Bifurcations of viscous boundary layers in the half space

D. Bian[1], E. Grenier[1], G. Iooss[2]

Abstract

It is well-established that shear flows are linearly unstable provided the viscosity is small enough, when the horizontal Fourier wave number lies in some interval, between the so-called lower and upper marginally stable curves. In this article, we prove that, under a natural spectral assumption, shear flows undergo a Hopf bifurcation near their upper marginally stable curve. In particular, close to this curve, there exists space periodic traveling waves solutions of the full incompressible Navier-Stokes equations. For the linearized operator, the occurrence of an essential spectrum containing the entire negative real axis causes certain difficulties which are overcome. Moreover, if this Hopf bifurcation is super-critical, these time and space periodic solutions are linearly and nonlinearly asymptotically stable.

1 Introduction

In this paper, we consider the incompressible Navier-Stokes equations in the half space $\Omega = \mathbb{R} \times \mathbb{R}_+$

$$\partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu - \nu \Delta u^\nu + \nabla p^\nu = f^\nu, \tag{1}$$

$$\nabla \cdot u^\nu = 0, \tag{2}$$

together with the Dirichlet boundary condition

$$u^\nu = 0 \quad \text{when} \quad y = 0 \tag{3}$$

and address the classical question of the linear and nonlinear stability of shear flows for these equations.

A shear flow is a stationary solution of (1,2,3) of the form

$$U(y) = (U_s(y), 0), \qquad f^\nu = (-\nu \Delta U_s, 0),$$

where $U_s(y)$ is a smooth function, vanishing at $y = 0$ and converging exponentially fast at infinity to some constant U_+.

[1]School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China. Email: biandongfen@bit.edu.cn and emmanuelgrenier@bit.edu.cn

[2]Laboratoire J.A.Dieudonné, I.U.F., Université Côte d'Azur, Parc Valrose 06108 Nice Cedex 02, France

The question of the linear stability of such shear flows is one of the most classical question in Fluid Mechanics, which has been intensively studied since the pioneering work of Lord Rayleigh at the end of the 19^{th} century. The situation has been progressively understood in the 20^{th} century thanks to the works of L. Prandtl, Orr, Sommerfeld, Schlichting and C.C. Lin to only quote a few names. We in particular refer to [8, 15, 16] for a detailed presentation of the approach in physics.

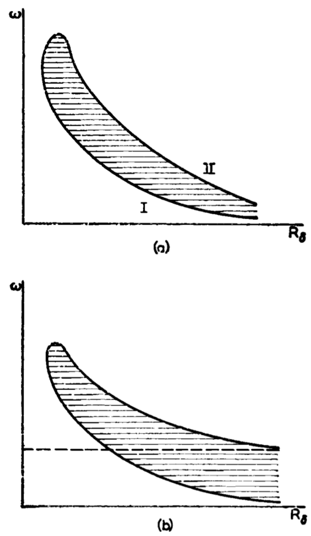

Figure 1: Stability of shear flows: horizontally, the Reynolds number (inverse of ν), vertically the horizontal wave length α of the perturbation. In grey, the unstable area. Top sub-figure: Euler-stable profile. Bottom sub-figure: Euler-unstable profile. From [14].

Roughly speaking, shear flows can be classified into two categories:

- Some are spectrally unstable for the Euler equations, namely in the case $\nu = 0$ (see the bottom sub-figure of Figure 1). According to Rayleigh criterium, such shear flows have inflection points.

- Other are spectrally stable for the Euler equations, which is the case if they have no inflection points, for instance if they are convex or concave (see the top sub-figure of Figure 1).

In both cases, provided the Reynolds number is large enough, namely provided the viscosity ν is small enough, the shear layer is linearly unstable with respect to perturbations whose horizontal wavenumber α lies in some interval $[\alpha_-(\nu), \alpha_+(\nu)]$ for some increasing functions $\alpha_\pm(\nu)$. The function $\alpha_-(\nu)$ (respectively $\alpha_+(\nu)$) is called the lower (respectively upper) marginal stability curve.

Let us fix $\nu = \nu_0$ small enough, which corresponds to a large enough Reynolds number. Then if $\alpha > \alpha_+(\nu_0)$, we note that the harmonics of α, namely all the positive multiple of α, remain larger than the basic harmonic $\alpha_+(\nu_0)$ and thus are all stable. As a consequence, we expect that the shear layer U is linearly and nonlinear stable with respect to perturbations which are $2\pi/\alpha$ periodic.

However, if $\alpha < \alpha_+(\nu_0)$ and close to $\alpha_+(\nu_0)$, a linear instability appears and, following [16] (section 5.3), we expect a bifurcation: small perturbations will grow exponentially till the nonlinear term saturates them, leading to a bifurcated solution, which is a traveling wave of small amplitude.

The aim of this article is to investigate mathematically this physical scenario. We first formalize the spectral situation depicted on Figure 1, which leads to the following set of assumptions and then study the bifurcation arising at $\alpha_+(\nu)$.

We take the Fourier transform in the x variable, with dual Fourier variable α. We denote by $Sp_{\alpha,\nu}$ the spectrum of the linearized Navier-Stokes equations after this Fourier transform, and by $\lambda(\alpha,\nu) \in Sp_{\alpha,\nu}$ the eigenvalue (if it exists and is unique) with the largest real part.

In this article, we assume that, when the viscosity ν is small enough, there exist two smooth and increasing functions $\alpha_\pm(\nu)$, defined for ν small enough, with $\alpha_-(\nu) < \alpha_+(\nu)$, such that

(A1) for $\alpha_-(\nu) < |\alpha| < \alpha_+(\nu)$, $\lambda(\alpha,\nu)$ exists, is unique, and satisfies

$$\Re\lambda(\alpha, \lambda) > 0.$$

Moreover, λ is simple, with corresponding eigenvector $\zeta(x, y)$ of the form

$$\zeta(x, y) = \nabla^\perp \Big[e^{i\alpha x} \psi_{\alpha,\nu}(y) \Big],$$

for some smooth stream function $\psi_{\alpha,\nu}(y)$ which is exponentially decreasing at infinity,

(A2) for $|\alpha| > \alpha_+(\nu)$, close to $\alpha_+(\nu)$, $\lambda(\alpha,\nu)$ exists, is unique and satisfies

$$\Re\lambda(\alpha, \nu) < 0,$$

(A3) for $|\alpha| = \alpha_+(\nu)$, $\lambda(\alpha,\nu)$ is well defined and purely imaginary. Moreover, defining $\nu_0(\alpha)$ by $\alpha = \alpha_+(\nu_0)$

$$(\partial_\nu \Re\lambda)(\alpha,\nu)|_{\nu=\nu_0(\alpha)} > 0.$$

The mathematical proof of the existence of an unstable mode for some values of α has been initiated in [9], where it is established that any shear flow is spectrally unstable for the incompressible Navier-Stokes equations provided the viscosity is small enough, a first result later improved by [6] and [2]. In this last paper, we proved that strictly convex or concave flows satisfy the set of assumptions (A1), (A2) and (A3) under the additional assumptions $U_s'(0) \neq 0$ and $U_+ \neq 0$.

When we cross the upper marginally stable curve $\alpha = \alpha_+(\nu)$, two eigenvalues cross the imaginary axis (corresponding to $\pm\alpha$): we are exactly in the situation of an Hopf bifurcation. This paper focuses on the study of this bifurcation.

Let us now state our main result in an informal way. We refer to Theorems 18, 22 and 27 for detailed statements, including precise definitions of the function spaces.

Theorem 1. *Let us assume (A1), (A2) and (A3). Let $\nu_0 > 0$ and let $\alpha = \alpha_+(\nu_0)$, let us consider perturbations which are $2\pi/\alpha$ periodic in x. Then, at $\nu = \nu_0$, the system undergoes a Hopf bifurcation:*

- *for $\nu < \nu_0$, the shear flow U is linearly and nonlinearly stable,*
- *if the Hopf bifurcation is subcritical, there exists a time and space periodic solution of (1,2,3) for $\nu < \nu_0$ sufficiently close to ν_0, and this time and space periodic solution is linearly unstable,*
- *if the Hopf bifurcation is supercritical, there exists a time and space periodic solution of (1,2,3) for $\nu > \nu_0$, sufficiently close to ν_0, and this time and space periodic solution is linearly and nonlinearly stable.*

There exist many works dealing with Hopf bifurcations on Navier-Stokes equations (see [11] for detailed references), however, all these works consider bounded domains or periodic domains (for Couette-Taylor flow [7] or Bénard-Rayleigh convection for example). Recently, some works deal with parallel flows, like for example Poiseuille flow in [5] at section 6.4, where a sub-critical Hopf bifurcation of a traveling wave is completely treated (see also [1] for the Hopf bifurcation of a shear flow between two plates).

None of these works deal with an half plane domain ($x \in \mathbb{R}, y \in \mathbb{R}^+$), assuming periodicity only in the x direction, the y direction being unbounded, which is a problem of serious physical interest. As the domain is unbounded in the y direction, the essential spectrum of the linearized operator goes up to zero and there is no spectral gap. The classical tools of bifurcation theory can not be applied and we have to design a new approach. Moreover, in cases where the shear flow U is stable, perturbation do not decay exponentially fast, but only like polynomially fast, like t^{-1}.

The super-criticality or sub-criticality of the Hopf bifurcation depends on the sign of a coefficient which unfortunately can only be numerically studied. In the case of exponential profiles $U_s(y) = 1 - e^{-y}$, numerical evidence [3] indicates that the Hopf bifurcation is supercritical.

The paper is organized as follows: in section 2, we introduce the various function spaces, the various linear operators and investigate their spectra and their related semi-groups. In section 3 we prove the stability of the shear flow U above the upper marginal stability curve. In section 4 we investigate the bifurcation and prove the existence of a time periodic solution to the Navier Stokes equations. In section 5, we focus on the linear and non linear stability of this periodic solution. The proofs of the various lemmas are detailed in the Appendix.

Notations

In all this paper, η will be a sufficiently small positive number. We define the one-dimensional periodic torus of period $2\pi/\alpha$ to be $\mathbb{T} = \mathbb{R}/(2\pi/\alpha)\mathbb{Z}$. We define the Banach space $C^0_\eta(\mathbb{R}^+)$ by its norm

$$\|u\|_{C^0_\eta} = \sup_{y \in \mathbb{R}^+} |u(y) e^{\eta y}|.$$

The components of a two dimensional vector field v will be denoted by $v = (v^x, v^y)$. We denote by D the derivative with respect to y. We denote by $u_n(y)$ the n^{th} Fourier component of a function $u(x, y)$.

By a slight abuse of language, we will also denote the vector $(U_+, 0)$ by U_+.

2 Preliminaries

2.1 Function spaces

We first observe that if $v = (v^x, v^y)$ is a two dimensional divergence free vector field which is independent on x and vanishes at $y = 0$, then, using the incompressibility condition, $v^y = 0$.

Let us define the Banach space

$$\mathcal{X}_\eta = \mathring{L}^2_\eta \oplus \mathring{L}^\infty,$$

so that $v \in \mathcal{X}_\eta$ can be written as

$$\begin{aligned} v &= \widetilde{v} + v_0, \ \widetilde{v} \in \mathring{L}^2_\eta, \\ v_0 &= (v_0^x, 0), \ v_0^x \in L^\infty(\mathbb{R}^+), \end{aligned}$$

where we defined

$$\mathring{L}^2_\eta = \Big\{ \widetilde{v} \in L^2_\eta; \ \nabla \cdot \widetilde{v} = 0, \ \widetilde{v}^y|_{y=0} = 0 \Big\},$$

with

$$L^2_\eta = \Big\{ \widetilde{u} \in [L^2[\mathbb{T}, C^0_\eta(\mathbb{R}^+)]]^2; \ \int_0^{2\pi/\alpha} \widetilde{u}(x,y)\, dx = 0, \ \|\widetilde{u}\|_{L^2_\eta} < \infty \Big\},$$

$$\mathring{L}^\infty = \Big\{ (v_0^x, 0); \ v_0^x \in L^\infty(\mathbb{R}^+) \Big\},$$

$$\begin{aligned} \|\widetilde{u}\|^2_{L^2_\eta} &= \sum_{|n|\geq 1} \|u_n\|^2_{C^0_\eta}, \\ \|v_0\|_{\mathring{L}^\infty} &= \|v_0^x\|_{L^\infty}. \end{aligned}$$

Note that we do not assume any decay in y on v_0. By analogy, we define the Banach spaces H^2_η, $\mathring{H}^2_\eta$ and

$$\mathcal{Z}_\eta = \mathring{H}^2_\eta \oplus \mathring{W}^{2,\infty},$$

where

$$\mathring{H}^2_\eta = \Big\{ \widetilde{v} \in H^2_\eta; \ \nabla \cdot \widetilde{v} = 0, \ \widetilde{v}|_{y=0} = 0, \ \|\widetilde{v}\|_{\mathring{H}^2_\eta} < \infty \Big\},$$

$$H^2_\eta = \Big\{ \widetilde{u} \in [H^2[\mathbb{T}, C^2_\eta(\mathbb{R}^+)]]^2; \ \int_0^{2\pi/\alpha} \widetilde{u}(x,y)\, dx = 0, \ \|\widetilde{u}\|_{H^2_\eta} < \infty \Big\},$$

$$\mathring{W}^{2,\infty} = \Big\{ (v_0^x, 0);\ v_0^x|_{y=0} = 0,\ v_0^x \in W^{2,\infty}(\mathbb{R}^+) \Big\},$$

with the corresponding norms

$$\|\widetilde{v}\|^2_{H^2_\eta} = \sum_{|n|\geq 1} \|D^2 v_n\|^2_{C^0_\eta} + n^2 \|Dv_n\|^2_{C^0_\eta} + n^4 \|v_n\|^2_{C^0_\eta}$$

and

$$\|v_0\|_{\mathring{W}^{2,\infty}} = \|v_0^x\|_{W^{2,\infty}},$$

so that $\mathcal{Z}_\eta \hookrightarrow \mathcal{X}_\eta$ densely.

We also define the intermediary Banach space

$$\mathcal{Y}_\eta = \mathring{H}^1_\eta \oplus \mathring{W}^{1,\infty} \hookrightarrow \mathcal{X}_\eta,$$

with

$$\begin{aligned} \mathring{H}^1_\eta \quad = \quad & \Big\{ \widetilde{v} \in [H^1[\mathbb{T}, C^1_\eta(\mathbb{R}^+)]]^2; \\ & \int_0^{2\pi/\alpha} \widetilde{v}(x,y)\, dx = 0,\ \nabla \cdot \widetilde{v} = 0, \widetilde{v}^y|_{y=0} = 0,\ \|\widetilde{v}\|_{\mathring{H}^1_\eta} < \infty \Big\}, \end{aligned}$$

$$\mathring{W}^{1,\infty} = \Big\{ (v_0^x, 0);\ v_0^x \in W^{1,\infty}(\mathbb{R}^+) \Big\}$$

and the norm

$$\|\widetilde{v}\|^2_{\mathring{H}^1_\eta} = \sum_{|n|\geq 1} n^2 \|v_n\|^2_{C^0_\eta} + \|Dv_n\|^2_{C^0_\eta}.$$

2.2 The Helmholtz decomposition in $\mathcal{X}_\eta$

Let $u \in L^2_\eta \oplus (L^\infty)^2$. Then u can be decomposed in $u = \widetilde{u} + u_0$ with $\widetilde{u} \in L^2_\eta$ and $u_0 \in [L^\infty(\mathbb{R}^+)]^2$. We further decompose $\widetilde{u}$ in

$$\widetilde{u} = \widetilde{v} + \nabla \widetilde{\phi}$$

with $\widetilde{v} \in \mathring{L}^2_\eta$ and

$$\Delta \widetilde{\phi} = \nabla \cdot \widetilde{u}, \quad \frac{\partial \widetilde{\phi}}{\partial y}|_{y=0} = \widetilde{u}^y|_{y=0} \in L^2(\mathbb{T}),$$

where $\nabla \cdot \widetilde{u}$ is understood in the distribution sense. The component u_0 can be decomposed as

$$u_0 \quad = \quad \begin{pmatrix} v_0^x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ D\phi_0 \end{pmatrix},\ \begin{pmatrix} v_0^x \\ 0 \end{pmatrix} \in \mathring{L}^\infty,$$

with $D\phi_0 = u_0^y \in L^\infty(\mathbb{R}^+)$, so that $\widetilde{v} + v_0 \in \mathcal{X}_\eta$. We note that v_0 and ϕ_0 are independent of x. We now define the projection Π on divergence free vector fields by

Lemma 2. *Let $\eta < \alpha/2$. Let $u \in L^2_\eta \oplus (L^\infty)^2$, which may be decomposed in*

$$u = \widetilde{u} + u_0,\ \ \widetilde{u} \in L^2_\eta,\ \ u_0 \in (L^\infty(\mathbb{R}^+))^2.$$

We can further decompose $\widetilde{u}$ in $\widetilde{u} = \widetilde{v} + \widetilde{\nabla\phi}$ where $\widetilde{v} \in \mathring{L}^2_\eta$. We define the projection Πu by

$$\Pi u = \Pi \widetilde{u} + (\Pi u)_0$$

where $\Pi\widetilde{u} = \widetilde{v}$ and $(\Pi u)_0 = (v_0^x, 0)$. The projection Π is bounded from L^2_η to $\mathring{L}^2_\eta$ and from $(L^\infty)^2$ to $\mathring{L}^\infty$

The projector Π is highly classical, however our function spaces are not the usual ones. We thus detail the proof of this Lemma in Appendix 6.1.

2.3 Linear operators

We define the linear operators $\boldsymbol{L}_{(\nu)}, \boldsymbol{L}^{(0)}_{(\nu)}, \boldsymbol{L}^{(1)}$ by

$$\begin{aligned} \boldsymbol{L}_{(\nu)} &= \boldsymbol{L}^{(0)}_{(\nu)} + \boldsymbol{L}^{(1)}, \\ \boldsymbol{L}^{(0)}_{(\nu)} v &= \nu \Pi \Delta v, \quad \boldsymbol{L}^{(1)} v = -\Pi\Big[(U\cdot\nabla)v + (v\cdot\nabla)U\Big], \end{aligned}$$

and further decompose $\boldsymbol{L}^{(0)}_{(\nu)}$ in

$$\boldsymbol{L}^{(0)}_{(\nu)} = \boldsymbol{L}^{(0)} + (\nu - \nu_0)\boldsymbol{L}'$$

with

$$\boldsymbol{L}^{(0)} v = \nu_0 \Pi \Delta v,\ \ \boldsymbol{L}' v = \Pi\Delta v.$$

Note that $\boldsymbol{L}'$ and $\boldsymbol{L}^{(0)}_{(\nu)}$ are Stokes operators. We easily obtain the following Lemma.

Lemma 3. *Assume that $U \in C^1(\mathbb{R}^+)$ with*

$$|U(y)| + |DU(y)| \le M < \infty, \quad y \in \mathbb{R}^+,$$

then, the linear operator $\boldsymbol{L}^{(1)}$ cancels on $\mathring{W}^{1,\infty}$. For $\eta < \alpha/2$ and $v = \widetilde{v} + v_0 \in \mathcal{Z}_\eta$ with $\widetilde{v} \in \mathring{H}^2_\eta$, $v_0 \in \mathring{W}^{2,\infty}$, then

$$\boldsymbol{L}^{(1)} v = \boldsymbol{L}^{(1)}\widetilde{v} \in \mathring{H}^1_\eta.$$

Moreover for $u \in \widetilde{u} + u_0 \in \mathcal{Y}_\eta$ *with* $\widetilde{u} \in \mathring{H}^1_\eta$, $u_0 \in \mathring{W}^{1,\infty}$, *then*

$$\boldsymbol{L}^{(1)} u = \boldsymbol{L}^{(1)} \widetilde{u} \in \mathring{L}^2_\eta.$$

There exists $C > 0$ *such that we have the estimates*

$$\|\boldsymbol{L}^{(1)} v\|_{\mathring{H}^1_\eta} \leq C \|\widetilde{v}\|_{\mathring{H}^2_\eta},$$

$$\|\boldsymbol{L}^{(1)} u\|_{\mathring{L}^2_\eta} \leq C \|\widetilde{u}\|_{\mathring{H}^1_\eta}.$$

The main observation is that

$$(U \cdot \nabla) v + (v \cdot \nabla) U = U_s \partial_x v + \begin{pmatrix} v^y D U_s \\ 0 \end{pmatrix},$$

cancels when v is independent of x and $v^y = 0$, which is the case in $\mathring{W}^{1,\infty}$.

2.4 Study of $\boldsymbol{L}^{(0)}_{(\nu)}$ and $\boldsymbol{L}_{(\nu)}$

We now turn to the study of the spectrum and of the resolvent of the operator $\boldsymbol{L}^{(0)}_{(\nu)}$. This operator is the classical Stokes operator, however our spaces are not the usual ones, thus we have to restart its study from the beginning.

Lemma 4. *The spectrum of* $\boldsymbol{L}^{(0)}_{(\nu)} = \Pi \nu \Delta$ *acting in* $\mathring{L}^\infty$ *is only formed by an essential spectrum which equals*

$$Sp|_{\mathring{L}^\infty} \boldsymbol{L}^{(0)}_{(\nu)} = (-\infty, 0] = \mathbb{R}^-,$$

and we have the estimates

$$\left\| (\boldsymbol{L}^{(0)}_{(\nu)} - \lambda \mathbb{I})^{-1} \right\|_{\mathcal{L}(\mathring{L}^\infty)} \leq \begin{cases} \sqrt{2} |\lambda|^{-1} & \text{when } \Re \lambda \geq 0, \\ (|\lambda| \cos \theta / 2)^{-1} & \text{when } \lambda = |\lambda| e^{i\theta} \text{ with } \Re \lambda < 0. \end{cases}$$

For $\lambda \in (-\infty, 0]$, *the range of the operator* $\boldsymbol{L}^{(0)}_{(\nu)} - \lambda \mathbb{I}$ *is not closed in* $\mathring{L}^\infty$.

Now consider the linear operator $\boldsymbol{L}^{(0)}_{(\nu)} = \Pi \nu \Delta$ *acting in* $\mathring{L}^2_\eta$. *For any* $\delta > 0$ *such that* $0 < \delta < \pi/6$, *and* $\lambda \in \mathbb{C}$ *such that*

$$\begin{aligned} 0 &\leq \arg(\lambda + \nu \alpha^2) \leq \frac{2\pi}{3} - \delta, \\ 0 &< \varepsilon_0 \leq |\lambda + \nu \alpha^2|, \end{aligned}$$

there exists $C > 0$ *such that, for* $\eta > 0$ *small enough,*

$$\left\| (\boldsymbol{L}^{(0)}_{(\nu)} - \lambda \mathbb{I})^{-1} \right\|_{\mathcal{L}(\mathring{L}^2_\eta)} \leq \frac{C}{|\lambda + \nu \alpha^2|}. \tag{4}$$

The spectrum of $\boldsymbol{L}^{(0)}_{(\nu)}$ on $\mathring{L}^2_\eta$ is thus contained in an angular sector of $\mathbb{C}$ defined by

$$\frac{2\pi}{3} \le \arg(\lambda + \nu\alpha^2) \le \pi.$$

The proof of this Lemma is detailed in Appendix 6.2.

With this inequality, λ is only on the upper complex plane ? Shouldn't it be

$$\frac{2\pi}{3} \le \arg(\lambda + \nu\alpha^2) \le \frac{4\pi}{3},$$

with a similar change in the Lemma ?

$$-2\pi/3 + \delta \le \arg(\lambda + \nu\alpha^2) \le 2\pi/3 - \delta.$$

To be changed everywhere.

Remark 5. *If we chose to include the exponential decay $e^{-\eta y}$ for the 0-Fourier mode, this would imply that*

$$Sp|_{\mathring{L}^\infty_\eta} \boldsymbol{L}^{(0)}_{(\nu)} \subset \Big\{ \lambda \in \mathbb{C};\ |\Im\lambda| \le 4\nu\eta^2(\nu\eta^2 - \Re\lambda) \Big\}$$

which is a parabolic region centered on, and containing, the negative real axis, bounded on the right side by $\Re\lambda = \nu\eta^2$.

Moreover, we have the following corollaries.

Lemma 6. *The linear operator $\boldsymbol{L}^{(0)}_{(\nu)} = \Pi\nu\Delta$ acting in $\mathring{L}^2_\eta$ has a bounded inverse in $\mathcal{L}(\mathring{L}^2_\eta, \mathring{H}^2_\eta)$. In $\mathring{H}^2_\eta$, the norms $\|\cdot\|_{\mathring{H}^2_\eta}$ and $\|\boldsymbol{L}^{(0)}_{(\nu)}\cdot\|_{\mathring{L}^2_\eta}$ are equivalent.*

This Lemma is a direct consequence of Lemma 4 since

$$\|\widetilde{v}\|_{\mathring{H}^2_\eta} \le M\|\boldsymbol{L}^{(0)}_{(\nu)}\widetilde{v}\|_{\mathring{L}^2_\eta} \le Mc\|\Delta\widetilde{v}\|_{\mathring{L}^2_\eta} \le Mc\|\widetilde{v}\|_{\mathring{H}^2_\eta}.$$

Lemma 7. *There exists $M > 0$ such that the spectrum of $\boldsymbol{L}_{(\nu)}$ is included in*

$$\Big\{|\lambda + \nu\alpha^2| \le M\Big\} \cup \Big\{\frac{2\pi}{3} \le \arg(\lambda + \nu\alpha^2) \le \pi\Big\} \cup (-\infty, 0].$$

In the domain, $|\lambda + \nu\alpha^2| > M$ and $0 \le \arg(\lambda + \nu\alpha^2) \le 2\pi/3 - \delta$ for some positive δ, we have

$$\Big\|(\boldsymbol{L}_{(\nu)} - \lambda\mathbb{I})^{-1}\Big\|_{\mathcal{L}(\mathring{L}^2_\eta)} \le \frac{C}{|\lambda|}. \tag{5}$$

We detail the proof of this lemma in Appendix 6.3.

