Some pseudo-Riemannian aspects of higher rank Teichmüller theory

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1 Introduction

Since the end of my doctoral thesis, in July 2015, my research has mainly concerned the study of the geometrical, dynamical and analytic aspect of representations of hyperbolic groups (mostly surface groups) into noncompact simple Lie groups. The techniques used in this study have been quite diverse, mixing complex algebraic geometry, geometric analysis and discrete groups action. The main goal of this thesis is to give a general overview of the results obtained.

1.1 Some context

Before diving into the technical results, we give a brief context.

1.1.1 An historical example: the Teichmüller space

In 1939, Oswald Teichmüller introduced in [Tei39] the space $\mathcal{T}(X)$, now called *Teichmüller space* of a Riemann surface X. This space has many different interpretations and lives at the crossroad of many different domains of mathematics, such as complex analysis, algebraic geometry, low dimensional topology and mathematical physics. These many facets provide the Teichmüller space with a fascinating geometry. We now describe some of these.

The original point of view. In Teichmüller's original definition, the space $\mathcal{T}(X)$ parametrizes quasiconformal deformations of a given Riemann surface X. This aspect was mainly developed by Lars Ahlfors and Lipman Bers after Teichmüller's death. Ahlfors proved for instance in [Ahl60] that $\mathcal{T}(X)$ is a complex analytic variety (for a closed Riemann surface), and that different Riemann surface structures X and Y on the same underlying smooth surface S gives biholomorphic varieties $\mathcal{T}(X)$ and $\mathcal{T}(Y)$. It is then common to identify all those spaces, and to denote by $\mathcal{T}(S)$ the resulting object.

Grothendieck's vision of moduli spaces. In a series of ten lectures given at Cartan's seminar in 1960 and 1961, Alexander Grothendieck [Gro60] introduced some algebraic construction of moduli problems. In this setting, the Teichmüller space $\mathcal{T}(S)$ of a closed surface appears as a *fine moduli space* representing the moduli functor of marked Riemann surfaces diffeomorphic to S. This point of view was the starting point of the theory of moduli spaces in algebraic geometry (see [AJP16] for a nice description of the underlying ideas).

Thurston's revolution. In the mid seventies, William Thurston developed an ambitious program to understand 3-dimensional topology. In Thurston's vision, low dimensional topology is inextricably linked with geometry, and hyperbolic geometry plays a prefered role in it's famous Geometrization conjecture.

Using uniformization Theorem, if S a closed oriented surface of negative Euler characteristic, a Riemann surface structure on S is equivalent to a hyperbolic metric. In Thurston's picture, $\mathcal{T}(S)$ thus parametrizes isotopy classes of hyperbolic structures on S. Using hyperbolic geometry, one can for instance constructs coordinates on $\mathcal{T}(S)$ as well as natural compactifications. The Mapping Class Group $\mathsf{MCG}(S)$ acts on $\mathcal{T}(S)$ and the quotient is canonically identified with the moduli space of curves.

Goldman's result. Since a Riemann surface structure X on S defines a biholomorphism f from $\widetilde{X}$ to the hyperbolic disk $\mathbf{H}^2$. This uniformization map f is equivariant under an action of $\pi_1(S)$ on $\mathbf{H}^2$, which is encoded by a discrete and faithful the morphism (called *holonomy*) from $\pi_1(S)$ to the group $\mathsf{PSL}(2,\mathbb{R})$ of biholomorphism of $\mathbf{H}^2$. This yields an embedding

$$\mathrm{hol} : \mathcal{T}(S) \longrightarrow \chi\big(S, \mathsf{PSL}(2,\mathbb{R})\big) ,$$

where $\chi(S, \mathsf{PSL}(2,\mathbb{R}))$ is the *character variety*, that is (roughly) the space of conjugacy classes of representations from $\pi_1(S)$ to $\mathsf{PSL}(2,\mathbb{R})$.

In his thesis, Goldman [Gol88] proved that the image of $\mathcal{T}(S)$ is a connected component of $\chi(S, \mathsf{PSL}(2,\mathbb{R}))$. He also proves that this component, now called the *Teichmüller component*, is the one maximizing a characteristic class (the so-called *Euler class*, see Subsection 2.3.5).

Let us now describe a surprising fact, sheding some lights on how intricate are the interactions between the above points of view. Since $\mathsf{PSL}(2,\mathbb{R})$ is *not* a complex group, the character variety $\chi(S, \mathsf{PSL}(2,\mathbb{R}))$ does not carry any natural complex structure. However, Ahlfors' result shows that the Teichmüller component is in fact complex analytic. On the other hand, Goldman proved in [Gol84] that the character variety is symplectic. It turns out that the symplectic structure on $\mathcal{T}(S)$ inherited from $\chi(S, \mathsf{PSL}(2,\mathbb{R}))$ is compatible with Ahlfors' complex structure. The resulting Kähler metric is called the *Weil-Petersson metric*.

1.1.2 Higher rank Teichmüller space: from $\mathsf{PSL}(2,\mathbb{R})$ to other Lie groups

We now describe a higher rank generalization of Teichmüller theory. As its rank 1 cousin, higher rank Teichmüller theory has many facets, and the mathematics involved in its study is a beautiful blend of discrete groups action, complex algebraic geometry and geometric analysis.

Hitchin's discovery. In 1992, Nigel Hitchin [Hit92] found a particular component of the character variety $\chi(S, \mathsf{PSL}(n,\mathbb{R}))$ for S a closed oriented surface of hyperbolic type (more generally for any G a split real simple Lie group of adjoint type, see Subsection 2.3.5 for more details). More specifically, given such a group G, there exists a preferred morphism, called *principal*, from $\mathsf{PSL}(2,\mathbb{R})$ to G. This morphism defines an embedding from $\chi(S, \mathsf{PSL}(2,\mathbb{R}))$ into $\chi(S, \mathsf{G})$. Hitchin's main result is that the connected component of $\chi(S, \mathsf{G})$ containing the image of the Teichmüller component (which is now called the *Hitchin component*) is smooth and contractible. In particular, all the representations in this component are irreducible. Hitchin's proof is based of Higgs bundle theory (see Subsection 2.1.6 for more details about Higgs bundles), and in particular, his techniques give no insight on the geometrical properties of the associated representations.

Anosov representations. In 2006, François Labourie [Lab06] introduced the notion of *Anosov representation* of a surface group into $\mathsf{PSL}(n,\mathbb{R})$ and proved that Hitchin representations into $\mathsf{PSL}(n,\mathbb{R})$ have this property. Since Anosov representations are in particular discrete and faithful, he obtained that the Hitchin component consists only on dicrete and faithful representations: this is what is now commonly called a *higher rank Teichmüller space*.

There is another famous family of higher rank Teichmüller spaces: the set of *maximal representations*. Those representations, defined for a Hermitian Lie group G, maximize a topological invariant introduced by Domingo Toledo in [DT87] that generalizes the Euler class. In [BIW10, BILW05], Marc Burger, Alessandra Iozzi, François Labourie and Anna Wienhard proved that maximal representations share some dynamical properties with Hitchin representations and are in particular discrete and faithful.

The notion of Anosov representation was broadly generalized by Olivier Guichard and Anna Wienhard in [GW12] (see also Kapovich-Leeb-Porti [KLP13]) to representations of hyperbolic groups into real semisimple Lie groups (see Subsection 2.3.1 for more details about Anosov representations). This general notion recovers in particular Labourie's original definition, maximal representations, as well as quasi-Fuchsian representations. Among other things, they proved that (most) Anosov representations are holonomy of geometric structure on some manifold. It has become clear in the past decade that, for hyperbolic groups, the notion of

Anosov representation is the correct generalization to higher rank of convex cocompact representations in rank 1.

Positivity. It is important to note that being Anosov is an open property, but fails to be closed. In particular, higher rank Teichmüller spaces do not coincide with the set of Anosov representations. The correct notion characterizing representations in higher rank Teichmüller spaces seems to be *positivity*, as introduced by Anna Wienhard and Olivier Guichard in [GW18] (generalizing the notion introduced by Lusztig [Lus94]). In fact, by a result of Olivier Guichard, François Labourie and Anna Wienhard [GLW21], and independently Jonas Beyrer and Beatrice Pozzetti [BP21], positivity is open and closed (in the correct representation space), and implies the Anosov property. This notion thus puts under the same framework Hitchin representations, maximal representations and all the exotic examples discovered by André Oliveira, Steve Bradlow, Brian Collier, Oscar García-Prada and Peter Gothen using Higgs bundles techniques [BCG$^+$21].

Toward a higher dimensional version of higher rank Teichmüller spaces? Given a hyperbolic group Γ, the existence of a (non trivial) connected component of the character variety $\chi(\Gamma, \mathsf{G})$ consisting only of discrete and faithful representations is quite surprising and seemed to be specific to the case of surface groups (as described above).

There is however 2 families of examples of higher dimensional version of this phenomenon. The first comes from Yves Benoist's work on divisible convex sets. He proved for instance that, for Γ a hyperbolic group of virtual cohomological dimension p, the set of representations acting properly discontinuously on some nonempty proper convex set in $\mathbf{P}(\mathbf{R}^{p+1})$ is a union of connected components of $\chi(\Gamma, \mathsf{PGL}(p+1, \mathbb{R}))$. Since those representations are in particular Anosov (see [DGK23]), the corresponding components consist only of discrete and faithful representations.

The other family is a recent striking result of Jonas Beyrer and Fanny Kassel [BK23], generalizing a result of Thierry Barbot [Bar15], in which they prove that for Γ as above, the space of representations of Γ into $\mathsf{PO}(p, q+1)$ that act properly discontinuously and cocompactly on some nonempty properly convex set in the signature (p, q) pseudo-hyperbolic space is a union of connected components.

Observe that the notion of positivity described above highly relies on the natural cyclic struture that exists on the boundary of a surface group. In particular, such a notion does not exists in higher dimension. In particular, the general picture in this setting remains quite mysterious!

1.1.3 Some general questions

We end this section by a list of general questions surrounding the topics of my research.

A complex structure? The name *higher rank Teichmüller theory* is quite confusing. In fact, what is generalized to higher rank is mainly the hyperbolic picture of Teichmüller theory (Thurston's point of view). Teichmüller's original point of view, that is the one of quasiconformal deformations, is the missing object in this higher rank picture. But the complex strcture on $\mathcal{T}(S)$ is obtained by considering the quasi-conformal picture. In particular, it is not clear whether all higher rank Teichmüller space are complex (or Kähler).

Mapping Class Group action. The mapping class group naturally acts on higher rank Teichmüller spaces, and this action is known to be properly discontinuous by the work of Labourie [Lab08] (see also [Wie06]). However, the structure of the quotient is not well understood in general.

Part of the famous Labourie's conjecture (see Subsection 2.3.6) was to describe the quotient as a bundle over the Teichmüller space. Unfortunately, since this conjecture is known to be false in general, the existence of a mapping class group equivariant projection from a higher rank Teichmüller space to the classical Teichmüller space is not even known to exist.

A universal object? An other drawback of generalizing only the hyperbolic side of Teichmüller's theory is that one can define those spaces only for closed surfaces (punctured, in the best situation).

Teichmüller's original definition only requires a Riemann surface of any type. One can for instance, as Bers did in [Ber65], define the Teichmüller space of the unit disk. This space is known as the *universal Teichmüller space*. It is an infinite dimensional complex Banach manifold which contains all the other Teichmüller spaces as complex submanifolds.

There have been different attempts to generalize this picture in higher rank (see [Tho19, Lab07a, LT23]), but the general picture is still not clear.

1.2 Overview of the results

1.2.1 The general philosophy

The general philosophy behind my research could be summarize in the following two general principles:

First: pseudo-Riemannian geometry is more natural than Riemannian geometry

A main challenge when one tries to extend classical Teichmüller theory to a higher rank Lie group G is that the geometry of the Riemannian symmetric space $\mathbf{Sym}(\mathsf{G})$ becomes very complicated. When G has rank one, $\mathbf{Sym}(\mathsf{G})$ is negatively curved, but for higher rank, $\mathbf{Sym}(\mathsf{G})$ is only nonpositively curved: there exists some copy of the Euclidean space $\mathbf{R}^k$ isometrically embedded in $\mathbf{Sym}(\mathsf{G})$ (the largest such k is called the real rank of G). This causes many important difficulties: among others, the group G does not act transitively on the unit tangent bundle and there are many non-equivalent notion of "boundary at infinity".

Instead of considering the Riemannian symmetric space $\mathbf{Sym}(\mathsf{G})$, one can in general consider a different symmetric space, which is only pseudo-Riemannian, but has rank one. The main advantage is that the geometry of such a space is in general easier to handle, and such a space has a natural boundary at infinity.

Note however that such a trick also brings new difficulties: a pseudo-Riemannian manifold is not a metric space, and the action of a discrete group on a pseudo-Riemannian symmetric space is not properly discontinuous in general. As a result, the orbit under a group action might have different behavior depending on the point we consider.

Let us illustrate this idea with the signature (p,q) pseudo-hyperbolic space. Given a real vector space V equipped with a signature $(p,q+1)$ quadratic form $\mathbf{q}$, we define

$$\mathbf{H}^{p,q} = \{x \in \mathbf{P}(V)\ ,\ \mathbf{q}(x) < 0\}\ .$$

This space is naturally equipped with a pseudo-Riemannian metric of signature (p,q) and curvature -1. The group G of orthogonal transformation of $(V,\mathbf{q})$, which is isomorphic to $\mathsf{O}(p,q+1)$, acts by isometry on $\mathbf{H}^{p,q}$ and turns it into a pseudo-Riemannian symmetric space. The geometry of $\mathbf{H}^{p,q}$ is quite natural: complete geodesics are given by the intersection of $\mathbf{H}^{p,q}$ with projective lines in $\mathbf{P}(V)$ and the boundary at infinity of $\mathbf{H}^{p,q}$ in $\mathbf{P}(V)$ is the space of isotropic lines in V (which is a flag manifold of $\mathsf{O}(p,q+1)$).

On the other hand, the Riemannian symmetric space $\mathbf{Sym}(\mathsf{G})$ is identified with the Grassmanian of positive definite p-planes in V. This space contains flats of dimension $\min(p,q+1)$ and its boundary in the Grassmanian of p-planes in V is a union of many G-orbits with non-isomorphic stabilizers. The geometry of $\mathbf{Sym}(\mathsf{G})$ is thus much more intricate that the one of $\mathbf{H}^{p,q}$. However, taking the orthogonal complement realizes points in $\mathbf{Sym}(\mathsf{G})$ as totally geodesic q-spheres in $\mathbf{H}^{p,q}$, and vice-versa. See Subsection 2.2 for more details.

This idea of considering pseudo-Riemannian symmetric spaces instead of their Riemannian analogue is not new and the first to do such was maybe Geoffrey Mess in his groundbreaking work [Mes07]. Many important mathematicians have also followed this principle, and to avoid forgetting some important name, we decided not trying to give an exhaustive list.

Second: if a representation is interesting, it must preserve some interesting object

Here, by "interesting object", we generally mean a solution to an elliptic problem (for instance a minimal surface, a maximal submanifold, a holomorphic curve...). Again, this idea is not new, and the celebrated Labourie's conjecture fits in this framework: it states that a Hitchin representations into G should preserve a unique minimal surface into **Sym**(G).

The advantage of preserving an “interesting object”, when it is unique, is that one can use it to describe the representation. This is where some analytic tools come into play. In some particular case, one can use the theory of Higgs bundles to give a complex analytic parametrization of the corresponding space of representaitons.

1.2.2 Some contributions

Asymptotic Plateau problem in $\mathbf{H}^{p,q}$. The pseudo-Riemannian geometry of $\mathbf{H}^{p,q}$ implies some strong restriction on the theory of spacelike submanifold (that is, submanifold with induced Riemannian metric) of dimension p. In particular, any connected, complete spacelike submanifold of dimension p is contractible and has a well-defined *asymptotic boundary*, which is a topological $(p-1)$-sphere in $\partial_\infty \mathbf{H}^{p,q}$ (see Subsection 2.2.5 for more details). The main contribution of my reasearch in the past 5 years can be summerized in the following result:

Theorem 1.1. *Let Λ be a topological $(p-1)$-sphere in $\partial_\infty \mathbf{H}^{p,q}$. Then the following assertions are equivalent*

i. Λ is the asymptotic boundary of some connected, complete spacelike submanifold of dimension p,

ii. Λ is the asymptotic boundary of a unique complete maximal submanifold of dimension p (here maximal means spacelike of vanishing mean curvature).

Remark 1.2. *i.* The main reason the theory of p-dimensional spacelike submanifolds in $\mathbf{H}^{p,q}$, and the corresponding Plateau problem, works so well is that the spacelike dimension is maximal (so the normal bundle is negative definite). In particular, one expects the analogue theory for spacelike k-dimensional submanifolds, for k less that p, to be much more difficult (this is already the case in the Riemannian hyperbolic space, where uniqueness and regularity become very tricky).

ii. The topological spheres arising as asymptotic boundaries of complete spacelike p-submanifolds are called *nonnegative spheres* (or nonpositive in [DGK18]) and can easily be characterized in terms of the signature of the linear spaces generated by triple of points.

History of the result. • For $q = 1$, this theorem was proved in the periodic case (that is, invariant under the cocompact action of a discrete group) by Lars Andersson, Thierry Barbot, François Béguin and Abdelghani Zeghib [ABBZ12]. It was also proved by Francesco Bonsante and Jean-Marc Schlenker [BS10] for $q = 1$ and no group action.

• For $p = 2$ and general q, it was proved by Brian Collier, Nicolas Tholozan and myself [CTT19] in the periodic case. The techniques we used relied on Higgs bundle theory, Anosov dynamics and pseudo-Riemannian geometry. We obtained various interesting corollaries:

Corollary 1.3. Let Σ be a closed oriented surface of hyperbolic surface and ρ a representation from $\pi_1(\Sigma)$ into $\mathsf{SO}_0(2, n+1)$. The following are equivalent

i. The representation ρ is maximal (up to reversing the orientation of Σ).

ii. The representation ρ acts cocompactly on a spacelike surface in $\mathbf{H}^{2,n}$.

iii. The representation ρ acts cocompactly on a unique maximal surface in $\mathbf{H}^{2,n}$ (that is a spacelike surface with zero mean curvature).

We also obtained

Corollary 1.4. Let Σ be as above and let ρ be a maximal representation of $\pi_1(\Sigma)$ into $\mathsf{SO}_0(2, n+1)$. Then,

i. ρ preserves a unique minimal surface in $\mathbf{Sym}(\mathsf{G})$.

ii. There is a Fuchsian representaton j of $\pi_1(\Sigma)$ whose length spectrum is dominated by the one of ρ.

iii. ρ is the holonomy of a geometric structure locally modelled on the space of isotropic 2-planes in $\mathbf{R}^{2,n+1}$ on a bundle over Σ.

iv. The space of maximal representations into $\mathsf{SO}_0(2, n+1)$ admits a parametrization by a complex analytic space which fibers over the Teichmüller space, and everything is equivariant under the mapping class group action.

See Section 3 for more details.

• For $p = 2$ and general q, with no group action, this theorem was proved by François Labourie, Mike Wolf and myself in [LTW23]. The proof relies on the theory of pseudo-holomorphic curves and is very specific to the case $p = 2$. See Subsection 4.2.

• For general p and q, and no group action, this result was proved by Graham Smith, Andrea Seppi and myself in [SST23]. In this setting, we do not have the interpretation in tems of pseudo-holomorphic curves, so we rely on some subtle analysis of elliptic operators on non-compact manifolds. See Subsection 4.4 for more details.

