Discrete De Rham method: summary of notations and formulas

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Part 1	Session 1	*The de Rham complex and its usefulness in PDEs*
	Session 2	*Low-order case, link with CW complexes*
	Session 3	*Design of the DDR complex in 2D*
Part 2	Session 1	*Construction of the DDR complex in 3D*
	Session 2	*Analytical properties: Poincaré inequalities, consistencies*
	Session 3	*Applications: magnetostatics and Stokes*

Table 1: Tentative contents of the sessions

1 De Rham complex

For $\Omega \subset \mathbb{R}^3$:

$$\mathbb{R} \xrightarrow{i_\Omega} \mathrm{H}^1(\Omega) \xrightarrow{\mathbf{grad}} \mathbf{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} \mathrm{L}^2(\Omega) \xrightarrow{0} \{0\},$$

where i_Ω injects real numbers as constant functions and

$$\mathrm{H}^1(\Omega) = \{q \in \mathrm{L}^2(\Omega) \,:\, \mathbf{grad}\, q \in \mathbf{L}^2(\Omega)\}, \tag{1.1}$$

$$\mathbf{H}(\mathbf{curl};\Omega) = \{\boldsymbol{v} \in \mathbf{L}^2(\Omega) \,:\, \mathbf{curl}\, \boldsymbol{v} \in \mathbf{L}^2(\Omega)\}, \tag{1.2}$$

$$\mathbf{H}(\mathrm{div};\Omega) = \{\boldsymbol{w} \in \mathbf{L}^2(\Omega) \,:\, \mathrm{div}\, \boldsymbol{w} \in \mathrm{L}^2(\Omega)\}. \tag{1.3}$$

Complex property: the image of an operator is contained in the kernel of the next one:

$$\begin{aligned} & \mathrm{Im}\, i_\Omega \subset \ker \mathbf{grad}\,, && \mathrm{Im}\, \mathbf{grad} \subset \ker \mathbf{curl}\,, && \mathrm{Im}\, \mathbf{curl} \subset \ker \mathrm{div}\,, \\ \text{that is: } & \mathbf{grad}\, i_\Omega = 0\,, && \mathbf{curl}\,\mathbf{grad} = \mathbf{0}\,, && \mathrm{div}\, \mathbf{curl} = 0. \end{aligned}$$

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Exactness properties: depending on the topology of Ω, these inclusions can become equalities.

$$\begin{aligned}
&\text{If } \Omega \text{ is connected:} && \ker \mathbf{grad} = \operatorname{Im} i_\Omega, && \text{(1.4a)}\\
&\text{If } \Omega \text{ does not have any tunnel:} && \ker \mathbf{curl} = \operatorname{Im} \mathbf{grad}, && \text{(1.4b)}\\
&\text{If } \Omega \text{ does not enclose any void:} && \ker \operatorname{div} = \operatorname{Im} \mathbf{curl}, && \text{(1.4c)}\\
&\text{Any } \Omega\text{:} && \operatorname{Im} \operatorname{div} = \mathrm{L}^2(\Omega). && \text{(1.4d)}
\end{aligned}$$

2 Magnetostatics model

Strong formulation. Find $\boldsymbol{A}$ (vector potential) and $\boldsymbol{H}$ (magnetic field) s.t.

$$\begin{aligned}
\mu \boldsymbol{H} - \mathbf{curl}\, \boldsymbol{A} &= \boldsymbol{0} && \text{in } \Omega && \text{(link field–potential)}, && \text{(2.1a)}\\
\mathbf{curl}\, \boldsymbol{H} &= \boldsymbol{J} && \text{in } \Omega && \text{(Ampère's law)}, && \text{(2.1b)}\\
\operatorname{div} \boldsymbol{A} &= 0 && \text{in } \Omega && \text{(Coulomb gauge)}, && \text{(2.1c)}\\
\boldsymbol{A} \times \boldsymbol{n} &= \boldsymbol{g} && \text{on } \partial\Omega && && \text{(2.1d)}
\end{aligned}$$

where $\mu : \Omega \to \mathbb{R}^+$ is the magnetic permeability and $\boldsymbol{J} \in \operatorname{Im} \mathbf{curl}$ the current density. This system assumes that Ω does not enclose any void.

Weak formulation. Find $(\boldsymbol{H}, \boldsymbol{A}) \in \mathbf{H}(\mathbf{curl}; \Omega) \times \mathbf{H}(\operatorname{div}; \Omega)$ such that

$$\begin{aligned}
a(\boldsymbol{H}, \boldsymbol{\zeta}) - b(\boldsymbol{\zeta}, \boldsymbol{A}) &= -\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{\zeta} && \forall \boldsymbol{\zeta} \in \mathbf{H}(\mathbf{curl}; \Omega),\\
b(\boldsymbol{H}, \boldsymbol{v}) + c(\boldsymbol{A}, \boldsymbol{v}) &= \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} && \forall \boldsymbol{v} \in \mathbf{H}(\operatorname{div}; \Omega),
\end{aligned} \tag{2.2}$$

with

$$\begin{aligned}
&a : \mathbf{H}(\mathbf{curl}; \Omega) \times \mathbf{H}(\mathbf{curl}; \Omega) \to \mathbb{R} && a(\boldsymbol{v}, \boldsymbol{\zeta}) \coloneqq \int_\Omega \mu \boldsymbol{v} \cdot \boldsymbol{\zeta},\\
&b : \mathbf{H}(\mathbf{curl}; \Omega) \times \mathbf{H}(\operatorname{div}; \Omega) \to \mathbb{R} && b(\boldsymbol{\zeta}, \boldsymbol{v}) \coloneqq \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\zeta},\\
&c : \mathbf{H}(\operatorname{div}; \Omega) \times \mathbf{H}(\operatorname{div}; \Omega) \to \mathbb{R} && c(\boldsymbol{w}, \boldsymbol{v}) \coloneqq \int_\Omega \operatorname{div} \boldsymbol{w} \operatorname{div} \boldsymbol{v}
\end{aligned}$$

Reference for the inf-sup analysis: [3, Lemma 2.1].

3 Mesh

Notations. $\mathcal{M}_h \coloneqq \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h \cup \mathcal{V}_h$, where:

- $\mathcal{T}_h$ are the polyhedral elements (notation T);
- $\mathcal{F}_h$ are the faces (notation F), and $\mathcal{F}_T$ = faces of $T \in \mathcal{T}_h$;
- $\mathcal{E}_h$ are the edges (notation E), and $\mathcal{E}_Y$ = edges of $Y \in \mathcal{T}_h \cup \mathcal{F}_h$;
- $\mathcal{V}_h$ the vertices (notations V with coordinates $\boldsymbol{x}_V$), and $\mathcal{V}_Y$ = vertices of $Y \in \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h$.

Each $Y \in \mathcal{T}_h \cup \mathcal{F}_h$ is topologically trivial (simply connected and connected boundary).

Orientations.
Intrinsic orientations:

- For each $F \in \mathcal{F}_h$, we fix a unit normal $\boldsymbol{n}_F$.
- For each $E \in \mathcal{E}_h$, we fix a unit tangent vector $\boldsymbol{t}_E$.

