $\Gamma_{\rm D}$ that we will consider in the subsequent analysis. The natural setting in our context is to define them as trace spaces of function spaces on the domain rather than intrinsically by chart and partition of unity. Consequently, the space $L^p(\Gamma_{\rm D})$ is defined as usual by the Lebesgue measure on the boundary, and for $s > 0$, the Sobolev spaces $W^{s,p}(\Gamma_{\rm D})$ (resp. the Besov spaces $B^s_{p,q}(\Gamma_{\rm D})$) are defined as the set of traces on $\Gamma_{\rm D}$ of functions in $W^{s+1/p,p}(\Omega)$ (resp. $B^{s+1/p}_{p,q}(\Omega)$) endowed with the quotient norm

(23) $$\|g\|_{W^{s,p}(\Gamma_{\rm D})} := \inf_{u \in W^{s+\frac{1}{p},p}(\Omega):\, \gamma_{\rm D}u = g} \|u\|_{W^{s+\frac{1}{p},p}(\Omega)},$$

for Sobolev spaces and

(24) $$\|g\|_{B^s_{p,q}(\Gamma_{\rm D})} := \inf_{u \in B^{s+\frac{1}{p}}_{p,q}(\Omega):\, \gamma_{\rm D}u = g} \|u\|_{B^{s+\frac{1}{p}}_{p,q}(\Omega)},$$

for Besov spaces. We will naturally set $W^{0,p}(\Gamma_{\rm D}) := L^p(\Gamma_{\rm D})$.

For negative subscripts, these spaces are obtained by duality. In particular for $\Gamma_{\rm D} = \Gamma$ and $s < 0$

$$W^{s,p}(\Gamma) := W^{-s,p'}(\Gamma)',\ B^s_{p,q}(\Gamma) := B^{-s}_{p',q'}(\Gamma)',\ \frac{1}{p} + \frac{1}{p'} = 1,\ \frac{1}{q} + \frac{1}{q'} = 1.$$

Direct and inverse estimates on $\Gamma_{\rm D}$. The derivation of such estimates on $\Gamma_{\rm D}$ has been already studied in [22] with the restriction $s \leq 1$ since the author considers intrinsic definitions of the function spaces. Using the same arguments, we can derive in our context inverse and direct estimates for the ranges $s < \tau(p) - 1/p$ and $s < n - 1/p$.

Proposition 2.2 *Assuming Hypothesis 2.1 and 2.5, for all* $g_j \in V_j(\varGamma_{\rm D})$ *and* $0 < s < \tau(p) - \frac{1}{p}$*, we have the inverse estimate*

$$\|g_j\|_{B^s_{p,p}(\varGamma_{\rm D})} \lesssim 2^{js}\|g_j\|_{L^p(\varGamma_{\rm D})}. \tag{25}$$

Proof. From the definition of the quotient norm, for $s > 0$ we have

$$\begin{aligned} \|g_j\|_{B^s_{p,p}(\varGamma_{\rm D})} &\leq \inf_{u_j \in V_j(\varOmega):\, \gamma_{\rm D} u_j = g_j} \|u_j\|_{B^{s+\frac{1}{p}}_{p,p}(\varOmega)} \\ &\lesssim 2^{j(s+\frac{1}{p})} \inf_{u_j \in V_j(\varOmega):\, \gamma_{\rm D} u_j = g_j} \|u_j\|_{L^p(\varOmega)} \\ &\lesssim 2^{js}\|g_j\|_{L^p(\varGamma_{\rm D})} \end{aligned}$$

where the inverse estimate (22) and the localness assumption (20) have been successively applied. □

Proposition 2.3 *Let* $p \in [1, \infty]$*, assuming Hypothesis 2.1, 2.2, 2.4, 2.5 and* $\tau(p) > 1/p$*, for all* $g \in B^s_{p,p}(\varGamma_{\rm D})$ *and* $0 < s < n - \frac{1}{p}$*, we have the direct estimate*

$$\|g - P_{j,\varGamma_{\rm D}} g\|_{L^p(\varGamma_{\rm D})} \lesssim 2^{-js}\|g\|_{B^s_{p,p}(\varGamma_{\rm D})}. \tag{26}$$

Proof. From the stability Assumption 2.4 of the projector, it suffices to estimate the best approximation error in $V_j(\varGamma_{\rm D})$. For $s > 0$, let us consider any $u \in B^{s+1/p}_{p,p}$ so that $\gamma_{\rm D} u = g$. From Theorem 2.2, Hypothesis 2.2 and 2.5 ensure that the multiscale decomposition defined by the biorthogonal projector $P_{j,\varOmega}$ is stable in $B^s_{p,p}$ for the range $0 < s < \tau(p)$. Then

$$\begin{aligned} \inf_{g_j \in V_j(\varGamma_{\rm D})} \|g - g_j\|_{L^p(\varGamma_{\rm D})} &\leq \|\gamma_{\rm D}(u - P_{j,\varOmega} u)\|_{L^p(\varGamma_{\rm D})} \\ &= \|\gamma_{\rm D} \sum_{l \geq j} Q_{l,\varOmega} u\|_{L^p(\varGamma_{\rm D})} \\ &\leq \sum_{l \geq j} \|\gamma_{\rm D} Q_{l,\varOmega} u\|_{L^p(\varGamma_{\rm D})} \\ &\lesssim \sum_{l \geq j} 2^{l/p}\|Q_{l,\varOmega} u\|_{L^p(\varOmega)} \\ &\lesssim 2^{-js}\left(\sum_{l \geq j} (2^{l(s+1/p)}\|Q_{l,\varOmega} u\|_{L^p(\varOmega)})^p\right)^{1/p} \end{aligned} \tag{27}$$

where we have used the localness Assumption 2.1 and the Cauchy-Schwartz inequality.

To estimate the above right-hand side we need the following inegality that derives from Hypothesis 2.2 (see e.g. Chap. III of [4]): for all $0 < t < n$

$$\left(\sum_{l\geq 0}(2^{jt}\|Q_{l,\Omega}u\|_{L^p(\Omega)})^p\right)^{1/p} \lesssim \|u\|_{B^t_{p,p}(\Omega)}. \tag{28}$$

Hence we have for all $u \in B^{s+1/p}_{p,p}$ of trace $\gamma_{\mathrm{D}}u = g$.

$$\inf_{g_j \in V_j(\Gamma_{\mathrm{D}})} \|g - g_j\|_{L^p(\Gamma_{\mathrm{D}})} \lesssim 2^{-js}\|u\|_{B^{s+1/p}_{p,p}(\Omega)},$$

and we conclude from the definition of the quotient norm on Γ_{D}. □

Direct estimates for $P^*_{j,\Omega}$. They are obtained in the same spirit as previously by combining the estimates for $P_{j,\Omega}$ and $P_{j,\Gamma_{\mathrm{D}}}$.

Proposition 2.4 *Let* $p \in [1,\infty]$, *assuming Hypothesis 2.1, 2.2, 2.3, 2.4, 2.5 and* $\tau(p) > 1/p$, *then, for all* $u \in B^s_{p,p}(\Omega)$ *with* $1/p < s < n$, *we have the direct estimate*

$$\|u - P^*_{j,\Omega}u\|_{L^p(\Omega)} \lesssim 2^{-js}\|u\|_{B^s_{p,p}(\Omega)}. \tag{29}$$

Proof. Let $g \in B^{s-1/p}_{p,p}(\Gamma_{\mathrm{D}})$ be the trace of u. The projection error rewrites

$$u - P^*_{j,\Omega}u = (I - P^0_{j,\Omega})(u - r_j \circ P_{j,\Gamma_{\mathrm{D}}}g).$$

where r_j is any lifting from $V_j(\Gamma_{\mathrm{D}})$ into $V_j(\Omega)$. Since the projectors $P^0_{j,\Omega}$ are assumed to be stable in $L^p(\Omega)$, it suffices to estimate

$$\begin{aligned}
&\inf_{r_j}\|u - r_j \circ P_{j,\Gamma_{\mathrm{D}}}g\|_{L^p(\Omega)}\\
&\quad= \inf_{u_j \in V_j(\Omega):\, \gamma_{\mathrm{D}}u_j = P_{j,\Gamma_{\mathrm{D}}}g} \|u - u_j\|_{L^p(\Omega)}\\
&\quad\leq \|u - P_{j,\Omega}u\|_{L^p(\Omega)}\\
&\qquad+ \inf_{u_j \in V_j(\Omega):\, \gamma_{\mathrm{D}}u_j = P_{j,\Gamma_{\mathrm{D}}}g} \|P_{j,\Omega}u - u_j\|_{L^p(\Omega)}.
\end{aligned}$$

The estimate of the first term in the right-hand side is obtained directly from Hypothesis 2.2. For the second term, we use again the localness Assumption 2.1, so that

$$\begin{aligned}
\inf_{u_j \in V_j(\Omega):\, \gamma_{\mathrm{D}}u_j = P_{j,\Gamma_{\mathrm{D}}}g} &\|P_{j,\Omega}u - u_j\|_{L^p(\Omega)}\\
&\sim 2^{-j/p}\|\gamma_{\mathrm{D}}P_{j,\Omega}u - P_{j,\Gamma_{\mathrm{D}}}g\|_{L^p(\Gamma_{\mathrm{D}})}\\
&\lesssim 2^{-j/p}\|g - P_{j,\Gamma_{\mathrm{D}}}g\|_{L^p(\Gamma_{\mathrm{D}})}\\
&\quad+2^{-j/p}\|\gamma_{\mathrm{D}}(u - P_{j,\Omega}u)\|_{L^p(\Gamma_{\mathrm{D}})}.
\end{aligned}$$

In the above right-hand side, the first term is estimated from Proposition 2.3 and the trace theorem while the second is treated like in (27). □

Projectors $P_{j,\Gamma_{\rm D}}$ unstable in L^p $(p<1)$. For $p<1$, the localness assumption is usually still true since the stability of the bases $\Phi_{j,\Omega}$ and $\Phi_{j,\Gamma_{\rm D}}$ is classically verified for $p<1$ from an argument of local linear independence of the basis functions. Hence, proposition 2.2 still applies in that case.

In contrast, the stability in L^p of the projectors $P_{j,\Gamma_{\rm D}}$ and $P^0_{j,\Omega}$ is no longer verified and consequently the proof of the direct estimates for $P^*_{j,\Omega}$ and $P_{j,\Gamma_{\rm D}}$ does not extend to $p<1$.

For the projectors $P^*_{j,\Omega}$ the direct estimate should be obtained classically from a local polynomial reproduction argument of the projectors $P^*_{j,\Omega}$ (following the techniques of Chap. III in [4]).

Then, the estimate for $P_{j,\Gamma_{\rm D}}$ can be deduced by the following proposition which avoids a direct proof that would require handling quotient norms.

Proposition 2.5 *Let $0<p<1$ and $r\in[1,\infty]$ so that $P_{j,\Gamma_{\rm D}}$ is stable in $L^r(\Gamma_{\rm D})$. We assume the localness Property 2.1, $\tau(p)>1/p$, and the direct estimate*

$$\|u-P^*_{j,\Omega}u\|_{L^p(\Omega)} \lesssim 2^{-jt}\|u\|_{B^t_{p,p}(\Omega)}$$

for all $\frac{1}{r}+\frac{d}{p}-\frac{d}{r}<t<n$, as well as the inverse estimates

$$\|u_j\|_{B^t_{p,p}(\Omega)} \lesssim 2^{jt}\|u_j\|_{L^p(\Omega)}$$

for all $u_j\in V_j(\Omega)$ and $0<t<\tau(p)$. Then, the direct estimate

$$\|g-P_{j,\Gamma_{\rm D}}g\|_{L^p(\Gamma_{\rm D})} \lesssim 2^{-js}\|g\|_{B^s_{p,p}(\Gamma_{\rm D})} \tag{30}$$

is verified for the range $(d-1)(\frac{1}{p}-\frac{1}{r})<s<n-\frac{1}{p}$.

Proof. Note that the condition $s>(d-1)(\frac{1}{p}-\frac{1}{r})$ implies that the space $B^s_{p,p}(\Gamma_{\rm D})$ is imbedded in $L^r(\Gamma_{\rm D})$ so that the projectors $P_{j,\Gamma_{\rm D}}$ are defined on this space. Also it results that $t=s+1/p$ satisfies $\frac{t}{d}-1/p>\frac{1/r}{d}-1/r$ so that $P_{j,\Gamma_{\rm D}}\circ\gamma_{\rm D}$ is defined on the space $B^{s+1/p}_{p,p}(\Omega)$ which is imbedded in $B^{1/r}_{r,r}(\Omega)$.

From Theorem 2.1, the direct and inverse estimates assumptions imply the stability of the multiscale decomposition induced by the biorthogonal projector $P^*_{j,\Omega}$ in $B^t_{p,p}(\Omega)$ for the range $\frac{1}{r}+\frac{d}{p}-\frac{d}{r}<t<\min(n,\tau(p))$. Thus we can apply the proof of Proposition 2.3 to $\|g-P_{j,\Gamma_{\rm D}}g\|_{L^p(\Gamma_{\rm D})}$, replacing $P_{j,\Omega}$ by $P^*_{j,\Omega}$ and the triangular inegality by the p-triangular inegality. For all $(d-1)(\frac{1}{p}-\frac{1}{r}) < s < n-1/p$ and $u \in B^{s+1/p}_{p,p}(\Omega)$ so that $\gamma_{\rm D} u = g$, we have

$$
\begin{aligned}
\|g - P_{j,\Gamma_{\rm D}} g\|_{L^p(\Gamma_{\rm D})} &= \|\gamma_{\rm D}(u - P^*_{j,\Omega} u)\|_{L^p(\Gamma_{\rm D})} \\
&= \|\gamma_{\rm D} \sum_{l \geq j} Q^*_{l,\Omega} u\|_{L^p(\Gamma_{\rm D})} \\
&\leq \left(\sum_{l\geq j} \|\gamma_{\rm D} Q^*_{l,\Omega} u\|^p_{L^p(\Gamma_{\rm D})} \right)^{1/p} \\
&\lesssim \left(\sum_{l\geq j} (2^{l/p} \|Q^*_{l,\Omega} u\|_{L^p(\Omega)})^p \right)^{1/p} \\
&\lesssim 2^{-js} \left(\sum_{l\geq j} (2^{l(s+1/p)} \|Q^*_{l,\Omega} u\|_{L^p(\Omega)})^p \right)^{1/p} \\
&\lesssim 2^{-js} \|u\|_{B^{s+1/p}_{p,p}(\Omega)}. \qquad \square
\end{aligned}
\tag{31}
$$

2.2.2 Stability properties of the bases $\Psi_{\Gamma_{\rm D}}$, Ψ^*_{Ω} and the lifting $\mathcal{R}$.

Theorem 2.3 *Let $p, q \in [1,\infty]$, assuming hypothesis 2.1, 2.2, 2.3, 2.4, 2.5 and $\tau(p) > 1/p$, the multiscale decomposition defined by the biorthogonal projectors $P^*_{j,\Omega}$ is stable in $B^s_{p,q}(\Omega)$ for the range $1/p < s < \min(\tau(p), n)$ i.e.*

$$
\|u\|_{B^s_{p,q}(\Omega)} \sim \|P^*_{0,\Omega} u\|_{L^p(\Omega)} + \| \left(2^{js} \|Q^*_{j,\Omega} u\|_{L^p(\Omega)}\right)_{j\geq 0} \|_{l^q}, \tag{32}
$$

and the decomposition defined by the projectors $P_{j,\Gamma_{\rm D}}$ is stable in $B^s_{p,q}$ $(\Gamma_{\rm D})$ for the range $0 < s < \min(\tau(p), n) - 1/p$ i.e.

$$
\|g\|_{B^s_{p,q}(\Gamma_{\rm D})} \sim \|P_{0,\Gamma_{\rm D}} g\|_{L^p(\Gamma_{\rm D})} + \| \left(2^{js} \|Q_{j,\Gamma_{\rm D}} g\|_{L^p(\Gamma_{\rm D})}\right)_{j\geq 0} \|_{l^q}. \tag{33}
$$

Proof. The spaces $B^s_{p,q}(\Gamma_{\rm D})$ are interpolation spaces from their definition as quotient spaces of the interpolation spaces $B^s_{p,q}(\Omega)$. Hence, these norm equivalences are direct applications of Theorem 2.1. □

In order to derive the stability of the lifting operator (19) we need to assume the stability in L^p (9) of the wavelet bases $\Psi^*_{j,\Omega}$ and $\Psi_{j,\Gamma_{\rm D}}$ at the level j satisfying the partition (18). Such properties will be classically obtained by an argument of local linear independence.