This case has some nice corollaries in the periodic case. Let Γ be a torsion free hyperbolic group with Gromov boundary homeomorphic to a $(p-1)$-sphere. The natural class of representations, acting properly discontinuously and cocompactly on some nonempty propertly convex set in $\mathbf{H}^{p,q}$ are called $\mathbf{H}^{p,q}$*-convex cocompact representations*. The boundary of such a convex is a nonnegative sphere. We obtain

Corollary 1.5. Let Γ be as above and ρ be a representation of Γ into $\mathsf{O}(p, q+1)$. If ρ is $\mathbf{H}^{p,q}$-convex cocompact, then ρ acts properly discontinuously and cocompactly on some maximal spacelike p-submanifold. In particular, Γ is the fundamental group of a closed p-manifold with contractible universal cover.

In a recent paper [BK23], Jonas Beyrer and Fanny Kassel proved a converse to the above result (their result is actually stronger: they only assume the representation preserves a weakly spacelike submanifold). Using our result, they prove that the set of $\mathbf{H}^{p,q}$-convex cocompact representations of Γ into $\mathsf{O}(p, q+1)$ is a union of connected components in $\mathrm{Hom}(\Gamma, \mathsf{O}(p, q+1))$. This gives a new family of higher dimensional higher rank Teichmüller spaces.

• In the paper [LT23], in collaboration with François Labourie, we introduced the notion of a *quasicircle* in $\partial_\infty \mathbf{H}^{2,n}$, and characterized those curves in terms both of cross-ratios and the geometry of the corresponding maximal surfaces. This gives a natural candidate for a universal space for maximal representations in $\mathsf{SO}_0(2, n+1)$. See Subsection 4.3 for more details.

The exceptional pseudo-hyperbolic space of dimension 6. The real split form G_2' of the exeptional complex group $\mathsf{G}_2(\mathbb{C})$ is naturally realized as a subgroup of $\mathsf{SO}_0(4,3)$. In particular, one obtains an action of G_2' on the pseudo-hyperbolic space $\mathbf{H}^{4,2}$. In turns out that this action is transitive and preserves a (non-integrable) almost complex structure J which is compatible with the pseudo-Riemannian metric.

"Interesting objects" in this geometry are holomorphic curves. We restrict our attention to what we call *alternating holomorphic curves* which are spacelike holomorphic curves with a naturally defined Frenet framing, which is constructued in an analogous way than for curves in $\mathbf{R}^3$.

In the paper [CT23], in collaboration with Brian Collier, we use Higgs bundle theory to study those alternating holomorphic curves, and more specifically, the representations arising in the periodic case.

Surprisingly, we find a natural invariant $d \in \{0, ..., 6g-6\}$ parametrizing connected components of equivariant alternating holomorphic curves. For $d = 6g-6$, the underlying representation is Hitchin in G_2', while for $d = 0$, it is Hitchin in $\mathsf{SL}(3,\mathbb{R})$, where $\mathsf{SL}(3,\mathbb{R})$ is embedded in G_2' as the stabilizer of a totally geodesic copy of $\mathbf{H}^{3,2}$ inside $\mathbf{H}^{4,2}$ (see Section 5 for more details).

This work is a very first step in the study of (equivariant or not) alternating holomorphic curves in $\mathbf{H}^{4,2}$. We do not know for instance if the underlying representations are Anosov, or if the curves extend to the boundary. Note that, one important interest of such study is that, unlike the asymptotic Plateau problem described above, our holomorphic curves do not saturate the spacelike dimension. It would be very interesting to understand however if stability (or uniqueness) still holds in this setting.

Compact components of relative character variety. This last work [TT21], in collaboration with Nicolas Tholozan, has a different flavor compared to what has been presented before.

In [DT19], Bertrand Deroin and Nicolas Tholozan discovered a surprising new phenomenon: for Σ the punctured sphere, the relative $\mathsf{PSL}(2,\mathbb{R})$-character variety of Σ (that is, the character variety obtain by fixing the holonomy around the punctures) has some compact components, whose representations are generically Zariski dense. The dynamical behavior of the corresponding representation is somehow opposite to the one of Anosov representation: any simple closed curve is mapped to an elliptic or a parabolic element, so in particular the orbit map is far from being a quasi-isometry.

Using the theory of parabolic Higgs bundles, we proved that for the Hermitian groups $\mathsf{SU}(p,q)$, $\mathsf{Sp}(2n,\mathbb{R})$ or $\mathsf{SO}^*(2n)$, the associated relative character variety also has compact components, and the representations have a similar dynamical behavior.

The general picture behind this phenomenon remains quite mysterious. It seems possible that the existence of such compact components is related to *Katz' middle convolution*, a construction in $\mathcal{D}$-module that gives isomorphisms between relative character varieties of the punctured sphere, but for groups with different ranks.

2 Preliminaries

2.1 Harmonic maps and Higgs bundles

2.1.1 Harmonic maps

Let (M,g) and (N,h) be two compact Riemannian manifolds. The *energy* of a smooth map f from M to N is defined by

$$E(f) = \frac{1}{2}\int_M \|\mathrm{d}f\|^2 \mathrm{dvol}_g ,$$

where we consider $\mathrm{d}f$ as a section of $\mathrm{T}^*M \otimes f^*\mathrm{T}N$ and the norm of $\mathrm{d}f$ is computed with respect to the metric $g^* \otimes f^*h$.

The map f will be called *harmonic* if it is a critical point of the energy functional, that is if for any deformation $(f_t)_{t\in(-\epsilon,\epsilon)}$ with $f_0 = f$, we have $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} E(f_t) = 0$.

Computing the variation of $\|\mathrm{d}f\|^2$ along a deformation yields

Proposition 2.1 (Euler-Lagrange equation)**.** *A map f from M to N is harmonic if and only if it satisfies*

$$d_\nabla^* df = 0 \tag{1}$$

where d_∇^* is the formal adjoint (with respect to the L^2 product) of the exterior differential d_∇ from $\Omega^0(M, f^*\mathrm{T}N)$ to $\Omega^1(M, f^*\mathrm{T}N)$.

Observe that equation (1) still makes sense for noncompact manifolds, and allows to define harmonic maps in a broader generality.

Example 2.2.

1. When $(N, h) = (\mathbf{R}, \mathrm{d}x^2)$, then harmonic maps are exactly harmonic functions.
2. When $(M, g) = (\mathbf{S}^1, \mathrm{d}\theta^2)$, then a harmonic map is a parametrized closed geodesic.
3. When (M, g) and (N, h) are Kähler, then holomorphic maps are harmonic.
4. Minimal immersion are exactly isometric harmonic maps.

The equation (1) is a quasilinear elliptic PDE of order 2. When (N, h) is non-positively curved, the celebrated result of Eells-Sampson [ES64] gives

Theorem 2.3 (Eells-Sampson)**.** *Let (M, g) and (N, h) be as above and assume (N, h) is non-positively curved. Then any homotopy class of map from M to N contains a harmonic map.*

Uniqueness of harmonic maps in a given homotopy class is a difficult problem. However, when (N, h) is non-positively curved, one can consider geodesic homotopy between to nearby maps. Hartman [Har67] proved

Theorem 2.4 (Hartman)**.** *If (N, h) is non-positively (respectively negatively) curved, the energy functional is convex (respectively strictly convex) on the space of maps from M to N.*

Corollary 2.5. Assume (N, h) is non-positively curved, then

1. harmonic maps from M to N are local minima of the energy,
2. if (N, h) is negatively curved, any homotopy class contains a unique harmonic representative.

2.1.2 The equivariant case

Consider a semi-simple real Lie group G with finite center and no compact factor. Then G acts transitively by isometry on a Hadamard manifold $\mathbf{Sym}(\mathsf{G})$, called its *(Riemannian) symmetric space* and the stabilizer of a point is a maximal compact subgroup of G.

Given a morphism ρ from $\pi_1(M)$ to G, a smooth map f from the universal cover $\widetilde{M}$ to $\mathbf{Sym}(\mathsf{G})$ is *ρ-equivariant* if

$$\forall x \in \widetilde{M}\,, \forall \gamma \in \pi_1(M)\,,\ \text{we have } f(\gamma \cdot x) = \rho(\gamma) \cdot f(x)\,,$$

where $\pi_1(M)$ acts on $\widetilde{M}$ by deck transformation. In such a case, the function $\|\mathrm{d}f\|^2$ is $\pi_1(M)$-invariant and thus desccends to an integrable function on M. It follows that the energy of a ρ-equivariant map is well-defined.

We call a morphism ρ from $\pi_1(M)$ to G *reductive* if its composition with the adjoint action of G on its Lie algebra is semi-simple as a linear representation. The equivariant version of Eells-Sampson' theorem was proved by Donaldson [Don87] for $\mathsf{G} = \mathsf{SL}_2(\mathbb{C})$ and Corlette [Cor88] in the general case:

Theorem 2.6 (Corlette-Donaldson)**.** *Consider a morphism ρ from $\pi_1(M)$ to* G*. There exists a ρ-equivariant harmonic map from $\widetilde{M}$ to* $\mathbf{Sym}(\mathsf{G})$ *if and only if ρ is reductive. If moreover the centralizer of ρ is trivial, then such a harmonic map is unique.*

2.1.3 Harmonic bundles

The notion of harmonic bundle gives a very efficient way to describe equivariant harmonic maps in a gauge theoretical language.

Fixing a point x in the symmetric space $\mathbf{Sym}(\mathsf{G})$ gives an identification between $\mathbf{Sym}(\mathsf{G})$ and the quotient G/K where K is the maximal compact subgroup of G fixing x. On the Lie algebra level, we obtain an $\mathrm{Ad}(\mathsf{K})$-invariant splitting

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} ,$$

where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebra of G and K respectively, and $\mathfrak{m}$ is the orthogonal of $\mathfrak{k}$ with respect to the Killing form on $\mathfrak{g}$. The orbit map $\mathsf{G} \to \mathbf{Sym}(\mathsf{G})$ then inherits the structure of a principal K-bundle. The Maurer-Cartan form $\omega \in \Omega^1(\mathsf{G}, \mathfrak{g})$ of G then decomposes as

$$\omega = A + \sigma \ , \ \text{where } A \in \Omega^1(\mathsf{G}, \mathfrak{k}) \ \text{ and } \sigma \in \Omega^1(\mathsf{G}, \mathfrak{m}) \ .$$

It turns out that A is a principal connection on $\mathsf{G} \to \mathbf{Sym}(\mathsf{G})$ and σ vanishes on vertical direction and so descends to a 1-form on $\mathbf{Sym}(\mathsf{G})$ with value in the vector bundle $\mathrm{Ad}(\mathfrak{m}) := (\mathsf{G} \times \mathfrak{m})/\mathsf{K}$ associated to the adjoint of K on $\mathfrak{m}$. The form σ, which is sometimes called the sewing form, identifies the tangent bundle of $\mathbf{Sym}(\mathsf{G})$ with $\mathrm{Ad}(\mathfrak{m})$. In this splitting, the *Maurer-Cartan equations*, which express the flatness of ω, give

$$\left\{ \begin{array}{lcl} F_A + \frac{1}{2}[\sigma \wedge \sigma] & = & 0 \\ d_A \sigma & = & 0 \end{array} \right. . \tag{2}$$

Given a map f from M to $\mathbf{Sym}(\mathsf{G})$, we obtain a triple (P, ∇, ψ) where

- $P = f^*\mathsf{G}$ is a principal K-bundle over M,
- $\nabla = f^*A$ is a principal connection on P and
- $\psi = f^*\sigma$ is an element of $\Omega^1(M, \mathrm{Ad}_P(\mathfrak{m}))$ that naturally identifies with $\mathrm{d}f$.

We will use the notation $(P, \nabla, \psi) = f^*(\mathsf{G}, A, \sigma)$. Pulling-back equation (2) we see that the triple (P, ∇, ψ) satisfies

$$\left\{ \begin{array}{lcl} F_\nabla + \frac{1}{2}[\psi \wedge \psi] & = & 0 \\ d_\nabla \psi & = & 0 \end{array} \right. .$$

The above equations can be thought of as an integrability condition in the following sense. A triple (P, ∇, ψ) as before over a simply connected manifold M can be realized as $f^*(\mathsf{G}, A, \sigma)$ for some $f : M \to \mathbf{Sym}(\mathsf{G})$ if and only if they satisfy the above equations.

Consider now a morphism ρ from $\pi_1(M)$ to G and a ρ-equivariant map f from $\widetilde{M}$ to $\mathbf{Sym}(\mathsf{G})$. The associated triple $f^*(\mathsf{G}, A, \sigma)$ descends to a triple (P, ∇, ψ) on M. One can then recover the (conjugacy class of the) representation ρ as the holonomy of the flat connection $D = \nabla + \psi$ on the principal G bundle P_{G} associated to the inclusion of K in G. The map f then corresponds to the reduction of structure group $P \subset P_{\mathsf{G}}$. By the Euler-Lagrange equation for the harmonic map that, the map f is harmonic if and only if the triple (P, ∇, ψ) satisfies

$$\left\{ \begin{array}{lcl} F_\nabla + \frac{1}{2}[\psi \wedge \psi] & = & 0 \\ d_\nabla \psi & = & 0 \\ d_\nabla^* \psi & = & 0 \end{array} \right. . \tag{3}$$

A triple (P, ∇, ψ) on M satisfying the above equation will be called a G-*harmonic bundle*. Corlette-Donaldson Theorem then provides a one-to-one correspondence between (conjugacy classes of) reductive representation from $\pi_1(M)$ to G and (equivalence classes of) G-harmonic bundles over M.

2.1.4 When M is a surface

Let us now assume that $M = \Sigma$ is an oriented surface, and let f a map from Σ to N.

Multiplying the metric on Σ by a factor e^u, where u is a smooth funciton on Σ, multiplies $\|\mathrm{d}f\|^2$ by e^{-2u} and the volume form on Σ by e^{2u}. As a result, the energy of f only depends on the conformal class of Σ, and so we can define the notion of harmonic map on a Riemann surface.

In the same way harmonic functions are intimately related with holomorphic functions, harmonic maps have a nice holomorphic interpretation.

Consider a Riemann surface structure X on Σ and denote by $\mathcal{K}_X$ the canonical bundle (that is the holomorphic cotangent bundle of X). Given a map f from X to N, we can write the complexification of $\mathrm{d}f$ as

$$\mathrm{d}f^{\mathbb{C}} = \partial f + \overline{\partial} f \quad \text{where} \quad \begin{cases} \partial f \in \Omega^{1,0}(X, f^*\mathrm{T}N^{\mathbb{C}}) \\ \overline{\partial} f \in \Omega^{0,1}(X, f^*\mathrm{T}N^{\mathbb{C}}) \end{cases} .$$

The Euler-Lagrange equation (1) is thus equivalent to

$$\overline{\partial}_\nabla \partial f = 0 \,, \tag{4}$$

where $\overline{\partial}_\nabla$ is the Dolbeaut operator on $\mathcal{K}_X \otimes f^*\mathrm{T}N^{\mathbb{C}}$ induced by the holomorphic structure on X and the pull-back of the Levi-Civita connection on N. As a result, a map f is harmonic if and only if the $(1,0)$-part of its differential is holomorphic, that is, is an element of $H^0\left(X, \mathcal{K}_X \otimes f^*\mathrm{T}N^{\mathbb{C}}\right)$.

The *Hopf differential* of a harmonic map f is the holomorphic quadratic differential defined by $\mathrm{Hopf}(f) = g_N^{\mathbb{C}}(\partial f, \partial f)$. It measures the lack of conformality of f: it vanishes exactly when f is a conformal harmonic map, that is a branched minimal immersion.

In the same spirit, one can describe a G-harmonic bundle over X as a triple (P, ∇, ϕ) where ϕ is the $(1,0)$-part of $\psi^{\mathbb{C}}$, where ψ was described in the previous subsection. Writing ϕ^* for the dual of ϕ (so the $(0,1)$-part of ψ), the harmonic bundle equations are equivalent to the so-called *Hitchin equations*

$$\begin{cases} F_\nabla + [\phi \wedge \phi^*] & = & 0 \\ \overline{\partial}_\nabla \phi & = & 0 \end{cases} . \tag{5}$$

2.1.5 Energy functional

The theory of harmonic maps furnishes a powerful tool to construct branched minimal immersion from a closed surface into non-positively curved manifold.

Fix a closed surface Σ and either a homotopy class of maps from Σ into a non-positively curved compact manifold (N, h) or a reductive representation ρ from $\pi_1(\Sigma)$ into G. In both case, one can associated to any Riemann surface structure X on Σ a unique harmonic map f_X valued in (N, h) or **Sym**(G) which is in the fixed homotopy class in the former case, or ρ-equivariant in the second case. This defines the *energy functional*

$$\begin{array}{rccc} \mathcal{E}: & \mathcal{T}(\Sigma) & \longrightarrow & \mathbb{R} \\ & X & \longmapsto & E(f_X) \end{array} ,$$

where $\mathcal{T}(\Sigma)$ is the Teichmüller space of Σ, that is the space of isotopy classes of complex structures on X. The following was proved by Sacks and Uhlenbeck in [SU82]

Theorem 2.7 (Sacks-Uhlenbeck)**.** *If X is a critical point of the energy functional $\mathcal{E}$, the underlying harmonic map f_X is conformal, so is a branched minimal immersion.*

Building of this result, Schoen and Yau [SY79] proved

Theorem 2.8 (Schoen-Yau)**.** *Let Σ be a closed oriented surface of hyperbolic type, M a compact Riemannian manifold of nonpositive curvature and f a continuous map from Σ to M whose induced map f_* on fundamental groups is injective. Then there is a branched minimal immersion h from Σ to M such that $h_* = f_*$.*

Sketch of proof. By Sacks-Uhlenbeck, it suffices to show that the corresponding energy functional $\mathcal{E}_f$ on $\mathcal{T}(\Sigma)$ is proper. The map h will then be realized as the harmonic map from a critical point of the energy functional.

To prove properness, consider a sequence $\{X_n\}_{n\in\mathbb{N}}$ in $\mathcal{T}(\Sigma)$ whose energy is bounded. We first show that the projection of $\{X_n\}_{n\in\mathbb{N}}$ in the moduli space is bounded: if not, then after extracting if necessary, there is an element $[\gamma] \in \pi_1(\Sigma)$ such that the length of γ with respect to X_n goes to zero. But by the collar lemma, such a curve γ is embedded in a very long and thin cylinder C_n in X_n. The harmonic map h_n then sends each curve homotopic to γ in C_n to a curve in M whose length is larger or equal to the geodesic in M homotopic to $f_*([\gamma])$. Mapping a very short curve to one of fixed length costs a lot of energy. Integrating over the cylinder implies that the energy of h_n on C_n tends to infinity.

The above argument now implies the existence of diffeomorphisms $\{\varphi_n\}_{n\in\mathbb{N}}$ of Σ such that each $h_n \circ \varphi_n$ is harmonic with uniformly bounded energy. The uniform bound on the energy implies the sequence $\{h_n \circ \varphi_n\}_{n\in\mathbb{N}}$ if equicontinuous, and so the class $[\varphi_n]$ of φ_n in the mapping class group can only take a finite number of values. The result follows. □

The above strategy was adapted by Labourie in the equivariant case in [Lab08]. A representation ρ from $\pi_1(\Sigma)$ to G is called *well-displacing* if there exists positive constant A, B such that for any γ in $\pi_1(\Sigma)$ we have

$$A^{-1}\tau_\Sigma(\gamma) - B \leq \tau_{\mathsf{G}}(\rho(\gamma)) \leq A\tau_\Sigma(\gamma) + B \, ,$$

where τ_Σ and τ_{G} are respectively the translation length of an element in $\pi_1(\Sigma)$ and $\mathbf{Sym}(\mathsf{G})$ (equipped with a word length metric and the Killing metric respectively). The notion of well-displacing representation plays the role of injectivity of $f_* : \pi_1(\Sigma) \to \pi_1(M)$ in the above result.

