- Numerical simulation of a stratigraphic model - single lithology case

Single lithology stratigraphic model

Our objective is to simulate the infill of sedimentary basins at large space and time scales.

- $\Omega = (0, L_x)$: horizontal extension of the basin
- $(0, t_f)$ is the time interval of the simulation with $t_f > 0$
- $h(x,t)$ is the sediment thickness for $(x,t) \in \Omega \times (0, t_f)$
- $h_{\text{sea}}(t)$ is the given sea level for $t \in (0, t_f)$
- $b(x,t) = h_{\text{sea}}(t) - h(x,t)$ is the bathymetry

The model accounts for the sediment thickness conservation of the sediments and for the boundary and initial conditions

$$\left\{ \begin{array}{l} \partial_t h(x,t) + \text{div}\ (\nabla \psi(b(x,t))) = 0 \text{ on } \Omega \times (0, t_f) \\ \\ h(x,0) = h_{\text{init}}(x) \text{ on } \Omega \\ \\ \nabla \psi(b(x,t)) \cdot \boldsymbol{n} = g_0 \text{ on } x = 0 \\ \nabla \psi(b(x,t)) \cdot \boldsymbol{n} = g_1 \text{ on } x = L_x \end{array} \right. \tag{1}$$

where $\psi(b) = \int_0^b k(u)\mathrm{d}u$ with $k > 0$ the diffusion coefficient of the sediments measuring their ability to be transported by gravity. This coefficient is modeled by a nonlinear function of the bathymetry as follows

$$k(b) = \left\{ \begin{array}{l} k^m \text{ if } b \geqslant 0 \\ k^c \text{ otherwise} \end{array} \right.$$

Finite Volume discretization

The model is discretized using a Two Points Flux Approximation (TPFA) on an unstructured orthogonal mesh. We obtain at each time step $n > 0$ and for each cell K

$$\left\{ \begin{array}{l} |K| \dfrac{h_K^n - h_K^{n-1}}{\Delta t^n} + \displaystyle\sum_{\sigma \in \mathcal{F}_K \cap \mathcal{F}_{\text{int}} = K|L} T_\sigma(\psi(b_L^n) - \psi(b_K^n)) + \sum_{\sigma \in \mathcal{F}_K \cap \mathcal{F}_{\text{ext}}} g_\sigma = 0 \\ \\ \text{with } \ T_{\sigma = K|L} = \dfrac{|\sigma|}{\text{dist}(x_K, x_L)} \\ \\ b_K^n = h_{\text{sea}}(t^n) - h_K^n. \end{array} \right. \tag{2}$$

The initial condition is computed by

$$h_K^0 = h_{\text{init}}(x_K).$$

Scilab implementation

Your work is to implement the scheme (2) by completing the given Scilab file.

The outputs will be the discrete values of $h(x,t)$ and $b(x,t)$.

- Compute the data structure needed for the implementation of the scheme for the given uniform 1D mesh with N cells of the domain $(0, L_x)$
- At each time step (inside the time loop) the nonlinear system (2) representing the scheme equations in all cells is denoted by
$$R(h^n) = 0$$
where the function $R : \mathbb{R}^N \to \mathbb{R}^N$ is called the "residual". To solve this nonlinear system, the Newton algorithm is used: set $\epsilon = 10^{-6}$, $h^{0,n} = h^{n-1}$, and for $l \geq 0$ until $\|R(h^{l,n})\| \leq \epsilon \|R(h^{0,n})\|$ compute
$$\frac{\partial R}{\partial h}(h^{l,n})dh = -R(h^{l,n}),$$
$$h^{l+1,n} = h^{l,n} + dh.$$
We underline that the Newton algorithm has a quadratic convergence if the initial solution $h^{0,n}$ is closed enough to the solution h^n, ie there exist $\alpha > 0$ and $\beta > 0$ such that if $\|h^{0,n} - h^n\| \leq \alpha$ then
$$\|R(h^{l+1,n})\| \leqslant \beta \ \|R(h^{l,n})\|^2. \tag{3}$$
Note also that if the Newton algorithm is not converged in $Newtmax$ iterations, then the time step is restarted using a reduced time step by a factor 2. If the time step is converged in less than $Newtmax$ iterations we can increase the time step by a factor 1.2 until the maximum time step is reached.
- Write the Scilab function computing the `residual` $R(h^n)$ given h^{n-1}. This computation is achieved using one loop over the cells, one loop over the inner faces, and one loop over the boundary faces.
- Write the Scilab function computing the `jacobian` $\dfrac{\partial R}{\partial h}(h^n)$. This computation is achieved using one loop over the cells and one loop over the inner faces.
- Compute and plot $h_s(x,t) = \min_{\{t \leqslant q \leqslant t_f\}} h(x,q)$ i.e. the sediment layers at each time t by taking into account the erosion