Theorem 2.4 *Let $p \in [1,\infty]$. Assuming the hypothesis of Theorem 2.3 and the stability in L^p of $\Psi^*_{j,\Omega}$ and $\Psi_{j,\Gamma_{\rm D}}$, the lifting (19) is stable from $B^s_{p,q}(\Gamma_{\rm D})$*

into $B^{s+1/p}_{p,q}(\Omega)$*, i.e.*

$$\|\mathcal{R}(g)\|_{B^{s+1/p}_{p,q}(\Omega)} \lesssim \|g\|_{B^s_{p,q}(\Gamma_{\rm D})},$$

for the range $0 < s < \min(\tau(p), n) - 1/p$.

Proof. This is a direct consequence of the norm equivalences (10) for both the domain and the boundary.

For $p < 1$, the above theorem extends under the assumptions of Proposition 2.5.

Theorem 2.5 *Let* $p, q \in]0,1[$*, under the assumptions of Proposition 2.5, we have for all* $\frac{1}{r} + \frac{d}{p} - \frac{d}{r} < s < \min(\tau(p), n)$

(34) $$\|u\|_{B^s_{p,q}(\Omega)} \sim \|P^*_{0,\Omega} u\|_{L^p(\Omega)} + \| \left(2^{js} \|Q^*_{j,\Omega} u\|_{L^p(\Omega)}\right)_{j \geq 0} \|_{l^q},$$

and for all $(d-1)(\frac{1}{p} - \frac{1}{r}) < s < \min(\tau(p), n) - 1/p$

(35) $$\|g\|_{B^s_{p,q}(\Gamma_{\rm D})} \sim \|P_{0,\Gamma_{\rm D}} g\|_{L^p(\Gamma_{\rm D})} + \| \left(2^{js} \|Q_{j,\Gamma_{\rm D}} g\|_{L^p(\Gamma_{\rm D})}\right)_{j \geq 0} \|_{l^q}.$$

If in addition the bases $\Psi^*_{j,\Omega}$ *and* $\Psi_{j,\Gamma_{\rm D}}$ *are stable in* L^p*, then the lifting (19) satisfies*

$$\|\mathcal{R}(g)\|_{B^{s+1/p}_{p,q}(\Omega)} \lesssim \|g\|_{B^s_{p,q}(\Gamma_{\rm D})}$$

for the range $(d-1)(\frac{1}{p} - \frac{1}{r}) < s < \min(\tau(p), n) - 1/p$.

Proof. Again this is a direct application of Theorem 2.1. □

3 Realization by domain decomposition

So far, there are mainly two constructions of biorthogonal wavelet bases on a domain Ω or a manifold $\Gamma_{\rm D}$. The first one, not considered in this paper, starts from $\mathbb{P}_1$ finite element spaces V_j defined on nested triangulations (in 2D), and looks for complement spaces W_j so that the primal and dual wavelet bases satisfy the desired properties of stability and supports (see for these approaches [27,15,8]).

The second construction starts from compactly supported biorthogonal wavelets on the real line as defined in [6]. Then, the construction on a fairly general domain Ω (or manifold) is achieved by the following steps

$$\mathbb{R} \rightarrow]0,1[\rightarrow]0,1[^d \rightarrow \Omega = \cup \Omega_i,\ \Omega_i \sim]0,1[^d,$$

where the first step is obtained by restriction and adaptations at the edges, the second step by tensor product, and the third step uses domain decomposition techniques. One of the main interest of this second approach is to benefit from the large class of biorthogonal wavelets on the line, with arbitrary orders of approximation and regularity of the MRA spaces as well as a construction on tensor product domains that preserves the nice properties that hold on the line.

In Sect. 3.1 we briefly recall the basic features of biorthogonal wavelets on the line and on the interval $]0,1[$. In particular we focus on a construction with homogeneous Dirichlet boundary conditions at the edges 0 or 1, which extends the construction of [2] in order to include the case of discontinuous generators on the line ϕ or $\tilde{\phi}$. Biorthogonal wavelet bases on $]0,1[^d$ are then classically derived by tensor products.

In Sect. 3.2.1 we recall the domain decomposition strategy introduced in [14], and in Sect. 3.2.1 we propose a new construction of the matching wavelets at the interfaces that differ from the one introduced in [2] and will be used in Sect. 3.3 for the construction of the lifting wavelets.

3.1 Biorthogonal wavelet bases on $]0,1[^d$

3.1.1 Biorthogonal wavelet bases on the line. The construction on the line ([6,16]) starts from a pair of compactly supported scaling functions $(\phi, \tilde{\phi})$ of supports $[-m_0, m_1]$ and $[-\tilde{m}_0, \tilde{m}_1]$ (with integer edges) satisfying (i) the two scale relations $\phi = \sum_{m=-m_0}^{m_1} \sqrt{2} h_m \phi(2.-m)$, $\tilde{\phi} = \sum_{m=-\tilde{m}_0}^{\tilde{m}_1} \sqrt{2}\tilde{h}_m \tilde{\phi}(2.-m)$ for finite masks or filters h and $\tilde{h}$, and (ii) the biorthogonality relations

$$\langle \phi, \phi(x-k) \rangle = \delta_k \text{ for all } k \in \mathbb{Z}.$$

A pair of primal and dual MRA spaces $(V_j(\mathbb{R}), \tilde{V}_j(\mathbb{R}))$ are spanned by the biorthogonal compactly supported bases

$$\begin{aligned} \Phi_j &= \{\phi_{j,k} := 2^{\frac{j}{2}} \phi(2^j.-k), k \in \mathbb{Z}\}, \\ \tilde{\Phi}_j &= \{\tilde{\phi}_{j,k} := 2^{\frac{j}{2}} \tilde{\phi}(2^j.-k), k \in \mathbb{Z}\}. \end{aligned} \tag{36}$$

It is shown in [6] that in that case the primal and dual wavelet spaces $(W_j(\mathbb{R}), \widetilde{W}_j(\mathbb{R}))$ are spanned by the biorthogonal compactly supported wavelet bases

$$\begin{aligned} \Psi_j &= \{\psi_{j,k} := 2^{\frac{j}{2}} \psi(2^j.-k), k \in \mathbb{Z}\}, \\ \tilde{\Psi}_j &= \{\tilde{\psi}_{j,k} := 2^{\frac{j}{2}} \tilde{\psi}(2^j.-k), k \in \mathbb{Z}\}, \end{aligned} \tag{37}$$

where $\psi := \sum \sqrt{2} g_m \phi(2.-m)$ and $\tilde{\psi} := \sum \sqrt{2}\tilde{g}_m \phi(2.-m)$ are the mother wavelets obtained from the wavelet filters $g_m = (-1)^m \tilde{h}_{1-m}$ and $\tilde{g}_m = (-1)^m h_{1-m}$.

In [6] such compactly biorthogonal generators $(\phi, \tilde{\phi})$ are built with arbitrary smoothness. We shall denote by $\tau(p)$ and $\tilde{\tau}(p)$ the supremum of their smoothness measured in $W^{s,p}$ (for $p \geq 1$ and in $B^s_{p,p}$ for $p \in]0,1[$) with $\tau(2) > 0$ and $\tilde{\tau}(2) > 0$. In addition, these generators have arbitrary order of approximations n and $\tilde{n}$ in the sense that the integer translates of ϕ and $\tilde{\phi}$ span the polynomials of $\mathbb{P}_{n-1}$ and $\mathbb{P}_{\tilde{n}-1}$.

The following proposition (see e.g. [4]) states the direct and inverse estimates in $V_j(\mathbb{R})$ and $\widetilde{V}_j(\mathbb{R})$ as well as the stability in L^p of the bases at the level j.

Proposition 3.1 *Let $p \in]0,\infty]$, if $\phi \in L^p$ then Φ_j and Ψ_j are L^p stable in the sense of (9).*

Let $r \in [1,\infty]$ so that $\phi \in L^r$ and $\tilde{\phi} \in L^{r'}$, then

$$(38) \qquad \|f - P_j f\|_{L^p} \lesssim 2^{-js}\|f\|_{B^s_{p,p}} \text{ for all } \frac{1}{p} - \frac{1}{r} < s \leq n.$$

If $p = r \in [1,\infty]$, these estimates are obtained in $W^{s,p}$ norms for the range $0 \leq s \leq n$ (in particular the projectors are stable in L^p).

Let $p \in]0,\infty]$ and assume $0 \leq s < \tau(p)$, then for all $f_j \in V_j(\mathbb{R})$

$$(39) \qquad \|f_j\|_{B^s_{p,p}} \lesssim 2^{js}\|f_j\|_{L^p}.$$

Again for $p \in [1,\infty]$ these estimates also hold in $W^{s,p}$ norms.

The same properties hold for the dual MRA spaces, projectors and bases with n and $\tau(p)$ replaced by $\tilde{n}$ and $\tilde{\tau}(p)$.

Note that the proof of the stability in L^p of Φ_j and Ψ_j derives from the local linear independence of the basis functions proved in [23].

From this proposition, the stability properties of the wavelet bases Ψ and $\tilde{\Psi}$ are readily obtained from theorems 2.1 for $p \in]0,1[$ and 2.2 for $p \geq 1$.

3.1.2 Biorthogonal wavelet bases on the interval. Starting from a pair of biorthogonal generators on the line $(\phi, \tilde{\phi})$, there are many constructions of biorthogonal wavelets on the interval $]0,1[$ (see [1,7,25,13]). All these constructions share the basic ideas introduced in [7] and [1] to retain for the definition of the MRA spaces V_j and $\widetilde{V}_j$

(i) the "interior" scaling functions on the line which supports are in $[\delta_0 2^{-j}, 1 - \delta_1 2^{-j}]$ for V_j and $[\tilde{\delta}_0 2^{-j}, 1 - \tilde{\delta}_1 2^{-j}]$ for $\widetilde{V}_j$, where $\delta = (\delta_0, \delta_1)$, $\tilde{\delta} = (\tilde{\delta}_0, \tilde{\delta}_1)$ are pairs of non negative integer parameters.

(ii) at the edges 0 and 1 only the n, for V_j, and $\tilde{n}$, for $\widetilde{V}_j$, truncated linear combinations of scaling functions that correspond to the reproduction on $]0,1[$ of the monomials of degrees $\alpha = 0,\ldots,n-1$, for V_j, and $\alpha = 0,\ldots,\tilde{n}-1$, for $\widetilde{V}_j$. Then, the optimal orders of approximation n and $\tilde{n}$ and the nestedness are preserved.

Let $\mathrm{CL} = (\mathrm{CL}_0, \mathrm{CL}_1)$ where $-1 \leq \mathrm{CL}_\varepsilon \leq n-2$, $\varepsilon = 0,1$, and similarly $\widetilde{\mathrm{CL}} = (\widetilde{\mathrm{CL}}_0, \widetilde{\mathrm{CL}}_1)$ with n replaced by $\tilde{n}$. This strategy enables in addition to take into account homogeneous boundary conditions at the edges 0 and 1 of orders CL for the primal MRA and $\widetilde{\mathrm{CL}}$ for the dual MRA. For that purpose it suffices to retain in the previous definition only the monomials of degrees $\alpha = \mathrm{CL}_\varepsilon + 1, \ldots, n-1$ and $\beta = \widetilde{\mathrm{CL}}_\varepsilon + 1, \ldots, \tilde{n}-1$, $\varepsilon = 0,1$ at the edges $\varepsilon = 0,1$. In these notations, CL_ε or $\widetilde{\mathrm{CL}}_\varepsilon = -1$ hence denotes no boundary conditions.

Recalling that $\operatorname{supp}\phi = [-m_0, m_1]$, the primal MRA $V_j^{\delta,\mathrm{CL}}$ is defined as follows.

$$\text{(40)}\quad \begin{cases} \varPhi_j^{\mathrm{int}\delta} := \Big\{\varphi_{j,k} = \phi_{j,k},\ k = m_0+\delta_0, \ldots, 2^j - m_1 - \delta_1\Big\}, \\ \varPhi_j^{(0),\delta_0,\mathrm{CL}_0} := \Big\{\varphi_{j,\alpha}^{(0),\delta_0} = \displaystyle\sum_{k=-m_1+1}^{m_0-1+\delta_0} \langle x^\alpha, \tilde{\phi}_{0,k}\rangle\, \phi_{j,k}, \\ \qquad\qquad \alpha = \mathrm{CL}_0 + 1, \ldots, n-1\Big\}, \\ \varPhi_j^{(1),\delta_1,\mathrm{CL}_0} := \Big\{\varphi_{j,\alpha}^{(1),\delta_1} = \displaystyle\sum_{k=-m_0+1}^{m_1-1+\delta_1} \langle (-1)^\alpha x^\alpha, \tilde{\phi}_{0,-k}\rangle\, \phi_{j,2^j-k}, \\ \qquad\qquad \alpha = \mathrm{CL}_1 + 1, \ldots, n-1\Big\}, \\ V_j^{\delta,\mathrm{CL}} := \mathrm{S}\,(\varPhi_j^{(0),\delta_0,\mathrm{CL}_0}) \oplus \mathrm{S}\left(\varPhi_j^{\mathrm{int}\delta}\right) \oplus \mathrm{S}\,(\varPhi_j^{(1),\delta_1,\mathrm{CL}_1}), \end{cases}$$

and similarly for the dual MRA $\widetilde{V}_j^{\tilde{\delta},\widetilde{\mathrm{CL}}}$ with $\tilde{\delta} = (\tilde{\delta}_0, \tilde{\delta}_1)$. The definitions of $V_j^{\delta,\mathrm{CL}}$ and $\widetilde{V}_j^{\tilde{\delta},\widetilde{\mathrm{CL}}}$ hold whenever $j \geq j_0$ for a coarse level j_0 so that there is no overlapping of the subscripts in the definitions of $\varPhi_j^{\mathrm{int}\delta}$ and $\tilde{\varPhi}_j^{\mathrm{int},\tilde{\delta}}$.