Theorem 2.9. *[Labourie] Let ρ be a reductive representation from $\pi_1(\Sigma)$ into G and let $\mathcal{E}_\rho$ be the corresponding energy functional. If ρ is well-displacing, then $\mathcal{E}_\rho$ is proper.*

2.1.6 Higgs bundles

We now give a brief overview of Higgs bundle theory, and refer to the classical references [Wen16, Gui18, Got14] for more details.

Let $\mathsf{K}^{\mathbb{C}}$ be a complex reductive Lie group whose maximal compact subgroup is isomorphic to K, and let X be a Riemann surface with canonical bundle $\mathcal{K}_X$. The adjoint action of K on $\mathfrak{m}$ extends to an action of $\mathsf{K}^{\mathbb{C}}$ on $\mathfrak{m}^{\mathbb{C}}$.

Definition 2.10. A G-*Higgs bundle* over a Riemann surface X is a pair $(\mathcal{P}, \phi)$ where $\mathcal{P}$ is a holomorphic $\mathsf{K}^{\mathbb{C}}$-bundle over X and ϕ, the Higgs field, is a holomorphic $(1,0)$-form with value in the adjoint bundle $\mathrm{Ad}_{\mathsf{K}^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}})$ – ie. ϕ is an element of $\mathrm{H}^0\left(X, \mathcal{K}_X \otimes \mathrm{Ad}_{\mathsf{K}^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}})\right)$.

Example 2.11. We now give some explicit examples of G-Higgs bundles:

(i) Let $\mathsf{G} = \mathsf{SL}(n,\mathbb{C})$. Then $\mathsf{K} = \mathsf{SU}(n)$ and $\mathsf{K}^{\mathbb{C}} = \mathsf{G}$. Using the standard representation of $\mathsf{SL}(n,\mathbb{C})$ on $\mathbb{C}^n$, we see that a holomorphic $\mathsf{K}^{\mathbb{C}}$-bundle is the same thing as a rank n holomorphic vector bundle $\mathcal{E}$ with a holomorphic volume form. In this setting, $\mathrm{Ad}_{\mathsf{K}^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}})$ is the bundle of traceless endomorphisms of $\mathcal{E}$. In particular, ϕ is a holomorphic 1-form with value in traceless endormorphisms of $\mathcal{E}$.

(ii) More generally, if G is a complex Lie group, then $\mathsf{K}^{\mathbb{C}} = \mathsf{G}$ and $\mathfrak{m}^{\mathbb{C}} = \mathfrak{g}$. In particular, a G-Higgs bundle is a holomorphic principal G-bundle together with a holomorphic 1-form valued in the adjoint bundle $\mathrm{Ad}_{\mathsf{G}}(\mathfrak{g})$.

(iii) For $\mathsf{G} = \mathsf{SU}(p,q)$, we have $\mathsf{K} = \mathsf{S}\big(\mathsf{U}(p)\times\mathsf{U}(q)\big)$ and $\mathsf{K}^{\mathbb{C}} = \mathsf{S}\big(\mathsf{GL}(p,\mathbb{C})\times\mathsf{GL}(q,\mathbb{C})\big)$. Using the standard representation, a holomorphic principal $\mathsf{K}^{\mathbb{C}}$-bundle corresponds to a holomorphic vector bundle $\mathcal{E}$ with a holomorphic volume form that splits holomorphically as $\mathcal{E} = \mathcal{U} \oplus \mathcal{V}$ where $\mathcal{U}$ and $\mathcal{V}$ have respectively rank p and q. In this picture, $\mathrm{Ad}_{\mathsf{K}^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}})$ is the vector bundle $\mathrm{Hom}(\mathcal{U},\mathcal{V}) \oplus \mathrm{Hom}(\mathcal{V},\mathcal{U})$. It follows that we can write

$$\phi = \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix} \ \text{ where } \ \begin{cases} \alpha \in H^0\big(X, \mathcal{K}_X \otimes \mathrm{Hom}(\mathcal{U},\mathcal{V})\big) \\ \beta \in H^0\big(X, \mathcal{K}_X \otimes \mathrm{Hom}(\mathcal{V},\mathcal{U})\big) \end{cases} .$$

To avoid writing matrices, we will write such a Higgs bundle using quiver diagrams:

$$(\mathcal{E},\phi) := \mathcal{U} \underset{\alpha}{\overset{\beta}{\rightleftarrows}} \mathcal{V} \ .$$

(iv) If $\mathsf{G} = \mathsf{SO}_0(p,q)$ then $\mathsf{K} = \mathsf{SO}(p)\times\mathsf{SO}(q)$ and $\mathsf{K}^{\mathbb{C}} = \mathsf{SO}(p,\mathbb{C})\times\mathsf{SO}(q,\mathbb{C})$. A holomorphic principal $\mathsf{K}^{\mathbb{C}}$-bundle then corresponds to a vector bundle $\mathcal{E}$ that splits as $\mathcal{E} = \mathcal{U}\oplus\mathcal{V}$ as above, but $\mathcal{U}$ and $\mathcal{V}$ are equipped with volume forms and quadratic forms $\mathbf{q}_{\mathcal{U}}$ and $\mathbf{q}_{\mathcal{V}}$ respectively, all these objects being holomorphic. The Higgs field ϕ then writes

$$\begin{pmatrix} 0 & \alpha^\dagger \\ \alpha & 0 \end{pmatrix} \ \text{ where } \ \alpha^\dagger = \mathbf{q}_{\mathcal{U}}^{-1}\circ\alpha^*\circ\mathbf{q}_{\mathcal{V}} \ .$$

Here, we see $\mathbf{q}_{\mathcal{U}}$ as a morphism from $\mathcal{U}$ to $\mathcal{U}^*$, similarly for $\mathbf{q}_{\mathcal{V}}$ and $\alpha^* \in H^0(X, \mathcal{K}_X \otimes \mathrm{Hom}(\mathcal{V}^*,\mathcal{U}^*))$ is the dual of α.

Consider a G-harmonic bundle (P,∇,ϕ) on X. The principal $\mathsf{K}^{\mathbb{C}}$-bundle $\mathcal{P}$ associated to P via the inclusion of K in $\mathsf{K}^{\mathbb{C}}$ carries a holomorphic structure induced by ∇. Using the second equation in (5), one sees that the pair $(\mathcal{P},\phi)$ is a G-Higgs bundle.

Conversely, given $(\mathcal{P},\phi)$ a G-Higgs bundle over X, a *metric* on $\mathcal{P}$ is a smooth reduction P_{K} to the maximal compact subgroup K of $\mathsf{K}^{\mathbb{C}}$. Such a metric uniquely defines

- A real subbundle $\mathrm{Ad}_{\mathsf{K}}(\mathfrak{m})$ of $\mathrm{Ad}_{\mathsf{K}^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}})$ (observe that the bundles $\mathrm{Ad}_{\mathsf{K}^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}})$ and $\mathrm{Ad}_{\mathsf{K}}(\mathfrak{m}^{\mathbb{C}})$ are naturally identified) ,
- a $(0,1)$-form ϕ^* with value in $\mathrm{Ad}_{\mathsf{K}^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}})$ such that $\psi := \phi + \phi^*$ takes values in $\mathrm{Ad}_{\mathsf{K}}(\mathfrak{m})$,
- a connection ∇ on P_{K} compatible with the holomorphic structure on $\mathcal{P}$.

Remark 2.12. In the vector bundle description given in Example 2.11, a metric on $\mathcal{P}$ is given by a Hermitian metric h on the underlying holomorphic vector bundle $\mathcal{E}$ (compatible with the extra structures). In this case, $\mathrm{Ad}_{\mathsf{K}}(\mathfrak{m})$ is the subbundle of $\mathrm{Ad}_{\mathsf{K}^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}})$ consisting of h-selfadjoint endomorphism of $\mathcal{E}$, ϕ^* is the h-adjoint of ϕ and ∇ is the Chern connection of h.

By equation (5), the triple (P, ∇, ϕ) is the a G-harmonic bundle if and only if

$$F_\nabla + [\phi \wedge \phi^*] = 0 \,.$$

In such a case, the connection $D = \nabla + \phi + \phi^*$ is a flat connection on the principal G-bundle P_{G} associated to P_{K} via the inclusion of K into G.

It turns out that not all G-Higgs bundle admits a metric solution to the Hitchin equations: existence of solutions is intimatelly related to the algebraic notion of *stability*. These notions are pretty involved to define for general Lie group G, so we will restrict to the case $\mathsf{G} = \mathsf{SL}(n, \mathbb{C})$ and work with vector bundles as in the the first item of Example 2.11.

Definition 2.13. Let $(\mathcal{P}, \phi)$ be a $\mathsf{SL}(n, \mathbb{C})$-Higgs bundle with underlying vector bundle $\mathcal{E}$. It is called

- *semistable* if any proper ϕ-invariant holomorphic subbundle $\mathcal{F}$ of $\mathcal{E}$ satisfies $\deg(\mathcal{F}) \leq 0$,
- *stable* if the above inequality is strict ,
- *polystable* if $\mathcal{E}$ splits holomorphically as $\mathcal{E} = \bigoplus_{i=1}^k \mathcal{E}_i$ where each $\mathcal{E}_i$ is ϕ-invariant of degree 0 and $(\mathcal{E}_i, \phi|_{\mathcal{E}_i})$ is stable.

The Kobayashi-Hitchin correspondence, first proved by Hitchin [Hit87a] for $\mathsf{SL}(2, \mathbb{C})$ and Simpson [Sim92] in the general case, gives

Theorem 2.14 (Kobayashi-Hitchin correspondence)**.** *A* G*-Higgs bundle admits a metric solution to the Hitchin equations if and only if it is semistable.*

2.1.7 Moduli space and non-abelian Hodge correspondence

Given a smooth principal bundle P over X, the *gauge group of* P is the group of automorphisms of P (covering the identity). This group naturally acts on the space of holomorphic structure on P. Two G-Higgs bundles over X are *equivalent* if they have the same underlying smooth principal bundle and they differ by the action of the corresponding gauge group. Using Geometric Invariant Theory, Simpson [Sim92] and Nitsure [Nit91] constructed a space $\mathcal{M}(X, \mathsf{G})$, called the *moduli space of* G*-Higgs bundles over* X, whose points parametrize equivalence classes of polystable G-Higgs bundles over X. The moduli space $\mathcal{M}(X, \mathsf{G})$ is a complex quasi-projective variety with a fascinating geometry. Let us describe some of its properties:

• Taking the L^2 norm of the Higgs field defines a perfect Morse-Bott function on $\mathcal{M}(X, \mathsf{G})$.

• Multiplying the Higgs field by $\lambda \in \mathbb{C}^*$ defines a action of $\mathbb{C}^*$ on $\mathcal{M}(X, \mathsf{G})$ which turns out to be algebraic. The fixed points of this action are called *variation of Hodge structures* and correspond to the critical points of the Morse function mentioned above.

• Applying $\mathrm{Ad}(\mathsf{K}^{\mathbb{C}})$-invariant homogeneous polynomials to the Higgs field defines the *Hitchin map*

$$H : \mathcal{M}(X, \mathsf{G}) \longrightarrow \bigoplus_{i=1}^{k} H^0(X, \mathcal{K}_X^{m_i}) \,,$$

where the m_i are the degrees of some homogeneous generators of the algebra $\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{\mathsf{K}^{\mathbb{C}}}$ of $\mathrm{Ad}(\mathsf{K}^{\mathbb{C}})$-invariant polynomials on $\mathfrak{m}^{\mathbb{C}}$. The Hitchin map is $\mathbb{C}^*$-equivariant and, when $\mathsf{G} = \mathsf{K}^{\mathbb{C}}$, the generic fibers of H are abelian varieties embedded in $\mathcal{M}(X, \mathsf{G})$ as Lagrangian submanifolds (the space $\mathcal{M}(X, \mathsf{G})$ is said to define an algebraic completely integrable system, see [Hit87b]).

Consider a reductive representation $\rho : \pi_1(\Sigma) \to \mathsf{G}$ and a Riemann surface structure X on Σ. Using Corlette-Donaldson's theorem, we can associate a harmonic bundle over X and so a polystable G-Higgs bundle. Conversely, given a polystable G-Higgs bundle over X, solving Hitchin equations yields a G-harmonic bundle whose underlying holonomy gives a reductive morphism from $\pi_1(\Sigma)$ to G. Keeping track of the equivalence relation gives the *non-abelian Hodge correspondence*:

Theorem 2.15 (Non-abelian Hodge correspondence)**.** *Given a Riemann surface structure X on Σ, the above correspondence defines a real analytic isomorphism between $\chi(\Sigma, \mathsf{G})$ and $\mathcal{M}(X, \mathsf{G})$.*

2.2 Pseudo-hyperbolic geometry

2.2.1 Pseudo-hyperbolic space

Let E be a real vector space of dimension $(p + q + 1)$ and equipped with a quadratic form $\mathbf{q}$ of signature $(p, q + 1)$. Consider the quadric

$$\mathbf{H}^{p,q}_{+} := \{x \in E \ , \ \mathbf{q}(x) = -1\} \ .$$

The tangent space to $\mathbf{H}^{p,q}_{+}$ at a point x is identified with the orthogonal $x^{\perp}$ to x with respect to $\mathbf{q}$. In particular, $\mathbf{q}$ restricts to a pseudo-Riemannian metric $\mathbf{g}$ of signature (p, q) on $\mathbf{H}^{p,q}_{+}$ which turns out to be geodesically complete of constant sectional curvature -1. The group $\mathsf{O}(\mathbf{q})$ of orthogonal transformation of $(E, \mathbf{q})$ acts on $\mathbf{H}^{p,q}_{+}$ preserving $\mathbf{g}$ and turns it into a pseudo-Riemannian symmetric space. We will often work with the projective model

$$\mathbf{H}^{p,q} = \{x \in \mathbf{P}(E) \ , \ \mathbf{q}(x) < 0\} \ .$$

Observe that the natural map from $\mathbf{H}^{p,q}_{+}$ to $\mathbf{H}^{p,q}$ is a 2-to-1 cover which is trivial if and only if $q = 0$ (that is in the case of the classical hyperbolic space). The metric $\mathbf{g}$ on $\mathbf{H}^{p,q}_{+}$ descends to a metric on $\mathbf{H}^{p,q}$ that we still denote by $\mathbf{g}$. The action of $\mathsf{O}(\mathbf{q})$ on $\mathbf{H}^{p,q}$ is not effective and descends to an action of the projective group $\mathsf{PO}(\mathbf{q}) = \mathsf{O}(\mathbf{q})/\{\pm \mathrm{Id}\}$.

2.2.2 Einstein Universe

The boundary of $\mathbf{H}^{p,q}$ in $\mathbf{P}(E)$ is called the *Eintein Universe* and corresponds to

$$\partial_{\infty}\mathbf{H}^{p,q} = \{x \in \mathbf{P}(E) \ , \ \mathbf{q}(x) = 0\} \ .$$

The group $\mathsf{PO}(\mathbf{q})$ acts transitively on $\partial_{\infty}\mathbf{H}^{p,q}$ preserving a conformal class of pseudo-Riemannian metric of signature $(p - 1, q)$ which turns out to be locally conformally flat. The stabilizer of a point is a parabolic subgroup of $\mathsf{PO}(\mathbf{q})$.

Similarly, if $\mathbf{P}_{+}(E)$ denotes the space of oriented lines in E, the boundary of $\mathbf{H}^{p,q}_{+}$ in $\mathbf{P}_{+}(E)$ is

$$\partial_{\infty}\mathbf{H}^{p,q}_{+} = \{x \in \mathbf{P}_{+}(E) \ , \ \mathbf{q}(x) = 0\} \ .$$

2.2.3 Spacelike immersion

Consider a connected manifold M of dimension p. An immersion ι from M to $\mathbf{H}^{p,q}$ is called *spacelike* if the induced metric is positive-definite. It is called *complete* if furthermore the induced metric is complete. Given such an immersion, the pull-back tangent bundle decomposes as

$$\iota^{*}\mathrm{T}\mathbf{H}^{p,q} = \mathrm{T}M \oplus \mathrm{N}M \ ,$$

where the *normal bundle* $\mathrm{N}M$ is the orthogonal complement of $\mathrm{T}M$ with respectively to $\iota^{*}\mathbf{g}$. The restriction g_{T} and g_{N} of $\iota^{*}\mathbf{g}$ to $\mathrm{T}M$ and $\mathrm{N}M$ respectively defines scalar product that are positive and negative definite respectively.

In this splitting, the pull-back $\iota^{*}\nabla$ of the Levi-Civita connection decomposes as

$$\iota^{*}\nabla = \begin{pmatrix} \nabla^{\mathrm{T}} & -\mathrm{B} \\ \mathrm{II} & \nabla^{\mathrm{N}} \end{pmatrix} ,$$

where

- ∇^{T} is the Levi-Civita connection of (M, g_{T}),
- ∇^{N} is a unitary connection on $(\mathrm{N}M, g_{\mathrm{N}})$,
- II is an element of $\Omega^1(M, \mathrm{Hom}(\mathrm{T}M, \mathrm{N}M))$ called the *second fundamental form*,
- B is an element of $\Omega^1(M, \mathrm{Hom}(\mathrm{N}M, \mathrm{T}M))$ called the *shape operator*,
- the fact that ∇ is torsion-free and $\mathbf{g}$-unitary gives for any vector field x, y on M and section n of $\mathrm{N}M$

$$\mathrm{II}(x, y) = \mathrm{II}(y, x) \text{ and } g_N(\mathrm{II}(x, y), n) = g_{\mathrm{T}}(y, \mathrm{B}(x, n)) ,$$

where we identified $\Omega^1(M, \mathrm{Hom}(\mathrm{T}M, \mathrm{N}M))$ with $\Omega^0(M, \mathrm{Hom}(\mathrm{T}M \otimes \mathrm{T}M, \mathrm{N}M))$.

Since $\mathbf{H}^{p,q}$ has constant sectional curvature, its curvature tensor is given by

$$R^{\nabla}(a, b)c = \mathbf{g}(a, c)b - \mathbf{g}(b, c)a ,$$

where a, b, c are vector fields on $\mathbf{H}^{p,q}$ and we used the convention $R^{\nabla}(a, b)c = \nabla_a \nabla_b c - \nabla_b \nabla_a c - \nabla_{[a,b]} c$. Pulling-back the equation above and projecting on $\mathrm{T}M$ and $\mathrm{N}M$ gives the *fundamental equations* :

Proposition 2.16 (Fundamental equations)**.** *Let ι be a spacelike immersion from M to $\mathbf{H}^{p,q}$. Given vector fields x, y, z, t on M and sections α, β of $\mathrm{N}M$, we have*

- ***Gauss equation:***

$$R^{\iota^*\nabla}(x, y, z, t) = R^{\mathrm{T}}(x, y, z, t) - g_{\mathrm{N}}(\mathrm{II}(x, t), \mathrm{II}(y, z)) + g_{\mathrm{N}}(\mathrm{II}(x, z), \mathrm{II}(y, t)) .$$

- ***Ricci equation:*** *for any section x, y of $\mathrm{T}M$ and α, β of $\mathrm{N}M$ we have*

$$g_{\mathrm{N}}\left(R^{\mathrm{N}}(x, y)\alpha, \beta\right) = g_{\mathrm{T}}(\mathrm{B}(y, \alpha), \mathrm{B}(x, \beta)) - g_{\mathrm{T}}(\mathrm{B}(x, \alpha), \mathrm{B}(y, \beta)) .$$

- ***Codazzi equation:***

$$d_{\nabla}\mathrm{II} = 0 ,$$

where we consider II *as an element of* $\Omega^1(M, \mathrm{Hom}(\mathrm{T}M, \mathrm{N}M))$ *and* ∇ *is the connection on* $\mathrm{Hom}(\mathrm{T}M, \mathrm{N}M)$ *induced from* ∇^{T} *and* ∇^{N}.

