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Surfaces K3 dans les espaces projectifs à poids, contractions de Mori de longueur sous-maximale

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Résumé en français.

Cette thèse s'articule en deux parties. La première est consacrée à la notion d'extensibilité des variétés projectives, dont la définition est la suivante : une variété projective non-dégénérée, c'est à dire non contenue dans un hyperplan de son espace projectif ambiant, est extensible si elle peut être obtenue par sections hyperplanes d'une variété projective de dimension plus grande qui n'est pas un cône. C'est en général une question difficile lorsque la variété de départ n'est pas intersection complète.

Nous considérons le cas particulier des courbes canoniques. C'est un fait standard que toute surface projective à singularités canoniques qui est extension de courbes canoniques est une surface K3. Réciproquement, toute section hyperplane lisse de surface K3 est une courbe canonique.

Le lien entre l'application de Gauß-Wahl d'une courbe canonique et son extensibilité est au centre de plusieurs travaux, dont ceux de Ciro Ciliberto, Joe Harris et Rick Miranda dans les années 80, et de plus récents, comme ceux de Ciro Ciliberto, Thomas Dedieu et Edoardo Sernesi. Dans un article paru en 2023, Thomas Dedieu et Edoardo Sernesi se sont intéressés à la liste des 14 espaces projectifs à poids de dimension 3 qui admettent des modèles projectifs dans lesquels ils sont extensions de courbes canoniques. Pour chacun d'entre eux, la dimension maximale de ses extensions était alors connue, mais il restait à donner une construction explicite d'extensions maximales pour 5 cas. C'est à cela qu'est consacré le premier chapitre de cette thèse, ainsi qu'à une caractérisation géométrique des courbes primitives contenues dans les sections hyperplanes de ces espaces projectifs à poids.

Le deuxième chapitre de cette thèse est voué à l'étude des contractions de Mori satisfaisant une condition de longueur. Étant donnée une variété lisse contenant des courbes projectives sur lesquelles le diviseur canonique est de degré négatif, la théorie de Mori donne l'existence de morphismes, appelés contractions de Mori élémentaires, qui contractent certaines de ces courbes. Ces morphismes sont les étapes élémentaires du programme de Mori, lequel vise à trouver un représentant birationnel de la variété fixée au départ dont le diviseur canonique est nef. La longueur d'une contraction de Mori élémentaire est une grandeur numérique qui obéit à une inégalité prouvée par Paltin Ionescu et Jaroslaw Wiśniewski (1989 et 1991). Cette inégalité fait intervenir la dimension de la variété de départ, celle du lieu exceptionnel et celle de la fibre générale contenue dans le lieu exceptionnel. Ceci donne lieu à la notion de contraction élémentaire de longueur maximale, et celle de longueur sous-maximale.

Andreas Höring et Carla Novelli ont prouvé dans un article de 2013 que le lieu exceptionnel d'une contraction de Mori élémentaire de longueur maximale admet une structure de fibré projectif à modification birationnelle près. Les résultats centraux de notre chapitre 2 exhibent une structure de fibré en quadriques, ou bien celle d'un fibré projectif, comme modèle birationnel du lieu exceptionnel d'une contraction de Mori élémentaire divisorielle de longueur sous-maximale.

Summary in English.

This thesis is articulated in two parts. The first one is dedicated to the extendability of projective varieties, which is defined as follows: a projective variety which is nondegenerate, meaning that it is contained in no hyperplane of its ambient space, is extendable if it can be obtained via hyperplane sections of a projective variety of larger dimension which is not a cone. It is in general a difficult question when the starting variety is not a complete intersection.

We consider the particular case of canonical curves. It is a common fact that a projective surface with canonical singularities which is an extension of a canonical curve is a K3 surface. Conversely, every smooth hyperplane section of a projective K3 surface is a canonical curve.

The link between the extendability of a canonical curve and the Gauß-Wahl map of said curve is at the center of several works, among which those of Ciro Ciliberto, Joe Harris and Rick Miranda from the 80s, and also of more recent ones, such as the works of Ciro Ciliberto, Thomas Dedieu and Edoardo Sernesi. In a paper published in 2023, Thomas Dedieu and Edoardo Sernesi investigated the list of the 14 weighted projective spaces of dimension 3 which each admit a projective model that is an extension of canonical curves. For each case, the maximal dimension of an extension was known via computations, but it remained to give an explicit construction of maximal extensions for 5 of these spaces. This is the focus of this thesis' first chapter, together with a characterization of the primitive curves contained in the hyperplane sections of these threefolds.

The second chapter of this thesis is dedicated to the study of Mori contractions with a certain length condition. Given a smooth variety containing projective curves on which the canonical divisor has negative degree, Mori theory ensures the existence of morphisms, called elementary Mori contractions, which contract some of these negative curves. These morphisms are the elementary steps of the Mori Program, which aims at finding a birational representative of the starting variety with nef canonical divisor. The length of an elementary Mori contraction is a numerical quantity which obeys an inequality proved by Paltin Ionescu and Jaroslaw Wiśniewski (1989 and 1991). This inequality is an upper bound involving the dimension of the total variety, that of the exceptional locus and that of a general fibre in the exceptional locus. It gives rise to the notion of an elementary Mori contraction of maximal length, and that of submaximal length.