In order to obtain biorthogonal MRA spaces, the parameters δ and $\tilde{\delta}$ are chosen to match the dimensions of V_j^δ and $\widetilde{V}_j^{\tilde{\delta}}$ separately at both edges i.e. for $\varepsilon = 0,1$

$$\text{(41)}\qquad m_\varepsilon + \delta_\varepsilon + -n + \mathrm{CL}_\varepsilon = \tilde{m}_\varepsilon + \tilde{\delta}_\varepsilon - \tilde{n} + \widetilde{\mathrm{CL}}_\varepsilon.$$

For $\varepsilon = 0,1$, let us consider the matrices

$$M^{\varepsilon,\mathrm{CL}_\varepsilon,\widetilde{\mathrm{CL}}_\varepsilon} = (\langle f_{j,k}^\varepsilon, \tilde{f}_{j,k'}^\varepsilon\rangle)_{k,k'=0,\ldots,\max(n-\mathrm{CL}_\varepsilon-1,\tilde{n}-\widetilde{\mathrm{CL}}_\varepsilon-1)-1},$$

where $f_{j,\alpha}^0 = \varphi_{j,\alpha+\mathrm{CL}_0+1}^{\delta_0,(0)}$ for $\alpha = 0,\ldots,n-\mathrm{CL}_0-2$, and $f_{j,k}^0 = \phi_{j,m_0+\delta_0+(k-n+\mathrm{CL}_0+1)}$ for $k = n-\mathrm{CL}_0-1,\ldots,\max(n-\mathrm{CL}_0-1,\tilde{n}-$

$\widetilde{\mathrm{CL}}_0 - 1) - 1$ (if $n - \mathrm{CL}_0 < \tilde{n} - \widetilde{\mathrm{CL}}_0$), and analogous definitions for $f^1_{j,k}$ $\tilde{f}^0_{j,k}$ and $\tilde{f}^1_{j,k}$.

Then, the MRA spaces admit local biorthogonal Riesz bases $\Phi_j^{\delta,\mathrm{CL}}$ and $\tilde{\Phi}_j^{\delta,\widetilde{\mathrm{CL}}}$ (cf. (11)) if and only if these matrices are non singular which will always be assumed in the following. The biorthogonal bases $\Phi_j^{\delta,\mathrm{CL}}$ and $\tilde{\Phi}_j^{\delta,\widetilde{\mathrm{CL}}}$ will include a fixed number $\max(n - \mathrm{CL}_\varepsilon - 1, \tilde{n} - \widetilde{\mathrm{CL}}_\varepsilon - 1) - 1$ of modified scaling functions at the edges $\varepsilon = 0, 1$ while all the other basis functions are scaling functions on the real line $\phi_{j,k}$.

The construction of local biorthogonal (in the sense of (11)) Riesz bases $(\Psi_j, \tilde{\Psi}_j)$ of the wavelet spaces $(W_j, \widetilde{W}_j)$ is addressed in [25] and in [13] using a different technique. In both cases, as for the scaling function bases, one ends up with a fixed number of modified wavelets at the edges while all the other wavelets are wavelets on the real line (37).

Proposition 3.1, stated on the line, still applies for this construction on the interval with the same ranges for the inverse and direct estimates (see e.g. [4]). Note that in the case $\mathrm{CL} \neq (-1,-1)$ (resp. $\widetilde{\mathrm{CL}} \neq (-1,-1)$) the direct estimates have to be written with the function spaces satisfying homogeneous boundary conditions of orders CL (resp. $\widetilde{\mathrm{CL}}$) at the edges.

Dirichlet homogeneous boundary conditions. In our setting we need to deal specifically with homogeneous Dirichlet boundary conditions, say for the primal MRA spaces, at the edges 0 or 1, i.e. $\mathrm{CL} = (-1,0), (-1,-1)$ or $(0,-1)$. In that particular case, a symmetric choice of the boundary conditions for the dual MRA i.e.

$$\widetilde{\mathrm{CL}} = \mathrm{CL}$$

enables to relate simply the biorthogonal wavelets bases of these spaces to those obtained for $\mathrm{CL} = \widetilde{\mathrm{CL}} = (-1,-1)$ i.e. without boundary conditions. This is specified by the following propositions which has been also derived in [2] in the case of continuous generators ϕ and $\tilde{\phi}$. It is stated for $\mathrm{CL} = (0,0)$ since both edges are treated separatly. The proof is postponed to the Appendix.

Let $(V_j^\delta, \widetilde{V}_j^{\tilde{\delta}})$ denote biorthogonal MRA spaces without boundary conditions so that $m_\varepsilon + \delta_\varepsilon - n = \tilde{m}_\varepsilon + \tilde{\delta}_\varepsilon - \tilde{n},\ \varepsilon = 0, 1$ and the matrices $M^{\varepsilon,-1,-1}$, $\varepsilon = 0, 1$ are assumed to be non singular.

Proposition 3.2 *Let* $\mathrm{CL_D} = (0,0)$*, we assume that the MRA spaces*

$$V_j^{\delta,\mathrm{CL_D}} \text{ and } \widetilde{V}_j^{\tilde{\delta},\mathrm{CL_D}}$$

are biorthogonal i.e. that the matrices $M^{\varepsilon,0,0}$, $\varepsilon = 0, 1$ *are non singular. Then, there exist biorthogonal bases*

(42) $$\begin{cases} \Phi_j^{\delta} = \{\varphi_{j,k}, k = 0, \ldots, \#\Delta_j - 1\} \\ \tilde{\Phi}_j^{\tilde{\delta}} = \{\tilde{\varphi}_{j,k}, k = 0, \ldots, \#\Delta_j - 1\} \end{cases}$$

of $(V_j^{\delta}, \widetilde{V}_j^{\tilde{\delta}})$ *so that*

(43) $$\begin{cases} \Phi_j^{\delta,\mathrm{CL_D}} = \{\varphi_{j,k}, k = 1, \ldots, \#\Delta_j - 2\} \\ \tilde{\Phi}_j^{\tilde{\delta},\mathrm{CL_D}} = \{\tilde{\varphi}_{j,k}, k = 1 \ldots, \#\Delta_j - 2\}. \end{cases}$$

are biorthogonal bases of $V_j^{\delta,\mathrm{CL_D}}$ *and* $\widetilde{V}_j^{\tilde{\delta},\mathrm{CL_D}}$.

Proposition 3.3 *Under the assumptions of the previous proposition, there exist biorthogonal wavelet bases* Ψ_j *and* $\tilde{\Psi}_j$ *of the spaces* $W_j = V_{j+1}^{\delta} \cap (\widetilde{V}_j^{\tilde{\delta}})^{\perp}$ *and* $\widetilde{W}_j = \widetilde{V}_{j+1}^{\tilde{\delta}} \cap (V_j^{\delta})^{\perp}$

(44) $$\begin{cases} \Psi_j = \{\psi_{j,k}, k = 0, \ldots, 2^j - 1\} \\ \tilde{\Psi}_j = \{\tilde{\psi}_{j,k}, k = 0 \ldots, 2^j - 1\} \end{cases}$$

so that

(45) $$\begin{cases} \Psi_j^{\mathrm{CL_D}} = \Big\{\frac{1}{\sqrt{2}}(\psi_{j,0} - \varphi_{j,0}),\ \psi_{j,k}, k = 1, \ldots, 2^j - 2, \\ \qquad\qquad \frac{1}{\sqrt{2}}(\psi_{j,2^j-1} - \varphi_{j,\#\Delta_j-1})\Big\} \\ \tilde{\Psi}_j^{\mathrm{CL_D}} = \Big\{\frac{1}{\sqrt{2}}(\tilde{\psi}_{j,0} - \tilde{\varphi}_{j,0}),\ \tilde{\psi}_{j,k}, k = 1, \ldots, 2^j - 2, \\ \qquad\qquad \frac{1}{\sqrt{2}}(\tilde{\psi}_{j,2^j-1} - \tilde{\varphi}_{j,\#\Delta_j-1})\Big\} \end{cases}$$

are biorthogonal bases of the wavelet spaces

$$W_j^{\mathrm{CL_D}} = V_{j+1}^{\delta,\mathrm{CL_D}} \cap (\widetilde{V}_j^{\tilde{\delta},\mathrm{CL_D}})^{\perp} \text{ and } \widetilde{W}_j^{\mathrm{CL_D}} = \widetilde{V}_{j+1}^{\tilde{\delta},\mathrm{CL_D}} \cap (V_j^{\delta,\mathrm{CL_D}})^{\perp}.$$

3.1.3 Tensor product wavelets. Starting from any biorthogonal wavelets on the interval $]0, 1[$, biorthogonal wavelets on the domain $\hat{\Omega} =]0, 1[^d$ are derived by tensor products. We will consider for simplicity $d = 2$ since any dimension is treated similarly.

Let $(\Phi_j, \tilde{\Phi}_j)$, $(\Psi_j, \tilde{\Psi}_j)$ denote biorthogonal scaling function and wavelet bases on the interval $]0, 1[$ spanning the MRA and wavelet spaces $(V_j, \widetilde{V}_j)$,

$(W_j, \widetilde{W}_j)$. Also we denote by $(P_j, \widetilde{P}_j)$ the corresponding biorthogonal projectors.

The tensor product bases

$$\begin{cases} \Phi_{j,\hat{\Omega}} := \Phi_j \otimes \Phi_j \\ \tilde{\Phi}_{j,\hat{\Omega}} := \tilde{\Phi}_j \otimes \tilde{\Phi}_j, \end{cases}$$

are biorthogonal and span the biorthogonal MRA spaces

$$\begin{cases} V_j(\hat{\Omega}) := V_j \otimes V_j, \\ \widetilde{V}_j(\hat{\Omega}) := \widetilde{V}_j \otimes \widetilde{V}_j \end{cases}$$

of $L^2(\hat{\Omega})$. They define the biorthogonal projectors $P_{j,\hat{\Omega}} := P_j \otimes P_j$ and $\widetilde{P}_{j,\hat{\Omega}} := \widetilde{P}_j \otimes \widetilde{P}_j$ stable in $L^2(\hat{\Omega})$.

The complement or wavelet spaces $(W_j(\hat{\Omega}), \widetilde{W_j}(\hat{\Omega}))$, i.e. the ranges of the projectors $Q_{j,\hat{\Omega}} = P_{j+1,\hat{\Omega}} - P_{j,\hat{\Omega}}$ and $\widetilde{Q}_{j,\hat{\Omega}} = \widetilde{P}_{j+1,\hat{\Omega}} - \widetilde{P}_{j,\hat{\Omega}}$, are spanned by the "canonical tensor product" biorthogonal wavelet Riesz bases defined by

$$\Psi_{j,\hat{\Omega}} = \Phi_j \otimes \Psi_j \,\cup\, \Psi_j \otimes \Phi_j \,\cup\, \Psi_j \otimes \Psi_j, \tag{46}$$

for the primal basis and similarly for the dual basis $\tilde{\Psi}_{j,\hat{\Omega}}$.

Again, Proposition 3.1 still applies in this multivariate case (see e.g. [4]) and the stability properties of the wavelet bases are deduced from theorem 2.1 for $p \in]0,1[$ and 2.2 for $p \geq 1$.

3.2 Domain decomposition

In these techniques, the domain Ω of $\mathbb{R}^d$ (or more generally a manifold of dimension d) is assumed to be decomposed into disjoint subdomains $(\Omega_i)_{i=1,\dots,N}$ so that

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i.$$

In addition, for each subdomain, there exists a regular one to one parametrization κ_i from $\overline{\hat{\Omega}}$ to $\overline{\Omega}_i$, where $\hat{\Omega}$ is the reference domain $]0,1[^d$. We shall denote by J_i the Jacobian $|\partial\kappa_i \circ \kappa_i^{-1}|$ defined on Ω_i.

Given biorthogonal wavelet bases $(\Psi_{\hat{\Omega}}, \tilde{\Psi}_{\hat{\Omega}})$ on the reference domain, their push-forwards $(\Psi_{\hat{\Omega}} \circ \kappa_i^{-1}, \tilde{\Psi}_{\hat{\Omega}} \circ \kappa_i^{-1})$ onto the subdomain Ω_i define biorthogonal wavelet bases on Ω_i with the same range of stability, where the duality pairing is induced by the equivalent scalar product $(f,g)_i = \int_{\hat{\Omega}} f \circ \kappa_i(\hat{x}) g \circ \kappa_i(\hat{x}) d\hat{x}$.

These bases have to be matched at the interfaces of the subdomains in order to define biorthogonal wavelet bases $(\Psi_\Omega, \tilde{\Psi}_\Omega)$ on the whole domain Ω. The regularity of the matching will crucially determine the range of stability of Ψ_Ω (and $\tilde{\Psi}_\Omega$), and impose (i) conforming assumptions on the domain sss decomposition $(\Omega_i, \kappa_i)_{i=1,\dots,N}$ and (ii) assumptions on the choice of the biorthogonal bases on the reference domain.

We follow here the strategy introduced in [18], [19], and in more generality in [14] and also in [2]. For each subdomain, biorthogonal MRA $(V_j(\Omega_i), \widetilde{V}_j(\Omega_i))$ are derived from tensor product MRA on the reference domain obtained from the same biorthogonal generators on the line $(\phi, \tilde{\phi})$. Assuming the C^0 conformity (specified below) of the domain decomposition, biorthogonal MRA $(V_j(\Omega), \widetilde{V}_j(\Omega))$ on the whole domain Ω are obtained by a C^0 matching at the interfaces for both the primal and dual MRA. The MRA spaces can include homogeneous Dirichlet boundary conditions on a subset $\Gamma_{\rm D}$ of $\Gamma := \partial\Omega$ satisfying some conformity assumption, provided that the same boundary conditions are retained for both the primal and dual MRA. This approach enables to cover in general the range of stability $-1/p < s < 1 + 1/p$ of the multiscale decomposition.