Extrinsic orientations:

- For $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_T$, $\omega_{TF} \in \{\pm 1\}$ such that $\omega_{TF}\boldsymbol{n}_F$ is the outer normal to T.
- For $F \in \mathcal{F}_T$ and $E \in \mathcal{E}_T$, $\boldsymbol{n}_{FE}$ unit normal to E in span(F) such that $(\boldsymbol{t}_E, \boldsymbol{n}_{FE}, \boldsymbol{n}_F)$ is right-handed in $\mathbb{R}^3$. $\omega_{FE} \in \{\pm 1\}$ such that $\omega_{FE}\boldsymbol{n}_{FE}$ is the outer normal on E to F in span(F).

If $E \in \mathcal{E}_T$ shared by two faces $F_1, F_2 \in \mathcal{F}_T$, we have $\omega_{TF_1}\omega_{F_1E} + \omega_{TF_2}\omega_{F_2E} = 0$.

4 Function spaces

Lebesgue and Sobolev. If $Y \in \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h$, $\mathrm{L}^2(Y)$ usual Lebesgue space and $\mathbf{L}^2(Y)$ = Lebesgue space of vector valued $Y \to \mathrm{span}(Y)$. Same for the Sobolev spaces $\mathrm{H}^l(Y)$ and $\mathbf{H}^l(Y)$.

Polynomial spaces. For $\ell \geq 0$ integer and $Y \in \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h$, we set

$$\mathcal{P}^\ell(Y) = \text{ restriction to } Y \text{ of polynomials } q : Y \to \mathbb{R} \text{ of total degree} \leq \ell.$$

This gives $\mathcal{P}^{-1}(Y) = \{0\}$. Vector-valued version: $\boldsymbol{\mathcal{P}}^\ell(Y)$ = restriction to Y of polynomials $\boldsymbol{v} : Y \to \mathrm{span}(Y)$.
$\pi^\ell_{\mathcal{P},Y} : \mathrm{L}^2(Y) \to \mathcal{P}^\ell(Y)$ is the L^2-orthogonal projector (corresponding vector-valued denoted by $\boldsymbol{\pi}^\ell_{\mathcal{P},Y}$).

Koszul complements. For all $Y \in \mathcal{T}_h \cup \mathcal{F}_h$, take $\boldsymbol{x}_Y \in Y$. For $F \in \mathcal{F}_h$, set

$$\boldsymbol{\mathcal{G}}^\ell(F) \coloneqq \mathbf{grad}_F\, \mathcal{P}^{\ell+1}(F), \qquad \boldsymbol{\mathcal{G}}^{\mathrm{c},\ell}(F) \coloneqq \varrho_{-\pi/2}(\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}^{\ell-1}(F), \tag{4.1a}$$

$$\boldsymbol{\mathcal{R}}^\ell(F) \coloneqq \mathbf{rot}_F\, \mathcal{P}^{\ell+1}(F), \qquad \boldsymbol{\mathcal{R}}^{\mathrm{c},\ell}(F) \coloneqq (\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}^{\ell-1}(F). \tag{4.1b}$$

Then we have the *non-orthogonal* decompositions

$$\boldsymbol{\mathcal{P}}^\ell(F) = \boldsymbol{\mathcal{G}}^\ell(F) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},\ell}(F) = \boldsymbol{\mathcal{R}}^\ell(F) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},\ell}(F). \tag{4.2}$$

The L^2-orthogonal projectors on these spaces are denoted by $\boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^{\ell}$, $\boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^{c,\ell}$, $\boldsymbol{\pi}_{\boldsymbol{\mathcal{R}},F}^{\ell}$, and $\boldsymbol{\pi}_{\boldsymbol{\mathcal{R}},F}^{c,\ell}$. For $T \in \mathcal{T}_h$, set

$$\boldsymbol{\mathcal{G}}^{\ell}(T) := \mathbf{grad}\,\mathcal{P}^{\ell+1}(T), \qquad \boldsymbol{\mathcal{G}}^{c,\ell}(T) := (\boldsymbol{x} - \boldsymbol{x}_T) \times \boldsymbol{\mathcal{P}}^{\ell-1}(T), \tag{4.3a}$$

$$\boldsymbol{\mathcal{R}}^{\ell}(T) := \mathbf{curl}\,\boldsymbol{\mathcal{P}}^{\ell+1}(T), \qquad \boldsymbol{\mathcal{R}}^{c,\ell}(T) := (\boldsymbol{x} - \boldsymbol{x}_T)\mathcal{P}^{\ell-1}(T). \tag{4.3b}$$

We have the decompositions

$$\boldsymbol{\mathcal{P}}^{\ell}(T) = \boldsymbol{\mathcal{G}}^{\ell}(T) \oplus \boldsymbol{\mathcal{G}}^{c,\ell}(T) = \boldsymbol{\mathcal{R}}^{\ell}(T) \oplus \boldsymbol{\mathcal{R}}^{c,\ell}(T) \tag{4.4}$$

with L^2-orthogonal projectors are denoted by $\boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},T}^{\ell}$, $\boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},T}^{c,\ell}$, $\boldsymbol{\pi}_{\boldsymbol{\mathcal{R}},T}^{\ell}$, and $\boldsymbol{\pi}_{\boldsymbol{\mathcal{R}},T}^{c,\ell}$.

Isomorphisms. The following are isomorphisms:

$$\mathbf{rot}_F : \mathcal{P}^{0,\ell}(F) \xrightarrow{\cong} \boldsymbol{\mathcal{R}}^{\ell-1}(F) \tag{4.5}$$

$$\mathrm{div}_F : \boldsymbol{\mathcal{R}}^{c,\ell}(F) \xrightarrow{\cong} \mathcal{P}^{\ell-1}(F), \qquad \mathrm{div} : \boldsymbol{\mathcal{R}}^{c,\ell}(T) \xrightarrow{\cong} \mathcal{P}^{\ell-1}(T), \tag{4.6}$$

$$\mathbf{curl} : \boldsymbol{\mathcal{G}}^{c,\ell}(T) \xrightarrow{\cong} \boldsymbol{\mathcal{R}}^{\ell-1}(T). \tag{4.7}$$

5 Integration by parts formulas

5.1 In 2D

Take F a flat face, $\mathbf{grad}_F$ and div_F the tangent gradient and divergence operators. With $\rho_{-\pi/2}$ rotation of angle $-\pi//2$ in F:

$$\begin{aligned} &\text{For } r : F \to \mathbb{R}: && \mathbf{rot}_F\, r := \varrho_{-\pi/2}(\mathbf{grad}_F\, r),\\ &\text{For } \boldsymbol{v} : F \to \mathrm{span}(F): && \mathrm{rot}_F\, \boldsymbol{v} := \mathrm{div}_F(\varrho_{-\pi/2}\boldsymbol{v}), \end{aligned}$$

where $\mathrm{span}(F) \approx \mathbb{R}^2$ is the tangent space to F.