2.2.4 Maximal submanifolds

Given a compact subset K of M, we can define the *volume* of a spacelike immersion ι of M into $\mathbf{H}^{p,q}$ by

$$\mathcal{V}_K(\iota) := \int_K \mathrm{dvol}_{\mathrm{T}} ,$$

where $\mathrm{vol}_{\mathrm{T}}$ is the volume of the metric g_{T} on M induced by ι. Given an infitinesimal deformation ξ of ι (so ξ is a section of $\mathrm{N}M$), the infinitesimal variation of $\mathcal{V}_K(\iota)$ is given by

$$\dot{\mathcal{V}}_K(\iota) = \int_K g_{\mathrm{N}}(\xi, \mathrm{H})\mathrm{dvol}_{\mathrm{T}} ,$$

where H is the *mean curvature* of ι, that is the trace of II with respect to g_{T}. It follows that a spacelike immersion is a critical point of the volume if and only if it has zero mean curvature. The second variation of the volume at a critical point is computed in [LTW23], and is given by

$$\ddot{\mathcal{V}}_K(\iota) = \int_K \left(p.g_{\mathrm{N}}(\xi, \xi) + \mathrm{tr}\left(g_{\mathrm{N}}(\nabla^{\mathrm{N}}\xi, \nabla^{\mathrm{N}}\xi) - g_{\mathrm{T}}(\mathrm{B}(.)\xi, \mathrm{B}(.)\xi)\right)\right) \mathrm{dvol}_{\mathrm{T}} .$$

Since g_{N} is negative and g_{T} positive, we obtain the inequality

$$\ddot{\mathcal{V}}_K(\iota) \leq p \int_K g_{\mathrm{N}}(\xi, \xi) \mathrm{dvol}_{\mathrm{T}} .$$

It follows that $\ddot{\mathcal{V}}_K(\iota)$ is negative as soon as ξ is non-zero on K. As a result, critical points of the volume are *stable* and correspond to local maxima. We thus call them *maximal submanifolds*.

2.2.5 Fermi charts

Consider an orthogonal decomposition $E = U \oplus V$ where U is a positive-definite linear subspace of dimension p and V negative-definite of dimension $q+1$. We denote $\langle .,. \rangle_U$ and $\langle .,. \rangle_V$ the positive-definite scalar product on U and V associated to $|\mathbf{q}|$. Let $\mathbf{B}^p$ be the open unit ball in U and $\mathbf{S}^q$ be the unit sphere in V. The map

$$\begin{array}{cccc} \psi : & \mathbf{B}^p \times \mathbf{S}^q & \longrightarrow & \mathbf{P}_+(U \oplus V) \\ & (u, v) & \longmapsto & \left[\left(\frac{2u}{1+\|u\|^2}, v \right) \right] \end{array}$$

defines a diffeomorphism onto $\mathbf{H}_+^{p,q}$ (seen as an open set in the space $\mathbf{P}_+(E)$ of oriented lines in E). The pull-back metric is given by

$$\psi^* \mathbf{g} = \phi \cdot (g_{\mathbf{S}^p} - g_{\mathbf{S}^q}) ,$$

where $g_{\mathbf{S}^p}$ is the spherical metric on $\mathbf{B}^p$ identified with an hemisphere of $\mathbf{S}^p$ via the stereographical projection and ϕ is a positive function on $\mathbf{B}^p \times \mathbf{S}^q$ that only depends on the first variable and goes to infinity on $\partial_\infty \mathbf{B}^p$. We call this model a *Fermi chart* and the composition of ψ^{-1} with the projection on $\mathbf{B}^p$ a *Fermi projection*. Since $\phi \cdot g_{\mathbf{S}^p}$ is the hyperbolic metric on $\mathbf{B}^p$, we get that a Fermi projection gives a proper submersion from $\mathbf{H}_+^{p,q}$ to $\mathbf{B}^p$ which increases the norm of tangent vectors.

Observe that ψ extends to a diffeomorphism from $\mathbf{S}^{p-1} \times \mathbf{S}^q$ to $\partial_\infty \mathbf{H}_+^{p,q}$ in which the conformal metric on $\partial_\infty \mathbf{H}_+^{p,q}$ is $[g_{\mathbf{S}^{p-1}} - g_{\mathbf{S}^q}]$.

One of the main application of Fermi charts is the following

Proposition 2.17. *Let ι be a spacelike immersion from M to $\mathbf{H}_+^{p,q}$. If ι is proper, or if the induced metric on M is complete, then $\iota(M)$ is a spacelike entire graph, that is, in any Fermi chart $\mathbf{B}^p \times \mathbf{S}^q$ there exists a smooth map f from $\mathbf{B}^p$ to $\mathbf{S}^q$ with $\|df\| < 1$ and whose graph is $\iota(M)$.*

Proof. Since a Fermi projection is proper and increases the norm of tangent vectors, we get that $\pi \circ \iota$ is a proper local diffeomorphism, thus a covering map. Since $\mathbf{B}^p$ is simply connected, $\pi \circ \iota$ is a global diffeomorphism. The condition $\|\mathrm{d}f\| < 1$ follows from the fact that $\iota(M)$ is spacelike. □

We denote by $\mathcal{E}(\mathbf{H}_+^{p,q})$ the set of spacelike entire graphs in $\partial_\infty \mathbf{H}_+^{p,q}$ that we endow with the topology of convergence C^∞ on every compact.

Since any 1-Lipschitz map $f : \mathbf{B}^p \to \mathbf{S}^q$ extends to a 1-Lipschitz map $\varphi : \mathbf{S}^{p-1} \to \mathbf{S}^q$, any spacelike entire graph M has a well-defined boundary $\partial_\infty M$ in $\partial_\infty \mathbf{H}_+^{p,q}$ which is the graph of a 1-Lipschitz map.

One can be more precise: if $\partial_\infty M$ is the graph of $\varphi : \mathbf{S}^{p-1} \to \mathbf{S}^q$, then the image of φ does not contain any antipodal points. In fact, since φ is 1-Lipschitz, its image contains antipodal points if and only if there exists $x \in \mathbf{S}^{p-1}$ such that $\varphi(-x) = -\varphi(x)$. But the only way to extend such a φ by f would be to map any geodesic in $\mathbf{B}^p$ between x and $-x$ by a geodesic between $\varphi(x)$ and $-\varphi(x)$ and such extension does not satisfy $\|\mathrm{d}f\| < 1$.

We will call an *admissible sphere* any topological $(p-1)$-sphere in $\partial_\infty \mathbf{H}_+^{p,q}$ which is the graph of a 1-Lispchitz map from $\mathbf{S}^{p-1}$ to $\mathbf{S}^q$ in some (equivalently any) Fermi chart, and whose image does not contain antipodal points. We denote by $\mathcal{B}(\partial_\infty \mathbf{H}^{p,q}_+)$ the set of admissible spheres endowed with Hausdorff topology. From the above discussion, we have a well-defined continuous *boundary map*

$$\partial_\infty : \mathcal{E}\left(\mathbf{H}^{p,q}_+\right) \longrightarrow \mathcal{B}(\partial_\infty \mathbf{H}^{p,q}_+) .$$

2.3 Anosov representations

We now quickly describe the theory of Anosov representations, focusing on examples provided by surface group representations. The notion of Anosov representation was first introduced by Labourie [Lab06] to give a dynamical interpretation of Hitchin representations. It was then broadly developed by Guichard and Wienhard in [GW12] (see also Kapovich Leeb and Porti in [KLP13]). The theory of Anosov representations has quickly become a very active area of research, with many important contributions.

2.3.1 Hyperbolic group

Let Γ be a finitely generated discrete group. Choosing a set of generators S of Γ defines a metric d_S on Γ by means of its Cayley graph. We say that Γ is *Gromov hyperbolic* if (Γ, d_S) is δ-hyperbolic for some $\delta \geq 0$. In such a case, the *boundary* $\partial_\infty\Gamma$ of Γ is the set of geodesic rays in (Γ, d_S) up to the equivalence relation of being at bounded distance.

Taking S' a different set of generators defines a different metric $d_{S'}$ on Γ. However, the two metrics d_S and $d_{S'}$ are quasi-isometric. In particular, they define homeomorphic boundaries.

The action of Γ on itself by left multiplication defines a continuous action on $\partial_\infty\Gamma$ which is *minimal* (every orbit is dense). Any element $\gamma \in \Gamma$ different from the identity has a "North-South dynamic", meaning that it has a unique attractive fixed point x_γ^+ and repealling point x_γ^-.

The main example of interest for us will be the fundamental group of a closed Riemannian manifold (M, g) of negative sectional curvature. In such a case, the action of Γ on the universal cover $\widetilde{M}$ is a quasi-isometry. It follows that in such a case $\partial_\infty\Gamma$ is naturally identified with $\partial_\infty\widetilde{M}$.

2.3.2 Lie theory

Cartan projection: Let G be a non-compact real semi-simple Lie group with finite center, K be a maximal compact subgroup and denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ the corresponding Cartan decomposition. A *Cartan subspace* is a linear abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$ of maximal dimension. The dimension of $\mathfrak{a}$ is called the *real rank* of G and K acts transitively on the set of Cartan subspaces in $\mathfrak{m}$.

Geometrically, the choice of K corresponds to choosing a point x in the symmetric space $\mathbf{Sym}(\mathsf{G})$ of G, while choosing $\mathfrak{a}$ corresponds to choosing a maximal flat in $\mathbf{Sym}(\mathsf{G})$ passing through x. More precisely, the exponential map sends $\mathfrak{a}$ to a totally geodesic submanifold of $\mathbf{Sym}(\mathsf{G})$ isometric to an Euclidean space.

Given $\alpha \in \mathfrak{a}^*$, we define

$$\mathfrak{g}_\alpha = \left\{x \in \mathfrak{g} \mid \forall a \in \mathfrak{a} \text{ we have } [a, x] = \alpha(a)x\right\} .$$

Such an element α is called a *restricted root* if it is nonzero and if $\mathfrak{g}_\alpha$ is different from $\{0\}$. Denote by Δ the set of roots. The corresponding root space decomposition is then

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha .$$

A choice of simple roots is given by choosing a subset Π in Δ such that any root is a linear combination of elements in Π with coefficient of the same sign. Such a choice define a subset of positive roots $\Delta^+ \subset \Delta$

corresponding to linear combination of elements in Π with nonnegative coefficients. The associated *positive Weyl chamber* is then

$$\mathfrak{a}_+ = \left\{ x \in \mathfrak{a} \mid \forall \alpha \in \Pi \text{ we have } \alpha(x) > 0 \right\} .$$

Given an element g in G, there exists $k, k' \in \mathsf{K}$ and a unique $\mu(g) \in \mathfrak{a}_+$ such that $g = k \exp(\mu(g))k'$. The element $\mu(g)$ is called the *Cartan projection* of g.

Parabolic subgroups: Given a subset θ of Π, let Δ^+_θ be the complementary set in Δ^+ of the positive roots spanned by $\Pi \setminus \theta$. Define P^+_θ and P^-_θ to be respectively the normalizer in G of

$$\mathfrak{u}^{\pm}_\theta = \bigoplus_{\alpha \in \Delta^+_\theta} \mathfrak{g}_{\pm\alpha} .$$

Definition 2.18. Let G and Π be as above.

i. A closed subgroup P of G is called *parabolic* if it is conjugated to P^+_θ or P^-_θ for some subset θ of Π.

ii. A pair $(\mathsf{P}^-, \mathsf{P}^+)$ of parabolic subgroups is called *opposite* if there exists a subset θ of Π such that $(\mathsf{P}^-, \mathsf{P}^+)$ is (simutaneously) conjugated to $(\mathsf{P}^+_\theta, \mathsf{P}^-_\theta)$.

iii. A subset θ of Π is called *symmetric* if P^+_θ and P^-_θ are conjugated.

When $\theta = \Delta^+$, we have $\Delta^+_\theta = \emptyset$ and the corresponding parabolic is called *minimal* (or *Borel subgroup*). When θ has a single element, the corresponding parabolic is called *maximal*.

A *flag variety* is a G-homogeneous space $\mathbf{Fl}(\mathsf{G})$ such that the stabilizer of a point is a parabolic subgroup. A pair $\left(\mathbf{Fl}^-(\mathsf{G}), \mathbf{Fl}^+(\mathsf{G})\right)$ of flag varieties is called *opposite* if the (generic) stabilizers define a pair of opposite subgroup. Finally, a pair of points (x_+, x_-) in opposite flag varieties $\left(\mathbf{Fl}^-(\mathsf{G}), \mathbf{Fl}^+(\mathsf{G})\right)$ is *transerve* if the stabilizer of (x_-, x_+) in G is reductive.

Example 2.19. Consider $\mathsf{G} = \mathsf{SL}(4, \mathbb{R})$ and let $\mathsf{K} = \mathsf{SO}(4)$. A Cartan subspace is then given by the space of traceless diagonal matrices. Any root has the form α_{ij} for $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$ where

$$\alpha_{ij} \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{pmatrix} = a_i - a_j ,$$

and the corresponding root space is the space of matrices $M = (m_{\alpha\beta})$ with $m_{\alpha\beta} = 0$ whenever $(\alpha, \beta) \neq (i, j)$. The associated Weyl chamber is

$$\mathfrak{a}^+ = \left\{ M = \mathrm{diag}(a_1, a_2, a_3, a_4) \mid a_1 > a_2 > a_3 > a_4 > 0 \right\} ,$$

and the Cartan projection consists of taking the logarithm of the singular values.

For $\theta := \{\alpha_2\}$, we have $\Delta^+_\theta = \{\alpha_2, \alpha_{13}, \alpha_{24}, \alpha_{14}\}$ and any M in P^+_θ has the form

$$M = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} .$$

Similarly, a matrix is in P^-_θ if and only if its transpose is in P^+_θ. In this example, one can see that θ is symmetric, the corresponding flag variety is identified with the Grassmannian of 2-planes in $\mathbf{R}^4$ and a pair (U, V) of 2-planes is transverse if and only if $\mathbf{R}^4 = U \oplus V$.

2.3.3 Definition of Anosov representations

Consider a Gromov hyperbolic group Γ with set of generators S and denote by d_S the corresponding distance on Γ. Fix G a non-compact real semi-simple Lie group with finite center and θ a subset of the set of simple roots Π. As above, denote by Δ_θ^+ the set of positive roots that do not belong to the span of $\Pi \setminus \theta$, and let $\mathsf{P}_\theta^\pm$ the corresponding parabolic subgroups.

Definition 2.20 (Anosov representations)**.** A representation ρ from Γ into G is P_θ*-Anosov* if there exists positive A, B such that

$$\forall \alpha \in \theta \, , \ \ \forall \gamma \in \Gamma \, , \ \alpha\big(\mu(\rho(\gamma))\big) > A \cdot d_S(\gamma, \mathrm{id}) - B \, .$$

Remark 2.21. Labourie's definition of Anosov representation is different from the above. Nevertheless, since the original definition relies on some flow associated to a representation, its definition is longer to state.

An important characterization of the P_θ-Anosov representations is the following

Proposition 2.22 ([GW12])**.** *A Zariski dense representation ρ from Γ to G is P_θ-Anosov if and only if there exists a pair of continuous ρ-equivariant map, called* boundary maps

$$\beta^\pm : \partial_\infty \Gamma \longrightarrow \mathsf{G}/\mathsf{P}_\theta^\pm \, ,$$

which are transverse *(that is for any distinct x, y in $\partial_\infty\Gamma$ the flags $\beta^+(x)$ and $\beta^-(y)$ are transverse) and* dynamic preserving *(that is for any γ in Γ, the points $\beta^\pm(x_\gamma^+)$ are attracting fixed point of $\rho(\gamma)$ in $\mathsf{G}/\mathsf{P}_\theta^\pm$).*

Example 2.23. Let us come back to the case $\mathsf{G} = \mathsf{SL}(4, \mathbb{R})$ and $\theta = \{\alpha_2\}$ as considered in the previous example. In this case, a matrix M is such that $\alpha_2(\mu(M))$ is large if and only if the ratio between the second and third singular value is large. In particular, one can find a dense open set U in the Grassmannian of 2-planes in $\mathbf{R}^4$ such that, for every $x \in U$, the orbit $\{M^n x\}_{n\in\mathbb{N}}$ accumulates to an "attractive point" of M.

In particular, a P_θ-Anosov representation $\rho : \Gamma \to \mathsf{G}$ will be such that, for large element $\gamma \in \Gamma$, the matrix $\rho(\gamma)$ will have large first 2 singular values compared to the third and fourth one. The boundary map β will be such that, for any $\gamma \in \Gamma \setminus \{\mathrm{id}\}$, $\beta(\gamma_+)$ is the attractive point of $\rho(\gamma)$.

2.3.4 The case $\mathsf{G} = \mathsf{PO}(p, q)$

The group $\mathsf{G} = \mathsf{PO}(p, q)$ is the group of projective transformation of $\mathbf{R}^{p,q}$, the vector space $\mathbf{R}^{p+q}$ equipped with a quadratic form $\mathbf{q}$ of signature (p, q). The Lie algebra $\mathfrak{g}$ consists of endomorphisms of $\mathbf{R}^{p,q}$ that are skewsymmetric with respect to $\mathbf{q}$. For $r = \min\{p, q\}$, consider a splitting

$$\mathbf{R}^{p,q} = L_1 \oplus \dots L_r \oplus L_r^\vee \oplus \dots L_1^\vee \oplus W \, ,$$

where

- each L_i and $L_i^\vee$ is an isotropic line,
- each pair (L_i, L_j) is orthogonal,
- each pair $(L_i^\vee, L_j^\vee)$ is orthogonal,
- each pair $(L_i, L_j^\vee)$ is orthogonal unless $i = j$,
- W is orthogonal to each L_i and $L_j^\vee$ (so it is definite of dimension $|p - q|$ and is positive if and only if $p > q$).

A Cartan subspace is thus given by diagonal matrices in the above splitting of the form

$$\mathfrak{a} = \left\{ M = \operatorname{diag}\left(\lambda_1, \dots, \lambda_r, -\lambda_r, \dots, -\lambda_1, 0_W\right) \ , \ \lambda_i \in \mathbb{R} \right\} \ .$$

When $p \neq q$, define $\alpha_i(M) = \lambda_i - \lambda_{i+1}$ for $i < r$ and $\alpha_r(M) = \lambda_p$. Then $\Pi = \{\alpha_1, ..., \alpha_r\}$ is a set of simple roots. The corresponding Weyl chamber is

$$\mathfrak{a}^+ = \left\{ M = \operatorname{diag}\left(\lambda_1, \dots, \lambda_r, -\lambda_r, \dots, -\lambda_1, 0_W\right) \ , \ \lambda_i > \lambda_{i+1} > 0 \right\} \ .$$

Consider now the case $\theta = \{\alpha_1\}$, which was extensively studied in [DGK18]. Such a θ is symmetric, and we denote P_θ by P_1. A P_1-Anosov representation is thus a morphism $\rho : \Gamma \to \mathsf{G}$ such that, for a long element $\gamma \in \Gamma$, the matrix $\rho(\gamma)$ has a large first eigenvalue, and the ratio between the two first eigenvalues is also large. Since the parabolic P_1 is the stabilizer in G of an isotropic line in $\mathbf{R}^{p,q}$, the boundary map is

$$\beta : \partial_\infty \Gamma \longrightarrow \partial_\infty \mathbf{H}^{p,q-1} \ .$$

Transversality means that any pair of different points (x, y) in $\partial_\infty \Gamma$ is mapped to a pair of non-orthogonal points in $\partial_\infty \mathbf{H}^{p,q-1}$. Dynamic preserving means that for each nonzero γ, there is a dense open set U in $\mathbf{P}(\mathbf{R}^{p,q})$ such that the restriction of $\rho(\gamma^n)$ to U converges to the constant map $\beta(x_\gamma^+)$.