One specific difficulty of these domain decomposition techniques is the explicit construction of the biorthogonal wavelet bases $\Psi_{j,\Omega}$ $\tilde{\Psi}_{j,\Omega}$ of the complement spaces, and in particular the matching wavelets at the interfaces of the decomposition. We introduce in Sect. 3.2.2 a new construction which differs from the one proposed in [2]. This analysis is based on a natural hierarchy of the interfaces. In addition, the same ideas will readily apply in Sect. 3.3 for the construction of the lifting wavelets of $\Psi^b_{j,\Omega}$.

C^0 conformity assumptions on the domain decomposition $(\Omega_i, \kappa_i)_{i=1,\dots,N}$. For $p = 0, \dots, d-1$, we call p-face of a subdomain Ω_i, the image $\kappa_i(\sigma)$ of a face σ of dimension p of $\hat{\Omega}$.

For each pair $(i,l) \in \{1, \dots, N\}^2$, the intersection $\bar{\Omega}_i \cap \bar{\Omega}_l$ is assumed to be either empty or a p-face of both subdomains Ω_i and Ω_l. These p-faces are the interfaces of the domain decomposition.

For any p-face $\sigma_{i,l}$ common to the subdomains Ω_i and Ω_l, so that $\sigma_{i,l} = \kappa_i(\hat{\sigma}) = \kappa_l(\hat{\sigma}')$, we assume that

$$\kappa_i^{-1} \circ \kappa_l \tag{47}$$

defines an affine isometry from $\hat{\sigma}'$ to $\hat{\sigma}$.

In addition, the boundary Γ of Ω splits into two disjoint open subsets $\Gamma_{\rm D}$ and $\Gamma_{\rm N}$ of Γ so that $\Gamma = \bar{\Gamma}_{\rm D} \cup \bar{\Gamma}_{\rm N}$, and for all $i = 1, \dots, N$,

$$\partial\Omega_i \cap \bar{\Gamma}_{\rm D},\ \partial\Omega_i \cap \bar{\Gamma}_{\rm N},$$

are either empty or the unions of p-faces of Ω_i.

3.2.1 Biorthogonal MRA on Ω. We consider a pair of biorthogonal generators on the line $(\phi, \tilde{\phi})$ where ϕ is continuous. We assume that these generators are symmetric in the sense that ϕ (and $\tilde{\phi}$) verifies $\phi(-x) = \phi(x)$ or $\phi(1-x) = \phi(x)$. This is for example the case for the commonly used family of spline biorthogonal wavelets [6].

The biorthogonal scaling functions and wavelets on the interval $]0,1[$ are built under the assumptions of Propositions 3.2 and 3.3 with the symmetric choice $\delta_0 = \delta_1$ and $\tilde{\delta}_0 = \tilde{\delta}_1$. Then, it is always possible to choose the bases $(\Phi_j^{\delta}, \tilde{\Phi}_j^{\tilde{\delta}})$ (42) and $(\Psi_j, \tilde{\Psi}_j)$ (44) so that in addition

$$(48)\quad \begin{cases} \varphi_{j,k}(1-x) = \varphi_{j,d_j-k}(x),\ \tilde{\varphi}_{j,k}(1-x) = \tilde{\varphi}_{j,d_j-k}(x), \\ \qquad\qquad k = 0, \dots, d_j \\ \psi_{j,k}(1-x) = \psi_{j,2^j-k-1}(x),\ \tilde{\psi}_{j,k}(1-x) = \tilde{\psi}_{j,2^j-k-1}(x), \\ \qquad\qquad k = 0, \dots, 2^j - 1 \end{cases}$$

where $d_j := \#\Delta_j - 1$. The biorthogonal bases with homogeneous Dirichlet boundary conditions

$$\mathrm{CL} = \widetilde{\mathrm{CL}} \in \mathcal{B} := \{(-1,-1), (-1,0), (0,-1), (0,0)\},$$

at the edges 0 or 1 are deduced from Propositions 3.2 and 3.3 and will be denoted by $(\Phi_j^{\delta,\mathrm{CL}}, \tilde{\Phi}_j^{\tilde{\delta},\mathrm{CL}})$ and $(\Psi_j^{\mathrm{CL}}, \tilde{\Psi}_j^{\mathrm{CL}})$.

In order to describe the matching of the scaling functions at the interfaces, it is convenient to replace the subscripts $k = 0, \dots, d_j$ by the disjoint points $\xi_{j,k} \in [0,1]$ so that $\xi_{j,0} = 0$ and $\xi_{j,d_j} = 1$. Then the scaling function bases on the interval rewrite

$$(49)\quad \begin{cases} \Phi_j^{\delta,\mathrm{CL}} = \left\{\varphi_{j,\xi_{j,k}},\ k \in \Delta_j^{\mathrm{CL}}\right\} \\ \tilde{\Phi}_j^{\delta,\mathrm{CL}} = \left\{\tilde{\varphi}_{j,\xi_{j,k}},\ k \in \Delta_j^{\mathrm{CL}}\right\}. \end{cases}$$

By tensor product techniques we obtain biorthogonal scaling function bases on the reference domain $\hat{\Omega}$ satisfying the boundary conditions of order

$\mathrm{cl} = (\mathrm{cl}^m, m = 1, \ldots, d) \in \mathcal{B}^d$. They are denoted by

$$(50) \qquad \begin{cases} \Phi^{\mathrm{cl}}_{j,\hat{\Omega}} = \bigotimes_{m=1}^{d} \Phi_j^{\delta,\mathrm{cl}^m} = \left\{ \varphi_{j,\hat{x}_{j,k}},\ k \in \Delta_j^{\mathrm{cl}} \right\} \\ \tilde{\Phi}^{\mathrm{cl}}_{j,\hat{\Omega}} = \bigotimes_{m=1}^{d} \tilde{\Phi}_j^{\delta,\mathrm{cl}^m} = \left\{ \tilde{\varphi}_{j,\hat{x}_{j,k}},\ k \in \Delta_j^{\mathrm{cl}} \right\}, \end{cases}$$

where $\hat{x}_{j,k} = (\xi_{j,k_1}, \ldots, \xi_{j,k_d})$ for all $k = (k_1, \ldots, k_d) \in \{0, \ldots, d_j\}^d$.

The scaling function bases on the subdomains Ω_i are the push-forwards of the bases defined on the reference domain denoted by

$$(51) \qquad \begin{cases} \Phi^{\mathrm{cl}}_{j,\Omega_i} = \left\{ \varphi^{(i)}_{j,x^i_{j,k}},\ k \in \Delta_j^{\mathrm{cl}} \right\} \\ \tilde{\Phi}^{\mathrm{cl}}_{j,\Omega_i} = \left\{ \tilde{\varphi}^{(i)}_{j,x^i_{j,k}},\ k \in \Delta_j^{\mathrm{cl}} \right\}, \end{cases}$$

where $x^i_{j,k} = \kappa_i(\hat{x}_{j,k}),\ k \in \{0, \ldots, d_j\}^d$ and $\varphi^{(i)}_{j,x^i_{j,k}} = \varphi_{j,\hat{x}_{j,k}} \circ \kappa_i^{-1}$, $\tilde{\varphi}^{(i)}_{j,x^i_{j,k}} = \tilde{\varphi}_{j,\hat{x}_{j,k}} \circ \kappa_i^{-1}$. They are biorthogonal for the scalar product $(.,.)_i$.

In order to take into account the homogeneous Dirichlet boundary conditions on Γ_{D}, we will denote by

$$\Delta_j^{(i)} = \{k \in \{0, \ldots, d_j\}^d :\ x^i_{j,k} \notin \partial\Omega_i \cap \bar{\Gamma}_{\mathrm{D}}\}$$

and by

$$(52) \qquad \begin{cases} \Phi_{j,\Omega_i} = \left\{ \varphi^{(i)}_{j,x^i_{j,k}},\ k \in \Delta_j^{(i)} \right\} \\ \tilde{\Phi}_{j,\Omega_i} = \left\{ \tilde{\varphi}^{(i)}_{j,x^i_{j,k}},\ k \in \Delta_j^{(i)} \right\}, \end{cases}$$

the scaling function biorthogonal bases including these boundary conditions. They span the biorthogonal MRA spaces

$$V_j(\Omega_i),\ \widetilde{V}_j(\Omega_i).$$

We consider the grids of points

$$(53) \qquad \mathcal{G}_j = \left\{ x^i_{j,k},\ k \in \{0, \ldots, d_j\}^d,\ i = 1, \ldots, N \right\}.$$

and

$$(54) \qquad \mathcal{G}_{j,\Omega} = \left\{ x^i_{j,k},\ k \in \Delta_j^{(i)},\ i = 1, \ldots, N \right\} \subset \mathcal{G}_j.$$

The primal MRA on Ω is defined by the C^0 matching

$$(55) \qquad V_j(\Omega) := (\Pi_{i=1}^N V_j(\Omega_i)) \cap C^0(\Omega).$$

Although the generator $\tilde{\phi}$ is not assumed to be continuous, the dual MRA is defined by the same matching conditions and the same boundary conditions. This is specified by the following proposition whose proof is elementary from the properties of the scaling function bases on the interval and the C^0 conformity of the decomposition. A detailed proof can be found in [2].

Proposition 3.4 *For all $x \in \mathcal{G}_{j,\Omega}$ let*

$$I(x) := \{i \in (1, \dots, N) : x \in \bar{\Omega}_i\}.$$

Then, the collection

$$\Phi_{j,\Omega} := \Big\{\varphi_{j,x} := \sum_{i \in I(x)} \chi_{\Omega_i} \varphi^{(i)}_{j,x},\ x \in \mathcal{G}_{j,\Omega}\Big\} \tag{56}$$

is a Riesz basis of $V_j(\Omega)$ and

$$\tilde{\Phi}_{j,\Omega} := \Big\{\tilde{\varphi}_{j,x} := \frac{1}{\# I(x)} \sum_{i \in I(x)} \chi_{\Omega_i} \tilde{\varphi}^{(i)}_{j,x},\ x \in \mathcal{G}_{j,\Omega}\Big\} \tag{57}$$

defines a biorthogonal Riesz basis (spanning the dual MRA spaces $\widetilde{V}_j(\Omega)$), for the equivalent scalar product

$$(f, g) := \sum_{i=1}^{n} (f|_{\Omega_i}, g|_{\Omega_i})_i. \tag{58}$$

In this definition, the biorthogonality is defined with respect to the scalar product $(.,.)$. Assuming a C^m conformity (with $m \geq 0$) of the domain decomposition $(\Omega_i, \kappa_i)_{i=1\dots,N}$, the Lebesgue measure defined by $(.,.)$ will induce an equivalent norm $W^{s,p}(\Omega)$ (or $B^s_{p,q}$) for the range $-(m + 1/p) < s < m + 1 + 1/p$ only (see [14] for details). This will in any case limit the range of stability of the wavelet basis Ψ_Ω and $\tilde{\Psi}_\Omega$.

With this limitation in mind, Proposition 3.1 on the line still holds, where the direct estimates have to be written in terms of function spaces satisfying Dirichlet homogeneous boundary condition on Γ_{D} and the inverse estimates are limited by both the regularity of the matching $m + 1 + 1/p$ and the generator $\tau(p)$. We refer to [2] for a detailed proof for $p = 2$ (that extends readily for any $p \geq 1$) and to [4] for $0 < p < 1$. Then, Theorems 2.1 and 2.2 apply directly with the above mentioned caution for negative smoothness.

3.2.2 Construction of the wavelet bases. We propose in this subsection a new technique for the construction of biorthogonal wavelet bases $\Psi_{j,\Omega}$ and $\tilde{\Psi}_{j,\Omega}$ of the complement spaces

$$W_j(\Omega) = V_{j+1}(\Omega) \cap (\widetilde{V}_j(\Omega))^{\perp} \text{ and } \widetilde{W}_j(\Omega) = \widetilde{V}_{j+1}(\Omega) \cap V_j(\Omega)^{\perp}.$$

In the first step of the construction we define a complement basis $\Psi^{\sharp}_{j,\Omega}$ of $\Phi_{j,\Omega}$ in $V_{j+1}(\Omega)$. Then it will suffice to subtract the biorthogonal projection $P_{j,\Omega}$ (also called coarse correction) in order to obtain a wavelet basis of $W_j(\Omega)$:

$$\Psi_{j,\Omega} = (I - P_{j,\Omega})\Psi^{\sharp}_{j,\Omega}. \tag{59}$$

The complement basis $\Psi^{\sharp}_{j,\Omega}$ is derived from the scaling function and wavelets bases Φ_{j,Ω_i} (52) and Ψ_{j,Ω_i} on each subdomain, in such a way that the coarse correction (59) will only apply at the interfaces of the domain decomposition.

A wavelet basis for $\widetilde{W}_j(\Omega)$ is obtained symmetrically and the biorthogonalization, i.e. the definition of new bases so that $\langle \Psi_{j,\Omega}, \tilde{\Psi}_{j,\Omega} \rangle = I$, will only involve the wavelets at the interfaces of the decomposition.

In addition, all the computations at the interfaces will be done according to a natural geometric hierarchy of the interfaces of the decomposition defined below.

Definition 3.1 Interfaces We call $(d-1)$-interface of the domain decomposition $(\Omega_i, \kappa_i)_{i=1,\ldots,N}$, a $(d-1)$-face common to two subdomains Ω_i and Ω_l. Recursively, for $0 \le p \le d-2$, $d \ge 2$, a p-interface is a p-face common to at least two $(p+1)$-interfaces. In order to complete the hierarchy, the d-interfaces will abusively denote the subdomains Ω_i, $i = 1 \ldots, N$.

For $d = 1, 2, 3$ we will use the following terminology: a 0-interface is called a vertex and a $(d-1)$-interface a face. For $d = 3$ a 1-interface is called a side.

Step 1. The basis $\Phi_{j+1,\Omega}$ is partitioned into p-blocks according to the hierarchy of the p-interfaces, each p-block corresponding to one p-interface. First we consider for $p = d$ all the scaling functions that vanish on all the q-interfaces for $q \le d-1$ i.e. the blocks related to each subdomain Ω_i, $i = 1, \ldots, N$. Then, for $p = d-1, \ldots 0$ fixed, we consider for any given p-interface the scaling functions non zero on this p-interface but vanishing on all the other p-interfaces and all the q-interfaces for $q \le p-1$.

We apply to each of these p-blocks, the canonical tensor product (in the reference domain, cf. (46)) p-dimensional two level ($j+1$ to j) decomposition with the same boundary conditions as the p-dimensional scaling function basis that is decomposed.

Proposition 3.5 *Excluding the functions written exclusively with univariate scaling functions (of scales j or $j+1$), we obtain a completion $\Psi^\sharp_{j,\Omega}$ of $\Phi_{j,\Omega}$ in $V_{j+1}(\Omega)$.*

Proof. By construction, $\Psi^\sharp_{j,\Omega}$ is a set of functions of $V_{j+1}(\Omega)$ so that $\#\Psi^\sharp_{j,\Omega} = \#\Phi_{j+1,\Omega} - \#\Phi_{j,\Omega}$. Let us prove that the collection $\Phi_{j,\Omega} \cup \Psi^\sharp_{j,\Omega}$ spans $V_{j+1}(\Omega)$. First the N d-blocks of $\Phi_{j+1,\Omega}$ are spanned as it is easily seen from the decomposition of these d-blocks for each subdomain Ω_i, $i = 1, \dots, N$.