IBP for the gradient/divergence:

$$\int_F \mathbf{grad}_F\, q_F \cdot \boldsymbol{v}_F = -\int_F q_F\, \mathrm{div}_F\, \boldsymbol{v}_F + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q_F(\boldsymbol{v}_F \cdot \boldsymbol{n}_{FE}),$$

IBP for the rotational:

$$\int_F \mathrm{rot}_F\, \boldsymbol{v}_F\; r_F = \int_F \boldsymbol{v}_F \cdot \mathbf{rot}_F\, r_F - \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E (\boldsymbol{v}_F \cdot \boldsymbol{t}_E) r_F.$$

5.2 In 3D

IBP for the gradient/divergence:

$$\int_T \mathbf{grad}\, q_T \cdot \boldsymbol{v}_T = -\int_T q_T \operatorname{div} \boldsymbol{v}_T + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F q_T(\boldsymbol{v}_T \cdot \boldsymbol{n}_F).$$

IBP for the curl:

$$\begin{aligned}\int_T \mathbf{curl}\, \boldsymbol{v}_T \cdot \boldsymbol{w}_T &= \int_T \boldsymbol{v}_T \cdot \mathbf{curl}\, \boldsymbol{w}_T + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F \boldsymbol{v}_T \cdot (\boldsymbol{w}_T \times \boldsymbol{n}_F) \\ &= \int_T \boldsymbol{v}_T \cdot \mathbf{curl}\, \boldsymbol{w}_T + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F (\boldsymbol{n}_F \times (\boldsymbol{v}_T \times \boldsymbol{n}_F)) \cdot (\boldsymbol{w}_T \times \boldsymbol{n}_F),\end{aligned}$$

5.3 Other useful formulas

If $\boldsymbol{n}_F$ is a normal vector to F and $\boldsymbol{\xi} \in \mathbb{R}^2$, $\boldsymbol{n}_F \times (\boldsymbol{\xi} \times \boldsymbol{n}_F) =: \boldsymbol{\xi}_{\mathrm{t},F}$ is the projection of $\boldsymbol{\xi}$ on $\operatorname{span}(F)$. It is the rotated by $\pi/2$ on that plane of $\boldsymbol{\xi} \times \boldsymbol{n}_F$.

If T is a polyhedron, F one of its face and $\boldsymbol{v} : T \to \mathbb{R}^3$, $r : T \to \mathbb{R}$:

$$(\mathbf{grad}\, r)_{|F} \times \boldsymbol{n}_F = \mathbf{rot}_F(r_{|F}), \tag{5.1}$$

$$(\mathbf{curl}\, \boldsymbol{v})_{|F} \cdot \boldsymbol{n}_F = \operatorname{div}_F(\boldsymbol{v}_{|F} \times \boldsymbol{n}_F) = \operatorname{rot}_F(\boldsymbol{n}_F \times (\boldsymbol{v}_{|F} \times \boldsymbol{n}_F)). \tag{5.2}$$

6 DDR complex

Main reference is [4], some proofs are only detailed in [3].

6.1 Spaces

Fix a polynomial degree $k \geq 0$.

Discrete $\mathrm{H}^1(\Omega)$ *space*:

$$\begin{aligned}\underline{X}^k_{\mathbf{grad},h} \coloneqq \Big\{ \underline{q}_h = \big((q_T)_{T\in\mathcal{T}_h}, (q_F)_{F\in\mathcal{F}_h}, (q_E)_{E\in\mathcal{E}_h}, (q_V)_{V\in\mathcal{V}_h}\big) \ :& \\ q_T \in \mathcal{P}^{k-1}(T) \text{ for all } T \in \mathcal{T}_h,& \\ q_F \in \mathcal{P}^{k-1}(F) \text{ for all } F \in \mathcal{F}_h,& \\ q_E \in \mathcal{P}^{k-1}(E) \text{ for all } E \in \mathcal{E}_h,& \\ \text{and } q_V \in \mathbb{R} \text{ for all } V \in \mathcal{V}_h. \Big\}&\end{aligned} \tag{6.1}$$

Index	Space	V	E	F	T
0	$\underline{\boldsymbol{X}}^k_{\mathbf{grad},h}$	$\mathbb{R} = \mathcal{P}^k(V)$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
1	$\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$		$\mathcal{P}^k(E)$	$\boldsymbol{\mathcal{R}}^{k-1}(F) \times \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F)$	$\boldsymbol{\mathcal{R}}^{k-1}(T) \times \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(T)$
2	$\underline{\boldsymbol{X}}^k_{\mathrm{div},h}$			$\mathcal{P}^k(F)$	$\boldsymbol{\mathcal{G}}^{k-1}(T) \times \boldsymbol{\mathcal{G}}^{\mathrm{c},k}(T)$
3	$\mathcal{P}^k(\mathcal{T}_h)$				$\mathcal{P}^k(T)$

Table 2: Polynomial components attached to each mesh vertex $V \in \mathcal{V}_h$, edge $E \in \mathcal{E}_h$, face $F \in \mathcal{F}_h$, and element $T \in \mathcal{T}_h$ for each of the DDR spaces.

Discrete $\mathbf{H}(\mathbf{curl};\Omega)$ *space*:

$$
\begin{aligned}
\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \coloneqq \Big\{ \underline{\boldsymbol{v}}_h = \big((\boldsymbol{v}_{\mathcal{R},T}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},T})_{T\in\mathcal{T}_h}, (\boldsymbol{v}_{\mathcal{R},F}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F})_{F\in\mathcal{F}_h}, (v_E)_{E\in\mathcal{E}_h} \big) \ :\ & \\
\boldsymbol{v}_{\mathcal{R},T} \in \boldsymbol{\mathcal{R}}^{k-1}(T) \text{ and } \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},T} \in \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(T) \text{ for all } T \in \mathcal{T}_h, & \\
\boldsymbol{v}_{\mathcal{R},F} \in \boldsymbol{\mathcal{R}}^{k-1}(F) \text{ and } \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F} \in \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F) \text{ for all } F \in \mathcal{F}_h, & \\
\text{and } v_E \in \mathcal{P}^k(E) \text{ for all } E \in \mathcal{E}_h \Big\}. &
\end{aligned} \tag{6.2}
$$

Discrete $\mathbf{H}(\mathrm{div};\Omega)$ *space*:

$$
\begin{aligned}
\underline{\boldsymbol{X}}^k_{\mathrm{div},h} \coloneqq \Big\{ \underline{\boldsymbol{w}}_h = \big((\boldsymbol{w}_{\mathcal{G},T}, \boldsymbol{w}^{\mathrm{c}}_{\mathcal{G},T})_{T\in\mathcal{T}_h}, (w_F)_{F\in\mathcal{F}_h} \big) \ :\ & \\
\boldsymbol{w}_{\mathcal{G},T} \in \boldsymbol{\mathcal{G}}^{k-1}(T) \text{ and } \boldsymbol{w}^{\mathrm{c}}_{\mathcal{G},T} \in \boldsymbol{\mathcal{G}}^{\mathrm{c},k}(T) \text{ for all } T \in \mathcal{T}_h, & \\
\text{and } w_F \in \mathcal{P}^k(F) \text{ for all } F \in \mathcal{F}_h \Big\}. &
\end{aligned} \tag{6.3}
$$

Discrete $\mathrm{L}^2(\Omega)$ *space*:

$$
\mathcal{P}^k(\mathcal{T}_h) \coloneqq \left\{ q_h \in \mathrm{L}^2(\Omega) \ :\ (q_h)_{|T} \in \mathcal{P}^k(T) \text{ for all } T \in \mathcal{T}_h \right\}.
$$

Restrictions. For $Y \in \mathcal{M}_h$ and $\bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$, $\underline{\boldsymbol{X}}^k_{\bullet,Y}$ is the space of restrictions to Y and the mesh entities on ∂Y of $\underline{\boldsymbol{X}}^k_{\bullet,h}$. If $\underline{x}_h \in \underline{\boldsymbol{X}}^k_{\bullet,h}$, we denote by $\underline{x}_Y$ the vector obtained by taking the components of $\underline{x}_h$ on Y and its boundary.