By transversality of β, the image of any triple of pairwise distinct points in ∂_∞ spans a vector space of signature $(2, 1)$ or $(1, 2)$. This motivates the following definition

Definition 2.24 (Positive map)**.** Let Λ be a set with at least 3 elements. A map β from Λ to $\partial_\infty \mathbf{H}^{p,q-1}$ is called *positive* (respectively *negative*) if the image of any triple of pairwise distinct points spans a vector space of signature $(2, 1)$ (respectively $(1, 2)$).

If β is the boundary map of a P_1-Anosov representation, the representation is called *positive* or *negative* accordingly.

Remark 2.25. In [DGK18], the authors use a different convention and call *negative* a set that is *positive* in our sense. The reason for their convention is that a positive set in our sense lift to a cone in $\mathbf{R}^{p,q}$ for which the scalar product of any two noncolinear vectors is negative.

When $\partial_\infty \Gamma$ is connected, it is proved in [DGK18] that any P_1-Anosov representation is either positive or negative. Moreover, changing $\mathbf{q}$ to $-\mathbf{q}$ switches positive and negative maps.

Given a positive P_1-Anosov representation ρ, the set $\beta(\partial_\infty \Gamma)$ has a well-defined *convex-hull*, which is a closed convex domain in $\mathbf{H}^{p,q-1}$ on which $\rho(\Gamma)$ acts properly discontinuously and cocompactly, that is, ρ is a $\mathbf{H}^{p,q-1}$*-convex cocompact representation*. More generally, we have

Theorem 2.26 (Danciger-Guéritaud-Kassel)**.** *Let Γ be a Gromov hyperbolic group and ρ a morphism from Γ to $\mathsf{PO}(p, q)$. Then ρ is positive P_1-Anosov if and only if it is $\mathbf{H}^{p,q-1}$-convex cocompact.*

Denote by $\mathbf{Isot}(\mathbf{R}^{p,q})$ the set of maximally isotropic subspaces of $\mathbf{R}^{p,q}$, so any element in $\mathbf{Isot}(\mathbf{R}^{p,q})$ has dimension $\min\{p, q\}$ and $\mathbf{Isot}(\mathbf{R}^{p,q})$ is a flag variety (when $p \neq q$). Given a P_1-Anosov representation $\rho : \Gamma \to \mathsf{PO}(p, q)$, define

$$\Omega_\rho := \left\{ V \in \mathbf{Isot}(\mathbf{R}^{p,q} \ , \ \mathbf{P}(V) \cap \beta(\partial_\infty \Gamma) = \emptyset \right\} \ .$$

Guichard and Wienhard proved in [GW12] that $\rho(\Gamma)$ acts properly discontinuously and cocompactly on Ω_ρ. In particular, if Ω_ρ is non-empty, then ρ is the holonomy of a geometric structure locally modelled on $\mathbf{Isot}(\mathbf{R}^{p,q})$. However, little is known in general about the topology of that quotient $\Omega_\rho / \rho(\Gamma)$.

2.3.5 Examples for surface groups

The main source of examples of Anosov representations comes from representation of surface groups, that is when Γ is the fundamental group of a closed oriented surface Σ of hyperbolic type. We present here two important constructions, linked with the so-called theory of *higher rank Teichmüller spaces*.

Teichmüller space The Teichmüller space of Σ is the space $\mathcal{T}(\Sigma)$ of isotopy classes of complex structures on Σ. By the uniformization theorem, any point X in $\mathcal{T}(\Sigma)$ is fully described by its (conjugacy class of) holonomy representation $\rho_X : \pi_1(\Sigma) \to \mathsf{PSL}(2,\mathbb{R})$. Such a representation is called *Fuchsian*: it is discrete, faithful and one recovers X as the quotient $\mathbf{H}^2/\rho_X(\pi_1(\Sigma))$. This yields to an embedding of $\mathcal{T}(\Sigma)$ into the character variety $\chi(\Sigma, \mathsf{PSL}(2,\mathbb{R}))$ whose image is a connected component.

Given a morphism $\rho : \pi_1(\Sigma) \to \mathsf{PSL}(2,\mathbb{R})$ and a smooth ρ-equivariant map $f : \widetilde{\Sigma} \to \mathbf{H}^2$, the pull-back $f^*\mathrm{vol}_{\mathbf{H}^2}$ of the area form by f descends to a closed 2-form on Σ. The *Euler class* is defined by

$$e(\rho, f) = \frac{1}{2\pi}\int_\Sigma f^*\mathrm{vol}_{\mathbf{H}^2} .$$

It turns out that $e(\rho, f)$ is an integer which only depends on the conjugacy class of ρ. It thus defines a continuous map

$$e : \chi(\Sigma, \mathsf{PSL}(2,\mathbb{R})) \longrightarrow \mathbb{Z} .$$

By the celebrated Milnor-Wood inequality, the Euler class takes values in the interval $[2-2g, 2g-2]$, where g is the genus of Σ. Goldman proved in [Gol88] that Fuchsian representations are exactly the ones maximizing the Euler class.

Theorem 2.27 (Goldman)**.** *Let $[\rho]$ be in $\chi(\Sigma, \mathsf{PSL}(2,\mathbb{R}))$. Then ρ is in $\mathcal{T}(\Sigma)$ if and only if $e([\rho])$ is equal to $2g-2$.*

Maximal representations Assume G is of Hermitian type, that is the symmetric space $\mathbf{Sym}(\mathsf{G})$ carries a G-invariant complex structure which is compatible with the Killing metric. For instance, $\mathsf{G} = \mathsf{Sp}(2n,\mathbb{R})$, $\mathsf{SO}(2,n)$, $\mathsf{SU}(p,q)$. Such an Hermitian Lie group has a preferred maximal parabolic subgroup, which stabilizes a point in the so-called *Shilov boundary*.

Replacing the area form of $\mathbf{H}^2$ by the Kähler form ω of $\mathbf{Sym}(\mathsf{G})$, one can mimic the definition of the Euler class to define the *Toledo invariant* of an element in $\chi(\Sigma, \mathsf{G})$. With the correct normalization, there exists a rational number ℓ_G such that the Toledo invariant is a continuous map

$$\tau : \chi(\Sigma, \mathsf{G}) \longrightarrow \ell_\mathsf{G}\mathbb{Z} .$$

Toledo proved in [DT87] an analogue of the Milnor-Wood inequality:

$$\forall [\rho] \in \chi(\Sigma, \mathsf{G}) , \ |\tau([\rho])| \leq (2g-2)\mathrm{rank}(\mathsf{G}) ,$$

where $\mathrm{rank}(\mathsf{G})$ is the real rank of G. Motivated by Goldman's result, one defines

Definition 2.28. A representation ρ from $\pi_1(\Sigma)$ to G is called *maximal* if $\tau(\rho)$ equals $(2g-2)\mathrm{rank}(\mathsf{G})$.

We now gather some important properties of maximal representations proved in [BIW10, BILW05]

Theorem 2.29 ([BILW05, BIW10])**.** *Let ρ be a maximal representation from $\pi_1(\Sigma)$ to G. Then,*

1. *ρ is P-Anosov, where P is the stabilizer of a point in the Shilov boundary of G.*
2. *There exists a reductive subgroup G_0 in G stabilizing a symmetric domain of tube type in $\mathbf{Sym}(\mathsf{G})$ such that the image of ρ lies in G_0.*

It follows that the subset $\chi^{\max}(\Sigma, \mathsf{G})$ of $\chi(\Sigma, \mathsf{G})$ consisting of maximal representations is a union of connected component and consist only of Anosov representations. Moreover, Goldman's theorem implies that $\chi^{\max}(\Sigma, \mathsf{PSL}(2,\mathbb{R}))$ is naturally identified with $\mathcal{T}(\Sigma)$. The space $\chi^{\max}(\Sigma, \mathsf{G})$ is an example of what is called a *higher rank Teichmüller space* (see [Wie18] for more details).

Hitchin representations We now describe another family of higher rank Teichmüller spaces. Let G be a real split semi-simple Lie group of adjoint type (for instance $\mathsf{G} = \mathsf{PSL}(n,\mathbb{R}), \mathsf{PSp}(2n,\mathbb{R}), \mathsf{PO}(n,n), \mathsf{PO}(n,n+1)$...). There exists a preferred morphism, called *principal*, from $\mathsf{PSL}(2,\mathbf{R})$ into G, which is unique up to conjugation. Post-composing with the principal morphism defines an inclusion

$$\iota_{\mathsf{G}} : \chi(\Sigma, \mathsf{PSL}(2,\mathbb{R})) \hookrightarrow \chi(\Sigma, \mathsf{G}) .$$

The *Hitchin component* $\mathrm{Hit}(\Sigma,\mathsf{G})$ is the connected component of $\chi(\Sigma,\mathsf{G})$ containing $\iota_{\mathsf{G}}(\mathcal{T}(\Sigma))$. This component was first studied by Hitchin in [Hit92]. Using non-Abelian Hodge correspondence, he proved

Theorem 2.30 (Hitchin)**.** *For any Riemann surface structure X on Σ, the component $\mathrm{Hit}(S,\mathsf{G})$ is diffeomorphic to $\bigoplus_{i=1}^{\mathrm{rank}(\mathsf{G})} H^0(X,\mathcal{K}_X^{m_i+1})$, where the m_i are the exponents of G and $\mathcal{K}_X$ is the canonical bundle of X.*

The notion of Anosov representations was originally introduced by Labourie to give a geometrical interpretation of Hitchin representations (that is, representaitons in $\mathrm{Hit}(\Sigma,\mathsf{G})$).

Theorem 2.31 (Labourie)**.** *Any Hitchin representation is B-Anosov, where B is a Borel subgroup of G.*

In particular, Hitchin components give another example of higher rank Teichmüller space. When G is both real split and of Hermitian type (for instance when $\mathsf{G} = \mathsf{PO}(2,3)$), then $\mathrm{Hit}(\Sigma,\mathsf{G})$ is a particular component of $\chi^{\max}(\Sigma,\mathsf{G})$.

2.3.6 Labourie's conjecture

The drawback of Hitchin's parametrization of $\mathrm{Hit}(\Sigma,\mathsf{G})$ is that it depends on the choice of a Riemann surface structure on Σ. In particular, we cannot expect to understand the action of the Mapping Class Group $\mathsf{MCG}(\Sigma)$ of Σ on $\mathrm{Hit}(\Sigma,\mathsf{G})$ and therefore possible structure (complex, Kähler...) on $\mathrm{Hit}(\Sigma,\mathsf{G})$ that is $\mathsf{MCG}(\Sigma)$-invariant.

However, if one can construct a projection $\pi : \mathrm{Hit}(\Sigma,\mathsf{G}) \to \mathcal{T}(\Sigma)$ which is $\mathsf{MCG}(\Sigma)$-equivariant and "natural" in some sense, one can hope to realize $\mathrm{Hit}(\Sigma,\mathsf{G})$ as a bundle over $\mathcal{T}(\Sigma)$ and obtain the desired structures in this way. This is a reason why he introduced his famous conjecture

Conjecture 1 (Labourie)**.** *Given a Hitchin representation ρ from $\pi_1(\Sigma)$ into G, there is a unique ρ-equivariant branched minimal immersion from $\widetilde{\Sigma}$ in* $\mathbf{Sym}(\mathsf{G})$.

Since branched minimal immersion are exactly conformal harmonic map and correspond to critical points of the energy functional $\mathcal{E}_\rho$ (see Subsection 2.1.5), Labourie's conjecture can be restated as

Conjecture 2 (Labourie)**.** *Given a Hitchin representation ρ from $\pi_1(\Sigma)$ into G, the energy functional $\mathcal{E}_\rho$ has a unique critical point.*

A polystable G-Higgs bundle $(\mathcal{P},\phi)$ corresponds to a conformal harmonic map if and only if it satisfies $\mathrm{tr}(\phi^2) = 0$. In Hitchin's parametrization, this corresponds to the vanishing of the coefficient in $H^0(X,\mathcal{K}_X^2)$ (which corresponds to a multiple of the Hopf differential of the underlying harmonic map).

Consider vector bundle $\pi : \mathcal{B} \to \mathcal{T}(\Sigma)$ whose fiber over X is $\bigoplus_{i=2}^{\mathrm{rank}(\mathsf{G})} H^0(X,\mathcal{K}_X^{m_i})$. Taking Hitchin's parametrization fiberwise yields a $\mathsf{MCG}(\Sigma)$-equivariant smooth map $\Psi : \mathcal{B} \longrightarrow \mathrm{Hit}(\Sigma,\mathsf{G})$. Labourie's conjecture is (roughly) equivalent to the fact that Ψ is a global diffeomorphism.

Building on a construction of Kim and Zhang [KZ17], Labourie [Lab17] proved that the L^2-metric on the dual bundle $\mathcal{B}^*$ is Griffiths negative, and thus carries a family of $\mathsf{MCG}(\Sigma)$-invariant Kähler metrics. Taking duals, such a construction would give a family of Kähler metric on $\mathrm{Hit}(\Sigma,\mathsf{G})$.

Since Anosov representations are well-displacing, if follows from Theorem 2.9 that the existence part of the conjecture is known in general. The uniqueness is the difficult part and known in the following cases:

- When $\mathsf{G} = \mathsf{PSL}(2,\mathbb{R})$, this is the Uniformization theorem.
- When $\mathsf{G} = \mathsf{PSL}(2,\mathbb{R}) \times \mathsf{PSL}(2,\mathbb{R})$, this is originally due to Schoen [Sch93]. The theory is deeply connected with the theory of maximal surfaces in $\mathbf{H}^{2,1}$, see [BBZ07, KS07].
- When $\mathsf{G} = \mathsf{PSL}(3,\mathbb{R})$ this is due to Loftin [Lof01] and independently Labourie [Lab07b]. This is linked with the theory of affine spheres.
- When G has rank 2 (so all the above cases, plus $\mathsf{G} = \mathsf{PSp}(4,\mathbb{R})$ or G_2'), this is due to Labourie [Lab17] and is linked with the theory of cyclic surfaces.

However, recent results of Markovic [Mar22], Markovic, Sagman and Smilie [MSS22] disproved it for $\mathsf{PSL}(2,\mathbb{R})^n$ with $n \geq 3$ and Sagman and Smilie [SS22] disproved it when $\mathrm{rank}(\mathsf{G})$ is at least 3.

3 Maximal representations in rank 2

This paper [CTT19] in collaboration with Brian Collier and Nicolas Tholozan is published at *Duke Mathematical Journal*. We study maximal representations of surface groups into a Hermitian Lie group of rank 2.

3.1 Main Theorem

The first remark is that, by the rigidity theorem of Burger, Iozzi and Weinhard (see the second item of Theorem 2.29), up to a compact factor, any maximal representation factor through a Hermitian Lie group of tube type. If G is a rank 2 Hermitian Lie group of tube type, then G is locally isomorphic to $\mathsf{PSp}(4,\mathbb{R})$, $\mathsf{SU}(2,2)$, $\mathsf{SO}^*(8)$ or $\mathsf{SO}_0(2,n)$ for some n at least 2. However, exceptionnal isomorphisms of Lie algebra give the following local isomorphisms

$$\mathsf{PSp}(4,\mathbb{R}) \cong \mathsf{SO}_0(2,3)\ ,\ \mathsf{SU}(2,2) \cong \mathsf{SO}_0(2,4)\ ,\ \mathsf{SO}^*(8) \cong \mathsf{SO}_0(2,6)\ .$$

In particular, rank 2 Hermitian Lie group of tube type are all locally isommorphic to some $\mathsf{SO}_0(2,n+1)$ for n at least 1. By a theorem of Danciger, Guéritaud and Kassel (see Theorem 2.26), those representations are exactly $\mathbf{H}^{2,n}$-convex cocompact representations. Our main theorem is

Theorem 3.1 ([CTT19])**.** *Let Σ be a closed oriented surface of hyperbolic type. A representation ρ from $\pi_1(\Sigma)$ into $\mathsf{SO}_0(2,n+1)$ is maximal if and only if there exists a ρ-equivariant maximal immersion from $\widetilde{\Sigma}$ into $\mathbf{H}^{2,n}$. Moreover, such an immersion is unique.*

3.2 Sketch of proof

3.2.1 Existence

The existence relies of Higgs bundle theory.

Fix X a Riemann surface structure on Σ. Since a holomorphic $\mathsf{SO}(2,\mathbb{C})$-bundle over X splits holomorphically as a direct sum of line bundles, the $\mathsf{SO}_0(2,n+1)$-Higgs bundle associated (via the non-Abelian Hodge correspondence) to a reductive representation $\rho : \pi_1(\Sigma) \to \mathsf{SO}_0(2,n+1)$ has the form

$$(\mathcal{E},\phi) = \mathcal{L} \underset{\alpha}{\overset{\beta}{\rightleftarrows}} \mathcal{W} \underset{\alpha^\dagger}{\overset{\beta^\dagger}{\rightleftarrows}} \mathcal{L}^{-1}\ ,$$

where $\mathcal{L}$ is a line bundle of nonnegative degree, $\mathcal{W}$ is a $\mathsf{SO}(n+1,\mathbb{C})$-bundle and $\alpha^\dagger$ (respectively $\beta^\dagger$) is the dual of α (respectively β) using the orthogonal structure on $\mathcal{W}$.

The representation ρ is maximal exactly when $\mathcal{L}$ has degree $(2g-2)$. Polystability implies that that the section α never vanishes. In such a case, the image of α in $\mathcal{W}$ is a non-isotropic line subbundle $\mathcal{I}$ which is thus a square root of the trivial bundle $\mathcal{O}_X$. Leting $\mathcal{V}$ be the orthogonal of $\mathcal{I}$ in $\mathcal{W}$, we get the following decomposition

$$(\mathcal{E},\phi) = \mathcal{I}\mathcal{K}_X \underset{1}{\overset{q_2}{\rightleftarrows}} \mathcal{I} \underset{1}{\overset{q_2}{\rightleftarrows}} \mathcal{I}\mathcal{K}_X^{-1}, \qquad \mathcal{I}\mathcal{K}_X^{-1} \xrightarrow{\eta} \mathcal{V} \xrightarrow{\eta^\dagger} \mathcal{I}\mathcal{K}_X ,$$

where $q_2 \in H^0(X,\mathcal{K}_X^2)$. One easily checks that $\mathrm{tr}(\phi^2) = 4q_2$. In particular, the underlying harmonic map is conformal exactly when $q_2 = 0$. It follows that if X is a critical point of the energy functional $\mathcal{E}_\rho$ (whose existence is granted by Theorem 2.9), then the underlying Higgs bundle is

$$(\mathcal{E},\phi) = \mathcal{I}\mathcal{K}_X \xrightarrow{1} \mathcal{I} \xrightarrow{1} \mathcal{I}\mathcal{K}_X^{-1}, \qquad \mathcal{I}\mathcal{K}_X^{-1} \xrightarrow{\eta} \mathcal{V} \xrightarrow{\eta^\dagger} \mathcal{I}\mathcal{K}_X .$$

Such a Higgs bundle is fixed by the cyclic subgroup $\mathbb{Z}_4$ of $\mathbb{C}^*$. Simpson's proved in [Sim09] that the solution to the Hitchin equations is then diagonal in the splitting $\mathcal{E} = \mathcal{I}\mathcal{K}_X \oplus \mathcal{I} \oplus \mathcal{I}\mathcal{K}_X^{-1} \oplus \mathcal{V}$.