Then, starting from $p = d-1$ to $p = 0$, for any given p-interface F, we shall show that the p-block $\Phi_{j+1,F} \subset \Phi_{j+1,\Omega}$ associated to F is spanned. For that, we factorize $\Phi_{j+1,F}$ as the p-dimensional (in the directions of F) scaling function basis (that is decomposed) times the remaining $(d-p)$-dimensional scaling function basis. Clearly, it suffices to show that we span the block $\bar{\Phi}_{j+1,F}$ obtained from $\Phi_{j+1,F}$ when writting the $(d-p)$ dimensional scaling functions at the scale j instead of $j+1$. Indeed $\Phi_{j+1,F} - 2^{(d-p)/2}\bar{\Phi}_{j+1,F}$ is spanned by the p'-blocks for $p' > p$ since these functions vanish at all the q-interfaces for $q \le p$. To conclude, we note that $\bar{\Phi}_{j+1,F}$ is spanned by construction, since its p-dimensional canonical decomposition gives rise to functions in $\Psi^\sharp_{j,\Omega} \cup \Phi_{j,\Omega}$. $\square$

Step 2. The completion $\tilde{\Psi}^\sharp_{j,\Omega}$ of $\tilde{\Phi}_{j,\Omega}$ in $\widetilde{V}_{j+1}(\Omega)$ is simply obtained from $\Psi^\sharp_{j,\Omega}$ adding the superscripts$\tilde{}$. Then, we have the following proposition.

Proposition 3.6 *The bases*

$$\begin{cases} \Psi_{j,\Omega} = (I - P_{j,\Omega})\,\Psi^\sharp_{j,\Omega}, \\ \tilde{\Psi}_{j,\Omega} = (I - \widetilde{P}_{j,\Omega})\,\tilde{\Psi}^\sharp_{j,\Omega}, \end{cases} \tag{60}$$

are local Riesz bases of $W_j(\Omega)$ and $\widetilde{W}_j(\Omega)$. In addition, the coarse corrections (60) and the biorthogonalization need to be done on a fixed number, independent of j, of functions for each p-interface, $p = d-1, \dots, 0$. Indeed, for a given p-interface, by tensor product, the computations reduce to those of a 0-interface in dimension $d-p$.

Proof. The proof is constructive.
The hierarchy of the p-interfaces for $p = d, \dots, 0$ defines a partition of the grid $\mathcal{G}_j$ (53). For a given p-interface F, we denote by $\mathcal{G}_j(F)$ the subset of $\mathcal{G}_j$ related to this partition. As we did for the scaling functions on the interval, it will be convenient to associate to each wavelet on the interval $\psi_{j,k}$ (or $\tilde{\psi}_{j,k}$), $k = 0, \dots, 2^j - 1$, (whatever the boundary conditions at the edges) a point $\eta_{j,k} \in [0,1]$. We require that $\eta_{j,0} = 0$, $\eta_{j,2^j-1} = 1$ while the other

points are chosen arbitrarily in $]0, 1[$, since only the interfaces will matter in the subsequent analysis.

Using the points $\eta_{j,k}$ together with the $\xi_{j,k}$ and the parametrizations κ_i, we can associate to each function of $\Psi^\sharp_{j,\Omega}$ (or $\tilde{\Psi}^\sharp_{j,\Omega}$) a point of Ω. To a given point x of an interface F corresponds one or several functions of $\Psi^\sharp_{j,\Omega}$ (and hence of $\Psi_{j,\Omega}$), that will be refered to as the functions of $\Psi^\sharp_{j,\Omega}$ and the wavelets "associated" to x.

Let $\mathcal{F}_p$ denote the set of p-interfaces and

$$\mathcal{F} = \bigcup_{p=0}^{d-1} \mathcal{F}_p.$$

Then we have the following properties: the coarse correction only applies to the functions of $\Psi^\sharp_{j,\Omega}$ associated to the points of

$$\bigcup_{F\in\mathcal{F}} \mathcal{G}_j(F) \,/\, \overline{\Gamma}_{\mathrm{D}}.$$

Let $x \in \bigcup_{F\in\mathcal{F}} \mathcal{G}_j(F)/\overline{\Gamma}_{\mathrm{D}} \subset \mathcal{G}_{j,\Omega}$. The coarse correction of any function $f \in \Psi^\sharp_{j,\Omega}$ associated to x simply writes

$$f - (f, \tilde{\varphi}_{j,x})\, \varphi_{j,x}. \tag{61}$$

In particular, this correction is local.

After coarse correction, only the primal and dual wavelets associated to the grid points

$$\bigcup_{F\in\mathcal{F}} \mathcal{G}_j(F)$$

need to be biorthogonalized. In addition, the biorthogonalization is done separately for each interface i.e. for each subset of functions associated to the grid points of $\mathcal{G}_j(F)$. We shall say that the corresponding wavelets are associated to the interface F.

Let $\Psi_{j,F}$ and $\tilde{\Psi}_{j,F}$ denote the sets of wavelets (before biorthogonalization) associated to a p-interface F, (hence with $\#\Psi_{j,F} = \#\Psi_{j,F} \sim 2^{jp}$). By tensor product in the directions of the interface, their construction by coarse correction reduces to the construction of wavelets associated to a 0-interface in dimension $d-p$ (i.e in the directions orthogonal to the interface). Also the inversion of the biorthogonalization matrix $C_F = (\Psi_{j,F}, \tilde{\Psi}_{j,F})$ reduces to the inversion of a matrix of dimension independent of j which is nothing but the biorthogonalization matrix of the wavelets associated to the 0-interface in dimension $d - p$.

Note that these matrices are non-singular since the non-singularity of the matrix $(\Psi_{j,\Omega}, \tilde{\Psi}_{j,\Omega})$ is deduced from the biorthogonality of the spaces $V_j(\Omega)$ and $\widetilde{V}_j(\Omega)$.

These properties prove in particular that the bases $\Psi_{j,\Omega}$ and $\tilde{\Psi}_{j,\Omega}$ are local Riesz bases before and after biorthogonalization. In fine, we end up with

- the (e.g. primal) wavelets associated to the p-interfaces, $p = d-1, \dots, 0$ that are obtained by tensor products from computations involving only the wavelets and scaling functions on the interval $[0,1]$ $\varphi_{j,0}$, φ_{j,d_j}, $\psi_{j,0}$, $\psi_{j,2^j-1}$ and the parametrizations κ_i.
- The remaining wavelets associated to each subdomain $\Omega_i, i = 1, \dots, N$, and belonging to Ψ_{j,Ω_i}. □

Remark 3.1 The choice of the biorthogonal wavelets at the interfaces is not unique as it clearly appears in the biorthogonalization step. If K_F and $\widetilde{K}_F$ denote the square matrices so that $\Psi_{j,F}^{new} = K_F \Psi_{j,F}$ and $\tilde{\Psi}_{j,F}^{new} = \widetilde{K}_F \tilde{\Psi}_{j,F}$, then they only need to verify the relation

$$K_F \, C_F \, (\widetilde{K}_F)^{\mathrm{T}} \, = \, I_F.$$

This non-uniqueness allows to impose additionnal properties for the wavelets at the interfaces such as symmetry or minimal supports that are studied in details in [3] in dimension 1 and 2.

The following subsection illustrates this construction for the interfaces of a general 2-dimensional domain decomposition. A similar analysis is carried out in [3] using the techniques described in [2].

3.2.3 Interface wavelets for a 2-dimensional domain. We need to build the wavelets associated to the 1-interfaces (faces F) and the 0-interfaces (vertices V). For these last wavelets we shall distinguish the *interior vertices* $V \in \Omega$, from the boundary vertices $V \in \Gamma$ which split into three subcases: $V \in \Gamma_{\mathrm{N}}$ (*Neumann-Neumann vertex*), $V \in \Gamma_{\mathrm{D}}$ (*Dirichlet-Dirichlet vertex*) and $V \in \bar{\Gamma}_{\mathrm{D}} \cap \bar{\Gamma}_{\mathrm{N}}$ (*Neumann-Dirichlet vertex*).

Face wavelets. Let $\hat{\Omega} =]0,1[^2$, $O = (0,0)$, $A = (1,0)$, $B = (0,1)$ and $\hat{I} = OA$. Given the domain decomposition $(\Omega_i, \kappa_i)_{i=1,\dots,N}$, the subdomains Ω_1 and Ω_2 are assumed to share the face $F = \bar{\Omega}_1 \cap \bar{\Omega}_2$. For conveniency in the notation we assume in addition that

$$\kappa_1(\hat{I}) = \kappa_2(\hat{I}) = F.$$

The construction of the wavelets at the face F reduces to the computation of a one dimensional vertex of a decomposition $(I_i, \xi_i)_{i=1,2}$ where $I_1 = \xi_1^{-1}(\hat{I})$, $I_2 = \xi_2^{-1}(\hat{I})$ and $V = \bar{I}_1 \cap \bar{I}_2 = \xi_1^{-1}(O) = \xi_2^{-1}(O)$. The functions of the completion associated to the vertex V are those written in terms of the scaling functions and wavelets $\varphi_{j,0} \circ \xi_i^{-1}$, $\psi_{j,0} \circ \xi_i^{-1}$, $i = 1, 2$, i.e.

$$\psi_{j,0} \circ \xi_1^{-1} - \varphi_{j,0} \circ \xi_1^{-1} \text{ and } \psi_{j,0} \circ \xi_2^{-1} - \varphi_{j,0} \circ \xi_2^{-1}.$$

Let $\varphi_{j,I_1 \cap I_2} := \frac{1}{\sqrt{2}}(\varphi_{j,0} \circ \xi_1^{-1} + \varphi_{j,0} \circ \xi_2^{-1})$ and $\tilde{\varphi}_{j,I_1 \cap I_2} := \frac{1}{\sqrt{2}}(\tilde{\varphi}_{j,0} \circ \xi_1^{-1} + \tilde{\varphi}_{j,0} \circ \xi_2^{-1})$ the biorthogonal scaling functions at the vertex V. The coarse correction on any f of the above two functions writes $f - (f, \tilde{\varphi}_{j,I_1 \cap I_2})\varphi_{j,I_1 \cap I_2}$. After biorthogonalization, we get for example the two primal wavelets

$$(62) \qquad \begin{cases} \psi^1_{j,I_1 \cap I_2} = \dfrac{1}{\sqrt{2}}(\psi_{j,0} \circ \xi_1^{-1} + \psi_{j,0} \circ \xi_2^{-1}), \\ \psi^2_{j,I_1 \cap I_2} = \dfrac{1}{\sqrt{2}}(\psi_{j,0} \circ \xi_1^{-1} \\ \qquad\qquad -\varphi_{j,0} \circ \xi_1^{-1} - \psi_{j,0} \circ \xi_2^{-1} + \varphi_{j,0} \circ \xi_2^{-1}). \end{cases}$$

and the same formula for the dual wavelets with the superscript $\tilde{}$.

The biorthogonal wavelets at the face F are then deduced by "canonical tensor product" (cf. (46)) from the univariate wavelets $\psi^1_{j,I_1 \cap I_2}$, $\psi^1_{j,I_1 \cap I_2}$ and the scaling function $\varphi_{j,I_1 \cap I_2}$ with the univariate scaling functions and wavelets $\varphi_{j,k}$, $\psi_{j,k}$ in the direction of the face. We obtain for the primal wavelets

$$(63) \begin{cases} \dfrac{1}{\sqrt{2}} \left(\psi_{j,k} \otimes \varphi_{j,0} \circ \kappa_1^{-1} \ + \ \psi_{j,k} \otimes \varphi_{j,0} \circ \kappa_2^{-1}\right) \\ \dfrac{1}{\sqrt{2}} \left(\varphi_{j,k} \otimes \psi_{j,0} \circ \kappa_1^{-1} \ + \ \varphi_{j,k} \otimes \psi_{j,0} \circ \kappa_2^{-1}\right) \\ \dfrac{1}{2} \left(\varphi_{j,k} \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_1^{-1} - \varphi_{j,k} \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_2^{-1}\right) \\ \dfrac{1}{\sqrt{2}} \left(\psi_{j,k} \otimes \psi_{j,0} \circ \kappa_1^{-1} \ + \ \psi_{j,k} \otimes \psi_{j,0} \circ \kappa_2^{-1}\right) \\ \dfrac{1}{2} \left(\psi_{j,k} \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_1^{-1} - \psi_{j,k} \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_2^{-1}\right) \end{cases}$$

and the same formula for the dual biorthogonal wavelets with the superscript $\tilde{}$.