We now decompose $L^{(1)}$ in

$$L^{(1)} = L^{(1,0)} + L^{(1,c)},$$

where

$$L^{(1,0)}v = -\Pi\Big[(U_+ \cdot \nabla)v\Big], \qquad L^{(1,c)}v = -\Pi\Big[\Big((U - U_+)\cdot\nabla\Big)v + (v\cdot\nabla)U\Big].$$

Lemma 8. *Assume that $U \in C^2(\mathbb{R}^+)$ satisfies*

$$|U(y) - U_+| + |DU(y)| + |D^2U(y)| \le ce^{-\gamma y},\ \ y \in \mathbb{R}^+,$$

then the linear operator $\boldsymbol{L}^{(1,c)}$ is relatively compact with respect to $\boldsymbol{L}^{(0)}_{(\nu)}$ acting in $\mathcal{X}_\eta$.

This Lemma is proved in Appendix 6.4 (see [13] for the definition of relative compactness of an operator with respect to another operator).

Lemma 9. *The essential spectrum of $L_{(\nu)}$ is the essential spectrum of $L^{(0)}_{(\nu)} + L^{(1,0)}$. It is the half line $(-\infty, 0]$ if restricted to $\mathring{L}^\infty$. The rest of the essential spectrum, for the operator restricted to $\mathring{L}^2_\eta$, is included in the region Σ_{U_+} defined by*

$$\Re\lambda < -\nu(\alpha^2 - \eta^2),$$

and either

$$\Re\lambda < -\frac{\nu(\Im\lambda)^2}{U_+^2 + 4\nu^2\eta^2} + \nu\eta^2$$

or

$$2\pi/3 \le \arg(\lambda + \nu\alpha^2) \le \pi.$$

The rest of the spectrum of $L_{(\nu)} = L^{(0)}_{(\nu)} + L^{(1)}$ is uniquely formed by isolated eigenvalues with finite multiplicities at finite distances of 0. See Figure 2.

Proof. The localization of the essential spectrum of $L^{(0)}_{(\nu)} + L^{(1,0)}$ is proved in Appendix 6.5. The result on the spectrum of $\boldsymbol{L}_{(\nu)}$ comes from the fact that it is the addition to $\boldsymbol{L}^{(0)}_{(\nu)} + L^{(1,0)}$ of the linear operator $\boldsymbol{L}^{(1,c)}$ which is a relatively compact perturbation, using the fact that $U(y) - U_+$ tends towards 0 exponentially as $y \to \infty$. We then apply theorem 5.35 in [13]. □

A corollary of the Lemmas above is the following estimate for λ far enough on the imaginary axis, which will be useful in the study of bifurcating periodic solutions.

Lemma 10. *Assume $\omega > 0$ and $\eta > 0$ small enough, then there exists $C > 0$ and $N > 0$ such that for $|n| > N$*

$$\left\| (in\omega - \boldsymbol{L}_{(\nu)})^{-1} \right\|_{\mathcal{L}(\mathcal{X}_\eta)} \leq \frac{C}{|n|}.$$

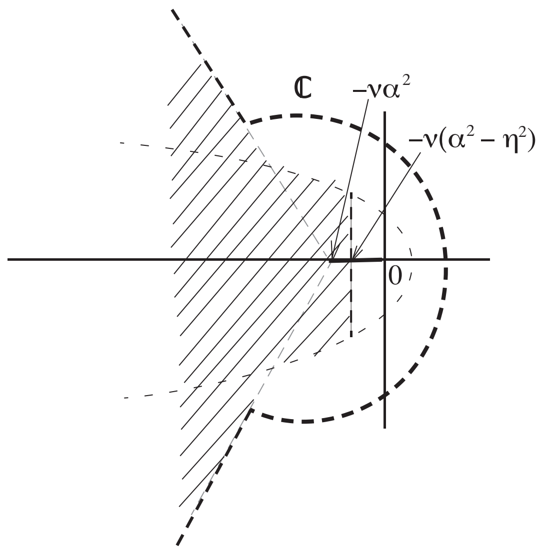

Figure 2: a) Location of the spectrum of $\boldsymbol{L}_{(\nu)}$ inside the region bounded by dashed line b) Location of the essential spectrum in the hatched region Σ_{U_+} and on half left real line.

2.5 Semi-groups

For the study of the linear and nonlinear stabilities of the basic flow U, we need to understand the behavior of the semi-group $e^{\boldsymbol{L}_{(\nu)} t}$ for $t > 0$. We start with estimates on the Stokes flow.

Lemma 11. *Let $\eta > 0$ be small enough. Assume $0 < \delta < \pi/6$. The linear operator $\boldsymbol{L}^{(0)}_{(\nu)}$ is the infinitesimal generator of a bounded semi-group $e^{\boldsymbol{L}^{(0)}_{(\nu)} t}$ in $\mathcal{L}(\mathcal{X}_\eta)$ and holomorphic for $t \in \mathbb{C}$ in a sector of angle $\pi/3 - 2\delta$ centered on $\mathbb{R}^+$. Moreover there exists $C > 0$ such that for any $t \geq 0$,*

$$\begin{aligned} \|e^{\boldsymbol{L}^{(0)}_{(\nu)} t}\|_{\mathcal{L}(\mathring{L}^2_\eta)} &\leq Ce^{-\nu\alpha^2 t}, \\ \|e^{\boldsymbol{L}^{(0)}_{(\nu)} t}\|_{\mathcal{L}(\mathring{L}^\infty)} &\leq C. \end{aligned} \tag{6}$$

Proof. This Lemma follows from the estimate (5) and from classical results on holomorphic semi-groups (see [13]). □

Now the operator $\boldsymbol{L}_{(\nu)}$ is a perturbation of $\boldsymbol{L}_{(\nu)}^{(0)}$ with the same essential spectrum as $\boldsymbol{L}_{(\nu)}^{(0)}+\boldsymbol{L}^{(0,1)}$, and with possible eigenvalues in a region bounded by the dashed curve in Figure 2. By assumptions (A1), (A2) and (A3), the spectrum of eigenvalues stays on the left half complex plane for α fixed and $\nu<\nu_0(\alpha)$. Moreover, for $\nu=\nu_0$, we have two simple isolated eigenvalues $\pm i\omega_0$ with no other eigenvalue of $\boldsymbol{L}_{(\nu_0)}=\boldsymbol{L}^{(0)}+\boldsymbol{L}^{(1)}$ as well on the imaginary axis as on the right side of the complex plane. It results from the bound of the spectrum found in previous section, and from the fact that eigenvalues are isolated, that there is a vertical line in the left complex plane bounding all other eigenvalues, at a finite distance from the imaginary axis, hence staying on the left side of the complex plane for ν close to ν_0.

As soon as we are able to justify the definition and obtain a good estimate of the semi-group generated by $\boldsymbol{L}_{(\nu)}$, it will result that the linear stability of the basic solution is determined by the sign of the real part of the eigenvalues perturbing the above two eigenvalues, meaning in this case that *spectral stability implies linear stability.*

Remark 12. *If we introduced the decay in $e^{-\eta y}$ in the 0- Fourier mode of the function spaces, we could not obtain an estimate such as (6), since we would have an exponential growing in $e^{\nu\eta^2 t}$ for the bound.*

For the study of the nonlinear stability of the basic flow U, we need to estimate $e^{\boldsymbol{L}_{(\nu)}t}$ in $\mathcal{L}(\mathcal{Y}_\eta,\mathcal{Z}_\eta)$ so that we can apply the semi-group to $B(u,u)\in\mathcal{Y}_\eta$. This is detailed in the next Lemmas.

Since the part of the operator $\boldsymbol{L}_{(\nu)}$ acting in $\mathring{L}^2_\eta$ is uncoupled from the $\mathring{L}^\infty$ part, we shall split the study in the two corresponding parts.

2.5.1 Study of $e^{\boldsymbol{L}_{(\nu)}t}$ in $\mathring{L}^\infty$ and in $\mathring{L}^\infty_\eta$

Lemma 13. *Assume that the eigenvalues λ of $\boldsymbol{L}_{(\nu)}$ are such that $\Re\lambda<\gamma<0$. Then, for any $v_0\in\mathring{W}^{2,\infty}_\eta$, where*

$$\|v_0\|_{\mathring{W}^{2,\infty}_\eta}=\|v_0\|_{\mathring{L}^\infty_\eta}+\|Dv_0\|_{\mathring{L}^\infty_\eta}+\|D^2v_0\|_{\mathring{L}^\infty_\eta},\quad \|v_0e^{\eta\cdot}\|_{\mathring{L}^\infty}=\|v_0\|_{\mathring{L}^\infty_\eta}$$

we have the estimate

$$\left\|e^{\boldsymbol{L}_{(\nu)}t}v_0\right\|_{\mathring{W}^{2,\infty}}\leq\frac{M}{1+t}\|v_0\|_{\mathring{W}^{2,\infty}_\eta},\ t\in[0,\infty). \tag{7}$$

Moreover, for $f_0 \in \mathring{W}^{1,\infty}_\eta$, we have

$$\left\| e^{\boldsymbol{L}_{(\nu)} t} f_0 \right\|_{\mathring{W}^{2,\infty}} \leq \frac{M}{\sqrt{t}(1+\sqrt{t})} \|f_0\|_{\mathring{W}^{1,\infty}_\eta}, \ t > 0. \tag{8}$$

Note that we have $\boldsymbol{L}^{(0)}_{(\nu)} + \boldsymbol{L}^{(1)} = \boldsymbol{L}^{(0)}_{(\nu)}$ on $\mathring{L}^\infty$ (since $\boldsymbol{L}^{(1)}$ cancels). Note also that we can choose $\gamma > 0$ with $0 < \gamma < \nu\alpha^2/2$ such that all the eigenvalues of the linearized operator $\boldsymbol{L}_{(\nu)}$ acting in $\mathring{L}^2_\eta$ are such that $\Re\lambda < -2\gamma$.

The proof, which relies on the study of Dunford's formula on the contour Γ described in Figure 3, is detailed in Appendix 6.6.

We observe on (7) that despite of the spectrum containing the full negatice real axis, we have a decay at infinity in t as soon as the semi-group operates on functions decaying to 0 exponentially as y goes to ∞, which is better than (6) in $\mathcal{L}(\mathring{L}^\infty)$. However we loose the decay in y for $e^{\boldsymbol{L}_{(\nu)} t} v_0$. To obtain (7) and (8) we make a direct proof.

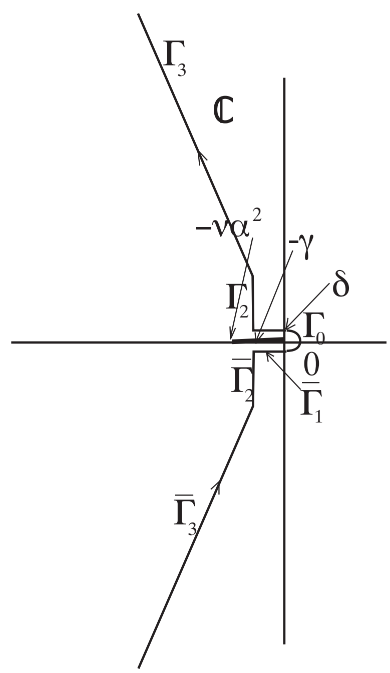

Figure 3: Contour Γ for the estimate of the semi-group used in Lemma 13.

2.5.2 Study of the semi-group $e^{\boldsymbol{L}_{(\nu)}t}$ in $\mathring{L}^2_\eta$

The following Lemma, which can be found in [12] chap VII, uses the holomorphy of the semi-group $e^{\boldsymbol{L}_{(\nu)}t}$:

Lemma 14. *As soon as there exists $M>0$ such that the following estimate holds*

$$\left\|(\mathbb{I}-\varepsilon\boldsymbol{L}^{(0)}_{(\nu)})^{-1}\right\|_{\mathcal{L}(\mathring{H}^1_\eta,\mathring{H}^2_\eta)}\leq M\varepsilon^{-1/2},\ \varepsilon\in(0,\delta_0),\ \delta_0>0, \tag{9}$$

then $\exists C>0$ such that

$$\left\|e^{\boldsymbol{L}^{(0)}_{(\nu)}t}\right\|_{\mathcal{L}(\mathring{H}^1_\eta,\mathring{H}^2_\eta)}\leq\frac{C}{t^{1/2}},\ t\in(0,1]. \tag{10}$$

In the Appendix 6.7, we will prove the following Lemma.

Lemma 15. *There exists $M>0$ such that the estimate (9) holds in $\mathcal{L}(\mathring{H}^1_\eta;\mathring{H}^2_\eta)$.*

Let us now consider the linear operator $\boldsymbol{L}_{(\nu)}=\boldsymbol{L}^{(0)}_{(\nu)}+\boldsymbol{L}^{(1)}$ acting in $\mathring{L}^2_\eta$. We have

$$\mathbb{I}-\varepsilon\boldsymbol{L}_{(\nu)}=(\mathbb{I}-\varepsilon\boldsymbol{L}^{(0)}_{(\nu)})\Big(\mathbb{I}-\varepsilon(\mathbb{I}-\varepsilon\boldsymbol{L}^{(0)}_{(\nu)})^{-1}\boldsymbol{L}^{(1)}\Big),$$

and we notice

$$\left\|\varepsilon(\mathbb{I}-\varepsilon\boldsymbol{L}^{(0)}_{(\nu)})^{-1}\boldsymbol{L}^{(1)}\right\|_{\mathcal{L}(\mathring{H}^2_\eta)}\leq\left\|\varepsilon(\mathbb{I}-\varepsilon\boldsymbol{L}^{(0)}_{(\nu)})^{-1}\right\|_{\mathcal{L}(\mathring{H}^1_\eta,\mathring{H}^2_\eta)}\|\boldsymbol{L}^{(1)}\|_{\mathcal{L}(\mathring{H}^2_\eta,\mathring{H}^1_\eta)}\leq M\varepsilon^{1/2},$$

hence, for ε small enough, $M\varepsilon^{1/2}<1/2$ and the operator $\mathbb{I}-\varepsilon(\mathbb{I}-\varepsilon\boldsymbol{L}^{(0)}_{(\nu)})^{-1}\boldsymbol{L}^{(1)}$ has a bounded (by 2) inverse in $\mathcal{L}(\mathring{H}^2_\eta)$. It results that

$$\left\|(\mathbb{I}-\varepsilon\boldsymbol{L}_{(\nu)})^{-1}\right\|_{\mathcal{L}(\mathring{H}^1_\eta,\mathring{H}^2_\eta)}\leq2\left\|(\mathbb{I}-\varepsilon\boldsymbol{L}^{(0)}_{(\nu)})^{-1}\right\|_{\mathcal{L}(\mathring{H}^1_\eta,\mathring{H}^2_\eta)}\leq\frac{2M}{\varepsilon^{1/2}}\text{ in }\mathcal{L}(\mathring{H}^1_\eta,\mathring{H}^2_\eta)$$

and, from Lemma 14 applied to $\boldsymbol{L}_{(\nu)}$, the following

Lemma 16. *Assume that the eigenvalues λ of $\boldsymbol{L}_{(\nu)}$ are such that $\Re\lambda<\gamma<0$, then, for $\widetilde{u}\in\mathring{H}^2_\eta$, we have*

$$\left\|e^{\boldsymbol{L}_{(\nu)}t}\widetilde{u}\right\|_{\mathring{H}^2_\eta}\leq Me^{-\gamma t}\left\|\widetilde{u}\right\|_{\mathring{H}^2_\eta},\ t\geq0.$$

Moreover, we have the estimate

$$\left\|e^{\boldsymbol{L}_{(\nu)}t}\widetilde{u}\right\|_{\mathcal{L}(\mathring{H}^1_\eta,\mathring{H}^2_\eta)}\leq M\Big(1+\frac{1}{t^{1/2}}\Big)e^{-\gamma t}\text{ for }t>0.$$

Proof. This results from the fact that the set of (isolated) eigenvalues of $\boldsymbol{L}_{(\nu)}$ is located as indicated in Lemma 7, and we notice that $\Re\lambda<\gamma$ is realized for $\nu>\nu_0$, due to the definition of $\nu_0(\alpha)$. □

To be detailed. Why for $t\geq1$.

2.6 Study of the quadratic term

Let us define projections $\widetilde{P}$ and P_0 in $\mathcal{X}_\eta$ (separating the 0-Fourier mode from the oscillating part): for any $u \in \mathcal{X}_\eta$

$$\begin{aligned} u &= \widetilde{P}u + P_0 u, \\ \widetilde{P}u &= \widetilde{u} \in \mathring{L}^2_\eta,\ P_0 u = u_0 \in \mathring{L}^\infty. \end{aligned}$$

The above projections naturally work in $\mathcal{Z}_\eta$ or $\mathcal{Y}_\eta$. The aim of this section is to show that the quadratic operator

$$\boldsymbol{B}(u,v) = -\frac{1}{2}\Pi\Big[(u\cdot\nabla)v + (v\cdot\nabla)u\Big]$$

is well defined in $\mathcal{Y}_\eta$ for u and v in $\mathcal{Z}_\eta$.

Lemma 17. *The quadratic operator* $\boldsymbol{B}$ *is bounded from* $\mathcal{Z}_\eta$ *to* $\mathcal{Y}_\eta$ *: there exists* $C>0$ *such that*

$$\|\boldsymbol{B}(u,v)\|_{\mathcal{Y}_\eta} \le C\|u\|_{\mathcal{Z}_\eta}\|v\|_{\mathcal{Z}_\eta}. \tag{11}$$

Proof. Let us decompose u, v in $\mathcal{Z}_\eta$

$$u = \widetilde{u} + u_0,\ v = \widetilde{v} + v_0$$

then

$$\begin{aligned} \widetilde{P}B(u,v) &= \widetilde{P}B(\widetilde{u},\widetilde{v}) + B(\widetilde{u},v_0) + B(\widetilde{v},u_0) \\ P_0B(u,v) &= P_0B(\widetilde{u},\widetilde{v}). \end{aligned} \tag{12}$$

Indeed,

$$(u_0\cdot\nabla)v_0 = u_0^y D v_0 = 0$$

since $u_0^y = 0$ for $u_0 \in \mathring{L}^\infty$. This implies that $P_0B(u_0,v_0)=0$.

Now, we observe that $(\widetilde{u}\cdot\nabla)v_0$ and $(u_0\cdot\nabla)\widetilde{v}$ satisfy

$$\Big\|(\widetilde{u}\cdot\nabla)v_0\Big\|^2_{L^2_\eta} = \sum_{|n|\ge 1}\|\widetilde{u}_n^y D v_0^x\|^2_{C^0_\eta} \le \|v_0\|^2_{W^{1,\infty}}\|\widetilde{u}\|^2_{L^2_\eta},$$

$$\begin{aligned} \Big\|(\widetilde{u}\cdot\nabla)v_0\Big\|^2_{H^1_\eta} &= \sum_{|n|\ge 1} n^2\|\widetilde{u}_n^y D v_0^x\|^2_{C^0_\eta} + \|D(\widetilde{u}_n^y D v_0^x)\|^2_{C^0_\eta} \\ &\le \|v_0\|^2_{W^{1,\infty}}\|\widetilde{u}\|^2_{H^1_\eta} + \|v_0\|^2_{W^{2,\infty}}\|\widetilde{u}\|^2_{L^2_\eta}, \end{aligned}$$

and analogously for $(u_0 \cdot \nabla)\widetilde{v}$. Hence, since Π is bounded from H^1_η to $\mathring{H}^1_\eta$, we obtain easily

$$\begin{aligned} \Big\|B(\widetilde{u}, v_0) + B(\widetilde{v}, u_0)\Big\|_{\mathring{H}^1_\eta} &\le C\Big(\|\widetilde{u}\|_{H^1_\eta}\|v_0\|_{W^{2,\infty}} + \|\widetilde{v}\|_{H^1_\eta}\|u_0\|_{W^{2,\infty}}\Big) \\ &\le C\|u\|_{\mathcal{Z}_\eta}\|v\|_{\mathcal{Z}_\eta}. \end{aligned}$$

Now, before considering $(\widetilde{u}\cdot\nabla)\widetilde{v}$, let us first consider the product of a scalar function f in H^1_η with a scalar g function in L^2_η, both with 0-average. We show that the product fg is in L^2_η. Indeed, for Fourier coefficients, we have

$$(fg)_n = \sum_{p,q=n-p} f_p g_q$$

and

$$\|(fg)_n\|_{C^0_\eta} \le \sum_{p\neq 0, q=n-p\neq 0} \frac{1}{p}\|pf_p\|_{C^0_\eta}\|g_q\|_{C^0_\eta},$$

so that

$$\|(fg)_n\|^2_{C^0_\eta} \le \left(\sum_{p'\neq 0}\frac{1}{p'^2}\right)\sum_{p,q=n-p}\|pf_p\|^2_{C^0_\eta}\|g_q\|^2_{C^0_\eta}.$$

Hence

$$\|fg\|^2_{L^2_\eta} \le C\sum_{n,p,q=n-p}\|pf_p\|^2_{C^0_\eta}\|g_q\|^2_{C^0_\eta} \le C\|f\|^2_{H^1_\eta}\|g\|^2_{L^2_\eta}.$$

Coming back to $(\widetilde{u}\cdot\nabla)\widetilde{v}$, with $\widetilde{u}$ and $\widetilde{v}$ in H^2_η, we deduce immediately that $\widetilde{P}(\widetilde{u}\cdot\nabla)\widetilde{v} \in H^1_\eta$, and the bound (11) for $\widetilde{P}B(\widetilde{u},\widetilde{v})$ holds immediately in $\mathring{H}^1_\eta$.

For the 0-Fourier mode, we have also

$$\|(fg)_0\|^2_{L^\infty_\eta} \le \left(\sum_{p'\neq 0}\frac{1}{p'^2}\right)\sum_p\|pf_p\|^2_{C^0_\eta}\|g_{-p}\|^2_{C^0_\eta} \le C\|f\|^2_{H^1_\eta}\|g\|^2_{L^2_\eta},$$

so that $P_0(\widetilde{u}\cdot\nabla)\widetilde{v}$ satisfies easily

$$\|P_0(\widetilde{u}\cdot\nabla)\widetilde{v}\|_{W^{1,\infty}_\eta} \le C\|\widetilde{u}\|_{H^2_\eta}\|\widetilde{v}\|_{H^2_\eta},$$

and estimate (11) holds. □

3 Nonlinear stability of U

In this section, we consider the nonlinear evolution problem with an initial data close to the basic flow U, assuming that the spectrum of the linearized operator $\boldsymbol{L}_{(\nu)}$ is situated on the left side of the imaginary axis, except the essential spectrum which contains the full negative real line. We prove the nonlinear stability of U.

Assume $\xi > 0$ and let us define the Banach space $\mathcal{Z}_{\eta,\xi}$ by

$$\mathcal{Z}_{\eta,\xi} = \Big\{ v = \widetilde{v} + v_0;\ \widetilde{v} \in C^0(\mathbb{R}^+, \mathring{H}^2_\eta),\ v_0 \in C^0(\mathbb{R}^+, \mathring{W}^{2,\infty}),$$

$$|||v|||_{\eta,\xi} = \sup_{t\geq 0}\Big(\|\widetilde{v}(t)e^{\xi t}\|_{\mathring{H}^2_\eta} + \|(1+t)v_0(t)\|_{\mathring{W}^{2,\infty}} \Big) < \infty \Big\}.$$

Note the exponential weight in time for the non zero Fourier component and the $(1+t)$ weight in time for the zero Fourier component. We now detail the stability part of the Theorem 1 of the introduction.

Theorem 18. *Let us assume $\eta > 0$ and assume that the set of eigenvalues λ of operator $\boldsymbol{L}_{(\nu)}$ satisfies*

$$\Re\lambda < -\xi < 0,$$

then there exists $\varepsilon > 0$ such that for

$$\|\widetilde{P}v(0)\|_{\mathring{H}^2_\eta} + \|P_0 v(0)\|_{\mathring{W}^2_\eta} \leq \varepsilon,$$

there is a unique solution $v \in \mathcal{Z}_{\eta,\xi}$ of the following differential equation in $\mathcal{X}_\eta$

$$\frac{dv}{dt} = \boldsymbol{L}_{(\nu)} v + B(v,v), \tag{13}$$

with $v_{t=0} = v(0) = \widetilde{P}v(0) + P_0 v(0)$. Moreover there exists $M > 0$ such that

$$\begin{aligned} |||\widetilde{P}v|||_{\eta,\xi} &\leq M\|\widetilde{P}v(0)\|_{\mathring{H}^2_\eta}, \\ |||P_0 v|||_{\eta,\xi} &\leq M\Big(\|P_0 v(0)\|_{\mathring{W}^2_\eta} + \|\widetilde{P}v(0)\|^2_{\mathring{H}^2_\eta}\Big). \end{aligned}$$

Remark 19. *Notice that the above Theorem says that*

$$\begin{aligned} \|\widetilde{P}v(t)\|_{\mathring{H}^2_\eta} &\leq Me^{-\xi t}\|\widetilde{P}v(0)\|_{\mathring{H}^2_\eta},\ t \geq 0, \\ \|P_0 v(t)\|_{\mathring{W}^{2,\infty}} &\leq \frac{M}{1+t}\Big(\|P_0 v(0)\|_{\mathring{W}^2_\eta} + \|\widetilde{P}v(0)\|^2_{\mathring{H}^2_\eta}\Big),\ t \geq 0. \end{aligned}$$

The slow decreasing of the 0-mode as $t \to \infty$ is due to the influence of the essential spectrum of $\boldsymbol{L}_{(\nu)}$ sitting on the full negative real line. Moreover, we may notice that the decay in y at ∞ is lost for the 0-mode for $t > 0$.