On the real bundle side, the above discussion iplies that the underlying flat bundle $E_\rho = \left(\widetilde{\Sigma}\times \mathbf{R}^{2,n+1}\right)/\pi_1(\Sigma)$ decomposes orthogonally as

$$E_\rho = \ell \oplus U \oplus V$$

where ℓ is a negative-definite line subbundle. The choice of ℓ in E_ρ thus corresponds to a ρ-equivariant map $u : \widetilde{X} \to \mathbf{H}^{2,n}$. Holomorphicity of ϕ implies that u harmonic and conformal, and so corresponds to a maximal surface.

3.2.2 Uniqueness

Uniqueness is obtained by a maximum principle. We assume the existence of two different ρ-equivariant maximal surfaces $u_1, u_2 : \widetilde{\Sigma} \to \mathbf{H}^{2,n}$ and, after lifting to $\mathbf{H}^{2,n}_+$, we consider the map

$$\beta : \begin{array}{ccc} \widetilde{\Sigma}\times\widetilde{\Sigma} & \longrightarrow & \mathbf{R} \\ (x,y) & \longmapsto & \langle u_1(x), u_2(y)\rangle \end{array} .$$

This map should be thought of as a "distance" function. The fact that the surfaces $u_i(\widetilde{\Sigma})$ are entire graph and share the same boundary implies that $\sup\beta \in (-1,0)$. By equivariance, β achieves its maximum at a point $p = (x,y) \in \widetilde{\Sigma}\times\widetilde{\Sigma}$. The Hessian of β then satisfies

$$\mathrm{Hess}_p\beta(w,w) = \big(\mathbf{q}(w_1) + \mathbf{q}(w_2)\big)\,\beta(p) + 2\langle w_1, w_2\rangle + \langle \mathrm{II}_1(w_1,w_1), y\rangle + \langle x, \mathrm{II}_2(w_2,w_2)\rangle ,$$

where $w = (w_1,w_2) \in \mathrm{T}_x\widetilde{\Sigma}\times\mathrm{T}_y\widetilde{\Sigma}$. By maximality of the u_i, we have $\mathrm{tr}\mathrm{II}_i = 0$. Choosing a unit vector, say w_1 corresponding to the largest eigenvalue of the last 2 terms, we get that for any unit vector w_2

$$\mathrm{Hess}_p\beta(w,w) \geq 2\beta(p) + 2\langle w_1, w_2\rangle .$$

Choosing $w_2 = \frac{\pi(w_1)}{\sqrt{\mathbf{q}(\pi(w_1))}}$, where $\pi : \mathbf{R}^{2,n+1} \to \mathrm{T}_x\widetilde{\Sigma}$ is the orthogonal projection, implies that $\mathrm{Hess}_p\beta(w,w) > 0$, giving a contradiction.

3.3 Consequences

The main theorem has many consequences we now describe.

3.3.1 An analogue of Labourie's conjecture

Corollary 3.2. For any maximal representation ρ of $\pi_1(\Sigma)$ into a rank 2 Hermitian Lie group G, there exists a unique ρ-equivariant minimal immersion from $\widetilde{\Sigma}$ into the symmetric space $\mathbf{Sym}(\mathsf{G})$.

Proof. By the existence part of the main theorem, any ρ-equivariant minimal surface in $\mathbf{Sym}(\mathsf{G})$ defines a ρ-equivariant maximal surface in $\mathbf{H}^{2,n}$ with same induced conformal structure. By the uniqueness part of Corlette-Donaldson theorem, two different equivariant minimal surfaces must have different induced conformal structure, thus giving different equivariant maximal surfaces in $\mathbf{H}^{2,n}$. Thus, uniqueness part of our main theorem implies the uniqueness of the minimal surface. □

3.3.2 Length spectrum domination

The dynamic of a maximal representation $\rho : \pi_1(\Sigma) \to \mathsf{SO}_0(2, n+1)$ is partly encoded in its *length spectrum*, which is the function $L_\rho : \pi_1(\Sigma) \to \mathbb{R}$ whose value on γ is the the logarithm of the spectral radius of $\rho(\gamma)$. Equivalently, if for any $x \in \mathbf{H}^{2,n}$ in convex hull of the boundary curve, we have

$$L_\rho(\gamma) = \lim_{n\to\infty} \frac{1}{n} \eth(x, \rho(\gamma)^n x) ,$$

where $\eth$ is the *spacial distance*, that is for any $a, b \in \mathbf{H}^{2,n}$ we have

$$\eth(a, b) = \begin{cases} d_{\mathbf{H}^2}(a, b) & \text{if } a, b \text{ lie in a hyperbolic plane} \\ 0 & \text{otherwise} \end{cases} .$$

We have

Corollary 3.3. Let ρ be a maximal representation of $\pi_1(\Sigma)$ into $\mathsf{SO}_0(2, n + 1)$. Then either ρ preserves a totally geodesic copy of $\mathbf{H}^2$ in $\mathbf{H}^{2,n}$ or there exists $\lambda > 1$ and a Fuchsian representation j such that

$$L_\rho \geq \lambda L_j .$$

Proof. The result follows from 2 inequalities. The first tells that given two points x, y in the maximal surface $u_\rho(\widetilde{\Sigma})$, we have

$$\eth(x, y) \geq d_g(x, y) ,$$

where g is the induced metric on $\widetilde{\Sigma}$. In fact, one can find a hyperbolic plane passing through x and y, since the corresponding Fermi projection (see Subsection 2.2.5) increases distances, we get the inequality.

For the second, Gauss equation (see Proposition 2.16) implies that the sectional curvature K_g of $(\widetilde{\Sigma}, g)$ is larger than or equal to -1. Then, by Schwarz-Pick lemma, the uniformization of the maximal surface is strictly length increasing, unless $K_g = -1$ everywhere, in which case the second fundamental form vanishes everywhere and the maximal surface in totally geodesic (so is a copy of $\mathbf{H}^2$). Thus, if $u(\widetilde{\Sigma})$ is not totally geodesic, there exists $\lambda > 1$ such that $g \geq \lambda g_{hyp}$. Combining the two inequalities gives the result. □

3.3.3 Geometric structures

Another consequence of the existence of a ρ-invariant maximal surface is that we understand the topology of the quotient $\Omega_\rho/\rho(\pi_1(\Sigma))$, where Ω_ρ is described in Subsection 2.3.4.

Corollary 3.4. Given a maximal representation ρ from $\pi_1(\Sigma)$ into $\mathsf{SO}_0(2, n+1)$, the quotient of Ω_ρ by $\rho(\pi_1(\Sigma))$ is homeomorphic to a bundle over Σ with fiber $\mathbf{Isot}(\mathbf{R}^{2,n})$.

Proof. Given a point $x \in u(\widetilde{\Sigma})$, the space $\mathbf{Isot}(x^\perp)$ of isotropic 2-planes in $x^\perp$ is diffeomorphic to $\mathbf{Isot}(\mathbf{R}^{2,n})$. Moreover, a maximal surface is contained in the convex hull of its boundary, the geodesic from x to a point in the limit set $\beta(\partial_\infty \pi_1(\Sigma))$ is spacelike. In particular, $\mathbf{Isot}(x^\perp) \subset \Omega_\rho$.

Now, spacelike surfaces are in particular acausal: the geodesic joining two distinct points $x, y \in u(\widetilde{\Sigma})$ is spacelike. It follows that $x \oplus y$ has signature $(1,1)$, and so $x^\perp \cap y^\perp = (x \oplus y)^\perp$ has signature $(1, n)$. Since $\mathbf{R}^{1,n}$ does not contain any isotropic 2-plane, we get that $\mathbf{Isot}(x^\perp) \cap \mathbf{Isot}(y^\perp) = \emptyset$.

Consider the bundle $B \to \widetilde{\Sigma}$ whose fiber over p is $\mathbf{Isot}(u(p)^\perp)$. Then B is a manifold of same dimension as $\mathbf{Isot}(\mathbf{R}^{2,n+1})$ and the natural injection $\iota : B \to \Omega_\rho$ is ρ-equivariant, continuous and locally injective. By Invariance of the Domain, ι is a local homeomorphism. Compactness of $B/\pi_1(\Sigma)$ implies the result. □

4 Plateau problems in pseudo-hyperbolic spaces

The main result of [CTT19] suggested the study of asymptotic Plateau problems in $\mathbf{H}^{p,q}$. I thus started the (long) project of studying such Plateau problems, have written in 3 papers in this topic:

- the first [LTW23], in collaboration with François Labourie and Mike Wolf, is accepted for publication at *Annales Scientifiques de l'ENS*. The results are describe in Subsection 4.2.
- The second [LT23], in collaboration with François Labourie, is accepted for publication at *Inventiones Mathematicae*. The results are described in Subsection 4.3.
- The third [SST23], in collaboration with Andrea Seppi and Graham Smith, has recently been submitted. The results are described in Subsection 4.4.

4.1 The asymptotic Plateau problem

Recall from Subsection 2.2.5 that, given a connected spacelike p-dimensional submanifold M of $\mathbf{H}^{p,q}$, if the induced metric on M is complete, then M is contractible and has a well-defined *asymptotic boundary* $\partial_\infty M$ in $\partial_\infty \mathbf{H}^{p,q}$, which is an admissible sphere.

The asymptotic Plateau problem asks, given an admissible sphere Λ in $\partial_\infty \mathbf{H}^{p,q}$, whether Λ is the asymptotic boundary of some complete maximal p-dimensional submanifold of $\mathbf{H}^{p,q}$.

More conceptually, define

$$\mathcal{M}(\mathbf{H}^{p,q}) := \{\text{complete maximal } p\text{-dimensional submanifolds of } \mathbf{H}^{p,q}\} ,$$

that we see as a subset of the space of entire graphs $\mathcal{E}(\mathbf{H}^{p,q})$ with the induced topology (see Subsection 2.2.5). The boundary map from $\mathcal{E}(\mathbf{H}^{p,q})$ to the space $\mathcal{B}(\mathbf{H}^{p,q})$ of admissible spheres thus restricts to a continuous map

$$\partial_\infty : \mathcal{M}(\mathbf{H}^{p,q}) \longrightarrow \mathcal{B}(\mathbf{H}^{p,q}) .$$

The asymptotic Plateau problem thus studies the surjectivity of ∂_∞. Here is an history of the results

- The case $(p, q) = (2, 1)$ and the boundary is invariant under a $\mathbf{H}^{2,1}$-convex cocompact representation was proved by Barbot, Béguin and Zeghib in [BBZ07].

- The case $(p, q) = (n, 1)$ and the boundary is invariant under a $\mathbf{H}^{n,1}$-convex cocompact representation was proved by Andersson, Barbot, Béguin and Zeghib in [ABBZ12].
- The case $(p, q) = (n, 1)$ with no group action and with positive boundary was proved by Bonsante and Schlenker in [BS10]. They also proved uniqueness for $(p, q) = (2, 1)$ when the boundary is a quasicircle.
- The case $(p, q) = (2, n)$ and the boundary is invariant under a $\mathbf{H}^{2,n}$-convex cocompact representation was proved by Collier, Tholozan and myself in [CTT19].
- The case $(p, q) = (2, n)$ and general boundary was proved by Labourie, Wolf and myself in [LTW23].
- General (p, q) and general boundary was proved by Seppi, Smith and myself in [SST23].

4.2 The case $\mathbf{H}^{2,n}$

In this paper [LTW23], in collaboration with François Labourie and Mike Wolf, we solve the asymptotic Plateau problem for $\mathbf{H}^{2,n}$, namely we prove

Theorem 4.1 ([LTW23])**.** *The boundary map ∂_∞ from $\mathcal{M}(\mathbf{H}^{2,n})$ to $\mathcal{B}(\mathbf{H}^{2,n})$ is a homeomorphism. In particular, any nonnegative circle in $\partial_\infty \mathbf{H}^{2,n}$ is the aymptotic boundary of a unique complete maximal surface in $\mathbf{H}^{2,n}$.*

Observe that, if Σ is a closed oriented surface of genus at least 2 and $\rho : \pi_1(\Sigma) \to \mathsf{SO}_0(2, n+1)$ is a maximal representation, then the image of the boundary map $\beta : \partial_\infty \pi_1(\Sigma) \to \partial_\infty \mathbf{H}^{2,n}$ is a positive circle, thus is particular a nonnegative circle. As a result, the main result in [CTT19] follows from the above theorem.

4.2.1 Strategy of the proof

The strategy of the proof is very natural. It goes as follow:

Step 1. We first prove a *finite* Plateau problem: any spacelike curve γ (with some extra assumptions) inside $\mathbf{H}^{2,n}$ bounds a maximal disk. We prove it by a method of continuity: we consider a smooth path $\{\gamma_t\}_{t\in[0,1]}$ of spacelike curves where $\gamma_1 = \gamma$ and γ_0 contained in a totally geodesic copy of $\mathbf{H}^2$ and let

$$I := \{t \in [0,1]\ ,\ \gamma_t \text{ bounds a maximal disk}\}.$$

The set I is then

i. *nonempty*: for $t = 0$ the curve γ_0 is contained in a totally geodesic surface,

ii. *open*: this is a consequence of the stability of maximal surface and elliptic regularity,

iii. *closed*: this follows from a compactness theorem (see below).

Step 2. We prove the result when Λ is smooth and spacelike. To do so, we take a sequence $\{\gamma_k\}_{k\in\mathbb{N}}$ of radial curves converging to Λ. For each k, *Step 1* gives the existence of a maximal surface S_k bounded by γ_k. A compactness theorem allows us to conclude that, up to extracting, $\{S_k\}_{k\in\mathrm{N}}$ converges to a complete maximal surface with asymptotic boundary Λ.

Step 3. We approximate any nonnegative circle by a sequence of smooth ones. Using *Step 2*, we obtain a sequence of complete maximal surface that we prove subconverges (using again a compactness theorem).

Step 4. We prove uniqueness, adapting the proof in [CTT19] using Omori's maximum principle.

4.2.2 Compactness theorem and the main tool

The first 3 steps rely on the following compactness result

Theorem 4.2 (Compactness theorem)**.** *Let $\{\gamma_k\}_{k\in\mathbb{N}}$ be a sequence of circles in $\mathbf{H}^{2,n}$ or $\partial_\infty\mathbf{H}^{2,n}$, with good properties, that converges to some limit γ_0 in $\mathbf{H}^{2,n}$ or $\partial_\infty\mathbf{H}^{2,n}$. If for each k we have a maximal surface S_k bounded by γ_k, then up to extracting, the sequence $\{S_k\}_{k\in\mathbb{N}}$ converges to a maximal surface bounded by γ_0.*

The key tool to prove such a compactness is the theory of *holomorphic curves* of Gromov. We consider the Grassmanian bundle

$$\mathcal{G}\left(\mathbf{H}^{2,n}\right) := \left\{(x,P) \mid x\in\mathbf{H}^{2,n}\ ,\ P \text{ oriented positive 2-plane in } \mathrm{T}_x\mathbf{H}^{2,n}\right\}\ .$$

The Levi-Civita connection on $\mathbf{H}^{2,n}$ defines a splitting

$$\mathrm{T}_{(x,P)}\mathcal{G}\left(\mathbf{H}^{2,n}\right) = \mathrm{T}_x\mathbf{H}^{2,n}\oplus \mathrm{Hom}(P,P^\perp)$$

and the *holonomic distribution* is the distribution $\mathcal{W}$ on $\mathcal{G}\left(\mathbf{H}^{2,n}\right)$ defined by

$$\mathcal{W}_{(x,P)} = P\oplus\mathrm{Hom}(P,P^\perp)\ .$$

Such a distribution carries an almost complex structure J defined by

$$J_{(x,P)}(u,\varphi) = (iu, \varphi\circ i)\ ,$$

where i is the rotation of angle $\frac{\pi}{2}$ in P.

Any spacelike immersion in $\mathbf{H}^{2,n}$ naturally lifts to $\mathcal{G}\left(\mathbf{H}^{2,n}\right)$. The key fact is that such an immersion is maximal if and only if its lift is J-holomorphic. As a result, we can use Gromov's theory of holomorphic curves to study geometric properties of maximal surfaces in $\mathbf{H}^{2,n}$. The main tool being Gromov-Schwarz lemma, from which our compactness theorem follows.

The main technical difficulty is that, since we aim to prove a finite Plateau problem, we have to consider holomorphic curves with boundary. The theory of holomorphic curves with boundary is much more subtle, and our compactness result, in the finite case, depends on technical properties of the boundary curves.

4.3 Quasicircles and quasiperiodic surfaces

Consider the space $\mathcal{M}^\bullet(\mathbf{H}^{2,n})$ of *pointed* complete maximal surfaces in $\mathbf{H}^{2,n}$, that is

$$\mathcal{M}^\bullet(\mathbf{H}^{2,n}) = \left\{(x,\Sigma)\mid \Sigma\in\mathcal{M}(\mathbf{H}^{2,n})\ ,\ x\in\Sigma\right\}\ .$$

The space $\mathcal{M}^\bullet(\mathbf{H}^{2,n})$ is a laminated topological space (leaves correspond to maximal surfaces) on which the group $\mathsf{G} = \mathsf{PO}_0(2,n+1)$ acts continuously, preserving the lamination. A consequence of our compactness theorem in [LTW23] is that the action of G on $\mathcal{M}^\bullet(\mathbf{H}^{2,n})$ is cocompact.

In particular, any continuous G-invariant function defined $\mathcal{M}^\bullet(\mathbf{H}^{2,n})$ admits global extrema. For instance, the function

$$K:\ \begin{array}{ccc}\mathcal{M}^\bullet(\mathbf{H}^{2,n}) & \longrightarrow & \mathbb{R}\\ (x,\Sigma) & \longmapsto & K_\Sigma(x)\end{array}\ ,$$

where $K_\Sigma(x)$ is the sectional curvature of Σ at the point x. It follows from Gauss equation that $K_\Sigma(x)\geq -1$. We prove

Theorem 4.3 ([LT23])**.** *Given any pointed complete maximal surface (x,Σ) in $\mathcal{M}^\bullet(\mathbf{H}^{2,n})$, we have $K_\Sigma(x)\leq 0$. Moreover, if $K_\Sigma(x)$ equals 0, then K_Σ is zero everywhere and Σ is a Barbot surface, that is Σ is the solution of the asymptotic Plateau problem whose boundary is a lightlike quadrilateral.*

This theorem yields a dichotomy: given a maximal surface Σ in $\mathcal{M}(\mathbf{H}^{2,n})$, either

i. there exists $\delta > 0$ such that $K_\Sigma \leq -\delta$, or

ii. the closure of the leaf corresponding to Σ in $\mathcal{M}^\bullet(\mathbf{H}^{2,n})$ contains a Barbot surface.

Surfaces satisfying the first condition are called *quasiperiodic*. In the paper [LT23], in collaboration with François Labourie, we introduce a cross-ratio on $\partial_\infty \mathbf{H}^{2,n}$ as well as the notion of quasisymmetric map and quasicircle (see below). We prove

Theorem 4.4 ([LT23])**.** *Let Σ be a complete maximal surface in $\mathbf{H}^{2,n}$. The following are equivalent*

i. Σ *is quasiperiodic,*

ii. *the asymptotic boundary $\partial_\infty \Sigma$ of Σ is a quasicircle,*

iii. Σ *uniformizes with the hyperbolic disk and the uniformization is biLipschitz.*

iv. Σ *is Gromov hyperbolic.*

Moreover, in such a case, the uniformization extends to a quasisymmetric map from $\partial_\infty \mathbf{H}^2$ to $\partial_\infty \Sigma$.