Interior vertex. Let $V \in \Omega$ common to M subdomains Ω_i, $i = 1, \ldots, M$. We denote by $I_{i,i+1} = \bar{\Omega}_i \cap \bar{\Omega}_{i+1}$, $i = 1, \ldots, M$, the interfaces where $M+1$ is identified to 1. Let $\hat{J} = OB$, for conveniency in the notation we assume that for $i = 1, \ldots, M$,

$$V = \kappa_i(O) \text{ and } \kappa_i(\hat{J}) = \kappa_{i+1}(\hat{I}) = I_{i,i+1}.$$

The functions of the completion $\Psi^{\sharp}_{j,\Omega}$ associated to the vertex V include for each subdomain Ω_i, $i = 1, \dots, M$, the function

$$f_{j,\Omega_i} = (\psi_{j,0} - \varphi_{j,0}) \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_i^{-1}, \tag{64}$$

and for each interface $I_{i,i+1}$, $i = 1, \dots, M$, the function

$$\begin{aligned} f_{j,I_{i,i+1}} = \varphi_{j,0} \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_i^{-1} \\ +(\psi_{j,0} - \varphi_{j,0}) \otimes \varphi_{j,0} \circ \kappa_{i+1}^{-1}, \end{aligned} \tag{65}$$

The coarse correction applied to any f of these $2M$ functions reduces to

$$\begin{aligned} f - \frac{1}{M} \left(\sum_{l=1}^{M} \int_{\hat{\Omega}} f|_{\Omega_l} \circ \kappa_l \, \tilde{\varphi}_{j,0} \otimes \tilde{\varphi}_{j,0} d\hat{x} \right) \\ \times \sum_{l=1}^{M} \varphi_{j,0} \otimes \varphi_{j,0} \circ \kappa_l^{-1}, \end{aligned} \tag{66}$$

which leads to the $2M$ primal wavelets, for $i = 1, \dots, M$

$$\begin{aligned} \psi_{j,\Omega_i} &= (\psi_{j,0} - \varphi_{j,0}) \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_i^{-1} - \frac{1}{M} \sum_{l=1}^{M} \varphi_{j,0} \otimes \varphi_{j,0} \circ \kappa_l^{-1}. \\ \psi_{j,I_{i,i+1}} &= \varphi_{j,0} \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_i^{-1} + (\psi_{j,0} - \varphi_{j,0}) \otimes \varphi_{j,0} \circ \kappa_{i+1}^{-1} \\ &\quad + \frac{2}{M} \sum_{l=1}^{M} \varphi_{j,0} \otimes \varphi_{j,0} \circ \kappa_l^{-1}. \end{aligned}$$

supported on the M subdomains. Simple linear combinations enable to better localize these wavelets on the subdomains. The linear combinations $\psi_{j,\Omega_i} + \psi_{j,\Omega_{i+1}} + \psi_{j,I_{i,i+1}}$ for $i = 1, \dots, M$ and $\psi_{j,\Omega_i} + \psi_{j,\Omega_{i+1}}$ for $i = 1, \dots, M-1$ define $2M-1$ linearly independent wavelets supported on two subdomains. They are complemented by the linear combination $\sum_{i=1}^{M} (\psi_{j,\Omega_i} + \psi_{j,I_{i,i+1}})$ supported on the M subdomains.

$$\Psi_{j,V} = \begin{cases} \displaystyle\sum_{l=0}^{1} (-1)^l (\psi_{j,0} - \varphi_{j,0}) \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_{i+l}^{-1}, \\ \qquad i = 1, \dots, M-1, \\ \psi_{j,0} \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_i^{-1} - (\psi_{j,0} - \varphi_{j,0}) \otimes \psi_{j,0} \circ \kappa_{i+1}^{-1}, \\ \qquad i = 1, \dots, M, \\ \displaystyle\sum_{i=1}^{M} \psi_{j,0} \otimes \psi_{j,0} \circ \kappa_i^{-1}. \end{cases} \tag{67}$$

The $2M$ dual wavelets are obtained just adding the superscripts $\tilde{}$ and the biorthogonalization matrix $(\Psi_{j,V}, \tilde{\Psi}_{j,V})$ is easily computed and only depends on M.

Neumann-Neumann vertex. We consider M subdomains sharing the vertex $V \in \Gamma_{\rm N}$. Let Ω_1 and Ω_M be the subdomains intersecting $\Gamma_{\rm N}$ and $I_{0,1} = \bar{\Omega}_1 \cap \Gamma_{\rm N}$, $I_{0,M} = \bar{\Omega}_M \cap \Gamma_{\rm N}$ their interfaces with $\Gamma_{\rm N}$. The subdomains share the 1-interfaces $I_{i,i+1} = \bar{\Omega}_i \cap \bar{\Omega}_{i+1}$, $i = 1, \ldots, M-1$. Again, we assume for conveniency in the notation that

$$\begin{cases} V = k_i(O), \ i = 1, \ldots, M, \\ k_1(\hat{I}) = I_{0,1}, \ k_M(\hat{J}) = I_{0,M}, \\ k_i(\hat{J}) = k_{i+1}(\hat{I}) = I_{i,i+1}, \ i = 1, \ldots, M-1. \end{cases}$$

The functions of the completion $\Psi^{\sharp}_{j,\Omega}$ associated to the vertex V include for each subdomain Ω_i, $i = 1, \ldots, M$, the function (64) and for each interface $I_{i,i+1}$, $i = 1, \ldots, M-1$, the function (65). Finally one function is associated to each face $I_{0,1}$ and $I_{0,M}$. For these last functions, by simple linear combinations we retain the wavelets

$$(\psi_{j,0} - \varphi_{j,0}) \otimes \psi_{j,0} \circ \kappa_1^{-1} \tag{68}$$

for $I_{0,1}$, and $\psi_{j,0} \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_M^{-1}$ for $I_{0,M}$, rather than the functions $(\psi_{j,0} - \varphi_{j,0}) \otimes \varphi_{j,0} \circ \kappa_1^{-1}$ and $\varphi_{j,0} \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_M^{-1}$ of $\Psi^{\sharp}_{j,\Omega}$. The coarse correction on these $2M+1$ functions is computed as previously and taking similar linear combinations as for the interior vertex we obtain the following collection of wavelets at the vertex V.

$$\Psi_{j,{\rm V}_{\rm N-N}} = \begin{cases} (\psi_{j,0} - \varphi_{j,0}) \otimes \psi_{j,0} \circ \kappa_1^{-1}, \\ \displaystyle\sum_{l=0}^{1} (-1)^l (\psi_{j,0} - \varphi_{j,0}) \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_{i+l}^{-1}, \\ \qquad\qquad i = 1, \ldots, M-1, \\ \psi_{j,0} \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_M^{-1}, \\ \psi_{j,0} \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_i^{-1} \\ -(\psi_{j,0} - \varphi_{j,0}) \otimes \psi_{j,0} \circ \kappa_{i+1}^{-1}, \\ \qquad\qquad i = 1, \ldots, M-1, \\ \displaystyle\sum_{i=1}^{M} \psi_{j,0} \otimes \psi_{j,0} \circ \kappa_i^{-1}. \end{cases} \tag{69}$$

These wavelets can be retained for the primal basis, and the set of biorthogonal dual wavelets is obtained by inversion of the biorthogonalization matrix $(\Psi_{j,\mathrm{V_{N-N}}}, \tilde{\Psi}_{j,\mathrm{V_{N-N}}})$ that depends only on M.

Dirichlet-Dirichlet vertex. The previous notation are reproduced. The functions of the completion $\Psi^{\sharp}_{j,\Omega}$ associated to the vertex V include for each subdomain Ω_i, $i = 1, \ldots, M$, the function (64) and for each interface $I_{i,i+1}$, $i = 1, \ldots, M-1$, the function (65). As announced in the proof of Proposition 3.6 there is no coarse correction since no scaling function is associated to the vertex V. Thus the $2M-1$ primal wavelets (and similarly the dual wavelets) are obtained directly from $\Psi^{\sharp}_{j,\Omega}$ and they only need to be biorthogonalized.

$$\Psi_{j,\mathrm{V_{D-D}}} = \begin{cases} (\psi_{j,0} - \varphi_{j,0}) \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_i^{-1}, \; i = 1, \ldots, M, \\ \varphi_{j,0} \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_i^{-1} \\ \qquad + (\psi_{j,0} - \varphi_{j,0}) \otimes \varphi_{j,0} \circ \kappa_{i+1}^{-1}, \; i = 1, \ldots, M-1. \end{cases}$$

Neumann-Dirichlet vertex. Again the notation are the same as previously with $I_{01} \subset \Gamma_{\mathrm{N}}$ and $I_{0M} \subset \Gamma_{\mathrm{D}}$. The functions of the completion $\Psi^{\sharp}_{j,\Omega}$ associated to the vertex V include for each subdomain Ω_i, $i = 1, \ldots, M$, the function (64), for the face $I_{0,1}$ the function (68) and for each interface $I_{i,i+1}$, $i = 1, \ldots, M-1$, the function (65).

Since there is no coarse correction, we obtain directly $2M$ primal (and similarly the dual) wavelets that only need to be biorthogonalized.

$$\Psi_{j,\mathrm{V_{N-D}}} = \begin{cases} (\psi_{j,0} - \varphi_{j,0}) \otimes \psi_{j,0} \circ \kappa_1^{-1}, \\ (\psi_{j,0} - \varphi_{j,0}) \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_i^{-1}, \; i = 1, \ldots, M, \\ \varphi_{j,0} \otimes (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_i^{-1} + \\ \qquad (\psi_{j,0} - \varphi_{j,0}) \otimes \varphi_{j,0} \circ \kappa_{i+1}^{-1}, \; i = 1, \ldots, M-1. \end{cases}$$

Remark 3.2 This hierarchical construction of the wavelets suggests a hierarchical and parallel programing of the wavelet transforms. These algorithms (for example for the primal bases) iterate the two level wavelet transforms $\Phi_{j+1,\Omega} \leftrightarrow \Phi_{j,\Omega} \cup \Psi_{j,\Omega}$.

A natural idea is to proceed in two steps that mimic the construction steps. For example the decomposition algorithm (and reversely for the recomposition) writes:

(i) decomposition according to

$$\Phi_{j+1,\Omega} \to \Phi_{j,\Omega} \cup \Psi^{\sharp}_{j,\Omega}$$

that "decouples" (and hence can be parallelized) on each p-interface for $p = d, \dots, 0$ and amounts on each p-block, in the directions of the p-interface, to a canonical tensor product wavelet transform in dimension p.

(ii) Corrections at the p-interfaces for $p = d-1, \dots, 0$ according to the formula of coarse correction and biorthogonalization.

3.3 Domain decomposition and liftings

We reproduce the notation and assumptions of Sect. 3.2. In particular the generators $(\phi, \tilde{\phi})$ on the line are symmetric and the primal generator ϕ is continuous. In the following analysis the dimension d is at least equal to 2 and the boundary $\varGamma_{\mathrm{D}}$ is not empty, otherwise the construction of the lifting becomes trivial.

The techniques of Sect. 3.2 enable to define the biorthogonal MRA "without boundary conditions"

$$\begin{cases} V_j(\varOmega) = \mathsf{S}(\varPhi_{j,\varOmega}) \\ \widetilde{V}_j(\varOmega) = \mathsf{S}(\tilde{\varPhi}_{j,\varOmega}). \end{cases}$$

We denote by $P_{j,\varOmega}$ the corresponding biorthogonal projector and by

$$\varPsi_{j,\varOmega}, \;\; \tilde{\varPsi}_{j,\varOmega},$$

the biorthogonal wavelet bases of the complement spaces $W_j(\varOmega), \widetilde{W}_j(\varOmega)$. Also, we can define the MRA with homogeneous Dirichlet boundary conditions on $\varGamma_{\mathrm{D}}$ (for the primal, and symmetrically for the dual)

$$\begin{cases} V_j^0(\varOmega) := V_j(\varOmega) \cap C^0_{\mathrm{D}}(\varOmega) = \mathsf{S}(\varPhi^0_{j,\varOmega}) \\ \widetilde{V}_j^0(\varOmega) = \mathsf{S}(\tilde{\varPhi}^0_{j,\varOmega}), \end{cases}$$

where $C^0_{\mathrm{D}}(\varOmega)$ denotes the space of continuous functions vanishing on $\varGamma_{\mathrm{D}}$. The corresponding biorthogonal projector is denoted by $P^0_{j,\varOmega}$ and the biorthogonal wavelet bases by

$$\varPsi^0_{j,\varOmega}, \;\; \tilde{\varPsi}^0_{j,\varOmega}.$$

The domain decomposition $(\varOmega_i, \kappa_i)_{i=1,\dots,N}$ of $\varOmega$ induces by restriction a domain decomposition of the boundary $\varGamma_{\mathrm{D}}$.

Let $\varGamma_{i,l}$ denote the $(d-1)$-faces of $\varOmega_i \cap \varGamma_{\mathrm{D}}$ images of $\hat{\sigma}_l$ by the parametrization κ_i. Also we denote by ρ_k the canonical affine isometries from the reference $(d-1)$-face $\hat{I} =]0,1[^{d-1} \times \{0\}$ to the $(d-1)$-faces $\hat{\sigma}_k$ of the reference domain $\hat{\varOmega} =]0,1[^d$. Then, the subdomains $\varGamma_{i,l}$ and the parametrizations

$$\kappa_i \circ \rho_l : \; \hat{I} \to \varGamma_{i,l}$$

define a domain decomposition of $\Gamma_{\rm D}$ denoted by

$$(\xi_i, \Gamma_i)_{i=1,\dots,N_{\rm D}}.$$

To prove the C^0 conformity of this decomposition it suffices to check the property (47): for all p-face common to the subdomains $\Gamma_{i,l} = \kappa_i \circ \rho_l(\hat\sigma)$ and $\Gamma_{i',l'} = \kappa_{i'} \circ \rho_{l'}(\hat\sigma')$, then

$$(\kappa_i \circ \rho_l)^{-1} \circ (\kappa_{i'} \circ \rho_{l'}) = \rho_l^{-1} \circ (\kappa_i^{-1} \circ \kappa_{i'}) \circ \rho_{l'}$$

is an affine isometry from $\hat\sigma'$ to $\hat\sigma$.

Thus, the techniques of Sect. 3.2 applied to this decomposition of $\Gamma_{\rm D}$ and the generators $(\phi, \tilde\phi)$, define the biorthogonal MRA

$$\begin{cases} V_j(\Gamma_{\rm D}) = \mathrm{S}(\Phi_{j,\Gamma_{\rm D}}) \\ \widetilde{V}_j(\Gamma_{\rm D}) = \mathrm{S}(\tilde\Phi_{j,\Gamma_{\rm D}}), \end{cases}$$

the associated biorthogonal projector $P_{j,\Gamma_{\rm D}}$ and the biorthogonal wavelet bases

$$\Psi_{j,\Gamma_{\rm D}}, \;\; \tilde\Psi_{j,\Gamma_{\rm D}}.$$

These MRA coincide with the "trace" MRA (defined rigorously for continuous generators ϕ and $\tilde\phi$). The main avantage of this definition is to prove directly the biorthogonality and the regularity $W^{s,p}(\Gamma_{\rm D})$ of these MRA. For the primal MRA, it is equal to $\min(\tau(p), 1+\frac{1}{p})$ and not only to $\min(\tau(p)-\frac{1}{p}, 1)$ as deduced from a direct application of the trace theorem.

The remaining is a direct application of the abstract setting developped in Sect. 2.2 which starts from the definition of the biorthogonal projector $P^*_{j,\Omega}$ onto $V_j(\Omega)$. We first discuss the fulfillment of Hypothesis 2.1 to 2.5 that determine the stability of the lifting operator. Then, we propose two constructions of the lifting wavelets of $\Psi^b_{j,\Omega}$. The first one is an extension of the construction developped in Sect. 3.2 and the second one shows that it is possible to pick up the lifting wavelets in $W_j(\Omega)$. This last choice will enable to obtain a range of stability of the lifting operator $\mathcal{R}$ including negative smoothness, although in that case the trace operator is no longer defined.

Lemma 3.1 *The localness Assumption 2.1 for the pair $(V_j(\Omega), \Gamma_{\rm D})$ is satisfied for all $p \in]0,\infty]$.*

Proof. This is a direct application of Lemma 2.1 and the L^p stability of the bases $\Phi_{j,\Omega}$ and $\Phi_{j,\Gamma_{\rm D}}$ for all $p \in]0,\infty]$, since the generator ϕ is continuous. □

We recall that the projectors $P_{j,\Gamma_{\rm D}}$ and $P^0_{j,\Omega}$ are uniformly stable in L^p for all $p \in [1,\infty]$ so that $\tilde\phi \in L^{p'}$ (cf. Proposition 3.1 on the line). Also, the inverse estimate on $V_j(\Omega)$ is obtained for the range $0 \leq s < \min(1 + 1/p, \tau(p))$, $p \in]0, \infty]$ and the direct estimate for $P_{j,\Omega}$ for the range $0 \leq s \leq n$ for all $p \in [1, \infty]$ such that $\tilde{\phi} \in L^{p'}$.