Proof. Using (12) and (11) shows that there exists $C > 0$ such that we have the estimates

$$\begin{aligned}\left\|\widetilde{P}B(\widetilde{v},\widetilde{v}) + 2B(\widetilde{v},v_0)\right\|_{\mathring{H}^1_\eta} &\leq C\Big(\|\widetilde{v}\|^2_{\mathring{H}^2_\eta} + \|\widetilde{v}\|_{\mathring{H}^2_\eta}\|v_0\|_{\mathring{W}^{1,\infty}}\Big),\\ \|P_0B(\widetilde{v},\widetilde{v})\|_{\mathring{W}^{1,\infty}_\eta} &\leq C\|\widetilde{v}\|^2_{\mathring{H}^2_\eta}.\end{aligned}$$

Now (13) becomes

$$\begin{aligned}\frac{d\widetilde{v}}{dt} &= \boldsymbol{L}_{(\nu)}\widetilde{v} + \widetilde{P}B(\widetilde{v},\widetilde{v}) + 2B(\widetilde{v},v_0),\\ \frac{dv_0}{dt} &= \boldsymbol{L}_{(\nu)}v_0 + P_0B(\widetilde{v},\widetilde{v}),\end{aligned}$$

which leads to the following integral formulation for $t \geq 0$

$$\widetilde{v}(t) = e^{\boldsymbol{L}_{(\nu)}t}\widetilde{P}v(0) + \int_0^t e^{\boldsymbol{L}_{(\nu)}(t-s)}\Big[\widetilde{P}B(\widetilde{v},\widetilde{v}) + 2B(\widetilde{v},v_0)\Big](s)ds, \quad (14)$$

$$v_0(t) = e^{\boldsymbol{L}_{(\nu)}t}P_0v(0) + \int_0^t e^{\boldsymbol{L}_{(\nu)}(t-s)}P_0B(\widetilde{v},\widetilde{v})(s)ds, \quad (15)$$

where we look for $\widetilde{v} + v_0 \in \mathcal{Z}_{\eta,\xi}$. Indeed we can solve the system above by the implicit function theorem with respect to $(\widetilde{v}, v_0)$ in the neighborhood of 0 for $v(0) = \widetilde{P}v(0) + P_0v(0)$ with

$$\begin{aligned}&\widetilde{P}v(0) \text{ sufficiently small in } \mathring{H}^2_\eta,\\ &P_0v(0) \text{ sufficiently small in } \mathring{W}^2_\eta.\end{aligned}$$

Using the bound of the imaginary parts of eigenvalues which are isolated in the left half complex plane, there exist $\xi_m > 0$ such that the eigenvalues λ of $\boldsymbol{L}_{(\nu)}$ satisfy

$$\sup \Re\lambda = -\xi_m < -\xi < 0.$$

We choose ξ_1 such that $-\xi_m < -\xi_1 < -\xi$. Then we have the estimates

$$\begin{aligned}\|e^{\boldsymbol{L}_{(\nu)}t}\|_{\mathcal{L}(\mathring{H}^2_\eta)} &\leq Ce^{-\xi_1 t}, \ t \geq 0,\\ \|e^{\boldsymbol{L}_{(\nu)}t}\|_{\mathcal{L}(\mathring{W}^{2,\infty}_\eta,\mathring{W}^{2,\infty})} &\leq \frac{C}{1+t}, \ t \geq 0,\\ \|e^{\boldsymbol{L}_{(\nu)}(t-s)}\|_{\mathcal{L}(\mathring{H}^1_\eta,\mathring{H}^2_\eta)} &\leq C\Big[1 + \frac{1}{\sqrt{t-s}}\Big]e^{-\xi_1(t-s)}, \ t > s,\\ \|e^{\boldsymbol{L}_{(\nu)}(t-s)}\|_{\mathcal{L}(\mathring{W}^{1,\infty}_\eta,\mathring{W}^{2,\infty})} &\leq \frac{C}{\sqrt{t-s}(1+\sqrt{t-s})}, \ t > s.\end{aligned}$$

We notice that

$$\begin{aligned} \|e^{\xi t} e^{\boldsymbol{L}_{(\nu)} t} \widetilde{P} v(0)\|_{\mathring{H}^2_\eta} &\leq C\|\widetilde{P} v(0)\|_{\mathring{H}^2_\eta}, \\ \|e^{\boldsymbol{L}_{(\nu)} t} P_0 v(0)\|_{\mathring{W}^{2,\infty}} &\leq \frac{C}{1+t}\|P_0 v(0)\|_{\mathring{W}^2_\eta}, \end{aligned}$$

that

$$\begin{aligned} &\left\| e^{\xi t} e^{\boldsymbol{L}_{(\nu)}(t-s)} \Big[\widetilde{P} B(\widetilde{v},\widetilde{v}) + 2B(\widetilde{v}, v_0)\Big](s)\right\|_{\mathring{H}^2_\eta} \\ \leq\; & MC\Big(1+\frac{1}{\sqrt{t-s}}\Big) e^{-(\xi_1-\xi)(t-s)} |||\widetilde{v}|||_{\eta,\xi} |||\widetilde{v}+v_0|||_{\eta,\xi}, \end{aligned}$$

and that

$$\left\| e^{\boldsymbol{L}_{(\nu)}(t-s)} P_0 B(\widetilde{v},\widetilde{v})(s)\right\|_{\mathring{W}^{2,\infty}} \leq \frac{MC}{\sqrt{t-s}(1+\sqrt{t-s})} e^{-2\xi s} |||\widetilde{v}|||^2_{\eta,\xi}.$$

Hence the right hand side of (14), and (15) is quadratic and C^1 in $(\widetilde{v}, v_0) \in \mathcal{Z}_{\eta,\xi}$. It results by the implicit function theorem, that for $(\widetilde{P}v(0), P_0 v(0))$ small enough in $\mathring{H}^2_\eta \oplus \mathring{W}^2_\eta$, there exists a unique solution $(\widetilde{v}, v_0) \in \mathcal{Z}_{\eta,\xi}$ in a neighborhood of 0. Moreover, directly from the system (14), (15), we have the estimate

$$\begin{aligned} \|e^{\xi t}\widetilde{v}(t)\|_{\mathring{H}^2_\eta} \leq C\|\widetilde{P}v(0)\|_{\mathring{H}^2_\eta} + MC\int_0^t (1+\frac{1}{(t-s)^{1/2}}) \\ \times e^{-(\xi_1-\xi)(t-s)} |||\widetilde{v}|||_{\eta,\xi} |||\widetilde{v}+v_0|||_{\eta,\xi} ds, \end{aligned}$$

hence

$$|||\widetilde{v}|||_{\eta,\xi} \leq C\|\widetilde{P}v(0)\|_{\mathring{H}^2_\eta} + K|||\widetilde{v}|||_{\eta,\xi} |||\widetilde{v}+v_0|||_{\eta,\xi},$$

using

$$\int_0^t \Big[1+\frac{1}{(t-s)^{1/2}}\Big] e^{-\delta(t-s)} ds \leq C_1(\delta) \text{ for } \delta > 0.$$

In the same way, we obtain

$$|||v_0|||_{\eta,\xi} \leq C\|P_0 v(0)\|_{\mathring{W}^{2,\infty}_\eta} + K|||\widetilde{v}|||^2_{\eta,\xi},$$

using

$$\int_0^t \frac{1}{\sqrt{t-s})(1+\sqrt{t-s})} e^{-\delta s} ds \leq \frac{C_2(\delta)}{1+t}, \text{ for } \delta > 0.$$

Hence,

$$|||\widetilde{v}|||_{\eta,\xi} \leq C\|\widetilde{P}v(0)\|_{\mathring{H}^2_\eta} + CK\|P_0 v(0)\|_{\mathring{W}^{2,\infty}_\eta} |||\widetilde{v}|||_{\eta,\xi} + K|||\widetilde{v}|||^2_{\eta,\xi} + K^2|||\widetilde{v}|||^3_{\eta,\xi},$$

so that for $v(0)$ such that

$$CK\|P_0 v(0)\|_{\mathring{W}^{2,\infty}_\eta} + 2CK\|\widetilde{P}v(0)\|_{\mathring{H}^2_\eta} + 4C^2K^2\|\widetilde{P}v(0)\|^2_{\mathring{H}^2_\eta} < \frac{1}{2},$$

we obtain

$$\begin{aligned} |||\widetilde{v}|||_{\eta,\xi} &\leq 2C\|\widetilde{P}v(0)\|_{\mathring{H}^2_\eta}, \\ |||v_0|||_{\eta,\xi} &\leq C\|P_0 v(0)\|_{\mathring{W}^{2,\infty}_\eta} + 4C^2K\|\widetilde{P}v(0)\|^2_{\mathring{H}^2_\eta} \end{aligned}$$

which ends the proof of the nonlinear stability. □

4 Study of the bifurcation

4.1 Setup

Let us define

$$s = \omega t$$

such that the time-periodic solution which we are looking for is now 2π periodic in s. The wave number ω is an unknown of the bifurcation, which must be determined.

Let us define the bifurcation parameter μ as

$$\mu = \nu - \nu_0$$

By assumption, $\mathbf{L} := \mathbf{L}_{(\nu_0)}$ has two purely imaginary simple eigenvalues $\pm i\omega_0$, associated to eigenvectors of the form

$$\zeta = e^{i\alpha x}\hat{v}(y), \quad \overline{\zeta} = e^{-i\alpha x}\overline{\hat{v}(y)} \tag{16}$$

and no other eigenvalues of non negative real part.

In what follows, we need to invert the linear system

$$\omega_0 \frac{dv}{ds} - \mathbf{L}\, v = f$$

in a space of vector functions which are 2π periodic in s and $2\pi/\alpha$ periodic in x. We expand f and v in Fourier series in time and space, k being the Fourier wave number in time and n the Fourier wave number in space. This leads to the sequence of equations

$$ik\omega_0 v_{k,n} - \mathbf{L}_n v_{k,n} = f_{k,n}$$

where $(k,n) \in \mathbb{Z}^2$, and where the linear operator $\mathbf{L}_n$ is defined in a space of vector functions of y by

$$\mathbf{L}_n v_n = \nu_0(D^2 - n^2\alpha^2)v_n + \begin{pmatrix} i\alpha n \\ D \end{pmatrix} q_n - in\alpha U v_n - \begin{pmatrix} v_n^y U' \\ 0 \end{pmatrix},$$
$$0 = in\alpha v_n^x + Dv_n^y,$$

where q_n is the related component of the pressure. We know by assumption that

- $\mathbf{L}_n - ik\omega_0$ is invertible for $n \neq 0$ and $k \neq \pm 1$,
- for $n = 1$, $\mathbf{L}_n - i\omega_0$ has a 1-dim kernel spanned by ζ,
- for $n = -1$, $\mathbf{L}_n + i\omega_0$ has a 1-dim kernel spanned by $\bar{\zeta}$,
- for $n \neq \pm 1$, $\mathbf{L}_n \mp i\omega_0$ is invertible,
- for $k \neq 0$, $\mathbf{L}_0 - ik\omega_0$ is invertible.

It remains to study the invertibility of $\mathbf{L}_0$.

4.2 Study of the inverse of $\mathbf{L}_0$

Lemma 20. *If $f_{0,0} \in L^\infty_\eta$, then the equation*

$$\mathbf{L}_0 v_{0,0} = \begin{pmatrix} f_{0,0} \\ 0 \end{pmatrix} \tag{17}$$

has a unique solution such that $v^x_{0,0}$ is bounded. Moreover,

$$\|v_{0,0}\|_{W^{2,\infty}} \leq C_\eta \|f_{0,0}\|_{L^\infty_\eta}$$

for some positive constant C_η.

Proof. The equation (17) leads to

$$\nu_0 D^2 v_{0,0} = f_{0,0}.$$

Taking care of the boundary condition $v^x_{0,0} = 0$ at $y = 0$ and its boundedness at infinity, we find

$$Dv^x_{0,0} = \frac{1}{\nu_0}\Big[-\int_y^\infty f_{0,0}(s)\,ds + A\Big],$$
$$v^x_{0,0} = \frac{1}{\nu_0}\int_0^y d\tau\Big[-\int_\tau^\infty f_{0,0}(s)\,ds + A\Big]$$

for some constant A to be determined so that $v^x_{0,0}$ is bounded and $Dv^x_{0,0}$ tends exponentially to 0 at infinity. This leads to $A = 0$ and to

$$v^x_{0,0}(y) = -\frac{1}{\nu_0}\int_0^y d\tau \int_\tau^\infty f_{0,0}(s)\, ds,$$

and, by Fubini's theorem,

$$v^x_{0,0}(y) = -\frac{1}{\nu_0}\Big[\int_0^y s f_{0,0}(s)\, ds + y\int_y^\infty f_{0,0}(s)\, ds\Big],$$

which ends the proof of the Lemma. $\square$

We note that

$$\lim_{y\to+\infty} v^x_{0,0}(y) = -\frac{1}{\nu_0}\int_0^{+\infty} s f_{0,0}(s)\, ds,$$

which in general does not vanish. Thus, in general, $v^x_{0,0}$ does not go to 0 at infinity, but $Dv^x_{0,0}$ and $D^2v^x_{0,0}$ decay exponentially fast at infinity. However $v_{0,0}$ only appears in the operator $(v\cdot\nabla)v$, and thus only in terms of the form $(v_{0,0}\cdot\nabla)v$ and $(v\cdot\nabla)v_{0,0}$ where v will be exponentially decaying.

4.3 The eigenvectors ζ and $\zeta^\star$

We look for $\zeta = e^{i\alpha x}\hat{v}(y) \in \mathcal{Z}_\eta$ where $\alpha \neq 0$ and $\hat{v} = (\hat{v}^x, \hat{v}^y) \in C^2_\eta$ satisfy

$$i\omega_0\hat{v} = \nu_0(D^2-\alpha^2)\hat{v} + \begin{pmatrix} i\alpha \\ D\end{pmatrix}\hat{q} - i\alpha U\hat{v} - \begin{pmatrix} \hat{v}^yU' \\ 0\end{pmatrix},$$
$$0 = i\alpha\hat{v}^x + D\hat{v}^y$$

with $\hat{v}^x = \hat{v}^y = 0$ for $y = 0$. This leads to $\hat{v}^y \in C^4_\eta$ and

$$\Big[i\omega_0 - \nu_0(D^2-\alpha^2) - i\alpha U\Big](D^2-\alpha^2)\hat{v}^y + i\alpha U''\hat{v}^y = 0 \qquad (18)$$

with $\hat{v}^y = D\hat{v}^y = 0$ for $y = 0$, which is the Orr-Sommerfeld equation.

By assumption, (18) has a non trivial smooth solution $\hat{v}^y(y)$, satisfying the boundary conditions, unique up to a multiplicative constant. Moreover, there exist $c_1 > 0$ and $\eta > 0$ such that

$$|\hat{v}^y(y)| + |D\hat{v}^y(y)| \le c_1 e^{-\eta y}$$

for $0 \le y < +\infty$. It results that there exists $c > 0$ and $\eta > 0$ such that

$$|\hat{v}(y)| + |D\hat{v}(y)| \le c e^{-\eta y}$$

for $0 \le y < +\infty$ (see [9]).

4.4 Pseudo-inverse of $\omega_0 d/ds - \mathbf{L}$

The aim of this section is to construct a pseudo inverse to the linear operator

$$\mathcal{L} = \omega_0 \frac{d}{ds} - \mathbf{L}$$

in a space of 2π periodic vector functions.

Let us define ζ^*, the eigenvector of $\mathbf{L}^\star$ for the eigenvalue $-i\omega_0$, such that $\langle \zeta, \zeta^\star \rangle = 1$ (see [13]). Let $\mathbb{T}_1 = \mathbb{R}/2\pi\mathbb{Z}$. Let $\mathcal{H}$ be a subspace of $L^2(\mathbb{T}_1, \mathcal{X}_\eta)$ such that, for every $v \in \mathcal{H}$,

$$\langle v, e^{is}\zeta^\star \rangle_{\mathcal{H}} = \langle v, e^{-is}\overline{\zeta^\star} \rangle_{\mathcal{H}} = 0$$

and such that $v_{00} = 0$. Let us define

$$H^\sharp = H^1(\mathbb{T}_1, \mathcal{X}_\eta) \cap L^2(\mathbb{T}_1, \mathcal{Z}_\eta),$$

together with its norm

$$\|v\|^2_{H^\sharp} = \|\partial_s v\|^2_{L^2(\mathbb{T}_1, \mathcal{X}_\eta)} + \|v\|^2_{L^2(\mathbb{T}_1, \mathcal{Z}_\eta)}.$$

Lemma 21. *For $f \in \mathcal{H}$, there exists a unique $v \in H^\sharp \cap \mathcal{H}$ such that*

$$\Big(\omega_0 \frac{d}{ds} - \mathbf{L}\Big) v = f.$$

This defines a pseudo-inverse $\mathcal{L}^{-1}$ to $\mathcal{L}$ which satisfies

$$\|v\|_{H^\sharp} \leq \|f\|_{\mathcal{H}}.$$

Proof. As $f \in \mathcal{H}$, f is of the form

$$f(x, y, s) = \sum_{(n,k)\in\mathbb{Z}^2} f_{k,n}(y) e^{i(ks + n\alpha x)}$$

with

$$\Big\langle f_{1,1} e^{i\alpha x}, \zeta^\star \Big\rangle = \Big\langle f_{-1,-1} e^{-i\alpha x}, \overline{\zeta^\star} \Big\rangle = 0, \quad f_{0,0} = 0.$$

Similarly, we look for $v \in \mathcal{H}$, namely for a function v of the form

$$v(x, y, s) = \sum_{(n,k)\in\mathbb{Z}^2} v_{k,n}(y) e^{i(ks + n\alpha x)}$$

with

$$\Big\langle v_{1,1} e^{i\alpha x}, \zeta^\star \Big\rangle = \Big\langle v_{-1,-1} e^{-i\alpha x}, \overline{\zeta^\star} \Big\rangle = 0, \quad v_{00} = 0.$$

This leads to

$$\Big(ik\omega_0 - \mathbf{L}_n\Big)v_{k,n} = f_{k,n} \tag{19}$$

for $(k,n) \neq \pm(1,1)$ and $(k,n) \neq (0,0)$.

Changed below

Using Lemma 10, we know that there exists k_0 and c such that if $|k| > k_0$ is large enough, the equation (19) has a unique solution such that

$$\|D^2 v_{k,n}\|_{C^0_\eta} + |n|\,\|Dv_{k,n}\|_{C^0_\eta} + (1+n^2)\|v_{k,n}\|_{C^0_\eta} \leq \|f_{k,n}\|_{C^0_\eta}, \tag{20}$$

and

$$\|v_{k,n}\|_{C^0_\eta} \leq \frac{c}{|k|}\|f_{k,n}\|_{C^0_\eta}$$

provided $(k,n) \neq \pm(1,1),(0,0)$.

For $n \neq \pm 1$, $n \neq 0$ and $|k| \leq k_0$, $\mathbf{L}_n$ is invertible by assumption in section 2.1, thus (20) is still valid (up to the change of the factor $c > 0$). For $n = \pm 1$, (20) is also valid since $f_{k,n}$ is orthogonal to $\zeta^\star$ and $\overline{\zeta^\star}$, and thus in the range of $ik\omega_0 - \mathbf{L}_n$. For $n = 0$, as $f_{0,0} = 0$, $v_{0,0} = 0$, and we just have to consider the case $k \neq 0$, where $ik\omega_0 - \mathbf{L}_0$ is invertible by Lemma 10, thus (20) is still valid.

Combining all these estimates, we get that the pseudo inverse of $\mathcal{L}$ is bounded from $\mathcal{H}$ to $H^\sharp$. □

4.5 Bifurcation of a time periodic solution

We are looking for a solution v, which is 2π periodic in s, $2\pi/\alpha$ periodic in x, of the non linear equation

$$\Big[\omega\frac{d}{ds} - \mathbf{L}\Big]v = \mu\mathbf{L}'v + B(v,v) \tag{21}$$

where

$$\mathbf{L}'v = \Pi\Delta v,$$

$$B(v,v) = -\Pi(v\cdot\nabla)v.$$

Let

$$H^{\sharp\sharp} = H^2(\mathbb{T}_1, \mathcal{X}_\eta) \cap H^1(\mathbb{T}_1, \mathcal{Z}_\eta).$$

We decompose $v \in H^{\sharp\sharp}$ as

$$v = \underline{v} + w,$$

$$\underline{v} = Ae^{is}\zeta + \overline{A}e^{-is}\bar{\zeta} + v_{0,0},$$

$$\langle w_{1,1}e^{i\alpha x}, \zeta^\star\rangle = \langle w_{-1,-1}e^{-i\alpha x}, \overline{\zeta^\star}\rangle = 0,$$

$$w_{0,0} = 0,$$

and define the projection $\mathfrak{P}_0$ by

$$w = \mathfrak{P}_0 v.$$

The projection $\mathfrak{P}_0$ is bounded in $H^1(\mathbb{T}_1, \mathcal{X}_\eta)$ and in $H^{\sharp\sharp}$ and commutes with the linear operator $(\omega\partial_s - \mathbf{L})$. In the following we use the property of our system invariant under two $SO(2)$ symmetries, hence commuting with the operator T_a representing the translation in time $s \to s+a$, and the operator $\tau_\beta : x \to x + \beta$ representing the translation in space.

We now prove the following bifurcation theorem, which precises Theorem 1 and gives an expansion in terms of the amplitude ε, of the bifurcation parameter μ, of the time frequency ω and of the time and space periodic solution near the bifurcation point.

Theorem 22. *For μ in a right or left neighborhood of 0, there exists a bifurcation time-periodic solution of (21) which is a traveling wave function $\widehat{V}_\varepsilon(\xi, y)$ where*

$$\xi = x + \frac{s}{\alpha}$$

under the form

$$\widehat{V}_\varepsilon = \varepsilon V_1 + \varepsilon^2 V_2 + \mathcal{O}(\varepsilon^3) \in \mathcal{Z}_\eta,$$

with

$$\begin{aligned}
V_1 &= (e^{i\alpha\xi}\widehat{\zeta} + e^{-i\alpha\xi}\overline{\widehat{\zeta}}) \ \in \mathring{H}^2_\eta, \\
V_2 &= \widehat{w}_{2,2}e^{2i\alpha\xi} + \overline{\widehat{w}_{2,2}}e^{-2i\alpha\xi} + v_{2,0},
\end{aligned}$$

$$\begin{aligned}
\mu(\varepsilon) &= \varepsilon^2\mu_2 + \mathcal{O}(\varepsilon^4), \\
\omega(\varepsilon) &= \omega_0 + \varepsilon^2\omega_2 + \mathcal{O}(\varepsilon^4),
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\mu_2\langle (D^2-\alpha^2)\widehat{\zeta}, \widehat{\zeta}^*\rangle - i\omega_2 &= c - b, \ \Re\langle (D^2-\alpha^2)\widehat{\zeta}, \widehat{\zeta}^*\rangle > 0, \\
\widehat{w}_{2,2}e^{2i\alpha\xi} &= (2i\omega_0 - \boldsymbol{L}^{(0)})^{-1}B(e^{i\alpha\xi}\widehat{\zeta}, e^{i\alpha\xi}\widehat{\zeta}) \in \mathring{H}^2_\eta, \\
v_{2,0} &= -2L_0^{-1}B(e^{i\alpha\xi}\widehat{\zeta}, e^{-i\alpha\xi}\overline{\widehat{\zeta}}) \in \mathring{W}^{2,\infty}, \\
c &= -\langle 2B(e^{i\alpha\xi}\widehat{\zeta}, v_{2,0}), e^{i\alpha\xi}\widehat{\zeta}^*\rangle, \\
b &= \langle 2B(e^{-i\alpha\xi}\overline{\widehat{\zeta}}, e^{2i\alpha\xi}\widehat{w}_{2,2}), e^{i\alpha\xi}\widehat{\zeta}^*\rangle,
\end{aligned}$$

We note that the bifurcation is supercritical if $\mu_2 > 0$, i.e. $\Re(c-b) > 0$, where

$$c - b = \Big\langle 4B\Big(\zeta, \mathbf{L}_0^{-1}B(\zeta,\overline{\zeta}) - 2B(\bar{\zeta}, (2i\omega_0 - L)^{-1}B(\zeta,\zeta)\Big), \zeta^\star\Big\rangle.$$

Proof. We use an adapted Lyapunov-Schmidt method (variant of the implicit function theorem). Using the decomposition $v = \underline{v} + w$, the system becomes

$$\Big(\omega_0 \frac{d}{ds} - \mathbf{L}\Big) w - \mathbf{L}_0 v_{0,0} = \mu \mathbf{L}' v - (\omega - \omega_0)\frac{dv}{ds} + B(v,v),$$

where we look for v such that $w \in \mathfrak{P}_0 H^{\sharp\sharp}$, $A \in \mathbb{C}$, $v_{0,0} \in W^{2,\infty}$ and for $(\mu, \omega - \omega_0)$ close to 0 in $\mathbb{R}^2$. After decomposition, this becomes

$$\Big(\omega_0 \frac{d}{ds} - \mathbf{L}\Big) w = \mu \mathfrak{P}_0 \mathbf{L}'(\underline{v} + w) - (\omega - \omega_0)\frac{dw}{ds} + \mathfrak{P}_0 B(\underline{v} + w, \underline{v} + w), \quad (22)$$

$$\Big\langle \mu \mathbf{L}'(\underline{v} + w) - (\omega - \omega_0)\frac{d\underline{v}}{ds} + B(\underline{v} + w, \underline{v} + w), \zeta^\star e^{is}\Big\rangle = 0, \quad (23)$$

$$\mathbf{L}_0 v_{0,0} + \mu \mathbf{L}' v_{0,0} + \Big[B(\underline{v} + w, \underline{v} + w)\Big]_{0,0} = 0, \quad (24)$$

where we look for solutions $(w, A, v_{0,0})$ in a neighborhood of 0 in

$$\mathfrak{P}_0 H^{\sharp\sharp} \times \mathbb{C} \times W^{2,\infty}.$$

We first solve (22) with respect to $w \in \mathfrak{P}_0 H^{\sharp\sharp}$ in function of $\underline{v}$, $\omega - \omega_0$, $v_{0,0}$ and μ in a neighborhood of 0. Moreover, the choice of our spaces (see (20)) implies that

$$\frac{dw}{ds}, \mathfrak{P}_0 B(v,v) \in H^1(\mathbb{T}_1, \mathcal{X}_\eta),$$

which allows to apply the implicit function theorem with respect to w. The principal part independent of w in the right hand side of (22) is

$$\mu \mathbf{L}' \underline{v} + \mathfrak{P}_0 B(\underline{v}, \underline{v})$$

so we find

$$w = W(A, \overline{A}, \omega - \omega_0, \mu, v_{0,0}) \in \mathfrak{P}_0 H^{\sharp\sharp},$$

with

$$\begin{aligned} W(A, \overline{A}, \omega - \omega_0, \mu, v_{0,0}) = {} & \Big(\omega_0 \frac{d}{ds} - \mathbf{L}\Big)^{-1}\Big[\mu \mathfrak{P}_0 \mathbf{L}' \underline{v} + \mathfrak{P}_0 B(\underline{v}, \underline{v})\Big] \\ & + O\Big(\|\underline{v}\|^3 + \Big[|\mu| + |\omega - \omega_0|\Big]\|\underline{v}\|^2 + |\mu|\,|\omega - \omega_0|\,\|\underline{v}\|\Big). \end{aligned}$$

We notice that for $A = 0$, *id est* for $\underline{v} = v_{0,0}$, we have

$$W(0, 0, \omega - \omega_0, \mu, v_{0,0}) = 0.$$

Using now T_a, we have

$$T_a(A\zeta e^{is}) = Ae^{ia}\zeta e^{is}, \qquad T_a\underline{v}(s) = \underline{v}(s + a),$$

hence, the commuting property leads to

$$T_a W(A, \overline{A}, \omega - \omega_0, \mu, v_{0,0}) = W(e^{ia}A, e^{-ia}\overline{A}, \omega - \omega_0, \mu, v_{0,0}).$$

Moreover, we note that, in using $\tau_{-a/\alpha}$

$$T_a\tau_{-a/\alpha} = Id, \text{ for any } a \in \mathbb{R} \tag{25}$$

on the $\underline{v}$ part.