6.2 Interpolators

Gradient space. $\underline{\boldsymbol{I}}^k_{\mathbf{grad},h} : \mathrm{C}^0(\overline{\Omega}) \to \underline{\boldsymbol{X}}^k_{\mathbf{grad},h}$ is such that

$$
\underline{\boldsymbol{I}}^k_{\mathbf{grad},h} q \coloneqq \big((\pi^{k-1}_{\mathcal{P},T} q_{|T})_{T\in\mathcal{T}_h}, (\pi^{k-1}_{\mathcal{P},F} q_{|F})_{F\in\mathcal{F}_h}, (\pi^{k-1}_{\mathcal{P},E} q_{|E})_{E\in\mathcal{E}_h}, (q(\boldsymbol{x}_V)_{V\in\mathcal{V}_h} \big). \tag{6.4}
$$

Curl space. $\underline{\boldsymbol{I}}^k_{\mathbf{curl},h} : \mathbf{C}^0(\overline{\Omega}) \to \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ is such that

$$\begin{aligned}\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{v} \coloneqq \big(&(\boldsymbol{\pi}^{k-1}_{\mathcal{R},T}\boldsymbol{v}_{|T}, \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},T}\boldsymbol{v}_{|T})_{T\in\mathcal{T}_h},\\ &(\boldsymbol{\pi}^{k-1}_{\mathcal{R},F}\boldsymbol{v}_{\mathrm{t},F}, \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},F}\boldsymbol{v}_{\mathrm{t},F})_{F\in\mathcal{F}_h},\\ &(\pi^k_{\mathcal{P},E}(\boldsymbol{v}_{|E}\cdot\boldsymbol{t}_E)_{E\in\mathcal{E}_h}\big)\end{aligned} \tag{6.5}$$

(recall that $\boldsymbol{v}_{\mathrm{t},F}$ is the tangent trace of $\boldsymbol{v}$ over F).

Divergence space. $\underline{\boldsymbol{I}}^k_{\mathrm{div},h} : \mathbf{H}^1(\Omega) \to \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ is such that

$$\underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{w} \coloneqq \big((\boldsymbol{\pi}^{k-1}_{\mathcal{G},T}\boldsymbol{w}_{|T}, \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{G},T}\boldsymbol{w}_{|T})_{T\in\mathcal{T}_h}, (\pi^k_{\mathcal{P},F}(\boldsymbol{w}_{|F}\cdot\boldsymbol{n}_F)_{F\in\mathcal{F}_h}\big). \tag{6.6}$$

L^2 *space.* $\pi^k_{\mathcal{P},h} : \mathrm{L}^2(\Omega) \to \mathcal{P}^k(\mathcal{T}_h)$ is the global L^2-orthogonal projector (we have $(\pi^k_{\mathcal{P},h}q)_{|T} = \pi^k_{\mathcal{P},T}q_{|T}$ for all $T \in \mathcal{T}_h$).

As for the spaces, restrictions to $Y \in \mathcal{M}_h$ are denoted by replacing h with Y.

6.3 Discrete vector calculus operators

6.3.1 In $\underline{X}^k_{\mathbf{grad},h}$: gradients and scalar potentials

Edge operators. Let $E \in \mathcal{E}_h$. The *edge gradient* is $G^k_E : \underline{X}^k_{\mathbf{grad},E} \to \mathcal{P}^k(E)$ such that, for $\underline{q}_E \in \underline{X}^k_{\mathbf{grad},E}$,

$$\int_E G^k_E\underline{q}_E\, r_E = -\int_E q_E r'_E + q_{V_2}r_E(\boldsymbol{x}_{V_2}) - q_{V_1}r_E(\boldsymbol{x}_{V_1}) \qquad \forall r_E \in \mathcal{P}^k(E), \tag{6.7}$$

with derivative r'_E taken along E in the direction $\boldsymbol{t}_E$, and V_1, V_2 vertices of E such that $\boldsymbol{t}_E$ points from V_1 to V_2.

The *scalar trace on* E is $\gamma^{k+1}_E : \underline{X}^k_{\mathbf{grad},E} \to \mathcal{P}^{k+1}(E)$ such that

$$\int_E \gamma^{k+1}_E\underline{q}_E\, z'_E = -\int_E G^k_E\underline{q}_E\, z_E + q_{V_2}z_E(\boldsymbol{x}_{V_2}) - q_{V_1}z_E(\boldsymbol{x}_{V_1}) \qquad \forall z_E \in \mathcal{P}^{0,k+2}(E) \tag{6.8}$$

where $\mathcal{P}^{0,k+2}(E)$ is the subspace of $\mathcal{P}^{k+2}(E)$ made of zero-average polynomials.

Face operators. Let $F \in \mathcal{F}_h$. The *face gradient* is $\mathsf{G}^k_F : \underline{X}^k_{\mathbf{grad},F} \to \boldsymbol{\mathcal{P}}^k(F)$ such that, for $\underline{q}_F \in \underline{X}^k_{\mathbf{grad},F}$,

$$\int_F \mathsf{G}^k_F\underline{q}_F \cdot \boldsymbol{w}_F = -\int_F q_F \,\mathrm{div}_F\, \boldsymbol{w}_F + \sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E \gamma^{k+1}_E\underline{q}_E(\boldsymbol{w}_F\cdot\boldsymbol{n}_{FE}) \qquad \forall \boldsymbol{w}_F \in \boldsymbol{\mathcal{P}}^k(F). \tag{6.9}$$

The *scalar trace on F* is $\gamma_F^{k+1} : \underline{X}_{\mathbf{grad},F}^k \to \mathcal{P}^{k+1}(F)$ such that

$$\int_F \gamma_F^{k+1} \underline{q}_F \operatorname{div}_F \boldsymbol{v}_F = -\int_F \mathsf{G}_F^k \underline{q}_F \cdot \boldsymbol{v}_F + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \gamma_E^{k+1} \underline{q}_E (\boldsymbol{v}_F \cdot \boldsymbol{n}_{FE}) \qquad \forall \boldsymbol{v}_F \in \mathcal{R}^{\mathrm{c},k+2}(F). \quad (6.10)$$

Element operators. Let $T \in \mathcal{T}_h$. The *element gradient* is $\mathsf{G}_T^k : \underline{X}_{\mathbf{grad},T}^k \to \mathcal{P}^k(T)$ such that, for $\underline{q}_T \in \underline{X}_{\mathbf{grad},T}^k$,

$$\int_T \mathsf{G}_T^k \underline{q}_T \cdot \boldsymbol{w}_T = -\int_T q_T \operatorname{div} \boldsymbol{w}_T + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F \gamma_F^{k+1} \underline{q}_F (\boldsymbol{w}_T \cdot \boldsymbol{n}_F) \qquad \forall \boldsymbol{w}_T \in \mathcal{P}^k(T). \quad (6.11)$$

