Course Finite Volume discretization of PDEs,
University Nice Sophia Antipolis

Exercize : finite volume discretization of the 1D Laplacian with Dirichlet and Neumann boundary conditions.

We consider the following problem

$$(P)\begin{cases} -u''(x) = f(x) \quad \text{on } (0,L), \\ u(0) = u_D, \\ -u'(L) = g, \end{cases}$$

which has a unique solution in $H^1(0,1)$ for all $u_D \in \mathbb{R}$, $g \in \mathbb{R}$, $f \in L^2(0,L)$.

We consider the following subdivision of the interval $(0,L)$ with $N+1$ points :

$$x_{1/2} = 0 < x_{3/2} < \cdots < x_{i-1/2} < x_{i+1/2} < \cdots < x_{N-1/2} < x_{N+1/2} = L.$$

Keeping the notations of the course, the finite volume discretization of the interval $(0,L)$ consists of the set of N cells $\kappa_i = (x_{i-1/2}, x_{i+1/2})$ for $i = 1, \cdots, N$, and of the cell centers $x_i = \frac{x_{i-1/2}+x_{i+1/2}}{2}$ for $i = 1, \cdots, N$. We also set $x_0 = 0$ and $x_{N+1} = L$, $h_{i+1/2} = |x_{i+1} - x_i|$ for $i = 0, \cdots, N$, and $h_i = |x_{i+1/2} - x_{i-1/2}|$ for $i = 1, \cdots, N$. Finally, we set $h = \max_{i=1,\cdots,N} h_i$.

(1) Let us consider the N discrete unknowns u_i approximating $u(x_i)$ for $i = 1, \cdots, N$. Write the discrete fluxes $f_{i+1/2}$ approximating $-u'(x_{i+1/2})$, $i = 0, \cdots, N$, and taking into account the boundary conditions for $i = 0$ and $i = N$.
Write the finite volume discretization of (P) consisting of N discrete conservation equations on the cells κ_i, $i = 1, \cdots, N$ using the previous fluxes.

(2) Write the square matrix A_h of size N and the right hand side $S_h \in \mathbb{R}^N$ such that the finite volume scheme is equivalent to $A_h U_h = S_h$ where $U_h \in \mathbb{R}^N$ is such that $U_{h,i} = u_i$, $i = 1, \cdots, N$.

(3) We consider a uniform mesh with $h = h_i = \frac{L}{N}$ for all $i = 1, \cdots, N$. Let us consider the fonction

$$u(x) = e^{\sin(\pi x)},$$

which is the exact solution of the previous problem with $u_D = u(0) = 1$, $g = -u'(L)$, and

$$f(x) = -u''(x) = \pi^2 \Big(\sin(\pi x) - \cos^2(\pi x)\Big) e^{\sin(\pi x)},$$

Program in scilab the functions u, u' and f, then compute $u_D = u(0)$ and $g = -u'(L)$. Program in scilab the vector $X \in \mathbb{R}^N$ containing the N cell centers sucth that $X(i) = x_i$, $i = 1, \cdots, N$.

(4) Implement the square matrix A_h and right hand side S_h of the discrete FV problem taking into account that $h = h_i = \frac{L}{N}$ for all $i = 1, \cdots, N$.
Compute the discrete solution using the scilab backslash command for solving linear systems. Plot the exact solution and the FV solution on the same figure for $L = 1$.

(5) Let us set $L = 1$ and let us define $e_i = u(x_i) - u_i$ for all $i = 0, \cdots, N$, setting $u_0 = u_D$. Plot in log 2 scale the discrete L^2 and H_0^1 errors

$$\|e_h\|_{L^2(0,L)} = \sqrt{\sum_{i=1}^{N} h(e_i)^2},$$

and

$$\|e_h\|_{1,h} = \sqrt{\sum_{i=0}^{N-1} \frac{(e_i - e_{i+1})^2}{h_{i+1/2}}},$$

as a function of h for increasing values of $N = 10, 20, 40, 80, 160, 320, 640, 1280$. What do you conclude about the convergence rates of the FV volume scheme ?