Replacing w by W in equations (23) and (24), we obtain an infinite dimensional system

$$\begin{aligned} F(A, \overline{A}, \omega - \omega_0, \mu, v_{0,0}) = 0, \\ \mathbf{L}_0 v_{0,0} + \mu\mathbf{L}' v_{0,0} + [B(\underline{v} + W, \underline{v} + W)]_{0,0} = 0. \end{aligned}$$

We notice that, applying T_a for any $a \in \mathbb{R}$,

$$F(e^{ia}A, e^{-ia}\overline{A}, \omega - \omega_0, \mu, v_{0,0}) = e^{ia}F(A, \overline{A}, \omega - \omega_0, \mu, v_{0,0}),$$

so that A is in factor in F and only occurs by $|A|^2$. Moreover

$$\Big\langle \mu\mathbf{L}'(\underline{v} + W), \zeta^\star e^{is} \Big\rangle = A\mu\langle \Delta\zeta, \zeta^\star\rangle + O(|\mu|\,\|W\|),$$

$$\Big\langle (\omega - \omega_0)\frac{d\underline{v}}{ds}, \zeta^\star e^{is} \Big\rangle = iA(\omega - \omega_0),$$

$$\Big\langle B(\underline{v}, \underline{v}), \zeta^\star e^{is} \Big\rangle = 2A\Big\langle B(\zeta, v_{0,0}), \zeta^\star \Big\rangle,$$

hence, it is clear that F is of the form

$$0 = A\mu\langle \Delta\zeta, \zeta^\star\rangle - iA(\omega - \omega_0) + bA|A|^2 + 2A\langle B(\zeta, v_{0,0}), \zeta^\star\rangle \tag{26}$$

$$+ O\Big(|A|\,|\mu|^2 + |\mu|\,|A|^2 + |\mu|\,|A|\,|v_{0,0}| + |A|\,|v_{0,0}|^2 + |A|^4\Big) \tag{27}$$

where

$$b = 2\Big\langle B\Big(\overline{\zeta}, (2i\omega_0 - L)^{-1}B(\zeta, \zeta)\Big), \zeta^\star \Big\rangle. \tag{28}$$

Finally, looking for $A \neq 0$ leads to

$$0 = \mu\langle \Delta\zeta, \zeta^\star\rangle - i(\omega - \omega_0) + b|A|^2 + 2\langle B(\zeta, v_{0,0}), \zeta^\star\rangle \tag{29}$$
$$+ O\Big(|\mu|^2 + (|\mu| + |\omega - \omega_0|)|A|^2 + |A|^4\Big). \tag{30}$$

Let us notice that

$$\langle \mathbf{L}'\zeta, \zeta^\star\rangle = \langle \Delta\zeta, \zeta^\star\rangle = \partial_\nu \lambda(\alpha, \nu_0(\alpha)).$$

The $(0,0)$ mode equation (24) has a very specific form. Indeed, defining by P_0 the projection giving the 0-Fourier mode, we may observe that

$$B(P_0 v, P_0 v) = 0, \text{ for any } v \in W^{2,\infty},$$

hence the term $B(\underline{v} + W, \underline{v} + W)$ has the decay property in $e^{-\eta y}$. It results that $[B(\underline{v} + W, \underline{v} + W)]_{0,0} \in L^\infty_\eta$. Moreover we observe that

$$\mathbf{L}_{0,0} + \mu\mathbf{L}' = (1 + \frac{\mu}{\nu_0})\mathbf{L}_{0,0},$$

hence, using Lemma **??**, (24) may be written in $W^{2,\infty}$ as

$$(1 + \frac{\mu}{\nu_0})v_{0,0} + \mathbf{L}_{0,0}^{-1}[B(\underline{v} + W, \underline{v} + W)]_{0,0} = 0,$$

which can be solved in $v_{0,0} \in W^{2,\infty}$ by the implicit function theorem, noticing that A only occurs with $|A|^2$. The principal part independent of $v_{0,0}$ in (24) is

$$\Big[B\Big(Ae^{is}\zeta + \bar{A}e^{-is}\bar{\zeta}, Ae^{is}\zeta + \bar{A}e^{-is}\bar{\zeta}\Big)\Big]_{0,0} = 2|A|^2 B(\zeta, \bar{\zeta}) \in C^0_\eta.$$

This leads to

$$v_{0,0} = -2|A|^2 \mathbf{L}_{0,0}^{-1} B(\zeta, \overline{\zeta}) + O\Big(|\mu|\,|A|^2 + |A|^4\Big).$$

Replacing $v_{0,0}$ by its expression in (29) leads to a two dimensional real system, solvable by implicit function theorem, with respect to μ and $\omega - \omega_0$. The equivariance of the system under the groups T_a and τ_β and (25) of the $\underline{v}$ part implies that (25) also holds on W, meaning that we obtain a travelling wave, with a function depending on (s, x) as $x + s/\alpha$. Defining

$$c = 4\Big\langle B(\zeta, L_{0,0}^{-1} B(\zeta, \overline{\zeta})), \zeta^\star\Big\rangle,$$

we finally obtain the bifurcating solution, parametrized by A (defined up to a phase shift)

$$\begin{aligned}
\mu &= \frac{\Re(c-b)}{\Re\langle\Delta\zeta,\zeta^\star\rangle}|A|^2 + O(|A|^4) \\
\omega-\omega_0 &= \Big(\Im(b-c) + \frac{\Re(c-b)\Im\langle\Delta\zeta,\zeta^\star\rangle}{\Re\langle\Delta\zeta,\zeta^\star\rangle}\Big)|A|^2 + O(|A|^4), \\
v_{0,0} &= -2|A|^2 L_{0,0}^{-1} B(\zeta,\overline{\zeta}) + O(|A|^4) \in W^{2,\infty}, \\
w &= A^2 e^{2is}(2i\omega_0 - L)^{-1} B(\zeta,\zeta) + c.c. + O(|A|^3) \in \mathfrak{P}_0 H^{\sharp\sharp}, \\
v &= Ae^{is}\zeta + \overline{A}e^{-is}\overline{\zeta} + v_{0,0} + w.
\end{aligned}$$

The theorem is proved as soon as we replace $|A|$ by ε. □

We notice that the phase of A is arbitrary, which corresponds to an arbitrary shift parallel to the x axis , or to shift in time. Moreover, by assumption,

$$\Re\langle\Delta\zeta,\zeta^\star\rangle > 0.$$

Then, the supercriticality or subcriticality of the bifurcation depends on the sign of $\Re(c-b)$ which needs to be computed.

5 Nonlinear stability of the bifurcating traveling wave

In all what follows the bracket $\langle\cdot,\cdot\rangle$ is understood as the duality product between $\mathcal{X}_\eta = \mathring{L}^2_\eta \oplus \mathring{L}^\infty$ and $\mathcal{X}^*_\eta = (\mathring{L}^2_\eta)^* \oplus \mathring{L}^1$.

5.1 Study of the linearized operator

We proved that the bifurcating periodic solution is in fact a travelling wave, function of $(\alpha x + \omega t, y)$. It is then natural to change coordinates in replacing x by

$$\xi = x + \frac{\omega t}{\alpha}.$$

With these coordinates, we define

$$\widehat{v}(\xi,y,t) = v(x,y,t),$$

so that the Navier-Stokes system becomes

$$\frac{d\widehat{v}}{dt} = \boldsymbol{L}_{\nu,\omega}\widehat{v} + B(\widehat{v},\widehat{v}),\ \nabla\cdot\widehat{v} = 0, \tag{31}$$

with

$$\boldsymbol{L}_{\nu,\omega}\widehat{v} = \boldsymbol{L}_{(\nu)}\widehat{v} - \frac{\omega}{\alpha}\frac{\partial \widehat{v}}{\partial \xi} = \boldsymbol{L}^{(0)}\widehat{v} + \mu \boldsymbol{L}'\widehat{v} - \frac{\omega}{\alpha}\frac{\partial \widehat{v}}{\partial \xi},$$

where

$$\boldsymbol{L}^{(0)} = \boldsymbol{L}^{(0)}_{(\nu_0)} + \boldsymbol{L}^{(1)}.$$

The linear operator $\boldsymbol{L}_{\nu,\omega}$ is acting in $\mathcal{X}_\eta$, with domain $\mathcal{Z}_\eta$. In the following we need to localize the essential spectrum of the linear operator

$$\boldsymbol{L}^{(0)}_{(\nu)} + \boldsymbol{L}^{(0,1)} - \frac{\omega}{\alpha}\frac{\partial}{\partial \xi} = \nu \Pi \Delta - \frac{\omega + \alpha U_+}{\alpha}\frac{\partial}{\partial \xi}$$

in $\mathring{\boldsymbol{L}}^2_\eta$. This is described in the following Lemma

Lemma 23. *There exists $\eta > 0$ small enough, such that for any λ in the spectrum of*

$$\nu \Pi \Delta - \frac{\omega + \alpha U_+}{\alpha}\frac{\partial}{\partial \xi}$$

acting in $\mathring{\boldsymbol{L}}^2_\eta$, we have $\lambda \in \Sigma_{U_+,\omega}$ where (see Figure 4)

$$\begin{aligned} \Sigma_{U_+,\omega} = \quad & \Big\{ \Re\lambda \leq -\nu(\alpha^2 - \eta^2) \Big\} \\ \cap \quad & \Big\{ \Big\{ 2\pi/3 \leq \arg(\lambda_0 + \nu\alpha^2) \leq \pi \Big\} \cup \Big\{ \Re\lambda \leq -\frac{\nu\alpha^2}{(\omega + \alpha U_+)^2 + 4\nu^2\alpha^2\eta^2}(\Im\lambda)^2 + \eta^2\nu \Big\} \Big\}. \end{aligned}$$

For λ outside of this region $\Sigma_{U_+,\omega}$, where

$$0 \leq \arg(\lambda_0 + \nu\alpha^2) \leq \frac{2\pi}{3} - \delta$$

with $0 < \delta < \pi/6$, and where we add δ_1 small to U_+, there exists $C > 0$ such that

$$\left\| \left(\nu\Pi\Delta - \frac{\omega + \alpha U_+}{\alpha}\frac{\partial}{\partial \xi} - \lambda \mathbb{I} \right)^{-1} \right\|_{\mathcal{L}(\mathring{\boldsymbol{L}}^2_\eta)} \leq \frac{C}{|\lambda + \nu\alpha^2|}.$$

The proof of this Lemma is identical to the proof of Lemma 9 for the essential spectrum, just changing U_+ in $U_+ + \omega/\alpha$. This enlarges the parabolic region.

Now we know that there is a bifurcating steady solution of (31) which is

$$\widehat{V}_\varepsilon(\xi, y) \stackrel{def}{=} V_\varepsilon(x, y, t),\ \omega = \omega(\varepsilon), \mu = \mu(\varepsilon),$$

as described in Theorem 22.

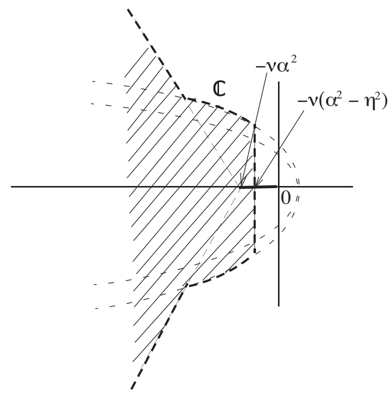

Figure 4: Hatched region $\Sigma_{U_+,\omega}$ and negative real line containing the spectrum of $\nu\Pi\Delta - (U_+ + \frac{\omega}{\alpha})\frac{\partial}{\partial\xi}$ and the essential spectrum of $\boldsymbol{L}_{\nu,\omega}$.

Now we introduce $\phi_0^*, \phi_1^* \in (\mathring{L}_\eta^2)^*$ defined by

$$\begin{aligned}\phi_0^* &= \frac{i}{2}(e^{i\alpha\xi}\widehat{\zeta^*} - e^{-i\alpha\xi}\overline{\widehat{\zeta^*}}),\\ \phi_1^* &= \frac{1}{2}(e^{i\alpha\xi}\widehat{\zeta^*} + e^{-i\alpha\xi}\overline{\widehat{\zeta^*}}),\end{aligned}$$

and we prove the following

Lemma 24. *For $\eta > 0$ small enough, there exists M such that the spectrum of $\boldsymbol{L}_{\nu,\omega}$ is first composed of an essential spectrum included in the region Σ_ω and in the half negative real line. The rest of the spectrum is a set of isolated eigenvalues λ with finite multiplicities, such that*

$$\lambda = \lambda_n^{(0)} - in\omega,$$

where $\lambda_n^{(0)}$ is an eigenvalue of $\boldsymbol{L}_{(\nu)}$ associated with the eigenvector $\zeta_n = e^{in\alpha}\widehat{v}_n \in \mathring{H}_\eta^2$.

Defining the linear operator

$$\mathcal{A}_\varepsilon = \boldsymbol{L}_{\nu,\omega} + 2B(\widehat{V}_\varepsilon, \cdot),$$

then, the essential spectrum of $\mathcal{A}_\varepsilon$ is formed by the half negative real line and a region perturbing at order ε^2 the region Σ_ε of figure 4.

For $\varepsilon = 0$, the operator

$$\mathcal{A}_0 = \boldsymbol{L}_{\nu,\omega}|_{\varepsilon=0} = \boldsymbol{L}_{(\nu_0)} - \frac{\omega_0}{\alpha}\frac{\partial}{\partial\xi}$$

has a double 0 *eigenvalue. For* $\varepsilon \neq 0$, 0 *is a simple eigenvalue of* $\mathcal{A}_\varepsilon$ *with eigenvector*

$$\begin{aligned} v_\varepsilon &= \frac{1}{\alpha\varepsilon}\frac{\partial \widehat{V}_\varepsilon}{\partial \xi} = i(e^{i\alpha\xi}\widehat{\zeta} - e^{-i\alpha\xi}\overline{\widehat{\zeta}}) + \mathcal{O}(\varepsilon) \ \in \mathring{H}^2_\eta \\ \langle v_\varepsilon, \phi_0^* \rangle &= 1,\ \langle v_\varepsilon, \phi_1^* \rangle = 0, \end{aligned}$$

and there is another simple eigenvalue $\sigma_\varepsilon = 2\Re(b-c)\varepsilon^2 + \mathcal{O}(\varepsilon^3)$ *in the neighborhood of* 0*. All other eigenvalues* λ *of* $\mathcal{A}_\varepsilon$ *are such that there exists* γ *independent of* ε*, with*

$$\Re\lambda < -\gamma < 0$$

for ε *close to* 0*. Moreover the eigenvector* u_ε *belonging to* σ_ε *has its order 1 part in* $\mathring{H}^2_\eta$*, while its order* ε *part has a component in* $\mathring{W}^{2,\infty}$*, and* u_ε *satisfies*

$$\langle u_\varepsilon, \phi_0^* \rangle = 0,\ \langle u_\varepsilon, \phi_1^* \rangle = 1.$$

Proof. The perturbation $-\frac{\omega}{\alpha}\frac{\partial}{\partial \xi}$ is in $\mathcal{L}(\mathcal{Z}_\eta, \mathring{H}^1_\eta)$ and is relatively bounded with respect to $\boldsymbol{L}^{(0)}$. The proof made for Lemma 7 applies for the perturbation operator $\boldsymbol{L}^{(1)} - \frac{\omega}{\alpha}\frac{\partial}{\partial \xi}$ showing that the spectrum of $\boldsymbol{L}_{\nu,\omega}$ is included in the region Σ defined by

$$|\lambda + \nu\alpha^2| < M,$$

$$\text{or } 2\pi/3 \leq |\arg(\lambda + \nu\alpha^2)| \leq \pi,\ \text{or } \lambda \in (-\infty, 0],$$

which is a right bounded region centered on the negative axis. This result is valid for the whole spectrum, including the essential spectrum and isolated eigenvalues.

Since the perturbation operator $\boldsymbol{L}^{(0,1)}$ is relatively compact with respect to $\boldsymbol{L}^{(0)}$ as well with respect to $\nu\Pi\Delta - \frac{\omega+\alpha U_+}{\alpha}\frac{\partial}{\partial \xi}$, we can assert that the essential spectrum of $\boldsymbol{L}_{\nu,\omega}$ is formed by $(-\infty, 0]$ corresponding to its part acting from $\mathring{W}^{2,\infty}$, and a part in $\Sigma_{U_+,\omega}$ corresponding to its part acting from $\mathring{H}^2_\eta$ (see Figure 4). The rest of the spectrum is composed of isolated eigenvalues with finite multiplicities, deduced from those of $\boldsymbol{L}_{(\nu)}$ in the following simple way. First we notice that there is no change in the subspace $\mathring{L}^\infty$ (0-Fourier mode), where the eigenvalues are the same for both operators. All other eigenvalues correspond to eigenvectors in $\mathring{H}^2_\eta$ of the form

$$e^{ni\alpha\xi}\widehat{\zeta}(y),\ n \neq 0,$$

so that an eigenvalue λ of $\boldsymbol{L}_{(\nu)}$ corresponds to an eigenvalue $\lambda - in\omega$ of $\boldsymbol{L}_{\nu,\omega}$, which does not change the real part of the eigenvalue.

Now about the linear operator $\mathcal{A}_\varepsilon$, we observe that the perturbation $2B(\hat{V}_\varepsilon, \cdot)$ is not relatively compact because of the 0 Fourier component of $\hat{V}_\varepsilon$, which does not decay as y goes to infinity. However, the part $2B(P_0\hat{V}_\varepsilon, \cdot)$ which is of order ε^2 only acts on non zero Fourier components, hence perturbs the spectrum (including the essential spectrum) at order ε^2 (see [13]). The rest of the perturbation $2B(\tilde{P}\hat{V}_\varepsilon, \cdot)$ is relatively compact, hence does not perturb the essential spectrum.

Now, we observe that 0 is a double eigenvalue of $\boldsymbol{L}_{\nu,\omega}|_{\varepsilon=0} = \boldsymbol{L}_{(\nu_0)} - \frac{\omega_0}{\alpha}\frac{\partial}{\partial\xi}$ with the 2-dimensional (on $\mathbb{R}$) eigenspace

$$a e^{i\alpha\xi}\widehat{\zeta} + \overline{a}e^{-i\alpha\xi}\overline{\widehat{\zeta}}. \tag{32}$$

Then differentiating

$$\mathbf{L}_{\nu,\omega}\widehat{V}_\varepsilon + B(\widehat{V}_\varepsilon, \widehat{V}_\varepsilon) = 0$$

with respect to ξ leads to

$$\boldsymbol{L}_{\nu,\omega}v_\varepsilon + 2B(\widehat{V}_\varepsilon, v_\varepsilon) = 0,$$

which shows that $v_\varepsilon \in \mathring{H}^2_\eta$ (defined in Lemma 24) belongs to the kernel of the linearized operator $\mathcal{A}_\varepsilon$, in particular for $\varepsilon = 0$, which corresponds to $a = i$ in (32). Let us look for eigenvalues close to 0 for ε close to 0. Here the problem is not standard since 0 is double, not isolated and lies in the essential spectrum of $\boldsymbol{L}_{\nu_0,\omega_0}$. However we can justify the following computation, identical to the computations used in the standard case. As soon as we can obtain the principal part of a potential eigenvector and of the potential eigenvalue, the rest of the expansion in powers of ε relies on the implicit function theorem, as in the computation of the bifurcating periodic solution. Now we check that

$$\langle v_\varepsilon, \phi_0^*\rangle = 1, \; \langle v_\varepsilon, \phi_1^*\rangle = 0,$$

due to the fact that higher order terms in v_ε occur with harmonics $e^{ni\xi}$, with $|n| > 1$. We notice that for any $u \in \mathring{H}^2_\eta$ we have in $\mathring{L}^2_\eta$,

$$\left\langle (\boldsymbol{L}_{(\nu_0)} - \frac{\omega_0}{\alpha}\frac{\partial}{\partial\xi})u, \phi_j^* \right\rangle = 0, \; j = 0, 1. \tag{33}$$

Indeed by definition of the duality product $\langle \mathring{L}^2_\eta, (\mathring{L}^2_\eta)^*\rangle$ we have

$$\left\langle \frac{\partial}{\partial\xi}u, \phi_j^* \right\rangle + \left\langle u, \frac{\partial}{\partial\xi}\phi_j^* \right\rangle = 0,$$

hence

$$\left\langle (\boldsymbol{L}_{(\nu_0)} - \frac{\omega_0}{\alpha}\frac{\partial}{\partial\xi})u, \phi_j^* \right\rangle = \left\langle u, (\boldsymbol{L}_{(\nu_0)}^* + \frac{\omega_0}{\alpha}\frac{\partial}{\partial\xi})\phi_j^* \right\rangle = 0$$

since

$$\left(\boldsymbol{L}_{(\nu_0)}^* + \frac{\omega_0}{\alpha}\frac{\partial}{\partial\xi}\right)\widehat{\zeta^*} = 0.$$

We may interpret (33) in saying that ϕ_j^* for $j = 1, 2$ are in the kernel of $(\boldsymbol{L}_{(\nu_0)}^* + \frac{\omega_0}{\alpha}\frac{\partial}{\partial\xi})$ It is known (see [13] p.185) that non isolated eigenvalues might not exist for the adjoint operator. Here we are saved by the fact that this occurs in $\overset{\circ}{H}{}_\eta^2$ where 0 is not in the essential spectrum of the reduced operator.