The *potential reconstruction* is $P_{\mathbf{grad},T}^{k+1} : \underline{X}_{\mathbf{grad},T}^k \to \mathcal{P}^{k+1}(T)$ such that

$$\int_T P_{\mathbf{grad},T}^{k+1} \underline{q}_T \operatorname{div} \boldsymbol{v}_T = -\int_T \mathsf{G}_T^k \underline{q}_T \cdot \boldsymbol{v}_T + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F \gamma_F^{k+1} \underline{q}_F (\boldsymbol{v}_T \cdot \boldsymbol{n}_F) \qquad \forall \boldsymbol{v}_T \in \mathcal{R}^{\mathrm{c},k+2}(T), \quad (6.12)$$

Discrete gradient. $\underline{\boldsymbol{G}}_h^k : \underline{X}_{\mathbf{grad},h}^k \to \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k$ such that, for all $\underline{q}_h \in \underline{X}_{\mathbf{grad},h}^k$,

$$\begin{aligned} \underline{\boldsymbol{G}}_h^k \underline{q}_h := \Big(&\big(\boldsymbol{\pi}_{\mathcal{R},T}^{k-1}(\mathsf{G}_T^k \underline{q}_T), \boldsymbol{\pi}_{\mathcal{R},T}^{\mathrm{c},k}(\mathsf{G}_T^k \underline{q}_T)\big)_{T\in\mathcal{T}_h}, \\ &\big(\boldsymbol{\pi}_{\mathcal{R},F}^{k-1}(\mathsf{G}_F^k \underline{q}_F), \boldsymbol{\pi}_{\mathcal{R},F}^{\mathrm{c},k}(\mathsf{G}_F^k \underline{q}_F)\big)_{F\in\mathcal{F}_h}, \\ &(G_E^k \underline{q}_E)_{E\in\mathcal{E}_h}\Big). \end{aligned} \quad (6.13)$$

6.3.2 In $\underline{\boldsymbol{X}}_{\mathbf{curl},h}^k$: discrete curls and vector potentials

Face operators. Let $F \in \mathcal{F}_h$. The *face curl* is $C_F^k : \underline{\boldsymbol{X}}_{\mathbf{curl},F}^k \to \mathcal{P}^k(F)$ such that, for $\underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}_{\mathbf{curl},F}^k$,

$$\int_F C_F^k \underline{\boldsymbol{v}}_F \, r_F = \int_F \boldsymbol{v}_{\mathcal{R},F} \cdot \mathbf{rot}_F \, r_F - \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E v_E r_F \qquad \forall r_F \in \mathcal{P}^k(F). \quad (6.14)$$

The *tangential trace* is $\boldsymbol{\gamma}_{\mathrm{t},F}^k : \underline{\boldsymbol{X}}_{\mathbf{curl},F}^k \to \mathcal{P}^k(F)$ such that, for $\underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}_{\mathbf{curl},F}^k$,

$$\int_F \boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{v}}_F \cdot (\mathbf{rot}_F \, r_F + \boldsymbol{w}_F) = \int_F C_F^k \underline{\boldsymbol{v}}_F \, r_F + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E v_E r_F + \int_F \boldsymbol{v}_{\mathcal{R},F}^{\mathrm{c}} \cdot \boldsymbol{w}_F \qquad \forall (r_F, \boldsymbol{w}_F) \in \mathcal{P}^{0,k+1}(F) \times \mathcal{R}^{\mathrm{c},k}(F). \quad (6.15)$$

Element operators. Let $T \in \mathcal{T}_h$. The *element curl* is $\mathsf{C}_T^k : \underline{\boldsymbol{X}}_{\mathbf{curl},T}^k \to \boldsymbol{\mathcal{P}}^k(T)$ such that, for $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}_{\mathbf{curl},T}^k$,

$$\int_T \mathsf{C}_T^k \underline{\boldsymbol{v}}_T \cdot \boldsymbol{w}_T = \int_T \boldsymbol{v}_{\mathcal{R},T} \cdot \mathbf{curl}\, \boldsymbol{w}_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{v}}_F \cdot (\boldsymbol{w}_T \times \boldsymbol{n}_F) \qquad \forall \boldsymbol{w}_T \in \boldsymbol{\mathcal{P}}^k(T). \quad (6.16)$$

The *vector potential reconstruction* is $\boldsymbol{P}_{\mathbf{curl},T}^k : \underline{\boldsymbol{X}}_{\mathbf{curl},T}^k \to \boldsymbol{\mathcal{P}}^k(T)$ such that

$$\int_T \boldsymbol{P}_{\mathbf{curl},T}^k \underline{\boldsymbol{v}}_T \cdot (\mathbf{curl}\, \boldsymbol{w}_T + \boldsymbol{z}_T) = \int_T \mathsf{C}_T^k \underline{\boldsymbol{v}}_T \cdot \boldsymbol{w}_T - \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{v}}_F \cdot (\boldsymbol{w}_T \times \boldsymbol{n}_F) + \int_T \boldsymbol{v}_{\mathcal{R},T}^{\mathrm{c}} \cdot \boldsymbol{z}_T \qquad \forall (\boldsymbol{w}_T, \boldsymbol{z}_T) \in \boldsymbol{\mathcal{G}}^{\mathrm{c},k+1}(T) \times \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(T). \quad (6.17)$$

Discrete curl. $\underline{\boldsymbol{C}}_h^k : \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \to \underline{\boldsymbol{X}}_{\mathrm{div},h}^k$ such that, for $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k$,

$$\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{v}}_h \coloneqq \left(\left(\boldsymbol{\pi}_{\mathcal{G},T}^{k-1}(\mathsf{C}_T^k \underline{\boldsymbol{v}}_T), \boldsymbol{\pi}_{\mathcal{G},T}^{\mathrm{c},k}(\mathsf{C}_T^k \underline{\boldsymbol{v}}_T) \right)_{T \in \mathcal{T}_h}, (C_F^k \underline{\boldsymbol{v}}_F)_{F \in \mathcal{F}_h} \right). \quad (6.18)$$

6.3.3 In $\underline{\boldsymbol{X}}_{\mathrm{div},h}^k$: divergence and vector potential

Element operators. Let $T \in \mathcal{T}_h$. The *element divergence* is $D_T^k : \underline{\boldsymbol{X}}_{\mathrm{div},T}^k \to \mathcal{P}^k(T)$ such that, for $\underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{X}}_{\mathrm{div},T}^k$,

$$\int_T D_T^k \underline{\boldsymbol{w}}_T \, q_T = -\int_T \boldsymbol{w}_{\mathcal{G},T} \cdot \mathbf{grad}\, q_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F w_F q_T \qquad \forall q_T \in \mathcal{P}^k(T). \quad (6.19)$$

The *vector potential reconstruction* is $\boldsymbol{P}_{\mathrm{div},T}^k : \underline{\boldsymbol{X}}_{\mathrm{div},T}^k \to \boldsymbol{\mathcal{P}}^k(T)$ such that

$$\int_T \boldsymbol{P}_{\mathrm{div},T}^k \underline{\boldsymbol{w}}_T \cdot (\mathbf{grad}\, r_T + \boldsymbol{z}_T) = -\int_T D_T^k \underline{\boldsymbol{w}}_T \, r_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F w_F \, r_T + \int_T \boldsymbol{w}_{\mathcal{G},T}^{\mathrm{c}} \cdot \boldsymbol{z}_T \qquad \forall (r_T, \boldsymbol{z}_T) \in \mathcal{P}^{0,k+1}(T) \times \boldsymbol{\mathcal{G}}^{\mathrm{c},k}(T). \quad (6.20)$$

Discrete divergence. $D_h^k : \underline{\boldsymbol{X}}_{\mathrm{div},h}^k \to \mathcal{P}^k(\mathcal{T}_h)$ such that, for $\underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}_{\mathrm{div},h}^k$,

$$(D_h^k \underline{\boldsymbol{w}}_h)_{|T} \coloneqq D_T^k \underline{\boldsymbol{w}}_T \qquad \forall T \in \mathcal{T}_h, \quad (6.21)$$

6.4 Discrete De Rham sequence

The global discrete De Rham sequence is:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}. \tag{6.22}$$

It has the **complex properties** [4, Theorem 1], and also satisfies the **exactness properties** provided $\boldsymbol{\Omega}$ has the correct topology, see [4, Theorem 2] and [2, Theorem 3] (actually, on any domain, the cohomology of this complex is isomorphic to the cohomology of the continuous de Rham complex [5]).

