WHAT IS a Cross Ratio?

François Labourie

Quadruples of distinct lines are endowed with a unique invariant, the *projective cross ratio*: two quadruples are equivalent under a linear transformation if and only if they have the same cross ratio. The projective cross ratio turns out to characterise the geometry of the projective line. In projective coordinates, the cross ratio is computed as a ratio involving four terms with some "crossing symmetries", hence its name.

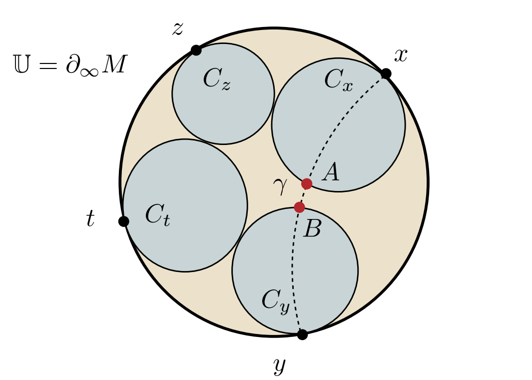
Consider four pairwise distinct lines (x, y, z, t) in the plane, all passing through the origin. There exist essentially unique coordinates so that x is generated by $(1, 0)$, y by $(0, 1)$, z by $(1, 1)$ and t by $(b, 1)$. Then, $b := \mathbf{b}(x, y, z, t)$ is the projective cross ratio of the four lines. The projective cross ratio satisfies a certain set $\mathcal{R}$ of functional rules We single out two of these rules: a *multiplicative cocycle rule* (1) on the first and second variables and an *additive rule* (2)

$$\begin{aligned}(1)\ \mathbf{b}(x, y, z, t) &= \mathbf{b}(x, w, z, t)\mathbf{b}(w, y, z, t),\\ (2)\ \mathbf{b}(x, y, z, t) &= 1 - \mathbf{b}(t, y, z, x).\end{aligned}$$

Conversely, *any set endowed with a function* **b** *of quadruples of points satisfying* $\mathcal{R}$ *can be realised as a subset of the projective line so that the function* **b** *is the restriction of the projective cross ratio.* This elementary though remarkable statement asserts that the projective cross ratio completely characterises the projective line. Consequently, we may define the projective cross ratio as a function on quadruples satisfying some functional rules.

The projective cross ratio has many descendants in algebraic geometry: invariants of configurations of planes, lines or flags. We shall not pursue this development here but rather restrict to real and complex projective lines and the relationships of the cross ratio with negatively curved manifolds, hyperbolic dynamics and Teichmüller Theory.

One can describe the real hyperbolic plane as a metric extension of the real projective line. In the complex line $\mathbb{C}$, seen as an affine chart of the complex projective line, a circle is the set of complex lines intersecting a given real plane and thus is in bijection with a real projective line. In the *Poincaré disk model*, the *hyperbolic plane* is the disk bounded by the unit circle $\mathbb{U}$ which we identify with a real projective line. The *geodesics* of the hyperbolic plane are circles orthogonal to $\mathbb{U}$. A pair of points in the real projective line $\mathbb{U}$ then determines a unique geodesic. The cross ratio is related to the hyperbolic distance as follows. An *horosphere centred at a point* x of the projective line is a circle in the Poincaré model tangent to $\mathbb{U}$ at the point x. Let γ be the unique geodesic joining two points A and B in the hyperbolic plane; let x and y be the end points at infinity of γ; let C_x and C_y be the horospheres centred at x and y and passing through A and B respectively; let finally t and z the centres of two horospheres tangent to each other as well as to C_x and C_y respectively. Then the hyperbolic distance between A and B is the logarithm of the projective cross ratio of the four points y, z, x, t. This construction allows us to derive the hyperbolic distance from the cross ratio and vice versa.



Conversely, the real projective line appears as the *boundary at infinity* of the hyperbolic plane. In the Poincaré disk model, two oriented geodesics end up at the same point in $\mathbb{U}$ if they are *asymptotic*, that is, eventually remain at finite distance from each other. This permits the extension of these ideas to *Hadamard surfaces*. A two dimensional Riemannian manifold M is a *Hadamard surface* if it is simply connected, negatively curved and complete. This *boundary* *at infinity* $\partial_\infty M$ of M is the set of equivalence classes of asymptotic oriented geodesics. For the hyperbolic plane, the boundary at infinity is the projective line $\mathbb{U}$. A *horosphere* is now the limit of metric spheres passing through a given point but whose centres go to infinity.