Now, since we perturb a double eigenvalue, we need to find another eigenvalue (necessarily real) close to 0. For this search, we make the Ansatz

$$\begin{aligned} \sigma &= \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + ... \\ u &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + ... \in \mathcal{Z}_\eta \end{aligned}$$

and identify powers of ε in the identity

$$\left[\boldsymbol{L}_{(\nu_0)} + \mu\boldsymbol{L}' - \frac{\omega}{\alpha}\frac{\partial}{\partial\xi} + 2B(\widehat{V}_\varepsilon, \cdot)\right]u = \sigma u.$$

Order ε^0 leads to

$$\boldsymbol{L}_{(\nu_0)}u_0 - \frac{\omega_0}{\alpha}\frac{\partial u_0}{\partial\xi} = 0,$$

which gives

$$u_0 = ae^{i\alpha\xi}\widehat{\zeta} + \overline{a}e^{-i\alpha\xi}\overline{\widehat{\zeta}},\ a \in \mathbb{C},$$

where we notice that $a = i$ corresponds to the already known solution v_ε. At order ε, we obtain

$$\sigma_1 u_0 = \boldsymbol{L}_{(\nu_0)}u_1 - \frac{\omega_0}{\alpha}\frac{\partial u_1}{\partial\xi} + 2B(V_1, u_0).$$

Taking the duality product with $e^{i\alpha\xi}\widehat{\zeta^*}$ leads to

$$a\sigma_1 = \left\langle 2B(V_1, u_0), e^{i\alpha\xi}\widehat{\zeta^*} \right\rangle$$

which vanishes because of the periodicity in ξ and of factors like $e^{ni\alpha\xi}$, with n odd in the duality product. Hence $\sigma_1 = 0$ and

$$\begin{aligned} u_1 &= -\widetilde{\left(\boldsymbol{L}_{(\nu_0)} - \frac{\omega_0}{\alpha}\frac{\partial}{\partial\xi}\right)}^{-1} 2B(V_1, u_0) \\ &= 2a\widehat{w}_{2,2}e^{2i\alpha\xi} + 2\overline{a}\overline{\widehat{w}_{2,2}}e^{-2i\alpha\xi} + (a + \overline{a})v_{2,0}. \end{aligned}$$

At order ε^2 we obtain

$$\sigma_2 u_0 = \Big(\boldsymbol{L}_{(\nu_0)} - \frac{\omega_0}{\alpha}\frac{\partial}{\partial \xi}\Big)u_2 + \mu_2 \boldsymbol{L}' u_0 - \frac{\omega_2}{\alpha}\frac{\partial u_0}{\partial \xi} + 2B(V_1, u_1) + 2B(V_2, u_0).$$

Taking the duality product with $e^{i\alpha\xi}\widehat{\zeta}^*$ leads to

$$a\sigma_2 = a\mu_2 \Big\langle (D^2 - \alpha^2)\widehat{\zeta}, \widehat{\zeta}^* \Big\rangle - ia\omega_2 - (a + \overline{a})c + 2ab + \overline{a}b - ac.$$

The identity

$$\mu_2 \Big\langle (D^2 - \alpha^2)\widehat{\zeta}, \widehat{\zeta}^* \Big\rangle - i\omega_2 = c - b$$

reduces the above identity to

$$a\sigma_2 = (a + \overline{a})(b - c).$$

We know that $(a + \overline{a}) = 0$ is a solution at any order, which gives the eigenvalue 0 and the eigenvector v_ε. We deduce the other solution (which is defined up to a real factor) by choosing

$$\arg a = \theta, \;\; a = \frac{e^{i\theta}}{\cos\theta}$$

and

$$\sigma_2 = 2\Re(b - c).$$

Now

$$\langle u_0, \phi_0^* \rangle = 0, \;\; \langle u_0, \phi_1^* \rangle = 1.$$

We can go on the computation of σ_n using the Fredholm alternative as in the simple case, so that finally

$$\langle u_\varepsilon, \phi_0^* \rangle = 0, \;\; \langle u_\varepsilon, \phi_1^* \rangle = 1,$$

because of periodicity in ξ and factors as $e^{in\xi}$ with $|n| \neq 1$ for $u_\varepsilon - u_0$. This ends the proof of Lemma 24. □

5.2 Elimination of the Goldstone mode. Operator $\boldsymbol{A}_\varepsilon$.

The Goldstone mode is the eigenvector $\frac{\partial V_\varepsilon}{\partial \xi} = \alpha\varepsilon v_\varepsilon$. Using the invariance of the system (31) under translations in ξ, we can eliminate the 0 eigenvalue and obtain a system in the codimension 1 subspace

$$\langle u, \phi_0^* \rangle = 0.$$

Indeed, let us set

$$\widehat{v} = \tau_b(\widehat{V}_\varepsilon + \varepsilon u), \ \langle u, \phi_0^* \rangle = 0, \tag{34}$$

where τ_b represents the shift $\xi \to \xi + b$, not forgetting that $\langle v_\varepsilon, \phi_0^* \rangle = 1$. Then

$$\frac{d\widehat{v}}{dt} = \tau_b \frac{\varepsilon du}{dt} + \varepsilon \frac{db}{dt} \tau_b \Big(\alpha v_\varepsilon + \frac{\partial u}{\partial \xi} \Big)$$

and (31) becomes, after factoring out $\varepsilon \tau_b$

$$\frac{du}{dt} + \frac{db}{dt} \Big(\alpha v_\varepsilon + \frac{\partial u}{\partial \xi} \Big) = \boldsymbol{L}_{\nu,\omega} u + 2B(\widehat{V}_\varepsilon, u) + \varepsilon B(u,u). \tag{35}$$

Taking the duality product with ϕ_0^* leads to

$$\frac{db}{dt} \Big(\alpha + \Big\langle \frac{\partial u}{\partial \xi}, \phi_0^* \Big\rangle \Big) = \langle \boldsymbol{L}_{\nu,\omega} u, \phi_0^* \rangle + \Big\langle 2B(\widehat{V}_\varepsilon, u) + \varepsilon B(u,u), \phi_0^* \Big\rangle, \tag{36}$$

which may be written as

$$\frac{db}{dt} = g_\varepsilon(u). \tag{37}$$

Taking into account

$$\Big\langle \frac{\partial u}{\partial \xi}, \phi_0^* \Big\rangle = - \Big\langle u, \frac{\partial}{\partial \xi} \phi_0^* \Big\rangle = \alpha \langle u, \phi_1^* \rangle,$$

$$\langle \boldsymbol{L}_{\nu,\omega} u, \phi_0^* \rangle = \langle \mu \boldsymbol{L}' u, \phi_0^* \rangle + \frac{\omega - \omega_0}{\alpha} \langle u, \phi_1^* \rangle = \mathcal{O}(\varepsilon^2 ||\widetilde{u}||_{\mathring{L}^2_\eta}),$$

we observe that, for $||u||_{\mathcal{Z}_\eta}$ small enough,

$$g_\varepsilon(u) = \frac{\varepsilon}{\alpha} \Big\langle 2B(V_1, u), \phi_0^* \Big\rangle + \mathcal{O}\Big(\varepsilon^2 ||u||_{\mathcal{Z}_\eta} + \varepsilon ||u||^2_{\mathcal{Z}_\eta} \Big).$$

Now defining the projection Q_ε for any $v \in \mathcal{X}_\eta$ by

$$Q_\varepsilon v = v - \langle v, \phi_0^* \rangle v_\varepsilon,$$

then, since $v_\varepsilon \in \mathring{H}^2_\eta$, we observe that

$$P_0(Q_\varepsilon v) = P_0 v, \ Q_\varepsilon u_\varepsilon = u_\varepsilon.$$

The rest of (35), which is now independent of b, becomes

$$\frac{du}{dt} = \boldsymbol{A}_\varepsilon u + \varepsilon Q_\varepsilon B(u,u) - g_\varepsilon(u) Q_\varepsilon \frac{\partial u}{\partial \xi}, \tag{38}$$

where the linear operator $\boldsymbol{A}_\varepsilon$ is defined by

$$\boldsymbol{A}_\varepsilon = Q_\varepsilon(\boldsymbol{L}_{\nu,\omega} + 2B(\widehat{V}_\varepsilon, \cdot)) = Q_\varepsilon \mathcal{A}_\varepsilon.$$

We know that the eigenvector u_ε satisfies

$$\langle u_\varepsilon, \phi_0^* \rangle = 0,$$

and

$$\Big(\boldsymbol{L}_{\nu,\omega} + 2B(\widehat{V}_\varepsilon, \cdot)\Big) u_\varepsilon = \sigma_\varepsilon u_\varepsilon,$$

so that we have

$$\boldsymbol{A}_\varepsilon u_\varepsilon = Q_\varepsilon \Big(\boldsymbol{L}_{\nu,\omega} + 2B(\widehat{V}_\varepsilon, \cdot)\Big) u_\varepsilon = \sigma_\varepsilon u_\varepsilon.$$

Remark 25. *We note that u_ε is not in $\mathring{H}^2$ since it contains a $0-$ Fourier mode. In fact we have*

$$u_\varepsilon = \widetilde{P} u_\varepsilon + P_0 u_\varepsilon, \;\; \langle u_\varepsilon, \phi_1^* \rangle = 1, \;\; P_0 u_\varepsilon = 2 v_{2,0} \varepsilon + \mathcal{O}(\varepsilon^2).$$

The rest of the spectrum of $\boldsymbol{A}_\varepsilon$ is the same as the spectrum of $\mathcal{A}_\varepsilon$ except the eigenvalues σ_ε and 0. The estimates obtained for the spectrum are similar to those for $\boldsymbol{L}_{\nu,\omega}$. Due to the perturbation of order ε, we can assert that all eigenvalues other than σ_ε, outside of the region where the essential spectrum is located, are in the region indicated on Figure 4, have finite multiplicities, are isolated and situated on the left of the line $\Re\lambda < -k < 0$. Observe that now the subspace with 0 average and the subspace with only the 0 Fourier mode are coupled by the term $B(\widehat{V}_\varepsilon, u)$:

$$\begin{aligned}
\widetilde{P} B(\widehat{V}_\varepsilon, u) &= \widetilde{P} B(\widetilde{P}\widehat{V}_\varepsilon, \widetilde{u}) + B(\widetilde{P}\widehat{V}_\varepsilon, P_0 u) + B(P_0 \widehat{V}_\varepsilon, \widetilde{u}) \\
P_0 B(\widehat{V}_\varepsilon, u) &= P_0 B(\widetilde{P}\widehat{V}_\varepsilon, \widetilde{u}),
\end{aligned}$$

so that the estimates on the semi group are more complicated if we wish to separate the subspaces.

5.3 Nonlinear stability

In the subspace $P_\varepsilon \mathcal{X}_\eta$ we need to solve the initial value problem

$$\frac{du}{dt} = \boldsymbol{A}_\varepsilon u + B_\varepsilon(u), \;\; u(0) \in \mathcal{Z}_\eta \tag{39}$$

with $u(0)$ small enough in norm, and where B_ε is analytic from $\mathcal{Z}_\eta$ to $\mathcal{Y}_\eta$ with

$$B_\varepsilon(u) = \varepsilon Q_\varepsilon B(u,u) - g_\varepsilon(u) Q_\varepsilon \frac{\partial u}{\partial \xi}.$$

Using that $Q_\varepsilon \widetilde{P} = \widetilde{P} Q_\varepsilon$, $P_0 Q_\varepsilon = P_0$ and $B(P_0 u, P_0 v) \equiv 0$, we have

$$\begin{aligned}
\widetilde{P} B_\varepsilon(\widetilde{u} + P_0 u) &= \varepsilon Q_\varepsilon \widetilde{P} B(\widetilde{u}, \widetilde{u}) + 2\varepsilon Q_\varepsilon B(\widetilde{u}, P_0 u) - g_\varepsilon(\widetilde{u} + P_0 u) Q_\varepsilon \frac{\partial \widetilde{u}}{\partial \xi}, \\
P_0 B_\varepsilon(\widetilde{u} + P_0 u) &= \varepsilon P_0 B(\widetilde{u}, \widetilde{u}),
\end{aligned}$$

hence, there exists $\delta > 0$ such that for $\|u\|_{\mathcal{Z}_\eta} \le \delta$

$$\|B_\varepsilon(u)\|_{\mathcal{Y}_{\eta,\eta}} \le c\varepsilon \|u\|^2_{\mathcal{Z}_\eta}, \tag{40}$$

$$|g_\varepsilon(u)| \le c\varepsilon \|u\|_{\mathcal{Z}_\eta},$$

where

$$\mathcal{Y}_{\eta,\eta} = \Big\{ f \in \mathcal{Y}_\eta;\ P_0 f \in \mathring{W}^{1,\infty}_\eta \Big\}.$$

From the properties of the operator $\boldsymbol{A}_\varepsilon$ we can prove that the spectrum of $e^{\boldsymbol{A}_\varepsilon t}$ is the union of an essential spectrum and of a bounded set of isolated eigenvalues of finite multiplicities. More precisely the essential spectrum is included in the union of the real interval $[0,1]$ with a set included in $\{\lambda \in \mathbb{C}; \lambda = e^{\sigma t}, \sigma \in \Sigma_\omega\}$ where Σ_ω is as in Figure 4. It results that in the case when the eigenvalue $\sigma_\varepsilon < -\kappa\varepsilon^2$, the spectral radius of $e^{\boldsymbol{A}_\varepsilon t}$ equals 1. Hence, we have

$$\|e^{\boldsymbol{A}_\varepsilon t}\|_{\mathcal{L}(\mathcal{Z}_\eta)} \le C, \text{ for } t \ge 0.$$

However, this estimate is not sufficient to avoid secular terms in the solution of the initial value problem (39).

Let us proceed in adapting the proof of Theorem 18. We obtain the following Lemma

Lemma 26. *Let us assume that $\sigma_\varepsilon < -\kappa\varepsilon^2$, then there exists $C(\varepsilon) > 0$ such that*

$$\|e^{\boldsymbol{A}_\varepsilon t}\|_{\mathcal{L}(\mathcal{Z}_{\eta,\eta}, \mathcal{Z}_\eta)} \le \frac{C(\varepsilon)}{1+t},\ t \ge 0 \tag{41}$$

$$\|\widetilde{P} e^{\boldsymbol{A}_\varepsilon t} f\|_{\mathcal{L}(\mathring{H}^2_\eta)} \le \frac{C(\varepsilon)}{\sqrt{t}(1+\sqrt{t})} \|\widetilde{P} f\|_{\mathring{H}^1_\eta} + \frac{C(\varepsilon)}{1+t} \|P_0 f\|_{\mathring{W}^{1,\infty}_\eta},\ t > 0 \tag{42}$$

$$\|P_0 e^{\boldsymbol{A}_\varepsilon t} f\|_{\mathcal{L}(\mathring{W}^{2,\infty})} \le \frac{C(\varepsilon)}{1+t} \|\widetilde{P} f\|_{\mathring{H}^1_\eta} + \frac{C(\varepsilon)}{\sqrt{t}(1+\sqrt{t})} \|P_0 f\|_{\mathring{W}^{1,\infty}_\eta},\ t > 0.$$

The proof is done in Appendix 6.9. The estimate (42), valid for all $t > 0$, shows the loss of regularity as $t \to 0$. We are now ready to prove the following

Theorem 27. *Assuming $\sigma_\varepsilon < 0$, choosing $\eta > 0$ and ε small enough, there exists $\delta > 0$ such that for $||u(0)||_{\mathcal{Z}_{\eta,\eta}} \leq \delta$ there is a unique solution u of*

$$\frac{du}{dt} = \boldsymbol{A}_\varepsilon u + B_\varepsilon(u), u(0) \in \mathcal{Z}_{\eta,\eta},\ t \geq 0. \tag{43}$$

Moreover, we have

$$||u(t)||_{\mathcal{Z}_\eta} \leq \frac{2C(\varepsilon)}{1+t}||u(0)||_{\mathcal{Z}_{\eta,\eta}},\ t \geq 0.$$

This stability result is completed by the solution $b(t)$ for the shift mode given by (37), leading to

$$|b(t)| \leq |b(0)| + \ln(1+t)C''(\varepsilon)\|u(0)\|_{\mathcal{Z}_{\eta,\eta}},\ t \geq 0.$$

Proof. The integral formulation of (43) is

$$u(t) = e^{\boldsymbol{A}_\varepsilon t}u(0) + \int_0^t e^{\boldsymbol{A}_\varepsilon (t-s)} B_\varepsilon\Big(u(s)\Big)\, ds. \tag{44}$$

Using the estimates (40), (41), (42), we obtain

$$\|u(t)\|_{\mathcal{Z}_\eta} \leq \frac{C(\varepsilon)}{1+t}\|u(0)\|_{\mathcal{Z}_{\eta,\eta}} + \int_0^t \frac{c\varepsilon C(\varepsilon)}{\sqrt{t-s}(1+\sqrt{t-s})}\|u(s)\|^2_{\mathcal{Z}_\eta}\, ds. \tag{45}$$

Let us define

$$|||u|||_t = \sup_{s\in[0,t]}\Big\|(1+s)u(s)\Big\|_{\mathcal{Z}_\eta},$$

then

$$\|(1+t)u(t)\|_{\mathcal{Z}_\eta} \leq C(\varepsilon)\|u(0)\|_{\mathcal{Z}_{\eta,\eta}} + K(\varepsilon)|||u|||_t^2, \tag{46}$$

where

$$\int_0^t \frac{c\varepsilon C(\varepsilon)(1+t)}{\sqrt{t-s}(1+\sqrt{t-s})(1+s^2)}ds \leq K(\varepsilon), \tag{47}$$

which can be checked by splitting the integral in $s < t/2$ and $s > t/2$. In view of (46), if $\|u(0)\|_{\mathcal{Z}_{\eta,\eta}}$ is small enough, then $\|(1+t)u(t)\|_{\mathcal{Z}_\eta}$ is bounded uniformly in time, by a constant time $\|u(0)\|_{\mathcal{Z}_{\eta,\eta}}$. The Lemma is proved. □

6 Appendix

6.1 Proof of Lemma 2

The Helmholtz decomposition is very classical, however, as our function spaces are not standard, we have to detail it. We take the Fourier transform in the horizontal variable of the decomposition

$$\widetilde{u} = \widetilde{v} + \nabla\phi, \quad \Delta\phi = \nabla\cdot\widetilde{u}, \quad \frac{\partial\phi}{\partial y}|_{y=0} = \widetilde{u}^y|_{y=0},$$

which gives

$$\begin{aligned} (D^2 - n^2\alpha^2)\phi_n &= in\alpha u_n^x + Du_n^y = g_n, \\ D\phi_n(0) &= \widetilde{u}_n^y(0) = h_n, \end{aligned}$$

where, by definition

$$|u_n(y)| \leq M_n e^{-\eta y}, \qquad ||u||^2_{L^2_\eta} = \sum_{|n|\geq 1} M_n^2 < \infty,$$

and Du_n^y is defined in the distribution sense. We easily obtain (for $n > 0$)

$$\phi_n(y) = -\frac{1}{2n\alpha}\int_0^\infty g_n(\tau)e^{-n\alpha|\tau-y|}d\tau - \frac{1}{2n\alpha}\int_0^\infty g_n(\tau)e^{-n\alpha(\tau+y)}d\tau - \frac{h_n}{n\alpha}e^{-n\alpha y}.$$

Replacing g_n by its expression gives, after an integration by parts,

$$\begin{aligned} \phi_n(y) &= \frac{1}{2}\int_0^y (u_n^y - iu_n^x)(\tau)e^{n\alpha(\tau-y)}d\tau - \frac{1}{2}\int_y^\infty (u_n^y + iu_n^x)(\tau)e^{n\alpha(\tau-y)}d\tau \\ &\quad -\frac{1}{2}\int_0^\infty (u_n^y + iu_n^x)e^{-n\alpha(\tau+y)}d\tau. \end{aligned}$$

We observe that $\phi_n \in C^1(\mathbb{R}^+)$ with

$$\begin{aligned} D\phi_n(y) &= \frac{\alpha n}{2}\int_0^y (iu_n^x - u_n^y)(\tau)e^{n\alpha(\tau-y)}d\tau - \frac{\alpha n}{2}\int_y^\infty (iu_n^x + u_n^y)(\tau)e^{n\alpha(\tau-y)}d\tau \\ &\quad +\frac{\alpha n}{2}\int_0^\infty (u_n^y + iu_n^x)(\tau)e^{-n\alpha(\tau+y)}d\tau - u_n^x(y), \end{aligned}$$

and using (48), provided that $\eta < \alpha/2$, ϕ_n is such that the following estimates hold, with c independent of n

$$|\phi_n(y)e^{\eta y}| \leq c\frac{M_n}{n}, \quad |D\phi_n(y)e^{\eta y}| \leq cM_n.$$

The result is that $\nabla\phi \in L^2_\eta$ and $\widetilde{v} \in \mathring{L}^2_\eta$, with $||\widetilde{v}||_{\mathring{L}^2_\eta} \leq c||u||_{L^2_\eta}$. The result on the component in $\mathring{L}^\infty$ is straightforward.

6.2 Proof of Lemma 4

6.2.1 Resolvent estimate and spectrum in $\mathring{L}^\infty$

For the part of the linear operator in $\mathring{L}^\infty$, we need to solve for $\lambda \notin (-\infty, 0]$,

$$\begin{aligned} \nu D^2 v_0^x - \lambda v_0^x &= f_0 \in L^\infty(\mathbb{R}^+) \\ v_0^x(0) &= 0, \; v_0^x \in W^{2,\infty}. \end{aligned}$$

Let us define s such that $\Re s > 0$ and $s^2 = \lambda/\nu$, then

$$\begin{aligned} v_0^x(y) &= -\frac{1}{2s\nu}\int_y^\infty f_0(\tau)e^{s(y-\tau)}d\tau - \frac{1}{2s\nu}\int_0^y f_0(\tau)e^{-s(y-\tau)}d\tau \\ &+\frac{1}{2s\nu}\int_0^\infty f_0(\tau)e^{-s(y+\tau)}d\tau, \end{aligned}$$

so that, we check that $v_0^x \in W^{2,\infty}$, and

$$|v_0^x(y)| \le \frac{\|f_0\|_{L^\infty}}{|s||\Re s|\nu},$$

hence for $\arg\lambda = \theta \in [0,\pi)$, if $\Re\lambda < 0$,

$$\frac{1}{|s||\Re s|\nu} = \frac{1}{\nu|s|^2\cos(\theta/2)} = \frac{1}{|\lambda|\cos(\theta/2)}.$$

If $\Re\lambda \ge 0$, then we have

$$\frac{1}{|s||\Re s|\nu} \le \frac{\sqrt{2}}{|\lambda|}$$

since in this case, $0 \le \theta \le \pi/2$. This proves the first estimate in Lemma 4.

Let us now concentrate on the case $\lambda < 0$. Then we have

$$v_0^x(y) = A\sin sy + \frac{1}{s\nu}\int_0^y f_0(\tau)\sin s(y-\tau)\,d\tau,$$

where $s^2 = -\lambda/\nu \ge 0$ and where A is arbitrary. Let us now show that the range of $\boldsymbol{L}_{(\nu)}^{(0)} - \lambda\mathbb{I}$ is not closed. Indeed, let us choose $f_0 \in L^\infty(\mathbb{R}^+)$ such that $f_0(y) = \chi(y)/y$, where $\chi(y)$ is the characteristic function of the set

$$\cup_{n\ge1}[\frac{2\pi n}{s}, \frac{2\pi n + \pi/2}{s}].$$

We obtain

$$\int_0^y f_0(\tau)\sin s(y-\tau)d\tau = \sin sy \int_0^y \chi\frac{\cos s\tau}{\tau}d\tau - \cos sy\int_0^y \chi\frac{\sin s\tau}{\tau}d\tau,$$

and

$$\sum_{1\le n\le N(y)} \frac{s}{(2\pi n+\pi/2)} < \int_0^y \chi\frac{\cos s\tau}{\tau}d\tau < \sum_{1\le n\le N(y)}\frac{s}{(2\pi n)}$$
$$\sum_{1\le n\le N(y)} \frac{s}{(2\pi n+\pi/2)} < \int_0^y \chi\frac{\sin s\tau}{\tau}d\tau < \sum_{1\le n\le N(y)}\frac{s}{(2\pi n)}$$

with $[sy] = 2\pi N(y)$,$[\cdot]$ being the integer part. As $y\to\infty$, series on both sides diverge, since the function $\int_0^y f_0(\tau)\sin s(y-\tau)d\tau$ behaves as

$$(\sin sy - \cos sy)\sum_{1\le n\le N(y)}\frac{s}{(2\pi n)},$$

hence the limit is not in $W^{2,\infty}(\mathbb{R}^+)$, showing that f_0 is not in the range of $\boldsymbol{L}^{(0)}_{(\nu)} - \lambda\mathbb{I}$. Now consider the sequence $\{f^{(N)}\}_{N\in\mathbb{N}}$ defined by

$$f^{(N)}(y) = f_0(y)\chi_{[0,N]}.$$

It is clear that $\{f^{(N)}\}$ is a Cauchy sequence in $L^\infty(\mathbb{R}^+)$ since for $M>N$

$$\|f^{(N)} - f^{(M)}\|_{L^\infty} \le \frac{1}{N}$$

and this series converges in $L^\infty(\mathbb{R}^+)$ towards f_0. Moreover the functions $f^{(N)}(y)$ lie in the range of $\boldsymbol{L}^{(0)}_{(\nu)} - \lambda\mathbb{I}$ since

$$v^x_{0N}(y) = \frac{1}{s\nu}\int_0^N f_0(\tau)\sin s(y-\tau)d\tau,$$

is the solution of $(\boldsymbol{L}^{(0)}_{(\nu)} - \lambda\mathbb{I})v_{0N} = f^{(N)}$ with $v^x_{0N}\in W^{2,\infty}$. Since f_0 is not in the range, this shows that the range is not closed.

For $\lambda = 0$, we obtain

$$v^x_0(y) = -\frac{1}{\nu}\Big[\int_0^y \tau f_0(\tau)d\tau + y\int_y^\infty f_0(\tau)d\tau\Big]$$

which shows that 0 is not eigenvalue. Moreover in choosing $f_0(y) = 1/(1+y)$ in L^∞ but not in the range, and choosing a Cauchy sequence $\{f^{(N)} = f_0\chi_{[0,N]}\}$ converging to f_0 in L^∞ and sitting in the range of $\boldsymbol{L}_{(\nu)}$, shows that the range is not closed. Hence $\lambda = 0$ lies in the essential spectrum.