Then, the following properties result directly from Propositions 2.2, 2.3 and 2.4.

Proposition 3.7 *Let* $p \in]0, \infty]$*, then*

$$\|g_j\|_{B^s_{p,p}(\Gamma_\mathrm{D})} \lesssim 2^{js} \|g_j\|_{L^p(\Gamma_\mathrm{D})}, \text{ for all } g_j \in V_j(\Gamma_\mathrm{D})$$

for all $0 < s < \min(\tau(p) - 1/p, 1)$.

Let $p \in [1, \infty]$ *so that* $\tilde{\phi} \in L^{p'}$*, then*

$$\begin{cases} \|g - P_{j,\Gamma_\mathrm{D}} g\|_{L^p(\Gamma_\mathrm{D})} \lesssim 2^{-js} \|g\|_{B^s_{p,p}(\Gamma_\mathrm{D})} \text{ for all } 0 < s < n - 1/p, \\ \|u - P^*_{j,\Omega} u\|_{L^p(\Omega)} \lesssim 2^{-js} \|u\|_{B^s_{p,p}(\Omega)} \text{ for all } 1/p < s < n. \end{cases}$$

For $p \in]0, 1[$, the direct estimate for P_{j,Γ_D} is obtained through Proposition 2.5. It requires the derivation of the direct estimate for $P^*_{j,\Omega}$, $p < 1$, which is obtained using the techniques of Sect. III of [4].

3.3.1 First construction of the lifting wavelets. In order to define explicitely the lifting operator (19), it remains to build a wavelet basis $\Psi^*_{j,\Omega}$ of $W^*_j(\Omega)$ stable in L^p and satisfying the properties 1.1.

For that purpose, we slightly modify the construction of Sect. 3.2. First, the hierarchy of the p-interfaces of the domain decomposition (Ω_i, κ_i) $_{i=1,\dots,N}$ is completed here by the hierarchy of the interfaces of the decomposition $(\Gamma_i, \xi_i)_{i=1,\dots,N_\mathrm{D}}$ of the boundary Γ_D. Second, we will subtract the projection $P^*_{j,\Omega}$ rather than $P_{j,\Omega}$.

Step 1. The basis $\Phi_{j+1,\Omega}$ is partitioned into p blocks corresponding to the hierarchy of the interfaces of the domain for the subset $\Phi^0_{j+1,\Omega}$, $p = d, \dots, 0$ *and* of the boundary Γ_D, $p = d - 1, \dots, 0$ for the remaining set of scaling functions denoted by $\Phi^b_{j+1,\Omega}$.

We apply to each of these p-blocks of $\Phi^0_{j+1}(\Omega)$ and $\Phi^b_{j+1}(\Omega)$ the tensor product two scale decomposition with the same boundary conditions.

Proposition 3.8 *Excluding the scaling functions (of scales j or $j+1$), we obtain the completion*

$$\Psi^{*,\sharp}_{j,\Omega} = \Psi^{0,\sharp}_{j,\Omega} \cup \Psi^{b,\sharp}_{j,\Omega}$$

of $\Phi_{j,\Omega}$ *in* $V_{j+1}(\Omega)$ *where* $\Psi^{0,\sharp}_{j,\Omega}$ *is the completion defined by Proposition 3.5 applied to the MRA* $V^0_j(\Omega)$*.*

Proof. It is similar to the proof of Proposition 3.5. □

Step 2. We apply the coarse correction

$$\Psi^*_{j,\Omega} = (I - P^*_{j,\Omega})\Psi^{*,\sharp}_{j,\Omega}.$$

On $\Psi^{0,\sharp}_{j,\Omega}$ the projection $P^*_{j,\Omega}$ reduces to $P^0_{j,\Omega}$ and we come back to the construction of Sect. 3.2 of the wavelet basis

$$\Psi^0_{j,\Omega} = (I - P^0_{j,\Omega})\Psi^{0,\sharp}_{j,\Omega}.$$

On $\Psi^{b,\sharp}_{j,\Omega}$, the projection $P^*_{j,\Omega}$ vanishes except at the p-interfaces of the decomposition of Γ_{D} for $p = d-2,\dots,0$. At these interfaces, $P^*_{j,\Omega}$ reduces to $r_j \circ P_{j,\Gamma_{\mathrm{D}}} \circ \gamma_{\mathrm{D}}$, for the choice of the lifting r_j involving only the scaling functions $\varphi_{j,0}$, φ_{j,d_j} in the reference interval.

As for the computations at the interfaces of the domain Ω, by tensor product, the computations of the lifting wavelets at the p-interfaces of the decomposition of Γ_{D} for $1 \le p \le d-2$ reduce to compute the lifting wavelets at a 0-interface in dimension $d-p$. The number of wavelets to build thus corresponds to the number of wavelets associated to a 0-interface of Γ_{D} (i.e. in dimension $d-p-1$).

This construction will be illustrated below in the case of a general 2-dimensional domain.

Lemma 3.2 *The wavelet basis $\Psi^*_{j,\Omega}$ is stable in L^p for all $p \in]0,\infty[$.*

Proof. The bases $\Psi^0_{j,\Omega}$ and $\Psi_{j,\Gamma_{\mathrm{D}}}$ are linearly locally independent respectively on a dyadic partition of Ω and its traces on Γ_{D}. Hence, the local linear independence also holds for the basis $\Psi^*_{j,\Omega} = \Psi^0_{j,\Omega} \cup \Psi^b_{j,\Omega}$. The stability is classically deduced since the generator ϕ is in L^p for all $p \in]0,\infty]$. □

We can now state the stability properties of the lifting operator (19).

Theorem 3.1 *Let $p \in [1,\infty]$ so that $\tilde{\phi} \in L^{p'}$. The lifting $\mathcal{R}$ verifies the stability property*

$$\|\mathcal{R}g\|_{B^{s+1/p}_{p,q}(\Omega)} \lesssim \|g\|_{B^s_{p,q}(\Gamma_{\mathrm{D}})}$$

for all $0 < s < \min(\tau(p) - 1/p, 1)$.
Let $p \in]0,1[$ and $r \in [1,\infty]$ so that $\tilde{\phi} \in L^{r'}$, then the lifting $\mathcal{R}$ is stable for the range $(d-1)(\frac{1}{p} - \frac{1}{r}) < s < \min(\tau(p) - 1/p, 1)$.

3.3.2 Second construction of the lifting wavelets. It is possible to define the lifting wavelets in $W_j(\Omega)$ in order to obtain additionnal stability properties of the lifting operator $\mathcal{R}$. It relies on the following lemma.

Lemma 3.3 *Let $w_j \in W_j(\Omega)$, then $w_j \in W_j^*(\Omega)$ if and only if $\gamma_{\mathrm{D}}\, w_j \in W_j(\Gamma_{\mathrm{D}})$.*

Proof. This is clearly a necessary condition, let us show that it is also sufficient. We shall show that $P^*_{j,\Omega} w_j = 0$. Since $\gamma_{\mathrm{D}} w_j \in V_{j+1}(\Gamma_{\mathrm{D}})$, the property $\gamma_{\mathrm{D}}\, w_j \in W_j(\Gamma_{\mathrm{D}})$ is equivalent to $P_{j,\Gamma_{\mathrm{D}}} \circ \gamma_{\mathrm{D}} w_j = 0$. Hence we have $P^*_{j,\Omega} w_j = P^0_{j,\Omega} w_j$ which vanishes since by construction $\widetilde{V}^0_j(\Omega) \subset \widetilde{V}_j(\Omega)$. □

Thus, if we can prove that $W_j(\Gamma_{\mathrm{D}}) \subset \gamma_{\mathrm{D}} W_j(\Omega)$, it will be possible to build the lifting wavelets in $W_j(\Omega)$ and to take avantage of the stability properties of the basis Ψ_Ω to obtain a larger range of stability of the lifting. We first show the following lemma.

Lemma 3.4 *Any function $\psi_{j,\Gamma_{\mathrm{D}}} \in V_{j+1}(\Gamma_{\mathrm{D}})$ written exclusively with the parametrizations ξ_l, $l = 1, \ldots, N_{\mathrm{D}}$, the scaling functions and the wavelets on the interval $\varphi_{j,0}, \varphi_{j,d_j}, \psi_{j,0}, \psi_{j,2^j-1}$, is the trace of a wavelet of $W_j(\Omega)$.*

Proof. The function $\psi_{j,\Gamma_{\mathrm{D}}}$ is associated to one (or several) 0-interface of the decomposition of Γ_{D} denoted by $V \in \Gamma_{\mathrm{D}}$. First, we lift the wavelet $\psi_{j,\Gamma_{\mathrm{D}}}$ in Ω using only the wavelets and scaling functions on the interval $\varphi_{j,0}, \varphi_{j,d_j}, \psi_{j,0}, \psi_{j,2^j-1}$ and the parametrizations κ_l. Let us denote by $\psi^\sharp_{j,\Omega}$ this function of $V_{j+1}(\Omega)$. This lifting function is already orthogonal to all the scaling functions of $\tilde{\Phi}_{j,\Omega}$ except to the scaling function $\tilde{\varphi}_{j,V}$ associated to the boundary vertex V. Let i be such that $V \in \Omega_i$ and let us assume for conveniency that $V = \kappa_i(O)$. We consider the function

$$\psi^0_{j,V} := \otimes^d (\psi_{j,0} - \varphi_{j,0}) \circ \kappa_i^{-1}$$

so that $\gamma \psi^0_{j,V} = 0$ and $(\psi^0_{j,V}, f) = 0$ for all $f \in \tilde{\Phi}_{j,\Omega} / \{\tilde{\varphi}_{j,V}\}$. Then

$$\psi_{j,\Omega} = \psi^\sharp_{j,\Omega} - \frac{(\psi^\sharp_{j,\Omega}, \tilde{\varphi}_{j,V})}{(\psi^0_{j,V}, \tilde{\varphi}_{j,V})} \psi^0_{j,V}$$

is a wavelet of $W_j(\Omega)$ of trace $\gamma_{\mathrm{D}}(\psi_{j,\Omega}) = \psi_{j,\Gamma_{\mathrm{D}}}$. □

The definition of lifting wavelets in $W_j(\Omega)$ is a consequence of the following proposition. Its proof is constructive.

Proposition 3.9 *The property $W_j(\Gamma_{\mathrm{D}}) \subset \gamma_{\mathrm{D}}\, W_j(\Omega)$ is always satisfied in the framework of domain decomposition of Sect. 3.2.*

Proof. Let $\psi_{j,\Gamma_{\mathrm{D}}} \in \Psi_{j,\Gamma_{\mathrm{D}}}$, we shall build explicitely a wavelet $\psi_{j,\Omega} \in W_j(\Omega)$ so that $\gamma_{\mathrm{D}}(\psi_{j,\Omega}) = \psi_{j,\Gamma_{\mathrm{D}}}$.

We consider successively for $p = d-1, \dots, 0$, the case of a wavelet $\psi_{j,\Gamma_{\mathrm{D}}}$ associated to a p-interface of the decomposition of Γ_{D}.

The case $p = d$ corresponds to the subdomains Γ_i of Γ_{D} for which the lifting is canonical, i.e. a tensor product in the reference domain $\hat{\Omega}$.

For $p = d-2, \dots, 0$, the analysis reduces by tensor product to a wavelet associated to a 0-interface in dimension $d-p-1$. This latter wavelet is written exclusively with the scaling functions and wavelets $\varphi_{j,0}, \varphi_{j,d_j}, \psi_{j,0}$, $\psi_{j,2^j-1}$ on the interval. Hence, the lifting in $W_j(\Omega)$ of the wavelet $\psi_{j,\Gamma_{\mathrm{D}}}$ is obtained by "tensor product" from similar computations as those displayed in the proof of Lemma 3.4 (that corresponds to $p = 0$). □

Theorem 3.2 *Let $p \in [1,\infty]$ so that $\tilde{\phi} \in L^{p'}$. For the choice of the lifting wavelets given by Proposition 3.9, the lifting $\mathcal{R}$ verifies the stability property*

$$\|\mathcal{R}g\|_{B^{s+1/p}_{p,q}(\Omega)} \lesssim \|g\|_{B^{s}_{p,q}(\Gamma)}$$

for all $-\min(\tilde{\tau}(p), 1/p) < s < \min(\tau(p) - 1/p, 1)$.

Proof. The wavelets of the collection $\Psi^b_{j,\Omega}$ are locally linearly independent since this is the case for their traces $\Psi_{j,\Gamma_{\mathrm{D}}}$. It results that the collection $\Psi^b_{j,\Omega}$ is stable in L^p for all $p \in]0,\infty]$. Then, the theorem is a direct consequence of the characterizations of Besov norms by the multiscale decompositions on the domain and on Γ_{D}. □

3.3.3 Lifting wavelets at the interfaces of Γ_{D} for a 2-dimensional domain. As for the wavelets at the interfaces of the domain decomposition of Ω, we define in this subsection the lifting wavelets for a general 2-dimensional domain. The task is simpler here since it will suffice to build these lifting wavelets at the 0-interfaces V (or vertices) of the decomposition of Γ_{D}. The lifting wavelets associated to the faces Γ_i of the decomposition of Γ_{D} are canonical tensor products on the reference domain.

The notation are the same as in Sect. 3.2.3 and we need to consider the two cases $V \in \Gamma_{\mathrm{D}}$ (*Dirichlet-Dirichlet vertex*) and $V \in \bar{\Gamma}_{\mathrm{D}} \cap \bar{\Gamma}_{\mathrm{N}}$ (*Neumann-Dirichlet vertex*). In both cases we give details of the constructions of Sect. 3.3.1 and Sect. 3.3.2.