Otal generalised the cross ratio to $\partial_\infty M$ by reversing the construction of the picture. Starting from four points in $\partial_\infty M$, draw four horospheres and define the *cross ratio* of x, y, z, t as the exponential of the distance between the two points A and B, with some sign convention. In general, the corresponding function satisfies all rules of $\mathcal{R}$ *except the additive relation* (2). Define a *cross ratio* as any function which satisfies this relaxed set of rules. Using these ideas, Otal proved that a metric of negative curvature on a surface is characterised by the length of closed geodesics. Bourdon used similar cross ratios to define a coarse geometry on the boundary at infinity of a general negatively curved metric space.

We now turn to dynamics and Teichmüller Theory. Assume that M is the universal cover of a closed surface S. Although $\partial_\infty M$ was defined from the metric geometry of M, it only depends on $\pi_1(S)$. Therefore we denote it $\partial_\infty\pi_1(S)$. The boundary at infinity is homeomorphic to a circle and admits an action of $\pi_1(S)$. Thus: *every negatively curved metric on S gives rise to a $\pi_1(S)$-invariant cross ratio on $\partial_\infty\pi_1(S)$.*

A cross ratio has a dynamical interpretation. Consider the quotient of the space of triples of pairwise distinct points of $\partial_\infty\pi_1(S)$ by the diagonal action of $\pi_1(S)$. This quotient, which we denote US, is compact. A cross ratio gives rise to a one parameter group of transformations $\{\phi_t\}$ on US, defined by $\phi_t(x, y, z) = (x, y, w)$ where $t = \mathbf{b}(x, y, z, w)$. The mutiplicative cocycle rule (1) translates into $\phi_{t+s} = \phi_t \circ \phi_s$. This construction recovers the geodesic flow when the cross ratio comes from a negatively curved metric. In general, there is an intimate relation between dynamics and the cross ratio.

What is the space $\mathcal{M}_S$ of all cross ratios on $\partial_\infty\pi_1(S)$? The Fricke Space of hyperbolic structures on S, usually identified with the Teichmüller Space $\mathcal{T}(S)$ of complex structures on S by the Uniformisation Theorem, naturally sits in $\mathcal{M}_S$ as the subset of projective cross ratios. Every such cross ratio identifies $\partial_\infty\pi_1(S)$ with the projective line and thus defines a representation of $\pi_1(S)$ in $\mathrm{PSL}(2, \mathbb{R})$ and a hyperbolic structure on S. Similarly, a space of representations of $\pi_1(S)$ in $\mathrm{PSL}(n, \mathbb{R})$, called the *Hitchin component*, has been identifed by the author as the space of cross ratios satisfying rules generalising the additive rule (2). Finally, $\mathcal{M}_S$ also contains the space of all negatively curved metrics on S.

The space $\mathcal{M}_S$ is suspected to have an interesting structure generalising the *Poisson structure* on Hitchin components described by Goldman. Recall that a Poisson structure on a set Y is a Lie algebra structure on a set of functions on Y, such that the Lie bracket satisfies a Leibniz rule with respect to multiplication. This notion arises from classical mechanics and leads to quantum mechanics. By construction, every quadruple of points (x, y, z, t) of the boundary at infinity defines a function on $\mathcal{M}_S$ by $\mathbf{b} \mapsto \mathbf{b}(x, y, z, t)$. These functions, when restricted to Fricke space, yield a natural class of functions whose Poisson brackets have been computed by Wolpert and Penner. These functions were later quantised – that is represented as operators on Hilbert spaces – by Chekhov, Fock, Penner. A more sophisticated construction led Fock and Goncharov to quantise Hitchin components.

In other directions, cross ratios are instrumental in generalising many properties of classical Teichmüller theory, such as McShane identities, to a *Higher Teichmüller-Thurston theory* that is, the study of Hitchin components.

The complex projective cross ratio strongly relates to hyperbolic 3-dimensional geometry. Two recent and beautiful examples are W. Neumann's study of Hilbert third problem in Hyperbolic Geometry, and the *Quantum Hyperbolic Geometry* developed by Baseilhac, Bonahon, Benedetti, Kashaev etc.

The simple functional rules satisfied by the ubiquitous cross ratio, are flexible enough to describe various geometric and dynamical situations. Yet they are rigid enough to carry important information about dynamics, Poisson structures and surface group representations.

References

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