6.2.2 Resolvent estimates in $\mathring{L}^2_\eta$

Let us now study the linear system

$$(\boldsymbol{L}^{(0)}_{(\nu)} - \lambda)v = f \in \mathring{L}^2_\eta,$$

namely Stokes' equation, where we look for $v \in \mathring{H}^2_\eta$. Using the Fourier series for f and v leads for $n \neq 0$ to

$$\begin{aligned} \nu\Big(D^2 - n^2\alpha^2 - \frac{\lambda}{\nu}\Big)v_n + \begin{pmatrix} in\alpha \\ D \end{pmatrix} q_n &= \begin{pmatrix} f_n \\ g_n \end{pmatrix}, \\ in\alpha v^x_n + Dv^y_n &= 0, \end{aligned} \tag{48}$$

with boundary condition $v_n|_{y=0} = 0$, where

$$in\alpha f_n + Dg_n = 0.$$

By definition of $\mathring{L}^2_\eta$, $\|(f_n, g_n)\|_{C^0_\eta} \leq M_n$ with $\sum_{n\neq 0} M^2_n = \|(f, g)\|^2_{L^2_\eta}$. This leads to

$$\begin{aligned} (D^2 - n^2\alpha^2)\Big[D^2 - \Big(n^2\alpha^2 + \frac{\lambda}{\nu}\Big)\Big]v^y_n &= \frac{1}{\nu}(D^2 - n^2\alpha^2)g_n, \\ v^y_n &= Dv^y_n = 0 \text{ for } y = 0, \end{aligned} \tag{49}$$

leading to explicit expressions for v^x_n and v^y_n. With

$$s^2_n = n^2\alpha^2 + \frac{\lambda}{\nu},$$

we obtain

$$v^y_n(y) = w_n(y) - \frac{w_n(0)}{2}\Big[\frac{n\alpha + s_n}{n\alpha - s_n}(e^{-s_n y} - e^{-n\alpha y}) + e^{-s_n y} + e^{-n\alpha y}\Big], \tag{50}$$

where

$$w_n(y) = -\frac{1}{2\nu s_n}\int_0^\infty g_n(\tau)e^{-s_n|\tau - y|}d\tau - \frac{1}{2\nu s_n}\int_0^\infty g_n(\tau)e^{-s_n(\tau+y)}d\tau. \tag{51}$$

We have the following useful Lemma:

Lemma 28. *There exists $C > 0$ such that for any δ with $0 < \delta < \pi/6$, any $n \neq 0$ and any $\lambda \in \mathbb{C}$ such that $0 \leq \arg(\lambda + \nu\alpha^2) \leq 2\pi/3 - \delta$ we have*

$$|s_n| \geq \Re s_n \geq C\Big(\alpha\sqrt{n^2 - 1} + |\lambda + \nu\alpha^2|^{1/2}\Big).$$

Proof. Using

$$s_n^2 = (n^2-1)\alpha^2 + \frac{\lambda+\nu\alpha^2}{\nu},$$

we have, with $\theta = \arg(\lambda+\nu\alpha^2)$,

$$|s_n|^4 = \Big[(n^2-1)\alpha^2 + \frac{\cos\theta}{\nu}|\lambda+\nu\alpha^2|\Big]^2 + \frac{1}{\nu^2}|\lambda+\nu\alpha^2|^2\sin^2\theta,$$

hence

$$|s_n|^4 = (n^2-1)^2\alpha^4 + \frac{1}{\nu^2}|\lambda+\nu\alpha^2|^2 + 2(n^2-1)\alpha^2\frac{1}{\nu}|\lambda+\nu\alpha^2|\cos\theta,$$

which implies

$$\begin{aligned} |s_n|^4 &\geq \Big[(n^2-1)^2\alpha^4 + \frac{1}{\nu^2}|\lambda+\nu\alpha^2|^2\Big] \text{ for } \cos\theta \geq 0, \\ &\geq (1-|\cos\theta|)\Big[(n^2-1)^2\alpha^4 + \frac{1}{\nu^2}|\lambda+\nu\alpha^2|^2\Big] \text{ for } \cos\theta < 0. \end{aligned}$$

Hence, in all cases

$$|s_n|^2 \geq \cos\frac{\theta}{2}\Big\{(n^2-1)\alpha^2 + \frac{1}{\nu}|\lambda+\nu\alpha^2|\Big\}, \tag{52}$$

and, as $\theta/2 \leq \pi/3$,

$$|s_n| \geq \frac{1}{2}\Big[(n^2-1)^{1/2}\alpha + \frac{1}{\nu^{1/2}}|\lambda+\nu\alpha^2|^{1/2}\Big] \geq C\Big(\alpha\sqrt{n^2-1} + |\lambda+\nu\alpha^2|^{1/2}\Big),$$

where the constant C depends on ν. Now, we also have

$$(\Re s_n)^2 = \frac{1}{2}\Big[\Re s_n^2 + |s_n|^2\Big] = \frac{1}{2}\Big[(n^2-1)\alpha^2 + \frac{1}{\nu}|\lambda+\nu\alpha^2|\cos\theta + |s_n|^2\Big].$$

Using (52), we obtain

$$(\Re s_n)^2 \geq \frac{1}{2}\Big[\Big(1+\cos\frac{\theta}{2}\Big)(n^2-1)\alpha^2 + \Big(\cos\frac{\theta}{2}+\cos\theta\Big)\frac{1}{\nu}|\lambda+\nu\alpha^2|\Big].$$

Since $\cos\theta/2 + \cos\theta \geq \cos(\pi/3-\delta/2) - \cos(\pi/3+\delta) > 0$, it results that, as above (adapting the constant C),

$$\Re s_n \geq C\Big(\alpha\sqrt{n^2-1} + |\lambda+\nu\alpha^2|^{1/2}\Big),$$

which ends the proof. □

Now, from (51) we have the estimate

$$|w_n(y)e^{\eta y}| \le \frac{3M_n}{2\nu|s_n|(\Re s_n - \eta)},$$

and assuming that λ is such that $|\lambda+\nu\alpha^2| \ge \varepsilon_0^2$, we choose $\eta > 0$ such that

$$\eta < C\frac{\varepsilon_0}{2},$$

so that

$$\Re s_n - \eta > C\Big(\alpha\sqrt{n^2-1} + |\lambda+\nu\alpha^2|^{1/2} - \frac{\varepsilon_0}{2}\Big) > \frac{C}{2}\Big(\alpha\sqrt{n^2-1} + |\lambda+\nu\alpha^2|^{1/2}\Big).$$

Finally we have (using $|n| \neq 0$)

$$|w_n(y)e^{\eta y}| \le \frac{cM_n}{(n^2-1)\alpha + |\lambda+\nu\alpha^2|}, \tag{53}$$

hence

$$||w||_{L^2_\eta} \le \frac{c_1}{|\lambda+\nu\alpha^2|}||g||_{L^2_\eta}.$$

Coming back to (50) we observe that

$$\Big|\frac{n\alpha+s_n}{n\alpha-s_n}\Big| = \frac{|s_n+n\alpha|^2}{|\alpha^2-|\lambda+\nu\alpha^2||}$$

and assuming that $|\lambda+\nu\alpha^2| \neq \alpha^2$ when λ is real , it is clear that

$$\frac{|s_n+n\alpha|^2}{|s_n|(\Re s_n-\eta)} \le c_2 \text{ independent of } n \text{ and } \lambda.$$

It results that

$$||\widetilde{v}^y||_{L^2_\eta} \le \frac{c_1}{|\lambda+\nu\alpha^2|}||g||_{L^2_\eta}.$$

Now we have

$$v_n^x = -\frac{Dv_n^y}{in\alpha},$$

hence

$$v_n^x(y) = -\frac{Dw_n(y)}{in\alpha} - iw_n(0)\frac{s_n}{n\alpha-s_n}(e^{-n\alpha y} - e^{-s_n y}). \tag{54}$$

It is easy to check that, after integration by parts,

$$Dw_n(y) = -\frac{1}{2\nu s_n}\int_0^\infty Dg_n(\tau)e^{s_n|y-\tau|}d\tau + \frac{1}{2\nu s_n}\int_0^\infty Dg_n(\tau)e^{-s_n(y+\tau)}d\tau,$$

and since

$$f_n = -\frac{Dg_n}{in\alpha},\ |f_n(y)e^{\eta y}| \leq M_n,$$

we obtain

$$\Big|\frac{Dw_n(y)}{in\alpha}e^{\eta y}\Big| \leq \frac{3M_n}{2\nu|s_n|(\Re s_n - \eta)}. \tag{55}$$

Now collecting (54), (55), (53) for $w_n(0)$, and

$$\frac{1}{|n\alpha - s_n|(\Re s_n - \eta)} = \frac{|n\alpha + s_n|}{(\Re s_n - \eta)|\alpha^2 - |\lambda + \nu\alpha^2||} \leq c_4,$$

we obtain, with a constant c independent of λ

$$||v||_{L^2_\eta} \leq \frac{c}{|\lambda + \nu\alpha^2|}\Big(||f||_{L^2_\eta} + ||g||_{L^2_\eta}\Big),$$

which is the result stated in Lemma 4.

6.3 Proof of Lemma 7

We first prove that the linear operator $\boldsymbol{L}^{(1)}$ is relatively bounded with respect to $\boldsymbol{L}^{(0)}_{(\nu)}$, namely that for $v \in \mathring{H}^2_\eta$

$$\|\boldsymbol{L}^{(1)}v\|_{\mathring{L}^2_\eta} \leq a\|v\|_{\mathring{L}^2_\eta} + b\|\boldsymbol{L}^{(0)}_{(\nu)}v\|_{\mathring{L}^2_\eta}, \tag{56}$$

where we can assume b small. Indeed

$$(\boldsymbol{L}^{(1)}v)_n = -\Pi_n\left[in\alpha U v_n + U'\begin{pmatrix} v_n^y \\ 0 \end{pmatrix}\right],$$

thus

$$\|(\boldsymbol{L}^{(1)}v)_n\|_{C^0_\eta} \leq c(|n|+1)M_n$$

where

$$\sup|\widetilde{v}_n(y)|e^{\eta y} = M_n,$$

hence

$$\|(\boldsymbol{L}^{(1)}v)_n\|^2_{C^0_\eta} \leq 2c^2\Big(\varepsilon^2 n^4 + \frac{1}{4\varepsilon^2}\Big)M_n^2,$$

and since

$$\begin{aligned}
\|v\|^2_{\mathring{L}^2_\eta} &= \sum_{|n|\geq 1} M_n^2, \\
\|v\|^2_{\mathring{H}^2_\eta} &= \sum_{|n|\geq 1} n^4 M_n^2 + n^2\|Dv_n\|^2_{C^0_\eta} + \|D^2 v_n\|^2_{C^0_\eta},
\end{aligned}$$

we obtain

$$\|\boldsymbol{L}^{(1)}v\|^2_{\mathring{L}^2_\eta} \le 2c^2\varepsilon^2\|v\|^2_{\mathring{H}^2_\eta} + \frac{c^2}{2\varepsilon^2}\|v\|^2_{\mathring{L}^2_\eta}.$$

Thanks to the equivalence between $\|\boldsymbol{L}^{(0)}_{(\nu)}v\|_{\mathring{L}^2_\eta}$ and $\|v\|_{\mathring{H}^2_\eta}$

$$\|v\|_{\mathring{H}^2_\eta} \le K\|\boldsymbol{L}^{(0)}_{(\nu)}v\|_{\mathring{L}^2_\eta},$$

we obtain (56) with $b = Kc\varepsilon$ and $a = c/2\varepsilon$ which may be large.

Now, using (56) on $v = (\lambda - \boldsymbol{L}^{(0)}_{(\nu)})w$ together with the bounds obtained in Lemma 4, we have

$$\begin{aligned} & a\Big\|(\lambda - \boldsymbol{L}^{(0)}_{(\nu)})^{-1}\Big\|_{\mathcal{L}(\mathring{L}^2_\eta)} + b\Big\|\boldsymbol{L}^{(0)}_{(\nu)}(\lambda - \boldsymbol{L}^{(0)}_{(\nu)})^{-1}\Big\|_{\mathcal{L}(\mathring{L}^2_\eta)} \\ \le \quad & \frac{cC}{2\varepsilon|\lambda+\nu\alpha^2|)} + Kc\varepsilon\Big(1 + \frac{C|\lambda|}{|\lambda+\nu\alpha^2|}\Big), \end{aligned}$$

provided

$$0 \le \arg(\lambda+\nu\alpha^2) \le 2\pi/3 - \delta.$$

It is clear that

$$a\Big\|(\lambda - L^{(0)}_{(\nu)})^{-1}\Big\|_{\mathcal{L}(\mathring{L}^2_\eta)} + b\Big\|L^{(0)}_{(\nu)}(\lambda - L^{(0)}_{(\nu)})^{-1}\Big\|_{\mathcal{L}(\mathring{L}^2_\eta)} < 1$$

for ε such that

$$Kc\varepsilon\Big(2 + \frac{\nu\alpha^2\varepsilon}{cC}\Big) < \frac{1}{2}$$

and imposing

$$|\lambda+\nu\alpha^2| > \frac{cC}{\varepsilon}. \tag{57}$$

Applying the theorem 3.17 p.214 in [13], we deduce that the set of λ satisfying (57) is included in the resolvent set of $\boldsymbol{L}^{(0)}_{(\nu)} + \boldsymbol{L}^{(1)} = \boldsymbol{L}_{(\nu)}$ when restricted to the subspace $\mathring{L}^2_\eta$.

Now in the subspace $\mathring{L}^\infty$, we have $\boldsymbol{L}^{(1)}|_{\mathring{L}^\infty} = 0$, hence the result holds. This means that the spectrum of eigenvalues of $\boldsymbol{L}_{(\nu)}$ is located in a right bounded region as indicated in the Lemma.

6.4 Proof of Lemma 8

Let us consider a sequence $\{v^{(n)}\}_{n\in\mathbb{N}} \subset \mathcal{Z}_\eta$ such that $v^{(n)}$ and $\boldsymbol{L}^{(0)}_{(\nu)}v^{(n)}$ are bounded in $\mathcal{X}_\eta$, then the linear operator $\boldsymbol{L}^{(1,c)}$ is relatively compact with respect to $\boldsymbol{L}^{(0)}_{(\nu)}$ if there exists a subsequence $\{v^{(p_n)}\}_{p_n\in\mathbb{N}}$ such that $\boldsymbol{L}^{(1,c)}v^{(p_n)}$ converges in $\mathcal{X}_\eta$.

We know that $\boldsymbol{L}^{(1,c)} \in \mathcal{L}(\mathcal{Z}_\eta, \mathcal{Y}_\eta)$ acts separately in $\mathcal{L}(\mathring{H}^2_\eta, \mathring{H}^1_\eta)$ and cancels in $\mathcal{L}(\mathring{W}^{2,\infty}, \mathring{W}^{1,\infty})$, so it is sufficient to work in $\mathcal{L}(\mathring{H}^2_\eta, \mathring{H}^1_\eta)$.

Here we use an additional property of $\boldsymbol{L}^{(1,c)}$: the function $U(y)$ tends exponentially to U_+ for $y \to \infty$ as $e^{-\gamma y}$. The $m-th$ Fourier component

$$\Big[((U-U_+)\cdot\nabla)v + (v\cdot\nabla)U\Big]_m(y) = (U(y)-U_+)im\alpha v_m(y) + v^y_m(y)U'(y)$$

is bounded by

$$Ke^{-(\gamma+\eta)y}\Big(1+|m|\Big)|v_m(y)e^{\eta y}| \leq Ke^{-(\gamma+\eta)y}\Big(1+|m|\Big)||v_m||_{C^0_\eta},$$

where

$$\sum_{|m|\geq 1}(1+m^2)\|v_m\|^2_{C^0_\eta} = \|v\|^2_{\mathring{H}^1_\eta}.$$

It appears, from the properties of the projection Π described in Appendix 6.1, that

$$\boldsymbol{L}^{(1,c)} \in \mathcal{L}(\mathring{H}^1_\eta, \mathring{L}^2_{\eta+\gamma}) \cap \mathcal{L}(\mathring{H}^2_\eta, \mathring{H}^1_{\eta+\gamma}).$$

Let us show that the identity map $\mathring{H}^1_{\eta+\gamma} \hookrightarrow \mathring{L}^2_\eta$ is compact, which is sufficient for our purpose.

We define the space $\mathring{H}^1_{\eta,N}$,and similarly $\mathring{L}^2_{\eta,N}$, where

$$\mathring{H}^1_{\eta,N} = \Big\{\widetilde{v} \in [H^1[(\mathbb{T}, C^0_\eta(0,N))]]^2;\ \int_0^{2\pi}\widetilde{v}(x,y)dx = 0,\ \nabla\cdot\widetilde{v} = 0,\ \widetilde{v}^y|_{y=0} = 0\Big\}.$$

Let us consider a sequence $\{v^{(n)}\}_{n\in\mathbb{N}}$ bounded in $\mathring{H}^1_{\eta+\gamma}$, and the corresponding sequence $\{v^{(n)N}\}_{n\in\mathbb{N}}$ in $\mathring{H}^1_{\eta+\gamma,N}$, where

$$v^{(n)N}_m(y) = \begin{cases} v^{(n)}_m(y) \text{ for } y\in[0,N] \\ 0 \text{ for } y > N \end{cases}.$$

Then

$$m^2\|v^{(n)}_m - v^{(n)N}_m\|^2_{C^0_\eta} + \|Dv^{(n)}_m - Dv^{(n)N}_m\|^2_{C^0_\eta} \leq 4M^2_m e^{-2\gamma N},$$

where

$$M_m^2 = m^2 \|v_m^{(n)}\|_{C_\eta^0}^2 + \|Dv_m^{(n)}\|_{C_\eta^0}^2.$$

Since the interval $[0, N]$ is bounded, and the functions are bounded and equicontinuous in y, there is a subsequence $\{v^{(p_n^N)}\}_n$ converging in $\mathring{L}^2_{\eta,N}$ when $n \to \infty$.

Now, for any N we have

$$\|v^{(p_n^N)} - v^{(p_q^N)}\|_{\mathring{L}^2_{\eta,N}} \leq K e^{-\gamma N} \varepsilon_{p_n} \text{ for } q > p,$$

and $\varepsilon_{p_n} \to 0$ as $n \to \infty$ since $\{v^{(p_n^N)}\}_n$ is a Cauchy sequence in $\mathring{L}^2_{\eta,N}$. Now extracting the diagonal subsequence $\{v^{(p_N^N)}\}_{N\to\infty}$ we obtain a Cauchy sequence $\{v^{(p_N^N)}\}$ in $\mathring{L}^2_\eta$, hence converging.

6.5 Proof of Lemma 9

To prove Lemma 9, let us proceed as in Appendix 6.2.2. We have explicitly the solution v of

$$\nu \Pi \Delta v - \Pi\Big[U_+ \frac{\partial v}{\partial x}\Big] - \lambda v = f \in \mathring{L}^2_\eta,\ v(0) = 0,$$

with formulas (48), (49) in Appendix 6.2.2. The result is based on good estimates given at Lemma 28, for s_n and $\Re s_n - \eta$ where now

$$s_n^2 = n^2\alpha^2 + \frac{\lambda - in\alpha U_+}{\nu},\ \Re s_n > 0,$$

and $\eta > 0$ is small enough. We first observe that if λ is such that $s_n = 0$ or $\Re s_n = \eta$ then λ is situated in a very specific region which can be avoided for a finite number of values $1 \leq |n| \leq N_0$. Then it will be sufficient to obtain lower estimates for $|s_n|$ and $\Re s_n - \eta$ for values of n such that $|n| > N_0$.

First consider λ such that $s_n = 0$. This implies

$$\lambda_r = -\nu\alpha^2 n^2,\ \lambda_i = n\alpha U_+,$$

hence

$$\lambda_r = -\frac{\nu}{U_+^2}(\lambda_i)^2, \tag{58}$$

which means that λ belongs to a parabola P_1, truncated by $\lambda_r \leq -\nu\alpha^2$ since $|n| \geq 1$.

Now consider λ such that $\Re s_n = \eta$. This implies

$$A + \sqrt{A^2 + B^2} = 2\eta^2,$$

with

$$A = n^2\alpha^2 + \frac{\lambda_r}{\nu},\ B = \frac{\lambda_i - n\alpha U_+}{\nu}.$$

Hence

$$B^2 = 4\eta^4 - 4A\eta^2,$$

which leads to

$$(\lambda_i - n\alpha U_+)^2 = 4\nu^2\eta^4 - 4\nu\eta^2(\lambda_r + \nu\alpha^2 n^2)$$

$$\lambda_r = -\frac{(\lambda_i - n\alpha U_+)^2}{4\nu\eta^2} - \nu(\alpha^2 n^2 - \eta^2)$$

which is a set of parabolas in $\mathbb{C}$ with the following envelope (varying the parameter $|n| \geq 1$)

$$\lambda_r = -\frac{\nu}{U_+^2 + 4\nu^2\eta^2}\lambda_i^2 + \nu\eta^2.$$

It results that if λ is such that $\Re s_n = \eta$ then

$$\lambda_r \leq -\frac{\nu}{U_+^2 + 4\nu^2\eta^2}\lambda_i^2 + \nu\eta^2,\ \lambda_r \leq -\nu(\alpha^2 - \eta^2), \tag{59}$$

which corresponds again to a parabolic region, truncated by $\lambda_r \leq -\nu(\alpha^2 - \eta^2) < 0$ for $\eta < \alpha$. It is clear that the parabola (58) is included in this region.

Let us choose λ outside of region (59) and such that

$$0 \leq \arg(\lambda + \nu\alpha^2) \leq 2\pi/3 - \delta,$$

and follow the method used in Appendix 6.2.2. We have for $\cos\theta > \frac{C}{N_0}$ where $\frac{|nU_+|}{(n^2-1)\alpha} \leq \frac{C}{N_0}$ for $|n| \geq N_0$,

$$|s_n|^4 \geq [(n^2-1)^2\alpha^4 + \frac{1}{\nu^2}|\lambda + \nu\alpha^2|^2] + \frac{n^2\alpha^2 U_+^2}{\nu^2},$$

and for $\cos(2\pi/3 - \delta) \leq \cos\theta \leq \frac{C}{N_0}$,

$$|s_n|^4 \geq (1 - |\cos\theta| - \frac{C}{N_0})[(n^2-1)^2\alpha^4 + \frac{1}{\nu^2}|\lambda + \nu\alpha^2|^2] + \frac{n^2\alpha^2 U_+^2}{\nu^2},$$

so that in all cases

$$|s_n|^2 \geq (\cos\theta/2 - \delta_1)[(n^2-1)\alpha^2 + \frac{1}{\nu}|\lambda + \nu\alpha^2|],$$

where δ_1 is such that

$$(\cos^2\theta/2 - \frac{C}{2N_0})^{1/2} = \cos\theta/2 - \delta_1,\ \delta_1 < \frac{C}{N_0},$$

using $\cos\theta/2 > \cos(\pi/3 - \delta/2) > 1/2$.

Now we have

$$(\Re s_n)^2 \geq \frac{1}{2}\{(1+\cos\theta/2-\delta_1)(n^2-1)\alpha^2 + (\cos\theta/2+\cos\theta-\delta_1)\frac{1}{\nu}|\lambda+\nu\alpha^2|\}.$$

We have now

$$\begin{aligned}\cos\theta/2+\cos\theta-\delta_1 &\geq \cos(\pi/3-\delta/2)-\cos(\pi/3+\delta)-\delta_1\\ &\geq \frac{\sqrt{3}\delta}{4}-\delta_1 \text{ for } \delta \text{ small enough.}\end{aligned}$$

Hence choosing N_0 large enough, such that

$$\frac{C}{N_0} < \frac{\sqrt{3}\delta}{8},$$

we obtain the required estimates of Lemma 28 in Appendix 6.2.2.

Now, the inverse of $\boldsymbol{L}^{(0)}_{(\nu)} + \boldsymbol{L}^{(1,0)} - \lambda$ is bounded in $\mathring{L}^2_\eta$ by $C/|\lambda+\nu\alpha^2|$ provided that we adapt the constant C in order to take care of the Fourier components with $|n| < N_0$.

The Lemma 9 is then proved.

6.6 Proof of Lemma 13

Let $f \in L^\infty_\eta$. Then $v = (\lambda - \boldsymbol{L}^{(0)}_{(\nu)})^{-1} f$ is explicitely given by

$$\begin{aligned}v(y) &= \frac{1}{2s\nu}\int_y^\infty f(\tau)e^{s(y-\tau)}d\tau + \frac{1}{2s\nu}\int_0^y f(\tau)e^{-s(y-\tau)}d\tau \\ &\quad -\frac{1}{2s\nu}\int_0^\infty f(\tau)e^{-s(y+\tau)}d\tau.\end{aligned} \tag{60}$$

We want to estimate$\|e^{\boldsymbol{L}^{(0)}_{(\nu)}t} f\|_{L^\infty}$, using

$$e^{\boldsymbol{L}^{(0)}_{(\nu)}t} f = \frac{1}{2i\pi}\int_\Gamma e^{\lambda t}\Big(\lambda - \boldsymbol{L}^{(0)}_{(\nu)}\Big)^{-1} f d\lambda, \tag{61}$$

where the contour Γ is detailed on Figure 3, where the limit $\delta = 0$ is expected.

The estimate obtained at Lemma 4 is not sufficient, and does not use the decay at ∞ of $f(y)$. Now we have, with $s^2 = \lambda/\nu$, $\Re s > 0$

$$\|(\lambda - \boldsymbol{L}^{(0)}_{(\nu)})^{-1} f\|_{L^\infty} \leq \frac{3}{2\nu|s||\Re s - \eta|}\|f\|_{L^\infty_\eta} \text{ if } |\Re s - \eta| \neq 0, \tag{62}$$

$$\|(\lambda - \boldsymbol{L}^{(0)}_{(\nu)})^{-1} f\|_{L^\infty} \leq \frac{3}{2\nu|s|\eta}\|f\|_{L^\infty_\eta} \text{ for } \frac{\eta}{2} \leq \Re s,$$

where we notice that $\Re s - \eta$ may be < 0. In fact, we use estimate (62) only for large $|\lambda|$. For bounded $|\lambda|$ we use next result.