4.3.1 Quasisymmetric maps

We introduce a cross-ratio b on $\partial_\infty \mathbf{H}^{2,n}$ defined

$$\mathrm{b}(a,b,c,d) = \frac{\langle a_0, b_0\rangle \langle c_0, d_0\rangle}{\langle a_0, d_0\rangle \langle b_0, c_0\rangle},$$

where (a,b,c,d) is a quadruple of points in $\partial_\infty \mathbf{H}^{2,n}$ with (a,d) and (b,c) transverse and a_0, b_0, c_0, d_0 are nonzero vectors on the lines a, b, c and d respectively. This cross-ratio naturally generalizes the usual cross-ratio $[\cdot,\cdot,\cdot,\cdot]$ of $\mathbf{P}(\mathbf{R}^2)$.

Recall that a positive triple in $\mathbf{P}(\mathbf{R}^2)$ is a triple of pairwise distinct points, while a triple in $\partial_\infty \mathbf{H}^{2,n}$ is positive if it spans a space of signature $(2,1)$. Fix positive triples κ_0 and τ_0 in $\mathbf{P}(\mathbf{R}^2)$ and $\partial_\infty \mathbf{H}^{2,n}$ respectively. We define

Definition 4.5. A continuous map β from $\mathbf{P}(\mathbf{R}^2)$ to $\partial_\infty \mathbf{H}^{2,n}$ is called

- *positive* if it sends positive triples triples to positive triples.
- *Quasisymmetric* if it is positive and the exists $A, B > 1$ such that for any x, y, z, t in $\mathbf{P}(\mathbf{R}^2)$ we have

$$A^{-1} < [x,y,z,t] < A \implies B^{-1} < \mathrm{b}(\beta(x),\beta(y),\beta(z),\beta(t)) < B\ .$$

- *Normalized* if β maps κ_0 to τ_0.

A *quasicircle* is a positive circle in $\partial_\infty \mathbf{H}^{2,n}$ that admits a quasisymmetric parametrization.

Our notion of quasisymmetric maps from $\mathbf{P}(\mathbf{R}^2)$ to $\partial_\infty \mathbf{H}^{2,n}$ is a natural generalization of quasisymmetric homeomorphisms of the circle.

4.3.2 Sketch of proof

We start by proving a compactness theorem for quasisymmetric maps:

Theorem 4.6 ([LT23]). *For any $A, B > 1$, the space of normalized (A, B)-quasisymmetric maps from $\mathbf{P}(\mathbf{R}^2)$ to $\partial_\infty \mathbf{H}^{2,n}$ is compact.*

The fixed triples κ_0 and τ_0 define "visual metrics" on $\mathbf{P}(\mathbf{R}^2)$ and $\partial_\infty \mathbf{H}^{2,n}$ which allow us to define the Hölder norm on the space of maps from $\mathbf{P}(\mathbf{R}^2)$ to $\partial_\infty \mathbf{H}^{2,n}$. To prove the above theorem, we show that for any (A, B), there exists α in $(0, 1)$ and a bounded set K in $C^{0,\alpha}(\mathbf{P}(\mathbf{R}^2), \partial_\infty \mathbf{H}^{2,n})$ such that any normalized (A, B)-quasisymmetric map is in K. This is done using the conformal geometry of $\partial_\infty \mathbf{H}^{2,n}$.

This compactness theorem easily implies that surfaces bounded by quasicircles are quasiperiodic. The converse implication is much more difficult. To prove it, we study some analogue of horofunctions and Gromov products in the pseudo-hyperbolic setting.

4.3.3 An analogue of the universal Teichmüller space

The *universal Teichmüller space* $\mathcal{T}(\mathbf{H}^2)$, introduced by Bers in [Ber65], is the set of quasisymmetric homeomorphisms of $\mathbf{P}(\mathbf{R}^2)$ fixing the positive triple $\kappa_0 = (0, 1, \infty)$. It is a complex Banach manifold, and composition defines a group structure on $\mathcal{T}(\mathbf{H}^2)$.

Given a Riemann surface structure X on a closed surface Σ of genus at least 2, the Teichmüller space $\mathcal{T}(X)$ of X (seen as the space of quasiconformal deformations of X) naturally embeds in $\mathcal{T}(\mathbf{H}^2)$ as a complex submanifold. Surprisingly, the induced complex structure on $\mathcal{T}(X)$ only depends on Σ and coincides with the usual complex structure on $\mathcal{T}(\Sigma)$. This is the main reason why $\mathcal{T}(\mathbf{H}^2)$ is refered to the *universal* Teichmüller space.

Define $\mathcal{QS}_n$ to be the quotient of the space of quasisymmetric maps from $\mathbf{P}(\mathbf{R}^2)$ to $\partial_\infty \mathbf{H}^{2,n}$ by the action of $\mathsf{G} = \mathsf{PO}(2, n+1)$ by postcomposition. For $n = 0$, we have a natural identification between $\mathcal{QS}_0$ and $\mathcal{T}(\mathbf{H}^2)$. We propose $\mathcal{QS}_n$ as an analogue of the universal Teichmüller space for maximal representations in rank 2. Here are some properties of this space

- The universal Teichmüller space $\mathcal{T}(\mathbf{H}^2)$ acts by precomposition on $\mathcal{QS}_n$. The quotient is identified with the space Λ_n of G-orbits of quasicircles in $\partial_\infty \mathbf{H}^{2,n}$. We can thus see the natural fibration $p : \mathcal{QS}_n \to \Lambda_n$ as a principal $\mathcal{T}(\mathbf{H}^2)$-bundle.

- Since quasicircles come with a preferred parametrization (the one extending the uniformization of the maximal surface), the fibration p described above admits a natural section. This yields a projection $\pi : \mathcal{QS}_n \to \mathcal{T}(\mathbf{H}^2)$.

- the space $\mathcal{QS}_n$ is *universal* in the following sense. Given a closed surface Σ of genus at least 2, an a maximal representation $\rho : \pi_1(\Sigma) \to \mathsf{G}$, we get a boundary map $\beta_\rho : \partial_\infty \Sigma \to \partial_\infty \mathbf{H}^{2,n}$. Uniformizing the ρ-invariant maximal surface gives an identification between $\partial_\infty \pi_1(\Sigma)$ and $\mathbf{P}(\mathbf{R}^2)$ such that β_ρ belongs to $\mathcal{QS}_n$. We thus get an injection ι_Σ for any Σ
$$\iota_\Sigma : \chi^{\max}(\Sigma, \mathsf{PO}(2, n+1)) \longrightarrow \mathcal{QS}_n .$$

- The classical Hitchin map extends to a *universal Hitchin map* from $\mathcal{QS}_n$ to $\mathcal{T}(\mathbf{H}^2) \times H^0_b(\mathcal{K}^4_{\mathbf{H}^2})$, where $H^0_b(\mathcal{K}^4_{\mathbf{H}^2})$ is the space of holomorphic quartic differential on $\mathbf{H}^2$ that are bounded with respect to the hyperbolic metric. Moreover, recent results of Li-Mochizuki [LM20] provides a natural section of this map that extends the usual Hitchin section.

4.4 The general case $\mathbf{H}^{p,q}$

With Andrea Seppi and Graham Smith, we completly solved the asymptotic Plateau problem as stated above:

Theorem 4.7 ([SST23]). *For any (p, q) the boundary map ∂_∞ from $\mathcal{M}(\mathbf{H}^{p,q})$ to $\mathcal{B}(\mathbf{H}^{p,q})$ is a homeomorphism.*

Observe that all the partial solutions to the asymptotic Plateau problems presented before follow from this theorem. Note also that, since p is general, the theory of holomorphic curves cannot be applied.

The strategy of proof is genuinely different from [LTW23]: instead of proving a finite Plateau problem and a limit of finte boundary going to infinity, we directly work with asymptotic boundary. As a result, the paper is much less technical. The price to pay is to work with weighted Sobolev spaces, but this technology also provides some fine results on the behavior of the maximal submanifolds.

Given $\alpha \in (0, 1)$, consider $\mathcal{B}^{2,\alpha}(\mathbf{H}^{p,q})$ to be the subset of nonnegative spheres that are graphs of $C^{2,\alpha}$ maps, and equip this subspace with the $C^{2,\alpha}$-topology. Similarly, denote by $\mathcal{M}^{2,\alpha}(\mathbf{H}^{p,q})$ the space $\partial_\infty^{-1}\left(\mathcal{B}^{2,\alpha}(\mathbf{H}^{p,q})\right)$ of complete maximal p-submanifolds with $C^{2,\alpha}$ boundary. Finally, denote by $\partial_\infty^{2,\alpha}$ the restriction of the boundary map to $\mathcal{M}^{2,\alpha}(\mathbf{H}^{p,q})$. The proof goes as follow

Step 1. The map ∂_∞ is proper. This compactness theorem easily follows from Schauder estimates and a global bound on the second fundamental form proved by Ishihara [Ish88].

Step 2. The map ∂_∞ is injective. The uniqueness is an adaptation of the case $(p, q) = (2, n)$. However, the higher dimensional case requires some extra work.

Step 3. This is, I believe, the real novelty of our approach: we prove that $\partial_\infty^{2,\alpha}$ is open. This step is analogous of the openness of I in Step 1. *ii*. in Subsection 4.2, but we work in the noncompact case. The main issue is that elliptic regularity (an particularly Rellich's theorem) fails in this setting. This is where the theory of weighted Sobolev comes into the game to bypass this issue.

Using the theory of weighted Sobolev spaces provides some interesting information about the geometry of elements in $\mathcal{M}^{2,\alpha}(\mathbf{H}^{p,q})$:

Proposition 4.8. *If M is an element in $\mathcal{M}^{2,\alpha}(\mathbf{H}^{p,q})$, then the norm of the second fundamental form of M at a point y decays as $e^{-d_M(x_0,y)}$ where x_0 is any base point and d_M is the distance induced on M. In particular, if Σ is in $\mathcal{M}^{2,\alpha}(\mathbf{H}^{2,n})$, then its second fundamental form II_Σ is in L^2.*

The L^2-norm of the second fundamental form of a non-compact surface is sometimes called the *renormalized area*. It has a deep connection with the celebrated AdS/CFT correspondence. The study of such minimal surfaces in hyperbolic 3-space has a long history (see [AM10, Bis20]). Our result gives the first examples of maximal surfaces in $\mathbf{H}^{2,n}$ with finite renormalized area.

We also obtain some interesting result about positive P_1-Anosov representations:

Corollary 4.9. Let Γ be a torsion free Gromov hyperbolic group with boundary homeomorphic to $\mathbf{S}^{p-1}$ and ρ a $\mathbf{H}^{p,q}$-convex cocompact representation of Γ into $\mathsf{PO}(p, q+1)$. Then $\rho(\Gamma)$ acts freely and properly discontinuously on a spacelike p-submanifold. In particular, Γ is the fundamental group of a closed p-manifold N_Γ with contractible universal cover.

Also, as in the case $(p, q) = (2, n)$ we obtain

Corollary 4.10. Let Γ be a torsion free Gromov hyperbolic group with boundary homeomorphic to $\mathbf{S}^{p-1}$, let ρ be a $\mathbf{H}^{p,q}$-convex cocompact representation of Γ into $\mathsf{PO}(p, q+1)$ and let Ω_ρ be the domain in $\mathbf{Isot}(\mathbf{R}^{p,q+1})$ constructed by Guichard and Wienhard. Then

- If $p > q$, then Ω_ρ is empty.

- If $p \leq q$, then the quotient $\Omega_\rho/\rho(\Gamma)$ is homeomorphic to a bundle over N_Γ with fiber homeomorphic to $\mathbf{Isot}(\mathbf{R}^{p,q})$.

The first item in the above was already proved, with different techniques, in [GW12].

4.5 Perspectives

Here is a list of questions and projects I believe should be investigated.

i. Our construction of $\mathcal{QS}_n$ in [LT23] is far from being fulfilling. For instance we put the C^0-topology on the different spaces considered in Subsection 4.3.3 . This is enough for our purpose, but it is important to remark that the topology on $\mathcal{T}(\mathbf{H}^2)$ induced by the Banach manifold structure is NOT the C^0-topology, but the *quasisymmetric topology*, which is much more subtle. It would be interesting to define a similar topology on $\mathcal{QS}_n$.

ii. Maximal surfaces in $\mathbf{H}^{2,n}$ give solutions to the Hitchin equations on some naturally defined $\mathsf{PO}(2, n+1)$-Higgs bundle on the underlying Riemann surface. In the equivariant case (for closed surfaces), the Hitchin-Kobayashi correspondence allows to come back: to some special type of $\mathsf{PO}(2, n+1)$-Higgs bundle over a closed Riemann surface, one obtain comlete equivariant maximal surfaces in $\mathbf{H}^{2,n}$ (see Section 3). The non-equivariant picture remains much less understood.

For instance, for quasiperiodic surfaces, one obtains $\mathsf{PO}(2, n+1)$-Higgs bundles on $\mathbf{H}^2$ with a harmonic metric with bounded geometry. It would be interesting to develop a theory of $\mathsf{PO}(2, n+1)$-Higgs bundles over $\mathbf{H}^2$ with "bounded geometry" in some sense. The hope would be to obtain nice moduli spaces with natural Kähler structure. This could be a possible approach to define a Banach space structure on $\mathcal{QS}_n$.

iii. We prove in [SST23] that any complete maximal surface in $\mathbf{H}^{2,n}$ with $C^{2,\alpha}$-boundary has finite renormalized area. However, a recent result of Bishop [Bis20] shows that, in $\mathbf{H}^3$, a minimal surface is renormalizable if and only if its boundary is a quasicircle in the Weil-Petersson class, that is admits a parametrization in the fractional Sobolev space $H^{3/2}$. It would be interesting to understand whether a similar phenomenon appears in our situation.

iv. We prove in [LT23] that quasicircles in $\partial_\infty \mathbf{H}^{2,n}$ are exactly asymptotic boundaries of Gromov hyperbolic complete maximal surface. It would be interesting to characterize positive spheres in $\partial_\infty \mathbf{H}^{p,q}$ whose associated complete maximal p-submanifold is Gromov hyperbolic. This would define an $\mathbf{H}^{p,q}$ analogue of the hyperbolic convex sets of Yves Benoist [Ben03].

v. We prove in [SST23] that $\mathsf{PO}(p, q+1)$ acts cocompactly on the space $\mathcal{M}^\bullet(\mathbf{H}^{p,q})$ of pointed complete maximal p-submanifolds of $\mathbf{H}^{p,q}$. It follows that the sectional curvature of complete maximal p-submanifolds in $\mathbf{H}^{p,q}$ are uniformly bounded. It would be interesting to know this bounds explicitely and see whether a rigidity result similar to Theorem 4.3 exists. Such a result may shed some light on the possible discrete groups that can act properly discontinuously and cocompactly on some elements in $\mathcal{M}^\bullet(\mathbf{H}^{p,q})$.

5 The exceptional pseudosphere of dimension 6

The last rank 2 real Lie group for which Labourie's conjecture hold is the split real form G_2' of the complex Lie group $\mathsf{G}_2(\mathbb{C})$. The group G_2' is identified with the subgroup of $\mathsf{SO}_0(4,3)$ that preserves a special non-integrable almost complex structure J on $\mathbf{H}^{4,2}$ (it is the noncompact analogue of the "famous" non-integrable almost complex structure of the 6-sphere). In a paper in collaboration with Brian Collier [CT23], we define a class of J-holomorphic curves in $\mathbf{H}^{4,2}$, called *alternating*, and propose these as a natural elliptic problem associated to representations in G_2'.

Given a closed oriented surface Σ of genus at least 2, we study pairs (ρ, f) where $\rho : \pi_1(\Sigma) \to \mathsf{G}_2'$ is a morphism and $f : \widetilde{\Sigma} \to \mathbf{H}^{4,2}$ is a ρ-equivariant alternating holomorphic curve. Considering those pairs up to the action of $\mathsf{G}_2' \times \mathrm{Diff}_0(\Sigma)$ defines $\mathcal{H}(\Sigma)$ the *moduli space of equivariant alternating holomorphic curves*. Taking the induced complex structure defines a projection $\pi : \mathcal{H}(\Sigma) \to \mathcal{T}(\Sigma)$. We prove

Theorem 5.1. *The space $\mathcal{H}(\Sigma)$ has the structure of a complex analytic space for which the Mapping Class Group acts holomorphically and the projection π is a surjective holomorphic map. Moreover, the space $\mathcal{H}(\Sigma)$ decomposes as*

$$\mathcal{H}(\Sigma) = \bigsqcup_{d=0}^{6g-6} \mathcal{H}_d(\Sigma) ,$$

where $\mathcal{H}_d(\Sigma)$ has complex dimension $8g-8+d$. For each d, the fiber of $\mathcal{H}_d(\Sigma)$ over a point X in $\mathcal{T}(\Sigma)$ is biholomorphic to

- *a rank $(2d-g+1)$ vector bundle over the $(6g-6-d)$-symmetric power of X, for d in $\{g, \dots, 6g-6\}$,*
- *a bundle over the $H^1(\Sigma, \mathbb{Z}_2)$-cover of the $2d$-symmetric power of X whose fiber is $(\mathbb{C}^{5g-5-d} \setminus \{0\})/\{\pm 1\}$ when $d \in \{0, ..., g-1\}$.*

For $d = 6g-6$, the underlying representations are Hitchin with value in G_2', while for $d = 0$ they are Hitchin in $\mathsf{PSL}(3, \mathbb{R})$ embedded in G_2' as the stabilizer of a positive definite vector in E.

We also prove that the *holonomy map* from $\mathcal{H}(\Sigma)$ to $\chi(\Sigma, \mathsf{G}_2')$ is an immersion.

5.1 Split octonions and G_2'

Denote by $\mathbb{H}$ the *quaternion algebra*, that is $\mathbb{H}$ is the associative $\mathbb{R}$-algebra spanned by $\{1, i, j, k\}$ as a vector space and equipped with the product completely determined by

$$i \cdot j = -j \cdot i = k \ , \ i^2 = j^2 = k^2 = -1 \ .$$

The *split ocotnions* is then the algebra $\mathbb{O}'$ given by $\mathbb{H} \oplus \mathbb{H}$ with product

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 \cdot a_2 + \overline{b_2} \cdot b_1, b_2 \cdot a_1 + b_1 \cdot \overline{a_2}) \ .$$

Observe that $\mathbb{O}'$ is non-associative, but the algebra generated by any 2 elements is associative ($\mathbb{O}'$ is alternative). The group G_2' is the defined as the group of algebra automorphisms of $\mathbb{O}'$.

A split octonion (a, b) is called *imaginary* if a belongs to the span of $\{i, j, k\}$. If E denotes the space of imaginary split octonions, we get a vector space decomposition

$$\mathbb{O}' = \mathrm{span}_{\mathbb{R}}\{1_{\mathbb{O}'}\} \oplus E \ .$$

Since any algebra automorphism of $\mathbb{O}'$ preserves the unit $1_{\mathbb{O}'}$, one easily checks that G_2' preserves E. The algebra structure on $\mathbb{O}'$ is fully encoded in the two following maps

$$\begin{array}{rccc} \langle .,. \rangle : & E \times E & \longrightarrow & \mathbb{R} \\ & (x, y) & \longmapsto & \mathfrak{R}(x \cdot y) \end{array}$$

$$\begin{array}{rccc} \wedge : & E \times E & \longrightarrow & E \\ & (x, y) & \longmapsto & \mathfrak{I}(x \cdot y) \end{array} ,$$

where $\mathfrak{R}$ and $\mathfrak{I}$ correspond to taking the real and imaginary part respectively, that is the first and second projection associated to the decomposition $\mathbb{O}' = \mathrm{span}_{\mathbb{R}}\{1_{\mathbb{O}'}\} \oplus E$. One easily checks that $\langle .,. \rangle$ is a quadratic form of signature $(4,3)$, while the *cross-product* $\wedge$ is skewsymmetric. In particular G_2', which is the subgroup of $\mathsf{GL}(E)$ preserving both $\langle .,. \rangle$ and $\wedge$, is a subgroup of $\mathsf{O}(4,3)$.