We have the commutation properties [4, Lemma 4]:

$$\underline{\boldsymbol{G}}^k_T(\underline{I}^k_{\mathbf{grad},T} q) = \underline{\boldsymbol{I}}^k_{\mathbf{curl},T}(\mathbf{grad}\, q) \qquad \forall q \in \mathrm{C}^1(\overline{T}), \tag{6.23}$$

$$\underline{\boldsymbol{C}}^k_T(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T} \boldsymbol{v}) = \underline{\boldsymbol{I}}^k_{\mathrm{div},T}(\mathbf{curl}\, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathbf{H}^2(T), \tag{6.24}$$

$$D^k_T(\underline{\boldsymbol{I}}^k_{\mathrm{div},T} \boldsymbol{w}) = \pi^k_{\mathcal{P},T}(\mathrm{div}\, \boldsymbol{w}) \qquad \forall \boldsymbol{w} \in \mathbf{H}^1(T). \tag{6.25}$$

In particular, the following diagram between the C^∞ and discrete de Rham sequences is commutative:

$$\begin{array}{ccccccc}
C^\infty(\overline{\Omega}) & \xrightarrow{\mathbf{grad}} & \boldsymbol{C}^\infty(\overline{\Omega}) & \xrightarrow{\mathbf{curl}} & \boldsymbol{C}^\infty(\overline{\Omega}) & \xrightarrow{\mathrm{div}} & C^\infty(\overline{\Omega}) \\
\downarrow \underline{I}^k_{\mathbf{grad},h} & & \downarrow \underline{\boldsymbol{I}}^k_{\mathbf{curl},h} & & \downarrow \underline{\boldsymbol{I}}^k_{\mathrm{div},h} & & \downarrow \pi^k_{\mathcal{P},h} \\
\underline{X}^k_{\mathbf{grad},h} & \xrightarrow{\underline{\boldsymbol{G}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}^k_h} & \underline{\boldsymbol{X}}^k_{\mathrm{div},h} & \xrightarrow{D^k_h} & \mathcal{P}^k(\mathcal{T}_h).
\end{array} \tag{6.26}$$

7 Design of schemes, analytical properties

7.1 L^2-inner products

For $\bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$, the (local) L^2-inner product on $\underline{X}^k_{\bullet,T}$ is

$$(\underline{x}_T, \underline{y}_T)_{\bullet,T} := \int_T P^\ell_{\bullet,T}\underline{x}_T \cdot P^\ell_{\bullet,T}\underline{y}_T + s_{\bullet,T}(\underline{x}_T, \underline{y}_T)$$

where $\ell = k+1$ for $\bullet = \mathbf{grad}$, $\ell = k$ for $\bullet \in \{\mathbf{curl}, \mathrm{div}\}$, and

$$\begin{aligned}
\mathrm{s}_{\mathbf{grad},T}(\underline{r}_T, \underline{q}_T) := {}& \sum_{F \in \mathcal{F}_T} h_F \int_F (P^{k+1}_{\mathbf{grad},T}\underline{r}_T - \gamma^{k+1}_F \underline{r}_F)(P^{k+1}_{\mathbf{grad},T}\underline{q}_T - \gamma^{k+1}_F \underline{q}_F) \\
& + \sum_{E \in \mathcal{E}_T} h_E^2 \int_E (P^{k+1}_{\mathbf{grad},T}\underline{r}_T - \gamma^{k+1}_E \underline{r}_E)(P^{k+1}_{\mathbf{grad},T}\underline{q}_T - \gamma^{k+1}_E \underline{q}_E),
\end{aligned} \tag{7.1}$$

$$
\begin{aligned}
&\mathrm{s}_{\mathbf{curl},T}(\underline{\boldsymbol{w}}_T,\underline{\boldsymbol{v}}_T) \coloneqq \\
&\qquad \sum_{F\in\mathcal{F}_T} h_F \int_F \left((\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{w}}_T)_{\mathrm{t},F} - \boldsymbol{\gamma}^k_{\mathrm{t},F}\underline{\boldsymbol{w}}_F\right)\cdot\left((\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T)_{\mathrm{t},F} - \boldsymbol{\gamma}^k_{\mathrm{t},F}\underline{\boldsymbol{v}}_F\right) \\
&\qquad + \sum_{E\in\mathcal{E}_T} h_E^2 \int_E \left(\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{w}}_T\cdot\boldsymbol{t}_E - w_E\right)\left(\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T\cdot\boldsymbol{t}_E - v_E\right)
\end{aligned}
\tag{7.2}
$$

and

$$
\mathrm{s}_{\mathrm{div},T}(\underline{\boldsymbol{w}}_T,\underline{\boldsymbol{v}}_T) \coloneqq \sum_{F\in\mathcal{F}_T} h_F \int_F \left(\boldsymbol{P}^k_{\mathrm{div},T}\underline{\boldsymbol{w}}_T\cdot\boldsymbol{n}_F - w_F\right)\left(\boldsymbol{P}^k_{\mathrm{div},T}\underline{\boldsymbol{v}}_T\cdot\boldsymbol{n}_F - v_F\right). \tag{7.3}
$$

Global L^2-products are

$$
(\underline{x}_h,\underline{y}_h)_{\bullet,h} = \sum_{T\in\mathcal{T}_h}(\underline{x}_T,\underline{y}_T)_{\bullet,T}.
$$

7.2 Scheme for the magnetostatics model

Find $(\underline{\boldsymbol{H}}_h,\underline{\boldsymbol{A}}_h)\in\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}\times\underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ such that

$$
\begin{aligned}
\mathrm{a}_h(\underline{\boldsymbol{H}}_h,\underline{\boldsymbol{\zeta}}_h) - \mathrm{b}_h(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{A}}_h) &= -\sum_{F\in\mathcal{F}_h^{\mathrm{b}}}\int_F \boldsymbol{g}\cdot\boldsymbol{\gamma}^k_{\mathrm{t},F}\underline{\boldsymbol{\zeta}}_h \qquad \forall\underline{\boldsymbol{\zeta}}_h\in\underline{\boldsymbol{X}}^k_{\mathbf{curl},h},\\
\mathrm{b}_h(\underline{\boldsymbol{H}}_h,\underline{\boldsymbol{v}}_h) + \mathrm{c}_h(\underline{\boldsymbol{A}}_h,\underline{\boldsymbol{v}}_h) &= \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{J}\cdot\boldsymbol{P}^k_{\mathrm{div},T}\underline{\boldsymbol{v}}_T \qquad \forall\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{X}}^k_{\mathrm{div},h}.
\end{aligned}
\tag{7.4}
$$

where $\mathcal{F}_h^{\mathrm{b}}$ are the boundary faces and

$$
\mathrm{a}_h(\underline{\boldsymbol{v}}_h,\underline{\boldsymbol{\zeta}}_h) \coloneqq \sum_{T\in\mathcal{T}_h}\mu_T(\underline{\boldsymbol{v}}_h,\underline{\boldsymbol{\zeta}}_h)_{\mathbf{curl},T}, \tag{7.5}
$$

$$
\mathrm{b}_h(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h) \coloneqq (\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h)_{\mathrm{div},h}, \qquad \mathrm{c}_h(\underline{\boldsymbol{w}}_h,\underline{\boldsymbol{v}}_h) \coloneqq \int_\Omega D^k_h\underline{\boldsymbol{w}}_h\ D^k_h\underline{\boldsymbol{v}}_h. \tag{7.6}
$$