6.6.1 Special estimates for bounded λ

Let us come back to

$$\lambda u - \nu D^2 u = f \in L^\infty_\eta, \ v(0) = 0. \tag{63}$$

and assume that λ is bounded. We first solve

$$\nu D^2 v = -f,$$

looking for v such that $v \to 0$ as $y \to \infty$. We find

$$v(y) = -\int_y^\infty ds \int_s^\infty \frac{f(\tau)}{\nu} d\tau,$$

so that

$$\|v(y)\| \leq \frac{e^{-\eta y}}{\eta^2 \nu}\|f\|_{L^\infty_\eta}, \ \|Dv(y)\| \leq \frac{e^{-\eta y}}{\eta \nu}\|f\|_{L^\infty_\eta}, \ \|D^2 v(y)\| \leq \frac{e^{-\eta y}}{\nu}\|f\|_{L^\infty_\eta}.$$

Let us now solve

$$\begin{aligned} \lambda w - \nu D^2 w &= -\lambda v \\ w(0) &= 0, \end{aligned}$$

leading to

$$\begin{aligned} w(y) &= \frac{-\lambda}{2s\nu}\int_y^\infty v(\tau) e^{s(y-\tau)} d\tau - \frac{\lambda}{2s\nu}\int_0^y v(\tau) e^{-s(y-\tau)} d\tau \\ &\quad + \frac{\lambda}{2s\nu}\int_0^\infty v(\tau) e^{-s(y+\tau)} d\tau, \end{aligned}$$

leading to

$$\begin{aligned}
\|w\|_{L^\infty} &\le \frac{3|s|}{\eta}\|v\|_{L^\infty_\eta} \le \frac{3|s|}{\eta^3\nu}\|f\|_{L^\infty_\eta},\\
\|Dw\|_{L^\infty} &\le \frac{3|s|}{\eta}\|Dv\|_{L^\infty_\eta} \le \frac{3|s|}{\eta^2\nu}\|f\|_{L^\infty_\eta},\\
\|D^2w\|_{L^\infty} &\le \frac{3|\lambda|}{\eta\nu}\|Dv\|_{L^\infty_\eta} \le \frac{3|\lambda|}{\eta^2\nu^2}\|f\|_{L^\infty_\eta},
\end{aligned}$$

i.e.

$$\|w\|_{W^{2,\infty}} \le \frac{3|s|}{\eta^3\nu}\Big[1+\eta+\eta|s|\Big]\|f\|_{L^\infty_\eta}.$$

Now we solve

$$\lambda w_1 - \nu D^2 w_1 = 0,\ \ w_1(0) = -v(0),$$

which leads to

$$w_1(y) = -v(0)e^{-sy},$$

hence

$$\|w_1\|_{W^{2,\infty}} \le \frac{1+|s|+|s|^2}{\eta^2\nu}\|f\|_{L^\infty_\eta},$$

The sum

$$u = v + w + w_1$$

satisfies (63) and we have the estimate

$$\Big\|(\lambda - L^{(0)}_{(\nu)})^{-1}f\Big\|_{W^{2,\infty}} \le \frac{4(1+|s|)^2}{\eta^3\nu}\|f\|_{L^\infty_\eta} \le c_0\|f\|_{L^\infty_\eta} \tag{64}$$

which is used for $\lambda = \nu s^2$ bounded by a certain A.

6.6.2 Study of the contour integral

On the part Γ_0 of Γ we have $\lambda = \delta e^{i\theta}$, $-\pi/2 \le \theta \le \pi/2$. For any $\delta > 0$

$$\begin{aligned}
\Big\|\frac{1}{2i\pi}\int_{\Gamma_0} e^{\lambda t}\Big(\lambda - \boldsymbol{L}^{(0)}_{(\nu)}\Big)^{-1} f d\lambda\Big|_{L^\infty} &\le \frac{c_0\|f\|_{L^\infty_k}}{2\pi}\int_{-\pi/2}^{\pi/2}\delta e^{\delta t}d\theta\\
&\le \frac{c_0}{2}\delta e^{\delta t}\|f\|_{L^\infty_k},
\end{aligned}$$

which goes to 0 as δ goes to 0, t being fixed.

On the part Γ_1 we have $\lambda = -\tau + i\delta$, $\tau \in [0, \gamma]$, with $\gamma > 0$ of order 1, so that $d\lambda = -d\tau$. Hence

$$\Big\|\frac{1}{2i\pi}\int_{\Gamma_1} e^{\lambda t}\Big(\lambda - \boldsymbol{L}^{(0)}_{(\nu)}\Big)^{-1} f d\lambda\Big\|_{L^\infty} \le \frac{c_0\|f\|_{L^\infty_k}}{2\pi}\int_0^\gamma e^{-\tau t} d\tau = \frac{c_0\|f\|_{L^\infty_k}}{2\pi}\frac{(1-e^{\gamma t})}{t}.$$

Now on the part Γ_2 of the contour Γ we have $\lambda = -\gamma + i\beta$, $\beta \in [\delta, A]$, so that $d\lambda = d\beta$, and the estimate (64) leads to

$$\Big\|\frac{1}{2i\pi}\int_{\Gamma_2} e^{\lambda t}\Big(\lambda - \boldsymbol{L}^{(0)}_{(\nu)}\Big)^{-1} f d\lambda\Big\|_{L^\infty} \le \frac{c_0\|f\|_{L^\infty_k} e^{-\gamma t}}{2\pi}(A - \delta).$$

It is then clear that the estimates for integral on $\overline{\Gamma_1}$ and $\overline{\Gamma_2}$ may be obtained in the same way and that the integral on $\Gamma_0 \cup \Gamma_1 \cup \overline{\Gamma_1} \cup \Gamma_2 \cup \overline{\Gamma_2}$ has a limit when $\delta \to 0$. Completing with the rest of Γ which is independent of δ, and which is bounded in a standard way for $t \in (0,1)$, and bounded for $t > 1$ by $ce^{-\delta_1 t}$, we obtain an estimate of the form (the essential part comes from the integral on Γ_1)

$$\|e^{\boldsymbol{L}^{(0)}_{(\nu)} t} f\|_{L^\infty} \le \Big\|\frac{1}{2i\pi}\int_{\Gamma} e^{\lambda t}\Big(\lambda - \boldsymbol{L}^{(0)}_{(\nu)}\Big)^{-1} f d\lambda\Big\|_{L^\infty} \le \frac{C}{1+t}\|f\|_{L^\infty_\eta}.$$

6.6.3 End of the proof

The rest of the proof of the first part of Lemma 13 results from the fact that $e^{\boldsymbol{L}^{(0)}_{(\nu)} t}$ commutes with $\boldsymbol{L}^{(0)}_{(\nu)}$ and from

$$D^2(\lambda - \boldsymbol{L}^{(0)}_{(\nu)})^{-1} f = (\lambda - \boldsymbol{L}^{(0)}_{(\nu)})^{-1} D^2 f$$

implying

$$\|D^2 e^{\boldsymbol{L}^{(0)}_{(\nu)} t} f\|_{L^\infty} \le \frac{C}{1+t}\|D^2 f\|_{L^\infty}.$$

Moreover the interpolation estimate

$$\|Dv\|_{L^\infty} \le \|D^2 v\|_{L^\infty} + \|v\|_{L^\infty},$$

is obtained in solving

$$D^2 v - v = g \in L^\infty,\ v(0) = 0.$$

Indeed this leads to

$$\begin{aligned} v(y) &= -\frac{1}{2}\int_0^\infty g(\tau)e^{-|\tau - y|}d\tau + \frac{1}{2}\int_0^\infty g(\tau)e^{-(\tau+y)}d\tau, \\ Dv(y) &= \frac{1}{2}\int_0^y g(\tau)e^{\tau - y}d\tau - \frac{1}{2}\int_y^\infty g(\tau)e^{y-\tau}d\tau - \frac{1}{2}\int_0^\infty g(\tau)e^{-(\tau+y)}d\tau, \end{aligned}$$

and we obtain

$$\|Dv\|_{L^\infty} \leq \|g\|_{L^\infty} \leq \|D^2 v\|_{L^\infty} + \|v\|_{L^\infty}.$$

Now taking $v = (\lambda - \boldsymbol{L}^{(0)}_{(\nu)})^{-1} f$ we have the first part of the Lemma.

Let $f \in W^{1,\infty}_\eta$. Then the solution $v(y)$ in $W^{2,\infty}_0$ of

$$(\lambda - \boldsymbol{L}^{(0)}_{(\nu)})v = f$$

with $s^2 = \lambda/\nu$, $\Re s > 0$, is given by (60) and after an integration by parts, we have

$$\begin{aligned} Dv(y) &= \frac{1}{2s\nu}\int_0^\infty f'(\tau)e^{-s|y-\tau|}d\tau + \frac{1}{2s\nu}\int_0^\infty f'(\tau)e^{-s(y+\tau)}d\tau, \\ D^2v(y) &= -\frac{1}{2\nu}\int_0^y f'(\tau)e^{-s(y-\tau)}d\tau + \frac{1}{2\nu}\int_y^\infty f'(\tau)e^{s(y-\tau)}d\tau \\ &\quad -\frac{1}{2\nu}\int_0^\infty f'(\tau)e^{-s(y+\tau)}d\tau. \end{aligned}$$

It is clear that we have the estimates

$$\|v\|_{W^{1,\infty}} \leq \begin{cases} c_0\|f\|_{W^{1,\infty}_\eta} \text{ for } |\lambda| \leq A \\ \frac{C}{|\lambda|}\|f\|_{W^{1,\infty}_\eta} \text{ for } \lambda \in \Gamma_3 \end{cases}, \tag{65}$$

$$\|D^2 v\|_{L^\infty} \leq \begin{cases} c_0\|f\|_{W^{1,\infty}_\eta} \text{ for } |\lambda| \leq A \\ \frac{C}{\sqrt{|\lambda|}}\|f\|_{W^{1,\infty}_\eta} \text{ for } \lambda \in \Gamma_3 \end{cases},$$

where $\lambda \in \Gamma_3$ is such that $\arg\lambda = 2\pi/3 - \delta$, hence

$$\Re\lambda \simeq -\frac{1}{2}|\lambda|,\ |d\lambda| = d|\lambda|,\ A \leq \Im\lambda \sim \frac{\sqrt{3}}{2}|\lambda| \leq \infty.$$

It results that for $\lambda \in \Gamma$ we have the estimates

$$\|v\|_{W^{2,\infty}} \leq \frac{C}{(1+|\lambda|)}\|f\|_{W^{2,\infty}_\eta}, \tag{66}$$

$$\|v\|_{W^{2,\infty}} \leq \frac{C}{1+\sqrt{|\lambda|}}\|f\|_{W^{1,\infty}_\eta}. \tag{67}$$

The estimate (66) implies the first part of the Lemma 13 already proved. Estimate (67) leads to the second estimate (8) in Lemma 13, where we notice that the behavior in $1/\sqrt{|\lambda|}$ for $|\lambda|$ large in the part Γ_3 of the integral (61) gives the factor $1/\sqrt{t}$ for t near 0, while the integral on Γ_1 gives a bound in $1/(1+t)$ as $t \to \infty$.

6.7 Proof of Lemma 15

Let us solve the system

$$(\mathbb{I} - \varepsilon \boldsymbol{L}^{(0)}_{(\nu)})u_\varepsilon = u \in \mathring{H}^1_\eta,\ u_\varepsilon \in \mathring{H}^2_\eta \tag{68}$$

hence in Fourier components

$$\begin{aligned}(D^2 - s_n^2)u_{\varepsilon,n} + \begin{pmatrix} in\alpha \\ D \end{pmatrix} q_n &= \frac{1}{\varepsilon\nu}\begin{pmatrix} u_n^x \\ u_n^y \end{pmatrix}, \\ in\alpha u^x_{\varepsilon,n} + Du^y_{\varepsilon,n} &= 0, u_{\varepsilon,n}|_{y=0} = 0,\end{aligned}$$

where

$$in\alpha u_n^x + Du_n^y = 0,$$
$$s_n^2 = n^2\alpha^2 + \frac{1}{\varepsilon\nu}.$$

We then obtain

$$\begin{aligned}(D^2 - n^2\alpha^2)[D^2 - s_n^2)]u^y_{\varepsilon,n} &= \frac{1}{\varepsilon\nu}(D^2 - n^2\alpha^2)u_n^y, \\ u^y_{\varepsilon,n} &= Du^y_{\varepsilon,n} = 0 \text{ for } y = 0,\end{aligned}$$

which allows to solve separately in $u^y_{\varepsilon,n}$ and $u^x_{\varepsilon,n}$. First, we obtain

$$u^y_{\varepsilon,n}(y) = w_n(y) - Dw_n(0)\frac{e^{-s_n y} - e^{-n\alpha y}}{s_n - n\alpha}, \tag{69}$$

where

$$w_n(y) = -\frac{1}{2\varepsilon\nu s_n}\int_0^\infty u_n^y(\tau)e^{-s_n|\tau - y|}d\tau + \frac{1}{2\varepsilon\nu s_n}\int_0^\infty u_n^y(\tau)e^{-s_n(\tau+y)}d\tau.$$

After an integration by parts, we obtain

$$\begin{aligned}w_n(y) &= \frac{1}{2\varepsilon\nu s_n^2}\int_0^y Du_n^y(\tau)e^{-s_n(y-\tau)}d\tau - \frac{1}{2\varepsilon\nu s_n^2}\int_y^\infty Du_n^y(\tau)e^{s_n(y-\tau)}d\tau \\ &\quad + \frac{1}{2\varepsilon\nu s_n^2}\int_0^\infty Du_n^y(\tau)e^{-s_n(y+\tau)}d\tau - \frac{1}{\varepsilon\nu s_n^2}u_n^y(y)\end{aligned}$$

from which we deduce that

$$|w_n(y)e^{\eta y}| \le \frac{3M_n}{2\nu\varepsilon s_n^2(s_n - \eta)} + \frac{M_n}{\varepsilon\nu n s_n^2}$$

where

$$\begin{aligned} n^2|u_n^y(y)e^{\eta y}|^2 + |Du_n^y(y)e^{\eta y}|^2 &\leq M_n^2, \\ \|u\|^2_{\overset{\circ}{H}{}^1_\eta} &= \sum_{|n|\geq 1} M_n^2. \end{aligned}$$

We also have

$$\begin{aligned} Dw_n(y) &= -\frac{1}{2\varepsilon\nu s_n}\int_0^y Du_n^y(\tau)e^{-s_n(y-\tau)}d\tau - \frac{1}{2\varepsilon\nu s_n}\int_y^\infty Du_n^y(\tau)e^{-s_n(\tau-y)}d\tau \\ &\quad -\frac{1}{2\varepsilon\nu s_n}\int_0^\infty Du_n^y(\tau)e^{-s_n(\tau+y)}d\tau, \end{aligned}$$

from which we deduce that

$$|Dw_n(y)e^{\eta y}| \leq \frac{3M_n}{2\nu\varepsilon s_n(s_n-\eta)},$$

and

$$\begin{aligned} D^2w_n(y) &= \frac{1}{2\varepsilon\nu}\int_0^y Du_n^y(\tau)e^{-s_n(y-\tau)}d\tau - \frac{1}{2\varepsilon\nu}\int_y^\infty Du_n^y(\tau)e^{-s_n(\tau-y)}d\tau \\ &\quad +\frac{1}{2\varepsilon\nu}\int_0^\infty Du_n^y(\tau)e^{-s_n(\tau+y)}d\tau, \end{aligned}$$

from which we deduce that

$$|D^2w_n(y)e^{\eta y}| \leq \frac{3M_n}{2\nu\varepsilon(s_n-\eta)}.$$

Now, we show that there exists $C > 0$ such that for $|n| \geq 1$

$$\max\left\{\frac{3n^2}{2\nu\varepsilon s_n^2(s_n-\eta)}, \frac{|n|}{\nu\varepsilon s_n^2}, \frac{|n|}{2\nu\varepsilon s_n(s_n-\eta)}, \frac{3}{2\nu\varepsilon(s_n-\eta)}\right\} \leq \frac{C}{\varepsilon^{1/2}}. \qquad (70)$$

Indeed, we notice that

$$\begin{aligned} \varepsilon s_n^2 &= \varepsilon n^2\alpha^2 + 1/\nu > \frac{1}{2}(\alpha|n|\sqrt{\varepsilon} + \nu^{-1/2})^2, \\ \sqrt{\varepsilon}(s_n-\eta) &> \frac{1}{\sqrt{2}}\Big((\alpha|n| - \eta\sqrt{2})\sqrt{\varepsilon} + \nu^{-1/2}\Big), \end{aligned}$$

so that, for η small enough, there exists $c > 0$ independent of n and ε such that

$$\sqrt{\varepsilon}s_n > \sqrt{\varepsilon}(s_n-\eta) > c\Big(|n|\sqrt{\varepsilon}+1\Big).$$

Then, we have (with $X = |n|\sqrt{\varepsilon}$)

$$\frac{3n^2}{2\nu\varepsilon s_n^2(s_n-\eta)} \leq \frac{3n^2\sqrt{\varepsilon}}{2\nu c^3(|n|\sqrt{\varepsilon}+1)^3} \leq \frac{C_1}{\sqrt{\varepsilon}}\frac{X^2}{(X+1)^3} \leq \frac{C}{\varepsilon^{1/2}},$$

$$\frac{|n|}{2\nu\varepsilon s_n(s_n-\eta)} \leq \frac{1}{2\nu c^2\sqrt{\varepsilon}}\frac{X}{(X+1)^2} \leq \frac{C}{\varepsilon^{1/2}},$$

$$\frac{3}{2\nu\varepsilon(s_n-\eta)} \leq \frac{3}{2c\nu\sqrt{\varepsilon}}\frac{1}{X+1} \leq \frac{C}{\varepsilon^{1/2}},$$

showing that (70) holds true. It then results that

$$n^4\|w_n\|^2_{C^0_\eta} + n^2\|Dw_n\|^2_{C^0_\eta} + \|D^2w_n\|^2_{C^0_\eta} \leq C_1\frac{M_n^2}{\varepsilon}.$$

Coming back to (69), we use (assuming from now on that $n > 0$)

$$|Dw_n(0)| \leq c_0\frac{M_n}{(X+1)^2},\ X = n\sqrt{\varepsilon},$$

and good estimates for the function of y defined by

$$b_n(y) \stackrel{def}{=} \frac{e^{-s_n y} - e^{-n\alpha y}}{s_n - n\alpha}.$$

Indeed, we have

$$\begin{aligned}
|e^{-(n\alpha-\eta)y} - e^{-(s_n-\eta)y}| &= e^{-(n\alpha-\eta)y}(1-e^{-(s_n-n\alpha)y}) \\
&\leq (s_n-n\alpha)ye^{-(n\alpha-\eta)y} \\
&\leq \frac{(s_n-n\alpha)}{(n\alpha-\eta)e},
\end{aligned}$$

hence there exists $C > 0$ such that

$$\|b_n\|_{C^0_\eta} \leq \frac{C}{n},$$

and

$$n^2\|Dw_n(0)b_n\|_{C^0_\eta} \leq c_0CM_n\frac{n}{(X+1)^2} \leq c_0C\frac{M_n}{\sqrt{\varepsilon}}.$$

Now

$$Db_n(y) = \frac{n\alpha e^{-n\alpha y} - s_n e^{-s_n y}}{s_n - n\alpha} = -e^{-s_n y} + \frac{n\alpha(e^{-n\alpha y} - e^{-s_n y})}{s_n - n\alpha}$$

hence there exists $C > 0$ such that

$$\|Db_n\|_{C^0_\eta} \leq C$$

and

$$n\|Dw_n(0)Db_n\|_{C^0_\eta} \leq C_1 M_n \frac{n}{(X+1)^2} \leq C_2 \frac{M_n X}{\sqrt{\varepsilon}(X+1)^2} \leq C_2 \frac{M_n}{\sqrt{\varepsilon}}.$$

In the same way, we have

$$D^2 b_n(y) = (s_n + n\alpha)e^{-s_n y} + \frac{n^2\alpha^2(e^{-s_n y} - e^{-n\alpha y})}{s_n - n\alpha}$$

leading to

$$\|D^2 b_n\|_{C^0_\eta} \leq C_3 \frac{X+1}{\sqrt{\varepsilon}},$$

hence

$$\|Dw_n(0)D^2 b_n\|_{C^0_\eta} \leq c_0 C_3 M_n \frac{1}{\sqrt{\varepsilon}(X+1)} \leq c_0 C_3 \frac{M_n}{\sqrt{\varepsilon}}.$$

Finally

$$n^4 \|u^y_{\varepsilon,n}\|^2_{C^0_\eta} + n^2 \|Du^y_{\varepsilon,n}\|^2_{C^0_\eta} + \|D^2 u^y_{\varepsilon,n}\|^2_{C^0_\eta} \leq C \frac{M_n^2}{\varepsilon}.$$

Let us now consider $u^x_{n,\varepsilon}$. We have

$$\begin{aligned}
(D^2 - s_n^2)u^x_{n,\varepsilon} &= -in\alpha q_n + \frac{u^x_n}{\nu_0 \varepsilon}, \; u^x_{n,\varepsilon}(0) = 0, \\
(D^2 - n^2\alpha^2)q_n &= 0, \; Dq_n(0) = -D^2 u^y_{\varepsilon,n}(0),
\end{aligned}$$

hence

$$q_n(y) = \frac{D^2 u^y_{\varepsilon,n}(0)}{n\alpha} e^{-n\alpha y},$$

and

$$\begin{aligned}
u^x_{n,\varepsilon}(y) &= \frac{1}{2\varepsilon\nu s_n^2} \int_0^y Du^x_n(\tau) e^{-s_n(y-\tau)} d\tau - \frac{1}{2\varepsilon\nu s_n^2} \int_y^\infty Du^x_n(\tau) e^{s_n(y-\tau)} d\tau \\
&\quad + \frac{1}{2\varepsilon\nu s_n^2} \int_0^\infty Du^x_n(\tau) e^{-s_n(y+\tau)} d\tau - \frac{1}{\varepsilon\nu s_n^2} u^x_n(y) \\
&\quad + \frac{iD^2 u^y_{\varepsilon,n}(0)}{2\nu s_n} \left(-b_n(y) + \frac{e^{-n\alpha y} + e^{-s_n y}}{s_n + n\alpha} \right).
\end{aligned}$$

It appears that the estimates on the part without $D^2u^y_{\varepsilon,n}(0)$ is the same as the estimates obtained for $w_n(y)$. Now we have in addition, as seen above

$$\left|\frac{iD^2u^y_{\varepsilon,n}(0)}{2\nu s_n}\right| \leq C\frac{M_n}{X+1},$$

$$n^2\|b_n\|_{C^0_\eta} + n\|Db_n\|_{C^0_\eta} + \|D^2b_n\|_{C^0_\eta} \leq C_1\frac{X+1}{\sqrt{\varepsilon}}.$$

Finally, we easily have

$$n^2\left\|\frac{e^{-n\alpha y}+e^{-s_ny}}{s_n+n\alpha}\right\|_{C^0_\eta} + n\left\|\frac{n\alpha e^{-n\alpha y}+s_ne^{-s_ny}}{s_n+n\alpha}\right|_{C^0_\eta} + \left\|\frac{n^2\alpha^2e^{-n\alpha y}+s_n^2e^{-s_ny}}{s_n+n\alpha}\right\|_{C^0_\eta}$$
$$\leq C_2\frac{X+1}{\sqrt{\varepsilon}},$$

hence finally

$$n^4\|u^x_{\varepsilon,n}\|^2_{C^0_\eta} + n^2\|Du^x_{\varepsilon,n}\|^2_{C^0_\eta} + \|D^2u^x_{\varepsilon,n}\|^2_{C^0_\eta} \leq C\frac{M_n^2}{\varepsilon},$$

which ends the proof of the Lemma 15.