5.2 Alternating holomorphic curves and sketch of proof

The cross-product on E defines an almost-complex structure on $\mathbf{H}_+^{4,2}$ we now describe. Recall that

$$\mathbf{H}_+^{4,2} = \{x \in E \mid \langle x, x \rangle = -1\} .$$

Given a point x in $\mathbf{H}^{4,2}$, the linear endomorphism L_x on E defined by $L_x(y) = x \wedge y$ preserves the quadratic form $\langle .,. \rangle$. By skewsymmetry of $\wedge$, the map L_x vanishes on the line spanned by x and thus restricts to an endomorphism J_x of $x^\perp = \mathrm{T}_x\mathbf{H}^{4,2}$. The condition $\langle x, x \rangle = -1$ implies that $J_x^2 = -\mathrm{Id}$. It follows that the corresponding section J of $\mathrm{End}\left(\mathrm{T}\mathbf{H}^{4,2}\right)$ is a (non-integrable) almost-complex structure on $\mathbf{H}^{4,2}$ which is compatible with the pseudo-hyperbolic metric. This realizes G_2' as the automorphism group of $(\mathbf{H}^{4,2}, J)$.

Given an oriented surface S, a J-holomorphic curve $f : S \to \mathbf{H}^{4,2}$ is called *alternating* if it is spacelike and, generically, the image of its second fundamental form is negative definite. In such a case, there is a well-defined rank 2 subbundle $\mathrm{N}S$ of $f^*\mathrm{T}\mathbf{H}^{4,2}$, called the *normal bundle*, which is negative definite and f^*J-invariant, such that the image of the second fundamental form of f lies in $\mathrm{N}S$. The *binormal bundle* $\mathrm{B}S$ is then defined as the orthogonal of $(\mathrm{T}S \oplus \mathrm{N}S)$ in $f^*\mathrm{T}\mathbf{H}^{4,2}$. This defines a *Frenet framing*

$$f^*\mathrm{T}\mathbf{H}^{4,2} = \mathrm{T}S \oplus \mathrm{N}S \oplus \mathrm{B}S .$$

Such a framing defines a lift of f into homogeneous space G_2'/T where T is a maximal compact torus. Such a surface is similar to the cyclic surfaces considered by Labourie in [Lab17].

Using this remark, we associate to every equivariant alternating holomorphic curve a G_2'-Higgs bundle on the underlying Riemann surface. This gives a description of $\mathcal{H}(\Sigma)$ as a family of G_2'-Higgs bundles parametrized by $\mathcal{T}(\Sigma)$. Adapting a construction of Simpson, we get an analytic structure on $\mathcal{H}(\Sigma)$. The topological description is obtained by studying the G_2'-Higgs bundles arising in this way, while the immersion property is proved adapting (and simplifying) the arguments of Labourie [Lab17].

5.3 Perspectives

This work is a first step in the study of alternating holomorphic curves in $(\mathbf{H}^{4,2}, J)$. Here are some ideas of future projects

i. We do not know whether any complete alternating holomorphic curve extends to the boundary $\partial_\infty\mathbf{H}^{4,2}$ (the difficulty comes from the fact that such a holomorphic curve does not saturate the spacelike directions). Proving such a fact could have two consequences

- provide a setting for the Asymptotic Plateau problem,
- give a powerful tool to study the Anosov properties of the corresponding representations.

ii. It should be possible to construct the moduli space of (equivariant or not) alternating holomorphic curves in $\mathbf{H}^{4,2}$ as it is done in symplectic geometry (see for instance [Gro85]). In this setting, the analytic structure on $\mathcal{H}(\Sigma)$ as well as the infinitesimal rigidity of such curves should be easier to prove.

6 Compact components of relative character variety

We now describe [TT21], a paper in collaboration with Nicolas Tholozan and published in *"Epijournal de Géométrie Algébrique"*.

When Σ is a punctured surface, that is is obtained by removing $s > 0$ points from a closed surface $\overline{\Sigma}$, it is natural to consider the *relative character variety*: for each puncture p_i, one fixes a loop δ_i around p_i and a conjugacy class h_i in G. For $h = (h_1, ..., h_s)$, the relative character variety $\chi_h(\Sigma, \mathsf{G})$ is the the subset of $\chi(\Sigma, \mathsf{G})$ consisting of representations ρ such that $\rho(\delta_i)$ belongs to h_i for all i. It turns out that, when G is a semi-simple Lie group, the character variety $\chi(\Sigma, \mathsf{G})$ of a punctured surface has a Poisson structure whose symplectic leaves are the relative character varieties.

The topology of $\chi_h(\Sigma, \mathsf{G})$ is in general very sensitive on the choice of h. With Nicolas Tholozan, we shed light on a surprising new phenomenon on existence of compact components in some relative character variety for non-compact G. We prove

Theorem 6.1. *Let G be either $\mathsf{SU}(p,q)$, $\mathsf{Sp}(2n,\mathbb{R})$ or $\mathsf{SO}^*(2n)$ and Σ the s-punctured sphere with s at least 3. Then there exists a s-tuple h of conjugacy classes of elements in G such that the relative character variety $\chi_h(\Sigma, \mathsf{G})$ has a compact component containing Zariski dense representations. Such a component is symplectomorphic to a decorated quiver variety.*

This theorem generalizes a result of Deroin and Tholozan [DT19] which holds when $\mathsf{G} = \mathsf{PSL}(2,\mathbb{R})$. We are also able to understand some of the dynamical properties of the corresponding representations (sometimes called *supra-maximal representations*). In many aspect, these representations behave like representations into a Lie compact group.

Theorem 6.2. *Let ρ be a representation in a compact component described in the above theorem. Then*

1. *for any Riemann surface structure X on Σ, there is a ρ-equivariant holomorphic map in the symmetric space of G.*
2. *For any simple closed curve γ, the complex eigenvalues of $\rho(\gamma)$ have modulus 1.*

The proof of the above theprems rely on the theory of parabolic Higgs bundles.

6.1 Parabolic Higgs bundles

Classical G-Higgs bundles describe, via the non-Abelian Hodge correspondence, representations of the fundamental group of a closed surface. Parabolic G-Higgs bundles then correspond to representations of the fundamental group of punctured surfaces.

Let X be a Riemann surface with finitely many cusps, $\overline{X}$ be a smooth compactification of X and let $D = \overline{X} \setminus X$ the corresponding divisor (we call elements in D the *punctures*). A *parabolic vector bundle* $\mathcal{E}_\bullet$ over X is a holomorphic vector bundle $\mathcal{E}$ over $\overline{X}$ together with, for each puncture $x \in D$, a choice of a *weighted flag* of $\mathcal{E}_x$, that is

$$\mathcal{E}_x = \mathcal{E}_{x,1} \supsetneq \mathcal{E}_{x,2} \supsetneq \cdots \supsetneq \mathcal{E}_{x,r} \supsetneq \{0\} \ , \ 0 \leq \alpha_1(x) < \alpha_2(x) \cdots < \alpha_r(x) < 1 \ .$$

For $i = 1, ..., r$, set

$$k_i(x) = \mathrm{rank}(\mathcal{E}_{x,i}) - \mathrm{rank}(\mathcal{E}_{x,i+1}) \ \text{(with } \mathrm{rank}(\mathcal{E}_{x,r+1}) = 0) \ \textit{and} \ |\alpha(x)| = \sum_{i=1}^{r} k_i(x)\alpha_i(x) \ .$$

The *parabolic degree* of $\mathcal{E}_\bullet$ is defined by

$$\deg(\mathcal{E}_\bullet) = \deg(\mathcal{E}) + \sum_{x\in D} |\alpha(x)| \ .$$

Example 6.3 (Parabolic line bundles over punctured Riemann sphere). Consider $X = \mathbf{P}^1 \setminus \{p_1, ..., p_s\}$ and $\overline{X} = \mathbf{P}^1$. We want to describe parabolic line bundles over X.

Since for any $d \in \mathbb{Z}$ there is a unique degree d line bundle $\mathcal{O}_{\mathbf{P}^1}(d)$ over $\mathbf{P}^1$, a parabolic line bundle $\mathcal{L}_\bullet$ over X is given by a parabolic structure on some $\mathcal{O}_{\mathbf{P}^1}(d)$. Since any flag in dimension 1 is trivial, such a parabolic structure is just given by the weigth $\alpha(x) \in [0, 1)$ for each $x \in \{p_1, ..., p_s\}$. We will denote such a parabolic bundle $\mathcal{L}_\bullet$ by $\mathcal{O}_{\mathbf{P}^1}\left(d + \sum_{j=1}^s \alpha^j p_j\right)$. The parabolic degree is equal to $\deg(\mathcal{L}_\bullet) = d + \sum_j \alpha^j$.

Definition 6.4. A *(strongly) parabolic Higgs bundle* over X is a pair $(\mathcal{E}_\bullet, \phi)$ where $\mathcal{E}_\bullet$ is a parabolic vector bundle over X and ϕ is a holomorphic section of $\mathcal{K}_{\overline{X}}(D) \otimes \mathrm{End}(\mathcal{E}_\bullet)$ such that for each x in D, the residue of ϕ at x sends $\mathcal{E}_{x,i}$ to $\mathcal{E}_{x,i+1}$ for all i.

Similarly, one can define parabolic G-Higgs bundle for semisimple real Lie groups G. For instance, a parabolic $\mathsf{SU}(p, q)$-Higgs bundle is a parabolic Higgs bundle $(\mathcal{E}_\bullet, \phi)$ such that

$$\det(\mathcal{E}_\bullet) = \mathcal{O}_{\overline{X}} \ , \ \mathcal{E}_\bullet = \mathcal{U}_\bullet \oplus \mathcal{V}_\bullet \ , \ \phi = \begin{pmatrix} 0 & \delta \\ \gamma & 0 \end{pmatrix} .$$

The number $\deg(\mathcal{V}_\bullet) - \deg(\mathcal{U}_\bullet)$ is called the *Toledo invariant*.

A parabolic Higgs bundle $(\mathcal{E}_\bullet, \phi)$ is *stable* if any ϕ-invariant subbundle $\mathcal{F}$ satisfies $\frac{\deg(\mathcal{F}_\bullet)}{\mathrm{rank}(\mathcal{F})} < \frac{\deg(\mathcal{E}_\bullet)}{\mathrm{rank}(\mathcal{E})}$ (where the parabolic structure on $\mathcal{F}$ is the one induced by the one on $\mathcal{E}$). It is called *semi-stable* if the previous inequality is large. Observe that, taking generic parabolic weights, a semi-stable parabolic bundle is automatically stable.

In [Sim90], Simpson extends the non-Abelian Hodge correspondance to the case of punctured surfaces. In this correspondence, stable strongly parabolic Higgs bundles of degree 0 correspond to representation of the punctured surface into $\mathsf{GL}(n, \mathbb{C})$ whose peripheral holonomy has complex eigenvalues of modulus 1. There is moreover an explicit relation between the parabolic weight of the underlying parabolic bundle and the argument of the eigenvalue of the peripheral holonomy. This relation is given by the famous Simpson's table [Sim90, p.720].

6.2 The example of Deroin and Tholozan

We now give a Higgs bundle interpretation of the main result of Deroin and Tholozan. First, we describe strongly parabolic $\mathsf{SU}(1, 1)$-Higgs bundles over $\mathbf{P}^1 \setminus \{p_1, ..., p_s\}$.

Consider two parabolic line bundles $\mathcal{L}_\bullet$ and $\mathcal{M}_\bullet$ where, using the notations of Example 6.3

$$\mathcal{L}_\bullet = \mathcal{O}_{\mathbf{P}^1}\left(l + \sum_{j=1}^s \alpha^j p_j\right) \ , \ \mathcal{M}_\bullet = \mathcal{O}_{\mathbf{P}^1}\left(m + \sum_{j=1}^s \beta^j p_j\right) .$$

The tensor product is then given by

$$\mathcal{L}_\bullet \otimes \mathcal{M}_\bullet = \mathcal{O}_{\mathbf{P}^1}\left(l + m + \sum_{j=1}^s (\alpha^j + \beta^j) p_j\right) .$$

It follows that, if $\mathcal{E}_\bullet = \mathcal{L}_\bullet \oplus \mathcal{M}_\bullet$, then $\det(\mathcal{E}_\bullet) = \mathcal{O}_{\mathbf{P}^1}$ if and only if we have

$$l + m = -s \ \text{ and } \ \forall j \in \{1, ..., s\} \text{ we have } \alpha^j + \beta^j = 1 .$$

Assume furthermore that for each j we have $\alpha^j < \frac{1}{2}$. Then $\alpha^j < \beta^j$ and $(\mathcal{E}_\bullet, \phi)$ is a strongly parabolic $\mathsf{SU}(1, 1)$-Higgs bundle if and only if $\phi = \begin{pmatrix} 0 & \delta \\ \gamma & 0 \end{pmatrix}$ with

$$\begin{cases} \gamma \in H^0\left(\mathbf{P}^1, \mathcal{K}_{\mathbf{P}^1}(D) \otimes \mathrm{Hom}(\mathcal{O}_{\mathbf{P}^1}(l), \mathcal{O}_{\mathbf{P}^1}(m))\right) \\ \delta \in H^0\left(\mathbf{P}^1, \mathcal{K}_{\mathbf{P}^1} \otimes \mathrm{Hom}(\mathcal{O}_{\mathbf{P}^1}(m), \mathcal{O}_{\mathbf{P}^1}(l))\right) \end{cases} .$$

Using $\mathcal{K}_{\mathbf{P}^1} = \mathcal{O}_{\mathbf{P}^1}(-2)$ and $l + m = -s$, we get

$$\begin{cases} \gamma \in H^0\left(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-2-2l)\right) \\ \delta \in H^0\left(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-2+2l+s)\right) \end{cases}.$$

In particular, if $-2+2l+s<0$, then $\delta = 0$. Such a strongly parabolic $\mathsf{SU}(1,1)$-Higgs bundle if stable if and only if $\deg(\mathcal{M}_\bullet) < 0$, that is $-s - l + \sum_{j=1}^{s}(1-\alpha^j) < 0$.

Recalling that $\alpha^j < \frac{1}{2}$ one sees that it is possible to satisfy both conditions. For instance, for s odd, let

$$l = -\frac{s-1}{2} \text{ and } \sum_{j=1}^{s} \alpha^j > \frac{s-1}{2}.$$

This condition is topological: it only depends on the weight α^j and the Toledo invariant of the Higgs bundle. Such a Higgs bundle is then fully described by $\gamma \in H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(s-3)\big)$, so the corresponding component is biholomorphic to $\mathbf{P}(\mathbb{C}^{s-2})$.

6.3 Sketch of proof

The proof in the general case will follow the same idea as the example of Deroin and Tholozan: one look for strongly parabolic $\mathsf{SU}(p,q)$-Higgs bundles $\left(\mathcal{U}_\bullet \oplus \mathcal{V}_\bullet, \begin{pmatrix} 0 & \delta \\ \gamma & 0 \end{pmatrix}\right)$ such that:

1. $\mathcal{V}_\bullet$ has negative parabolic degree (or equivalently, $\mathcal{U}_\bullet$ has positive parabolic degree),
2. the degree of $\mathcal{U}$ is negative enough to force δ to be equal to zero.

To facilitate the second item, one can for instance take the parabolic weights of $\mathcal{U}_\bullet$ to be less than the ones of $\mathcal{V}_\bullet$. This will impose that $\delta \in H^0\left(\mathbf{P}^1, \mathcal{K}_{\mathbf{P}^1} \otimes \mathrm{Hom}(\mathcal{V}, \mathcal{U})\right)$.

We prove that, for some very specific values of the degree of $\mathcal{U}$ and parabolic weights, it is possible to satisfy both conditions above. It follows that the corresponding Higgs bundles are all variation of Hodge structures, so the Hitchin map sends the entire component to 0. Properness of the Hitchin map implies compactness of the corresponding component.

When describing the Higgs bundles in the component, we realized that the underlying bundles $\mathcal{U}$ and $\mathcal{V}$ respectively have the form $\mathcal{O}_{\mathbf{P}^1}(u)^{\oplus p}$ and $\mathcal{O}_{\mathbf{P}^1}(v)^{\oplus q}$ for some fixed integers u and v. Choosing a basis of $H^0\big(\mathbf{P}^1, \mathcal{K}_{\mathbf{P}^1}(D) \otimes \mathrm{Hom}(\mathcal{O}_{\mathbf{P}^1}(u), \mathcal{O}_{\mathbf{P}^1}(v)\big)$, the component in the Higgs bundle moduli space is fully described by:

- γ, which is $(s-2)$-tuple of linear maps from $\mathbb{C}^p$ to $\mathbb{C}^q$ (here $(s-2)$ corresponds to the dimension of $H^0(\mathbf{P}^1, \mathcal{K}_{\mathbf{P}^1}(D)))$,
- the parabolic structure of $\mathcal{U}$ and $\mathcal{V}$ which is given by the choice of $2s$-flags,
- we have to quotient by the action of the automorphisms of $\mathcal{U}$ and $\mathcal{V}$, that is, by $\mathsf{GL}(p,\mathbb{C}) \times \mathsf{GL}(q,\mathbb{C})$.

This quotient is what algebraic geometer call a (decorated) quiver variety. Studying the corresponding stability conditions, we prove that the corresponding compact component is biholomorphic to this decorated quiver variety.

Finally, to obtain the dynamical properties of a given representation ρ in such component, observe that the condition $\delta = 0$ implies that the underlying harmonic map is holomorphic. Moreover, the condition $\delta = 0$ is independent on the choice of the Riemann surface structure: so for all Riemann surface structure on the s-punctured sphere Σ, there exists a ρ-equivariant holomorphic map f_X from $\mathbf{H}^2$ to $\mathbf{Sym}(\mathsf{SU}(p,q))$. But

Sym($\mathsf{SU}(p,q)$) is Kobayashi hyperbolic, thus equipped with the Kobayashi metric (which is biLipschitz to the Killing metric), the map f_X is contracting. Given a simple closed curve γ on Σ, one can find hyperbolic metric such that the length of γ is as small as one wants. It implies that the translation length of $\rho(\gamma)$ on **Sym**($\mathsf{SU}(p,q)$) is zero, so $\rho(\gamma)$ has all eigenvalues of modulus 1.

6.4 Perspective

Our result gives a large family of examples of compact components of relative character varieties for some Hermitian Lie groups G. However, the list is far from being exhaustive. Here is a list of important questions

- Does such a phenomenon exists only for the punctured sphere? The answer is known to be yes when $\mathsf{G} = \mathsf{PSL}(2,\mathbb{R})$ by the work of Mondello [Mon16], but the global picture is still mysterious.
- Does compact components exists only when the underlying Lie group is Hermitian?
- In his fundamental work on rigid local systems, Katz [Kat96] defined an operation, called *Katz's middle convolution*, giving homeomorphisms between relative character variety of the punctured spheres into general linear groups *of different rank*. Experts seem to believe (see for instance [Sim09]) that his operation preserves variations of Hodge structure. If this is the case, one might expect that the component we describe with Nicolas are obtained by applying Katz' middle convolution to relative character varieties in the compact group $\mathsf{SU}(n)$.

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