Using the exactness of the DDR complex and the Poincaré inequalities below, the uniform inf-sup property follows as the continuous case [2, Theorem 10]. The consistencies (see below) then yield the following error estimate [4, Theorem 12]:

$$
\begin{aligned}
&\|\underline{\boldsymbol{H}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{H}\|_{\mu,\mathbf{curl},1,h} + \|\underline{\boldsymbol{A}}_h - \underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{A}\|_{\mathrm{div},1,h}\\
&\qquad \lesssim h^{k+1}\left(|\,\mathbf{curl}\,\boldsymbol{H}|_{\mathbf{H}^{k+1}(\mathcal{T}_h)} + |\boldsymbol{H}|_{\mathbf{H}^{(k+1,2)}(\mathcal{T}_h)} + |\boldsymbol{A}|_{\mathbf{H}^{k+1}(\mathcal{T}_h)} + |\boldsymbol{A}|_{\mathbf{H}^{k+2}(\mathcal{T}_h)}\right),
\end{aligned}
\tag{7.7}
$$

where, here and in the following, $a \lesssim b$ means $a \le Cb$ with C depending only on the mesh regularity parameter and k.

7.3 Poincaré inequalities

References: [4, Theorem 3] and [2, Proposition 16, Theorem 18 and Theorem 20].

$$\|\underline{q}_h\|_{\mathbf{grad},h} \lesssim \|\underline{\boldsymbol{G}}_h^k \underline{q}_h\|_{\mathbf{curl},h} \qquad \forall \underline{q}_h \in \underline{X}_{\mathbf{grad},h}^k \text{ s.t. } \sum_{T\in\mathcal{T}_h} \int_T P_{\mathbf{grad},T}^{k+1} \underline{q}_h = 0. \tag{7.8}$$

$$\|\underline{\boldsymbol{w}}_h\|_{\mathrm{div},h} \lesssim \|D_h^k \underline{\boldsymbol{w}}_h\|_{\mathrm{L}^2(\Omega)} \qquad \forall \underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}_{\mathrm{div},h}^k \text{ s.t. } \underline{\boldsymbol{w}}_h \perp \ker D_h^k. \tag{7.9}$$

If Ω has a trivial topology:

$$\|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h} \lesssim \|\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{v}}_h\|_{\mathrm{div},h} \qquad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \text{ s.t. } \underline{\boldsymbol{v}}_h \perp \ker \underline{\boldsymbol{C}}_h^k. \tag{7.10}$$

7.4 Primal consistencies

Reference: [4, Section 6.1].

Define

$$|\boldsymbol{v}|_{\mathbf{H}^{(k+1,2)}(\mathcal{T}_h)} \coloneqq \left(\sum_{T\in\mathcal{T}_h} |\boldsymbol{v}|^2_{\mathbf{H}^{(k+1,2)}(T)} \right)^{1/2}$$

$$\text{with } |\boldsymbol{v}|_{\mathbf{H}^{(k+1,2)}(T)} \coloneqq \begin{cases} |\boldsymbol{v}|_{\mathbf{H}^1(T)} + h_T |\boldsymbol{v}|_{\mathbf{H}^2(T)} & \text{if } k = 0, \\ |\boldsymbol{v}|_{\mathbf{H}^{k+1}(T)} & \text{if } k \geq 1. \end{cases}$$

Consistency of the potential reconstructions:

$$\|P_{\mathbf{grad},T}^{k+1}(\underline{I}_{\mathbf{grad},T}^k q) - q\|_{\mathrm{L}^2(T)} \lesssim h_T^{k+2} |q|_{\mathrm{H}^{k+2}(T)} \qquad \forall q \in \mathrm{H}^{k+2}(T), \tag{7.11}$$

$$\|\boldsymbol{P}_{\mathbf{curl},T}^{k}(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k \boldsymbol{v}) - \boldsymbol{v}\|_{\mathbf{L}^2(T)} \lesssim h_T^{k+1} |\boldsymbol{v}|_{\mathbf{H}^{(k+1,2)}(T)} \qquad \forall \boldsymbol{v} \in \mathbf{H}^{\max(k+1,2)}(T), \tag{7.12}$$

$$\|\boldsymbol{P}_{\mathrm{div},T}^{k}(\underline{\boldsymbol{I}}_{\mathrm{div},T}^k \boldsymbol{w}) - \boldsymbol{w}\|_{\mathbf{L}^2(T)} \lesssim h_T^{k+1} |\boldsymbol{w}|_{\mathbf{H}^{k+1}(T)} \qquad \forall \boldsymbol{w} \in \mathbf{H}^{k+1}(T). \tag{7.13}$$

Consistency of the discrete vector calculus operators:

$$\|\mathsf{G}_T^k(\underline{I}_{\mathbf{grad},T}^k q) - \mathbf{grad}\, q\|_{\mathbf{L}^2(T)} \lesssim h_T^{k+1} |q|_{\mathrm{H}^{k+2}(T)} \qquad \forall q \in \mathrm{H}^{k+2}(T), \tag{7.14}$$

$$\|\mathsf{C}_T^k(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k \boldsymbol{v}) - \mathbf{curl}\, \boldsymbol{v}\|_{\mathbf{L}^2(T)} \lesssim h_T^{k+1} |\,\mathbf{curl}\, \boldsymbol{v}|_{\mathbf{H}^{k+1}(T)} \qquad \forall \boldsymbol{v} \in \mathbf{H}^2(T) \text{ s.t. } \mathbf{curl}\, \boldsymbol{v} \in \mathbf{H}^{k+1}(T), \tag{7.15}$$

$$\|D_T^k(\underline{\boldsymbol{I}}_{\mathrm{div},T}^k \boldsymbol{w}) - \mathrm{div}\, \boldsymbol{w}\|_{\mathrm{L}^2(T)} \lesssim h_T^{k+1} |\,\mathrm{div}\, \boldsymbol{w}|_{\mathrm{H}^{k+1}(T)} \qquad \forall \boldsymbol{w} \in \mathbf{H}^1(T) \text{ s.t. } \mathrm{div}\, \boldsymbol{w} \in \mathrm{H}^{k+1}(T). \tag{7.16}$$