6.8 Complementary estimate on $(\boldsymbol{L}_{\nu,\omega}-\lambda)^{-1}\widetilde{P}$

Let us consider $v \in \mathring{H}^2_\eta$ solution of

$$(\boldsymbol{L}_{\nu,\omega}-\lambda)v = f \in \mathring{L}^2_\eta,$$

for $\lambda \notin \Sigma_{U_+,\omega}$. We have

$$\boldsymbol{L}_{\nu,\omega} = \boldsymbol{L}^{(0)}_{\nu,\omega} + \boldsymbol{L}^{(1)}, \text{ with } \boldsymbol{L}^{(0)}_{\nu,\omega} = \nu\Pi\Delta - \frac{\omega}{\alpha}\frac{\partial}{\partial\xi},$$

and going to the Fourier components, we obtain

$$[(\boldsymbol{L}^{(0)}_{\nu,\omega})_n - \lambda]v_n + (\boldsymbol{L}^{(1)})_nv_n = f_n,$$

with

$$(\boldsymbol{L}^{(1)})_nv_n = -\Pi_n\left\{in\alpha U\begin{pmatrix}v^x_n\\v^y_n\end{pmatrix} + \begin{pmatrix}v^y_nDU\\0\end{pmatrix}\right\},$$

hence

$$||(\boldsymbol{L}^{(1)})_nv_n||_{C^0_\eta} \leq C(1+|n|)||v_n||_{C^0_\eta}.$$

Now, from the computation above, we know that for $\lambda \notin \Sigma_{U_+,\omega}$

$$\left\| [(\boldsymbol{L}^{(0)}_{\nu,\omega})_n - \lambda]^{-1} \right\|_{\mathcal{L}(C^0_\eta)} \leq \frac{C}{(n^2-1) + |\lambda + \nu\alpha^2|},$$

so that there exists N_0 such that for $|n| > N_0$

$$\left\| [(\boldsymbol{L}^{(0)}_{\nu,\omega})_n - \lambda]^{-1} (\boldsymbol{L}^{(1)})_n \right\|_{\mathcal{L}(C^0_\eta)} \leq 1/2,$$

hence, for $|n| > N_0$

$$\begin{aligned} ||v_n||_{C^0_\eta} &\leq 2||[(\boldsymbol{L}^{(0)}_{\nu,\omega})_n - \lambda]^{-1} f_n||_{C^0_\eta} \\ &\leq \frac{2C}{(n^2-1) + |\lambda + \nu\alpha^2|} ||f_n||_{C^0_\eta}. \end{aligned}$$

Now consider a contour Γ such as the one described at Figure 2, we can choose Γ such that there is no eigenvalue (bounded discrete set) of $(\boldsymbol{L}_{\nu,\omega})_n$ on it for $1 \leq |n| \leq N_0$, it results easily that for $\lambda \in \Gamma$ there exists $M > 0$ with

$$\begin{aligned} ||(\boldsymbol{L}_{\nu,\omega} - \lambda)^{-1} \widetilde{P}||_{\mathcal{L}(\mathring{H}^2_\eta, \mathring{H}^2_\eta)} &\leq \frac{M}{|\lambda + \nu\alpha^2|}, \\ ||(\boldsymbol{L}_{\nu,\omega} - \lambda)^{-1} \widetilde{P}||_{\mathcal{L}(\mathring{H}^1_\eta, \mathring{H}^2_\eta)} &\leq \frac{M}{|\lambda + \nu\alpha^2|^{1/2}}, \\ ||(\boldsymbol{L}_{\nu,\omega} - \lambda)^{-1} \widetilde{P}||_{\mathcal{L}(\mathring{L}^2_\eta, \mathring{H}^2_\eta)} &\leq M. \end{aligned}$$

We may observe that, for $\lambda \in \Gamma$, there exists $C > 0$ and $a > 0$ such that

$$|\lambda + \nu\alpha^2| \geq \frac{1}{C}(a^2 + |\lambda|).$$

6.9 Estimate for $(\lambda - \boldsymbol{A}_\varepsilon)^{-1}$ for $\lambda \in \Gamma$, and for $e^{\boldsymbol{A}_\varepsilon t}$

Let us estimate the solution of

$$(\lambda - \boldsymbol{A}_\varepsilon) u = f \in Q_\varepsilon \mathcal{X}_\eta,$$

where we look for $u \in Q_\varepsilon \mathcal{Z}_\eta$. We look for

$$\begin{aligned} u &= \beta u_\varepsilon + \widetilde{w} + u_0, \ \beta = \langle u, \phi_1^* \rangle, \ \langle u, \phi_0^* \rangle = 0, \\ \widetilde{P} u &= \beta \widetilde{P} u_\varepsilon + \widetilde{w}, \ P_0 u = \beta P_0 u_\varepsilon + u_0, \end{aligned}$$

with

$$\begin{aligned} f &= \eta u_\varepsilon + \widetilde{g} + f_0,\ \eta = \langle f, \phi_1^* \rangle,\ \langle f, \phi_0^* \rangle = 0, \\ \widetilde{P} f &= \eta \widetilde{P} u_\varepsilon + \widetilde{g},\ P_0 f = \eta P_0 u_\varepsilon + f_0, \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{A}_\varepsilon(\beta u_\varepsilon + \widetilde{w} + u_0) &= \beta \sigma_\varepsilon u_\varepsilon + \boldsymbol{A}_\varepsilon \widetilde{w} + \boldsymbol{A}_\varepsilon u_0, \\ \boldsymbol{A}_\varepsilon u_0 &= \boldsymbol{L}_\nu^{(0)} u_0 + 2 Q_\varepsilon B(\widehat{V}_\varepsilon, u_0), \\ \boldsymbol{A}_\varepsilon \widetilde{w} &= Q_\varepsilon [\boldsymbol{L}_{\nu,\omega} \widetilde{w} + 2B(\widehat{V}_\varepsilon, \widetilde{w})]. \end{aligned}$$

Then, we obtain the following system for the unknown $\beta, \widetilde{w}, u_0$:

$$(\lambda - \sigma_\varepsilon)\beta = \eta + \langle [\boldsymbol{L}_{\nu,\omega} - \boldsymbol{L}_{\nu_0,\omega_0} + 2B(\widehat{V}_\varepsilon, \cdot)](\widetilde{w} + u_0), \phi_1^* \rangle, \tag{71}$$

$$\begin{aligned} (\lambda - \widetilde{P}\boldsymbol{A}_\varepsilon)\widetilde{w} &= \widetilde{g} + 2\widetilde{P} Q_\varepsilon B(\widehat{V}_\varepsilon, u_0), & (72) \\ (\lambda - \boldsymbol{L}_\nu^{(0)}) u_0 &= f_0 + 2 P_0 B(\widehat{V}_\varepsilon, \widetilde{w}), & (73) \end{aligned}$$

and we need to estimate the solution all along the curve Γ described by λ. Now we fix the choice of Γ in such a way that γ which fixes the parts Γ_1 and Γ_2 is such that

$$\gamma = \kappa^2 \varepsilon^2 \sim |\sigma_\varepsilon|/2,$$

so that the curves $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ lie on the right of the real eigenvalue $\sigma_\varepsilon = -|\sigma_\varepsilon|$. We notice that the operator $\widetilde{P}\boldsymbol{A}_\varepsilon$ is

$$Q_\varepsilon(\boldsymbol{L}_{\nu,\omega} + 2\widetilde{P} B(\widehat{V}_\varepsilon, \cdot)),$$

and we notice that for $\varepsilon = 0$, it is the operator $Q_0 \boldsymbol{L}_{\nu_0,\omega_0}$ which has a simple eigenvalue 0, eliminated when it is acting in the subspace orthogonal to ϕ_1^*. The remaining spectrum is then a perturbation of order ε of the spectrum of $\boldsymbol{L}_{\nu,\omega}$ except the eigenvalue close to 0, i.e. its spectrum is "far" on the left from the imaginary axis. It results from estimates obtained at Appendix 6.8, that we have, with a certain $a > 0$ the estimates

$$\left\| (\lambda - \widetilde{P}\boldsymbol{A}_\varepsilon)^{-1} \right\|_{\mathcal{L}(\mathring{H}_\eta^2)} \leq \frac{C}{a^2 + |\lambda|},\ \text{for } \lambda \in \Gamma,$$

and in the same way

$$\left\| (\lambda - \widetilde{P}\boldsymbol{A}_\varepsilon)^{-1} \right\|_{\mathcal{L}(\mathring{H}_\eta^1, \mathring{H}_\eta^2)} \leq \frac{C}{a + |\lambda|^{1/2}},\ \text{for } \lambda \in \Gamma, \tag{74}$$

using the following property for $\lambda \in \Gamma$

$$\frac{1}{|\lambda+\nu\alpha^2|} \leq \frac{c}{a^2+|\lambda|}, \text{ for some } c, a > 0.$$

From (72), (73) we obtain immediately, for $\lambda \in \Gamma$,

$$\widetilde{w} = (\lambda - \widetilde{P}\boldsymbol{A}_\varepsilon)^{-1}\widetilde{g} + 2(\lambda - \widetilde{P}\boldsymbol{A}_\varepsilon)^{-1}\widetilde{P}Q_\varepsilon B(\widehat{V}_\varepsilon, u_0)$$

$$u_0 = (\lambda - \boldsymbol{L}_\nu^{(0)})^{-1} f_0 + 2(\lambda - \boldsymbol{L}_\nu^{(0)})^{-1} P_0 B(\widehat{V}_\varepsilon, \widetilde{w}).$$

Finally we obtain

$$\begin{aligned} \widetilde{w} - D_{\varepsilon,\lambda}\widetilde{w} &= (\lambda - \widetilde{P}\boldsymbol{A}_\varepsilon)^{-1}\widetilde{g} + 2(\lambda - \widetilde{P}\boldsymbol{A}_\varepsilon)^{-1}\widetilde{P}Q_\varepsilon B[\widehat{V}_\varepsilon, (\lambda - \boldsymbol{L}_\nu^{(0)})^{-1} f_0], \\ u_0 - E_{\varepsilon,\lambda}u_0 &= (\lambda - \boldsymbol{L}_\nu^{(0)})^{-1} f_0 + 2(\lambda - \boldsymbol{L}_\nu^{(0)})^{-1} P_0 B[\widehat{V}_\varepsilon, (\lambda - \widetilde{P}\boldsymbol{A}_\varepsilon)^{-1}\widetilde{g}], \end{aligned}$$

where we need to estimate the operator $D_{\varepsilon,\lambda}$ acting on $\widetilde{w}$:

$$D_{\varepsilon,\lambda}\widetilde{w} \stackrel{def}{=} 4(\lambda - \widetilde{P}\boldsymbol{A}_\varepsilon)^{-1}\widetilde{P}Q_\varepsilon B[\widehat{V}_\varepsilon, (\lambda - \boldsymbol{L}_\nu^{(0)})^{-1} P_0 B(\widehat{V}_\varepsilon, \widetilde{w})], \tag{75}$$

and the operator $E_{\varepsilon,\lambda}$ acting on u_0 :

$$E_{\varepsilon,\lambda}u_0 \stackrel{def}{=} 4(\lambda - \boldsymbol{L}_\nu^{(0)})^{-1} P_0 B[\widehat{V}_\varepsilon, (\lambda - \widetilde{P}\boldsymbol{A}_\varepsilon)^{-1}\widetilde{P}Q_\varepsilon B(\widehat{V}_\varepsilon, u_0)]. \tag{76}$$

6.9.1 Estimate for $D_{\varepsilon,\lambda}$

We have

$$\|D_{\varepsilon,\lambda}\|_{\mathcal{L}(\mathring{H}^2_\eta)} \leq c\varepsilon^2 \|(\lambda - \widetilde{P}\boldsymbol{A}_\varepsilon)^{-1}\|_{\mathcal{L}(\mathring{H}^1_\eta, \mathring{H}^2_\eta)} \|(\lambda - \boldsymbol{L}_\nu^{(0)})^{-1}\|_{\mathcal{L}(\mathring{W}^1_\eta, \mathring{W}^{2,\infty})},$$

hence, using (64) and (74),

$$\begin{aligned} \|D_{\varepsilon,\lambda}\|_{\mathcal{L}(\mathring{H}^2_\eta)} &\leq \frac{cC^2\varepsilon^2}{\sqrt{|\lambda|}(a+\sqrt{|\lambda|})} \leq C_1\varepsilon^2, \text{ for } \lambda \in \Gamma_3 \\ &\leq \frac{cC^2\varepsilon^2 c_0}{(a+\sqrt{|\lambda|})} \leq C_1\varepsilon^2, \text{ for } \lambda \in \Gamma_0 \cup \Gamma_1 \cup \Gamma_2. \end{aligned}$$

It results that for ε small enough, the operator $\mathbb{I} - D_{\varepsilon,\lambda}$ has a bounded inverse in $\mathcal{L}(\mathring{H}^2_\eta)$.

6.9.2 Estimate for $E_{\varepsilon,\lambda}$

In the same way, we have

$$\|E_{\varepsilon,\lambda}\|_{\mathcal{L}(\mathring{W}^{2,\infty})} \leq c\varepsilon^2 \|(\lambda - \boldsymbol{L}_\nu^{(0)})^{-1}\|_{\mathcal{L}(\mathring{W}_\eta^1, \mathring{W}^{2,\infty})} \|(\lambda - \widetilde{P}\boldsymbol{A}_\varepsilon)^{-1}\|_{\mathcal{L}(\mathring{H}_\eta^1, \mathring{H}_\eta^2)},$$

hence

$$\|E_{\varepsilon,\lambda}\|_{\mathcal{L}(\mathring{W}^{2,\infty})} \leq C_1\varepsilon^2, \text{ for } \lambda \in \Gamma.$$

It results that, for ε small enough, the operator $\mathbb{I} - E_{\varepsilon,\lambda}$ has a bounded inverse in $\mathcal{L}(\mathring{W}^{2,\infty})$.

6.9.3 Estimates related to $\widetilde{g}$ and f_0

we have now

$$\begin{aligned}
\|2\widetilde{P}Q_\varepsilon B[\widehat{V}_\varepsilon, (\lambda - \boldsymbol{L}_\nu^{(0)})^{-1} f_0]\|_{\mathring{H}_\eta^1} &\leq \frac{C\varepsilon}{\sqrt{|\lambda|}(1+\sqrt{|\lambda|}),} \|f_0\|_{\mathring{W}_\eta^{2,\infty}}, \text{ for } \lambda \in \Gamma_3 \\
&\leq \frac{c_0 C\varepsilon}{(1+\sqrt{|\lambda|}),} \|f_0\|_{\mathring{W}_\eta^{2,\infty}}, \text{ for } \lambda \in \Gamma_0 \cup \Gamma_1 \cup \Gamma_2, \\
\|2P_0 B[\widehat{V}_\varepsilon, (\lambda - \widetilde{P}\boldsymbol{A}_\varepsilon)^{-1}\widetilde{g}\|_{\mathring{W}_\eta^1} &\leq \frac{C\varepsilon}{a^2+|\lambda|} \|\widetilde{g}\|_{\mathring{H}_\eta^2}.
\end{aligned}$$

For $\widetilde{g} \in \mathring{H}_\eta^1$ and f_0 in $\mathring{W}_\eta^{1,\infty}$ we obtain instead

$$\begin{aligned}
\|2\widetilde{P}Q_\varepsilon B[\widehat{V}_\varepsilon, (\lambda - \boldsymbol{L}_\nu^{(0)})^{-1} f_0]\|_{\mathring{H}_\eta^1} &\leq \frac{C\varepsilon}{1+\sqrt{|\lambda|}} \|f_0\|_{\mathring{W}_\eta^{1,\infty}}, \\
\|2P_0 B[\widehat{V}_\varepsilon, (\lambda - \widetilde{P}\boldsymbol{A}_\varepsilon)^{-1}\widetilde{g}\|_{\mathring{W}_\eta^1} &\leq \frac{C\varepsilon}{a+\sqrt{|\lambda|}} \|\widetilde{g}\|_{\mathring{H}_\eta^1}.
\end{aligned}$$

6.9.4 Estimate related to β

For the component β of u we obtain

$$\begin{aligned}
\beta &= \frac{\eta}{\lambda - \sigma_\varepsilon} + \frac{\varepsilon}{\lambda - \sigma_\varepsilon} F(\widetilde{w} + u_0) \\
|F(\widetilde{w} + u_0)| &\leq C(\|\widetilde{w}\|_{\mathring{H}_\eta^1} + \|u_0\|_{\mathring{W}^{1,\infty}}),
\end{aligned}$$

where for $\lambda \in \Gamma$ we have $|\lambda - \sigma_\varepsilon| \geq \frac{1}{2}\kappa^2\varepsilon^2$ by construction.

6.9.5 Estimates for $e^{\boldsymbol{A}_\varepsilon t}$

We still use the contour Γ defined in Figure 3 with $\gamma = \kappa^2\varepsilon^2 \sim |\sigma_\varepsilon|/2$. We realize that

$$\begin{aligned}
\widetilde{P}e^{\boldsymbol{A}_\varepsilon t}(\eta u_\varepsilon + \widetilde{g} + f_0) &= \frac{1}{2i\pi}\int_\Gamma e^{\lambda t}\left(\widetilde{w} + \beta\widetilde{P}u_\varepsilon\right)d\lambda \\
P_0 e^{\boldsymbol{A}_\varepsilon t}(\eta u_\varepsilon + \widetilde{g} + f_0) &= \frac{1}{2i\pi}\int_\Gamma e^{\lambda t}(u_0 + \beta P_0 u_\varepsilon)d\lambda
\end{aligned}$$

where the explicit part of β gives

$$\frac{1}{2i\pi}\int_\Gamma e^{\lambda t}\frac{\eta}{\lambda - \sigma_\varepsilon}d\lambda = \eta e^{-|\sigma_\varepsilon|t}. \tag{77}$$

Now we use on Γ_3,

$$\begin{aligned}
\|\widetilde{w}\|_{\mathring{H}^2_\eta} &\leq \frac{C}{a^2+|\lambda|}\|\widetilde{g}\|_{\mathring{H}^2_\eta} + \frac{cC^2\varepsilon}{\sqrt{|\lambda|}(a+\sqrt{|\lambda|})(1+\sqrt{|\lambda|})}\|f_0\|_{\mathring{W}^{2,\infty}_\eta}, \\
\|u_0\|_{\mathring{W}^{2,\infty}} &\leq \frac{C}{\sqrt{|\lambda|}(1+\sqrt{|\lambda|})}\|f_0\|_{\mathring{W}^{2,\infty}_\eta} + \frac{cC^2\varepsilon}{(a^2+|\lambda|)(1+\sqrt{|\lambda|})}\|\widetilde{g}\|_{\mathring{H}^2_\eta},
\end{aligned}$$

and on $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$,

$$\begin{aligned}
\|\widetilde{w}\|_{\mathring{H}^2_\eta} &\leq \frac{C}{a^2+|\lambda|}\|\widetilde{g}\|_{\mathring{H}^2_\eta} + \frac{cc_0C^2\varepsilon}{(a+\sqrt{|\lambda|})(1+\sqrt{|\lambda|})}\|f_0\|_{\mathring{W}^{2,\infty}_\eta}, \\
\|u_0\|_{\mathring{W}^{2,\infty}} &\leq \frac{c_0C}{(1+\sqrt{|\lambda|})}\|f_0\|_{\mathring{W}^{2,\infty}_\eta} + \frac{cC^2\varepsilon}{(a^2+|\lambda|)(1+\sqrt{|\lambda|})}\|\widetilde{g}\|_{\mathring{H}^2_\eta},
\end{aligned}$$

and for $\widetilde{g} \in \mathring{H}^1_\eta$ and f_0 in $\mathring{W}^{1,\infty}_\eta$

$$\begin{aligned}
\|\widetilde{w}\|_{\mathring{H}^2_\eta} &\leq \frac{C}{a+\sqrt{|\lambda|}}\|\widetilde{g}\|_{\mathring{H}^1_\eta} + \frac{cC^2\varepsilon}{(1+\sqrt{|\lambda|})(a+\sqrt{|\lambda|})}\|f_0\|_{\mathring{W}^{1,\infty}_\eta}, \\
\|u_0\|_{\mathring{W}^{2,\infty}} &\leq \frac{C}{1+\sqrt{|\lambda|}}\|f_0\|_{\mathring{W}^{1,\infty}_\eta} + \frac{cC^2\varepsilon}{(a+\sqrt{|\lambda|})(1+\sqrt{|\lambda|})}\|\widetilde{g}\|_{\mathring{H}^1_\eta}.
\end{aligned}$$

It results that there exists $M(\varepsilon) > 0$ such that

$$\begin{aligned}
\left\|\frac{1}{2i\pi}\int_\Gamma e^{\lambda t}\widetilde{w}d\lambda\right\|_{\mathring{H}^2_\eta} + &\leq \frac{M(\varepsilon)}{1+t}\left[\|\widetilde{g}\|_{\mathring{H}^2_\eta} + \|f_0\|_{\mathring{W}^{2,\infty}_\eta}\right], \\
\left\|\frac{1}{2i\pi}\int_\Gamma e^{\lambda t}u_0 d\lambda\right\|_{\mathring{W}^{2,\infty}} &\leq \frac{M(\varepsilon)}{1+t}\left[\|\widetilde{g}\|_{\mathring{H}^2_\eta} + \|f_0\|_{\mathring{W}^{2,\infty}_\eta}\right],
\end{aligned}$$

and for $\widetilde{g} \in \mathring{H}^1_\eta$ and f_0 in $\mathring{W}^{1,\infty}_\eta$

$$\begin{aligned}
\Big\| \frac{1}{2i\pi} \int_\Gamma e^{\lambda t} \widetilde{w} d\lambda \Big\|_{\mathring{H}^2_\eta} &\leq \frac{M(\varepsilon)}{\sqrt{t}(1+\sqrt{t})} \|\widetilde{g}\|_{\mathring{H}^1_\eta} + \frac{M(\varepsilon)}{1+t} \|f_0\|_{\mathring{W}^{1,\infty}_\eta}, \\
\Big\| \frac{1}{2i\pi} \int_\Gamma e^{\lambda t} u_0 d\lambda \Big\|_{\mathring{W}^{2,\infty}} &\leq \frac{M(\varepsilon)}{\sqrt{t}(1+\sqrt{t})} \|f_0\|_{\mathring{W}^{1,\infty}_\eta} + \frac{M(\varepsilon)}{1+t} \|\widetilde{g}\|_{\mathring{H}^1_\eta}.
\end{aligned}$$

Now we also have

$$\frac{1}{2i\pi} \int_\Gamma e^{\lambda t} \beta d\lambda = \eta e^{-|\sigma_\varepsilon| t} + \frac{1}{2i\pi} \int_\Gamma e^{\lambda t} \frac{\varepsilon}{\lambda - \sigma_\varepsilon} F(\widetilde{w} + u_0) d\lambda,$$

with

$$\begin{aligned}
&\Big| \frac{1}{2i\pi} \int_\Gamma e^{\lambda t} \frac{\varepsilon}{\lambda - \sigma_\varepsilon} F(\widetilde{w} + u_0) d\lambda \Big| \\
\leq\ & \frac{c}{\varepsilon} \left(\Big\| \frac{1}{2i\pi} \int_\Gamma e^{\lambda t} \widetilde{w} d\lambda \Big\|_{\mathring{H}^1_\eta} + \Big\| \frac{1}{2i\pi} \int_\Gamma e^{\lambda t} u_0 d\lambda \Big\|_{\mathring{W}^{1,\infty}} \right).
\end{aligned}$$

Since it is sufficient to estimate $(\widetilde{w} + u_0)$ in $\mathring{H}^1_\eta \oplus \mathring{W}^{1,\infty}$, similarly, we have

$$\Big| \frac{1}{2i\pi} \int_\Gamma e^{\lambda t} \frac{\varepsilon}{\lambda - \sigma_\varepsilon} F(\widetilde{w} + u_0) d\lambda \Big| \leq \frac{cM(\varepsilon)}{\varepsilon(1+t)} \left(\|\widetilde{g}\|_{\mathring{H}^1_\eta} + \|f_0\|_{\mathring{W}^{1,\infty}_\eta} \right).$$

Finally, we obtain the following estimates for a certain constant $C(\varepsilon)$

$$\|e^{\boldsymbol{A}_\varepsilon t} u\|_{\mathcal{Z}_\eta} \leq \frac{C(\varepsilon)}{1+t} \|u\|_{\mathcal{Z}_{\eta,\eta}}.$$

In the same way, for $\widetilde{g} \in \mathring{H}^1_\eta$ and f_0 in $\mathring{W}^{1,\infty}_\eta$ we obtain

$$\begin{aligned}
\|\widetilde{P} e^{\boldsymbol{A}_\varepsilon t} u\|_{\mathcal{L}(\mathring{H}^2_\eta)} &\leq \frac{C(\varepsilon)}{\sqrt{t}(1+\sqrt{t})} \|\widetilde{P} u\|_{\mathring{H}^1_\eta} + \frac{C(\varepsilon)}{1+t} \|P_0 u\|_{\mathring{W}^{1,\infty}_\eta}, \\
\|P_0 e^{\boldsymbol{A}_\varepsilon t} u\|_{\mathcal{L}(\mathring{W}^{2,\infty})} &\leq \frac{C(\varepsilon)}{1+t} \|\widetilde{P} u\|_{\mathring{H}^1_\eta} + \frac{C(\varepsilon)}{\sqrt{t}(1+\sqrt{t})} \|P_0 u\|_{\mathring{W}^{1,\infty}_\eta}.
\end{aligned}$$

Acknowledgments

The authors would like to warmly thank Mariana Haragus for allowing our collaboration and her valuable help and comments. D. Bian is supported by NSFC under the contract 12271032.

Conflict of interest

The authors state that there is no conflict of interest.

Data availability

Data are not involved in this research paper.

References

[1] D. Bian, M. Haragus, E. Grenier: Bifurcations of flows in a strip, *preprint*, 2025.

[2] D. Bian, E. Grenier: Spectrum of Orr-Sommerfeld in the half space, *preprint*, 2025.

[3] D. Bian, E. Grenier: Onset of nonlinear instabilities in monotonic viscous boundary layers, *SIAM J. Math. Anal.* 56(3), 3703-3719, 2024.

[4] D. Bian, S. Dai, E. Grenier : Numerical investigation of the bifurcation of shear layers, *preprint*, 2025.

[5] F. Charru, G. Iooss, A. Léger: Instabilités et bifurcations en mécanique, Coll. Mécanique Théorique, Cépadues Ed 2018.

[6] Q. Chen, D. Wu, Z. Zhang: Tollmien-Schlichting waves near neutral stable curve, *preprint*, 2025.

[7] P. Chossat, G. Iooss: The Couette-Taylor problem. *Applied Mathematical Sciences*, 102. Springer-Verlag, New York, 1994.

[8] P. G. Drazin, W. H. Reid: Hydrodynamic stability, *Cambridge Monographs on Mechanics and Applied Mathematics*. Cambridge University, Cambridge–New York, 1981.

[9] E. Grenier, Y. Guo, and T. Nguyen: Spectral instability of characteristic boundary layer flows, *Duke Math. J.,* 165(16), 3085–3146, 2016.

[10] E. Grenier, T. Nguyen: L^∞ instability of Prandtl layers, *Ann. PDE*, 5(2), 2019.

[11] M. Haragus, G. Iooss: Local bifurcations, center manifolds, and normal forms in infinite dimensional dynamical systems, *Springer*, 2011.

[12] G.Iooss. Cours d'Orsay. vol 31, 1972-73. Université Paris XI.

[13] T.Kato. Perturbation Theory for Linear Operators, *Springer Verlag*, New-York, 1966

[14] L. Landau, L. Lifshitz. Fluids mechanics, Course of Theoretical Physics, Volume 6.

[15] W. H. Reid: The stability of parallel flows, *Developments in Fluid dynamics*, Vol 1, Academic Press, 1965.

[16] P.J. Schmid, D.S. Henningson: Stability and transition in shear flows, *Applied Mathematical Sciences*, 142, Springer-Verlag, New York, 2001.