Dirichlet-Dirichlet vertex. The decomposition of Γ_{D} induced by the domain decomposition is defined by the two subdomains $\Gamma_1 = I_{0,1}$, $\Gamma_M = I_{0,M}$ and the parametrizations

$$\xi_1 = \kappa_1|_{\hat{I}} \text{ and } \xi_M = \kappa_M \circ \mathcal{R}_{0,\frac{\pi}{2}}|_{\hat{I}}.$$

The two wavelets of $\Psi_{j,\Gamma_{\mathrm{D}}}$ associated to the vertex V are built as in Sect. 3.2.3 (62). We obtain for example the wavelets

$$(70)\qquad \begin{cases} \psi_{j,0}\circ\xi_1^{-1}+\psi_{j,0}\circ\xi_M^{-1} \\ (\psi_{j,0}-\varphi_{j,0})\circ\xi_1^{-1}-(\psi_{j,0}-\varphi_{j,0})\circ\xi_M^{-1}. \end{cases}$$

The functions of the completion $\Psi_{j,\Omega}^{\sharp,b}$ associated to the vertex V include the function

$$(\psi_{j,0}-\varphi_{j,0})\otimes\varphi_{j,0}\circ\kappa_1^{-1}$$

related to Γ_1 and

$$(71)\qquad \varphi_{j,0}\otimes(\psi_{j,0}-\varphi_{j,0})\circ\kappa_M^{-1}$$

related to Γ_M. The coarse projection $P_{j,\Omega}^*$ (cf. Step 2) applied to any of these two functions reduces to

$$\frac{1}{2}\sum_{i=1}^{M}\varphi_{j,0}\otimes\varphi_{j,0}\circ\kappa_i^{-1}.$$

Taking the linear combinaisons $(1,1)$ and $(1,-1)$ of the functions obtained after correction, we get the two lifting wavelets

$$\begin{cases} \psi_{j,0}\otimes\varphi_{j,0}\circ\kappa_1^{-1}+\varphi_{j,0}\otimes\psi_{j,0}\circ\kappa_M^{-1}+\displaystyle\sum_{i=2}^{M-1}\varphi_{j,0}\otimes\varphi_{j,0}\circ\kappa_i^{-1}, \\ (\psi_{j,0}-\varphi_{j,0})\otimes\varphi_{j,0}\circ\kappa_1^{-1}-\varphi_{j,0}\otimes(\psi_{j,0}-\varphi_{j,0})\circ\kappa_M^{-1}, \end{cases}$$

the traces of which on Γ_{D} are proportional to the wavelets (70).

The construction of lifting wavelets in $W_j(\Omega)$ in the framework of Proposition 3.9 is straightforward in 2-dimension. For example we retain

$$\begin{cases} \displaystyle\sum_{i=1}^{M}\psi_{j,0}\otimes\psi_{j,0}\circ\kappa_i^{-1}, \\ (\psi_{j,0}-\varphi_{j,0})\otimes\psi_{j,0}\circ\kappa_1^{-1}-\psi_{j,0}\otimes(\psi_{j,0}-\varphi_{j,0})\circ\kappa_M^{-1}, \end{cases}$$

the traces of which on Γ_{D} are proportional to the wavelets (70).

Neumann-Dirichlet vertex. The notation are the same as previously. We need to lift the single wavelet of Γ_{D} associated to the vertex V:

$$(72)\qquad \psi_{j,0}\circ\xi_M^{-1}.$$

The first construction starts from the function (71) of the completion $\Psi_{j,\Omega}^{\sharp,b}$ associated to the vertex V. Then we subtract the projection $P_{j,\Omega}^*$ that amounts to add $\sum_{i=1}^{M}\varphi_{j,0}\otimes\varphi_{j,0}\circ\kappa_i^{-1}$. It yields the wavelet

$$\varphi_{j,0}\otimes\psi_{j,0}\circ\kappa_M^{-1}+\sum_{i=1}^{M-1}\varphi_{j,0}\otimes\varphi_{j,0}\circ\kappa_i^{-1},$$

the trace of which on Γ_{D} is proportional to (72).

In the framework of Proposition 3.9, we easily check that the wavelet

$$\sum_{i=1}^{M} \psi_{j,0} \otimes \psi_{j,0} \circ \kappa_i^{-1}$$

is a lifting wavelet of $W_j(\Omega)$ the trace of which on Γ_{D} is proportional to (72).

Appendix

Proof of Proposition 3.2. The proof is constructive and treats separately both edges. We first biorthogonalize at the edges 0 and 1 the scaling function bases (cf. 40) of the MRA spaces $V_j^{\delta,\mathrm{CL}_\mathrm{D}}$ and $\widetilde{V}_j^{\tilde{\delta},\mathrm{CL}_\mathrm{D}}$.

Let $\Phi_j^{\delta,\mathrm{CL}_\mathrm{D}}$ and $\tilde{\Phi}_j^{\tilde{\delta},\mathrm{CL}_\mathrm{D}}$ be any choice of such biorthogonal bases. Next (for example at the edge 0), we look for the linear combinations

$$\left\{ \begin{aligned} \varphi_{j,0} &= \varphi_{j,0}^{\delta_0,(0)} + \sum_{k=1}^{\max(n,\tilde{n})-1} c_k \varphi_{j,k}, \\ \tilde{\varphi}_{j,0} &= \tilde{c}_0 \tilde{\varphi}_{j,0}^{\tilde{\delta}_0,(0)} + \sum_{k=1}^{\max(n,\tilde{n})-1} \tilde{c}_k \tilde{\varphi}_{j,k} \end{aligned} \right. \tag{73}$$

so that $\langle \varphi_{j,0}, \tilde{\varphi}_{j,k} \rangle = \langle \varphi_{j,k}, \tilde{\varphi}_{j,0} \rangle = \delta_k^0$, $k = 0, \ldots, \max(n, \tilde{n}) - 1$. Equivalently we obtain $c_k = -\langle \varphi_{j,0}^{\delta_0,(0)}, \tilde{\varphi}_{j,k} \rangle$, $k = 1, \ldots, \max(n, \tilde{n}) - 1$ and the vector $(\tilde{c}_k,\ k = 0, \cdots, \max(n, \tilde{n}) - 1)^{\mathrm{T}}$ is solution of the square system

$$\left\{ \begin{aligned} &\langle \tilde{\varphi}_{j,0}^{\tilde{\delta}_0,(0)}, \varphi_{j,k} \rangle \, \tilde{c}_0 + \tilde{c}_k = 0, \quad k = 1, \ldots, \max(n, \tilde{n}) - 1, \\ &\langle \tilde{\varphi}_{j,0}^{\tilde{\delta}_0,(0)}, \varphi_{j,0} \rangle \, \tilde{c}_0 \qquad\quad = 1. \end{aligned} \right.$$

which is unique from the non singularity assumption of the matrix $M^{0,-1,-1}$. Note in particular that $\langle \tilde{\varphi}_{j,0}^{\tilde{\delta}_0,(0)}, \varphi_{j,0} \rangle$ does not vanish as well as $\tilde{c}_0$. □

For the next proof, we shall need in addition the following lemma on the filters H^j of Φ_j^δ and $\widetilde{H}^j$ of $\tilde{\Phi}_j^{\tilde{\delta}}$ so that $\varphi_{j,k} = \sum_{m \in \Delta_{j+1}} H_{k,m}^j \varphi_{j+1,m}$ and $\tilde{\varphi}_{j,k} = \sum_{m \in \Delta_{j+1}} \widetilde{H}_{k,m}^j \tilde{\varphi}_{j+1,m}$.

Lemma 3.5 *The filters H^j and $\widetilde{H}^j$ verify the relations, for all $k = 0, \dots, \#\Delta_j - 1$*

$$\begin{cases} H^j_{k,0} = \frac{1}{\sqrt{2}}\delta^0_k, \;\; H^j_{k,\#\Delta_{j+1}-1} = \frac{1}{\sqrt{2}}\delta^{\#\Delta_j-1}_k, \\ \widetilde{H}^j_{k,0} = \frac{1}{\sqrt{2}}\delta^0_k, \;\; \widetilde{H}^j_{k,\#\Delta_{j+1}-1} = \frac{1}{\sqrt{2}}\delta^{\#\Delta_j-1}_k. \end{cases} \tag{74}$$

Proof. This is a direct consequence of the Definition (73), the property $\tilde{c}_0 \neq 0$ and the two scale relations for the scaling functions $\varphi^{\delta_\varepsilon,(\varepsilon)}_{j,\alpha}$ recalled below e.g. at the edge 0 (see e.g. [25] for details).

$$\varphi^{\delta_0,(0)}_{j,\alpha} = \frac{2^{-\alpha}}{\sqrt{2}}\varphi^{\delta_0,(0)}_{j+1,\alpha} + \sum_{m=m_0+\delta_0}^{m_1+2m_0+2\delta_0-2} \sum_{k=-m_1+1}^{m_0+\delta_0-1} \langle x^\alpha, \tilde{\phi}_{0,k}\rangle h_{m-2k}\, \phi_{j+1,m}. \qquad \square$$

Proof of Proposition 3.3. Classically (see e.g. [25] for details) the complement spaces W_j and $\widetilde{W}_j$ are biorthogonal (in the sense that they can be endowed with biorthogonal (uniform) Riesz bases). We shall prove that $W^0_j = W_j \cap V^{\delta,\mathrm{CL_D}}_{j+1}$ and $\widetilde{W}^0_j = \widetilde{W}_j \cap \widetilde{V}^{\delta,\mathrm{CL_D}}_{j+1}$ are biorthogonal subspaces of codimension 2.

Let us assume that result for the moment. We recall that in all cases the wavelet bases include a fixed number of modified wavelets at the edges and the remaining wavelets are wavelets on the line supported on $[0,1]$. From similar argument as in the proof of Proposition 3.2, we conclude that there exist biorthogonal wavelet bases Ψ_j and $\tilde{\Psi}_j$ (44) of W_j, $\widetilde{W}_j$ so that the collections

$$\begin{cases} \Psi^0_j = \{\psi_{j,k}, k = 1, \dots, 2^j - 2\} \\ \tilde{\Psi}^0_j = \{\tilde{\psi}_{j,k}, k = 1 \dots, 2^j - 2\} \end{cases} \tag{75}$$

define biorthogonal bases of the subspaces W^0_j and $\widetilde{W}^0_j$. Consequently the filters G^j and $\widetilde{G}^j$ so that $\Psi_j = G^j \Phi^\delta_{j+1}$ and $\tilde{\Psi}_j = \widetilde{G}^j \tilde{\Phi}^{\tilde{\delta}}_{j+1}$ verify for all $k = 1, \dots, 2^j - 2$ the relations

$$\begin{cases} G^j_{k,0} = G^j_{k,\#\Delta_{j+1}-1} = 0, \\ \widetilde{G}^j_{k,0} = \widetilde{G}^j_{k,\#\Delta_{j+1}-1} = 0. \end{cases} \tag{76}$$

Let us show that in addition the bases Ψ_j and $\tilde{\Psi}_j$ can be chosen so that

$$\begin{cases} G_{0,0} = G_{2^j-1,\#\Delta_{j+1}-1} = \frac{1}{\sqrt{2}}, \\ \widetilde{G}_{0,0} = \widetilde{G}_{2^j-1,\#\Delta_{j+1}-1} = \frac{1}{\sqrt{2}}. \end{cases} \tag{77}$$

Let us recall the following identity that rewrites the biorthogonality relations in terms of the filters:

$$\sum_{k\in\Delta_j} H^j_{k,m}\widetilde{H}^j_{k,m'} + \sum_{k\in\nabla_j} G^j_{k,m}\widetilde{G}^j_{k,m'} = \delta_m^{m'},\ m,m'\in\Delta_{j+1}$$

Written for $m=m'=0$ and $m=m'=\#\Delta_{j+1}-1$, from (76) and (74), it implies that

$$G_{0,0}\widetilde{G}_{0,0} = \frac{1}{2} \text{ and } G_{2^j-1,\#\Delta_{j+1}-1}\widetilde{G}_{2^j-1,\#\Delta_{j+1}-1} = \frac{1}{2}.$$

Hence, it suffices to renormalize $\psi_{j,0}$ by $\frac{1}{\sqrt{2}G_{0,0}}$ and $\tilde{\psi}_{j,0}$ by $\sqrt{2}G_{0,0} = \frac{1}{\sqrt{2}\widetilde{G}_{0,0}}$ to obtain the property (77).

Choosing such biorthogonal wavelet bases Ψ_j and $\tilde{\Psi}_j$, it is easily seen that $\Psi_j^{\mathrm{CL_D}}$ and $\tilde{\Psi}_j^{\mathrm{CL_D}}$ define biorthogonal bases of $W_j^{\mathrm{CL_D}}$ and $\widetilde{W}_j^{\mathrm{CL_D}}$.

Indeed, from the properties on the filters, on the one hand $\mathrm{S}(\Psi_j^{\mathrm{CL_D}}) \subset V_{j+1}^{\delta,\mathrm{CL_D}}$ and $\mathrm{S}(\tilde{\Psi}_j^{\mathrm{CL_D}}) \subset \widetilde{V}_{j+1}^{\tilde{\delta},\mathrm{CL_D}}$. On the other hand $\mathrm{S}(\Psi_j^{\mathrm{CL_D}}) \perp \widetilde{V}_j^{\tilde{\delta},\mathrm{CL_D}}$ and $\mathrm{S}(\tilde{\Psi}_j^{\mathrm{CL_D}}) \perp V_j^{\delta,\mathrm{CL_D}}$ and the bases are clearly biorthogonal.

It remains to prove the codimension 2 and the biorthogonality of the subspaces W_j^0 and $\widetilde{W}_j^0$. From the decoupling of the edges, we can argue separately on the boundary conditions at the edges 0 and 1. Let us consider for example homogeneous Dirichlet boundary conditions at the edge 0. If $W_j\cap V_{j+1}^{\delta,(0,-1)} = W_j$, then any function of $V_{j+1}^{\delta}\cap(\widetilde{V}_j^{\tilde{\delta}})^{\perp}$ belongs to $V_{j+1}^{\delta,(0,-1)}$. From the properties (74) of the filters H^j and $\widetilde{H}^j$, the function

$$\begin{aligned}\varphi_{j+1,0} - P_j\,\varphi_{j+1,0} &= (1-H_{0,0}\widetilde{H}_{0,0})\,\varphi_{j+1,0} + \cdots \\ &= \frac{1}{2}\,\varphi_{j+1,0} + \cdots\end{aligned} \tag{78}$$

contradicts this assumption which proves the codimension 1 of $W_j\cap V_{j+1}^{\delta,(0,-1)}$ in W_j.

From the biorthogonality of the wavelets on the line, to prove the biorthogonality of $W_j\cap V_{j+1}^{\delta,(0,-1)}$ and $\widetilde{W}_j\cap\widetilde{V}_{j+1}^{\tilde{\delta},(0,-1)}$ it suffices classically to prove that any function $f_{j+1}\in W_j\cap V_{j+1}^{\delta,(0,-1)}$ so that $f_{j+1}\perp \widetilde{W}_j\cap\widetilde{V}_{j+1}^{\tilde{\delta},(0,-1)}$ is equal to zero. Such a function f_{j+1} is in addition orthogonal to the space

$$\widetilde{V}_j^{\tilde{\delta}} \oplus (\widetilde{W}_j\cap\widetilde{V}_{j+1}^{\tilde{\delta},(0,-1)}) \oplus \mathrm{S}(\{\tilde{\varphi}_{j+1,0}\})$$

which, from (78) (written for $\tilde{\varphi}_{j+1,0}$), is equal to $\widetilde{V}_{j+1}^{\tilde{\delta}}$. Hence $f_{j+1}=0$. $\square$

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