Consistency of the inner products:

$$\left|\int_T q\, P^{k+1}_{\mathbf{grad},T}\underline{r}_T - (\underline{I}^k_{\mathbf{grad},T} q, \underline{r}_T)_{\mathbf{grad},T}\right| \lesssim h_T^{k+2}|q|_{\mathrm{H}^{k+2}(T)}\|\underline{r}_T\|_{\mathbf{grad},T} \qquad \forall q \in \mathrm{H}^{k+2}(T)\,,\ \forall \underline{r}_T \in \underline{X}^k_{\mathbf{grad},T}, \tag{7.17}$$

$$\left|\int_T \boldsymbol{v}\cdot \boldsymbol{P}^{k}_{\mathbf{curl},T}\underline{\boldsymbol{\zeta}}_T - (\underline{\boldsymbol{I}}^k_{\mathbf{curl},T} \boldsymbol{v}, \underline{\boldsymbol{\zeta}}_T)_{\mathbf{curl},T}\right| \lesssim h_T^{k+1}|\boldsymbol{v}|_{\mathbf{H}^{(k+1,2)}(T)}\|\underline{\boldsymbol{\zeta}}_T\|_{\mathbf{curl},T} \qquad \forall \boldsymbol{v} \in \mathbf{H}^{\max(k+1,2)}(T),\ \forall \underline{\boldsymbol{\zeta}}_T \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}, \tag{7.18}$$

$$\left|\int_T \boldsymbol{w}\cdot \boldsymbol{P}^{k}_{\mathrm{div},T}\underline{\boldsymbol{\xi}}_T - (\underline{\boldsymbol{I}}^k_{\mathrm{div},T} \boldsymbol{w}, \underline{\boldsymbol{\xi}}_T)_{\mathrm{div},T}\right| \lesssim h_T^{k+1}|\boldsymbol{w}|_{\mathbf{H}^{k+1}(T)}\|\underline{\boldsymbol{\xi}}_T\|_{\mathrm{div},T} \qquad \forall \boldsymbol{w} \in \mathbf{H}^{k+1}(T),\ \forall \underline{\boldsymbol{\xi}}_T \in \underline{\boldsymbol{X}}^k_{\mathrm{div},T}. \tag{7.19}$$

7.5 Adjoint consistencies

Reference: [4, Section 6.2].

For the gradient: For all $\boldsymbol{v} \in \mathbf{C}^0(\overline{\Omega}) \cap \mathbf{H}_0(\mathrm{div};\Omega)$ such that $\boldsymbol{v} \in \mathbf{H}^{\max(k+1,2)}(\mathcal{T}_h)$ and all $\underline{q}_h \in \underline{X}^k_{\mathbf{grad},h}$,

$$\left|\sum_{T\in\mathcal{T}_h}\left[(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T}\boldsymbol{v}_{|T}, \underline{\boldsymbol{G}}^k_T\underline{q}_T)_{\mathbf{curl},T} + \int_T \mathrm{div}\,\boldsymbol{v}\ P^{k+1}_{\mathbf{grad},T}\underline{q}_T\right]\right| \lesssim h^{k+1}|\boldsymbol{v}|_{\mathbf{H}^{(k+1,2)}(\mathcal{T}_h)}\|\underline{\boldsymbol{G}}^k_h\underline{q}_h\|_{\mathbf{curl},h}. \tag{7.20}$$

For the curl: For all $\boldsymbol{w} \in \mathbf{C}^0(\overline{\Omega}) \cap \mathbf{H}_0(\mathbf{curl};\Omega)$ such that $\boldsymbol{w} \in \mathbf{H}^{k+2}(\mathcal{T}_h)$ and all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$,

$$\begin{aligned}&\left|\sum_{T\in\mathcal{T}_h}\left[(\underline{\boldsymbol{I}}^k_{\mathrm{div},T}\boldsymbol{w}_{|T}, \underline{\boldsymbol{C}}^k_T\underline{\boldsymbol{v}}_T)_{\mathrm{div},T} - \int_T \mathbf{curl}\,\boldsymbol{w}\cdot \boldsymbol{P}^{k}_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T\right]\right| \\ &\qquad \lesssim h^{k+1}\left(|\boldsymbol{w}|_{\mathbf{H}^{k+1}(\mathcal{T}_h)} + |\boldsymbol{w}|_{\mathbf{H}^{k+2}(\mathcal{T}_h)}\right)\left(\|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}\right).\end{aligned} \tag{7.21}$$

For the divergence: For all $q \in \mathrm{C}^0(\overline{\Omega}) \cap H^1_0(\Omega)$ such that $q \in \mathrm{H}^{k+2}(\mathcal{T}_h)$ and all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$,

$$\left|\int_\Omega \pi^k_{\mathcal{P},h} q\ D^k_h\underline{\boldsymbol{v}}_h + \sum_{T\in\mathcal{T}_h}\int_\Omega \mathbf{grad}\,q\cdot \boldsymbol{P}^k_{\mathrm{div},T}\underline{\boldsymbol{v}}_T\right| \lesssim h^{k+1}|q|_{\mathrm{H}^{k+2}(\mathcal{T}_h)}\|\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}. \tag{7.22}$$

7.6 Stokes in curl–curl formulation

Reference: [1].

Strong formulation:

$$\begin{cases} \text{Find the velocity } \boldsymbol{u} : \Omega \to \mathbb{R}^3 \text{ and the pressure } p : \Omega \to \mathbb{R} \text{ such that} \\ \mathbf{curl}(\mathbf{curl}\,\boldsymbol{u}) + \mathbf{grad}\,p = \boldsymbol{f} \quad \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \\ \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \partial\Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \\ \int_\Omega p = 0. \end{cases} \tag{7.23}$$

Weak formulation:

$$\begin{cases} \text{Find } \boldsymbol{u} \in \mathbf{H}(\mathbf{curl};\Omega) \text{ and } p \in H^1(\Omega)\cap L^2_0(\Omega) \text{ such that} \\ \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} + \int_\Omega \mathbf{grad}\,p\cdot\boldsymbol{v} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \quad \forall \boldsymbol{v}\in\mathbf{H}(\mathbf{curl};\Omega) \\ \int_\Omega \mathbf{grad}\,q\cdot\boldsymbol{u} = 0 \quad \forall q \in H^1(\Omega)\cap L^2_0(\Omega). \end{cases} \tag{7.24}$$

DDR scheme: Let

$$\underline{X}^k_{\mathbf{grad},h,0} \coloneqq \left\{\underline{q}_h \in \underline{X}^k_{\mathbf{grad},h} \,:\, (\underline{q}_h, \underline{I}^k_{\mathbf{grad},h}1)_{\mathbf{grad},h} = 0\right\}.$$

Assuming $\boldsymbol{f} \in \boldsymbol{C}^0(\overline{\Omega})$, the DDR scheme reads:

$$\begin{cases} \text{Find } \underline{\boldsymbol{u}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \text{ and } \underline{p}_h \in \underline{X}^k_{\mathbf{grad},h,0} \text{ such that} \\ (\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{G}}^k_h\underline{p}_h, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},h} = (\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{f}, \underline{\boldsymbol{v}}_h)_{\mathbf{curl},h} \qquad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}, \\ \qquad -(\underline{\boldsymbol{G}}^k_h\underline{q}_h, \underline{\boldsymbol{u}}_h)_{\mathbf{curl},h} = 0 \qquad \forall \underline{q}_h \in \underline{X}^k_{\mathbf{grad},h,0}. \end{cases} \tag{7.25}$$

References

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