Course *Finite Volume discretization of PDEs*
University of Nice Sophia Antipolis

Exercize : finite volume discretization of the 1D convection diffusion equation.

Let us consider the following convection diffusion equation

$$\left\{\begin{array}{r}cu'(x) - \nu u''(x) = 0 \qquad \text{on } (0,1),\\ u(0) = u_D^0,\\ u(1) = u_D^1,\end{array}\right.$$

with $c \geq 0$ and $\nu > 0$. Using the notations of the course, we consider the following finite volume scheme $u_h \in V_h$ such that

$$f_{i+1/2} - f_{i-1/2} \quad = \quad 0,\ i = 1, \cdots, N, \tag{1}$$

setting $u_0 = u_D^0$, $u_{N+1} = u_D^1$ and with

$$f_{i+1/2} = \nu \frac{u_i - u_{i+1}}{h_{i+1/2}} + c(\theta_{i+1/2} u_i + (1 - \theta_{i+1/2}) u_{i+1}),\ i = 0, \cdots, N,$$

for given $\theta_{i+1/2} \in [1/2, 1]$.

(1) How can we call the discretization of the convection term for $\theta_{i+1/2} = 1$ and for $\theta_{i+1/2} = 1/2$ for all $i = 1, \cdots, N$.

(2) Give a condition on $\theta_{i+1/2}$, $i = 1, \cdots, N$, and c and h such that

$$u_i = \alpha_i u_{i+1} + (1 - \alpha_i) u_{i-1},$$

with $0 < \alpha_i < 1$ for all $i = 1, \cdots, N$. In the following we assume that this condition is satisfied.

(3) Show that the matrix of the scheme is an M-matrix and deduce that it admits a unique solution

(4) Prove that the solution of the scheme satisfies the following maximum principle

$$\min(u_D^0, u_D^1) \leq u_i \leq \max(u_D^0, u_D^1),$$

for all $i = 1, \cdots, N$.