Andreas Höring and Carla Novelli have proven in a paper from 2013 that the exceptional locus of an elementary Mori contraction of maximal length, in other words for which the upper bound is met, admits the structure of a projective bundle up to a birational modification. The central results of this thesis' chapter 2 exhibit the structure of a quadric bundle, or of a projective bundle, as a birational model for the exceptional locus of a divisorial elementary Mori contraction of submaximal length.

Contents

Introduction en Français

Cette thèse de doctorat est centrée autour de la notion de variété projective complexe. Un tel objet géométrique est par définition un espace annelé qui admet localement la structure d'une variété algébrique affine complexe, et admet globalement un plongement dans un espace projectif sur $\mathbf{C}$, dans lequel elle est le lieu des zéros d'un idéal de polynômes homogènes. L'un des principaux propos de la géométrie algébrique complexe est la classification de tels objects d'un point de vue géométrique. Plus précisément, une préoccupation principale réside en les relations qu'il y a entre la géométrie d'une variété donnée et ses équations polynomiales homogènes, de même qu'en l'identification de ses sous-variétés (par exemple, des courbes "spéciales" dans une variété de diension 2 ou plus) que l'on peut contracter par un morphisme.

Hors indications supplémentaires, toutes les variétés que nous considérerons dans cette thèse sont sur le corps des nombres complexes et munies de leur topologie de Zariski. Cette topologie est bien moins fine que celle à laquelle d'autres géomètres peuvent être habitués, mais c'est un bon cadre pour faire de la géométrie algébrique, puisqu'il s'agit de la plus simple qui rende les polynômes continus.

⋆ ⋆

Qu'est-ce qu'une intersection (in)complète ? Qu'est-ce qu'une extension de variété projective ?

Supposons que X est une variété plongée dans l'espae projectif $\mathbf{P}^N$. Elle est alors définie par un ensemble d'équations, autrement dit, elle est le lieu des zéros de polynômes homogènes en les coordonnées homogènes $\mathbf{x} = [x_0 : \cdots : x_N]$ de $\mathbf{P}^N$. La philosophie de la géométrie algébrique projective est que la donnée de l'image du plongement $X \hookrightarrow \mathbf{P}^N$ et celle de ses équations polynomiales sont équivalentes. À savoir, il est possible d'obtenir toutes les informations géométriques sur X à partir de l'idéal de polynômes par lequel elle est définie. Cependant, quand la codimension de X est grande, c'est un processus relativement difficile.

La situation la plus courante est celle d'une intersection incomplète, c'est à dire, une variété $X \subset \mathbf{P}^N$ qui est définie par un plus grand nombre d'équations polynomiales que sa codimension. Un aspect de cela est incarné par la dépendance algébrique entre les équations de X : si un ensemble d'équations pour X dans ce plongement est

$$f_1(\mathbf{x}) = \cdots = f_r(\mathbf{x}) = 0,$$

et I désigne l'idéal engendré par les f_i dans l'anneau $R = \mathbf{C}[x_0, ..., x_N]$, alors les polynômes homogènes f_i satisfont des relations algébriques "non-triviales" (naturellement, les relations triviales $f_i f_j - f_j f_i = 0$ sont toujours de rigueur). L'ensemble des relations entre les f_i constitue le noyau du morphisme d'anneaux $R^{\oplus r} \to R$ qui à chaque $(h_1, ..., h_r)$ associe $\sum_{i=1}^r f_i h_i$ et qui a pour image I. Cela prend la forme d'une suite exacte

$$M_0 \to R^{\oplus r} \xrightarrow{(f_1 \cdots f_r)} R \to R/I \to 0,$$

où M_0 est un R-module libre de type fini, le noyau de $R^{\oplus r} \to R$ étant finiment engendré par noetherianité. On pourrait également vouloir identifier le noyau de $M_0 \to R^{\oplus r}$, qui représente des "relations entre les relations" des f_i. Ce noyau est l'image d'un autre morphisme $M_1 \to M_0$, avec M_1 un autre R-module libre de type fini. On peut répéter ce processus : en fin de compte, par le théorème des syzygies de Hilbert, on obtient une suite exacte de longueur finie

$$0 \to M_s \to \cdots \to M_0 \to R^{\oplus r} \xrightarrow{(f_1 \cdots f_r)} R \to {}^{R}/_{I} \to 0,$$

où les M_i sont des R-modules libres de type fini. Les *syrygies* de I (ou de façon équivalente, les syzygies de X), suivant la définition qui en est donnée par D. Eisenbud dans [Ei05], désignent les éléments des noyaux des morphismes impliqués dans cette suite exacte.

Le fait que X n'est pas une intersection incomplète dans $\mathbf{P}^N$ se traduit par le fait que les f_i ne forment pas une suite régulière dans R, autrement dit, l'un des f_i est un diviseur de zéro dans ${}^{R}/_{(f_1,\ldots,f_{i-1})}$, et les syzygies sont non-triviales. Dans cette situation, l'idéal engendré par les f_i est, d'une certaine manière, plus compliqué qu'il ne le serait pour une intersection complète. En d'autres termes, ce n'est pas un procédé facile d'apprendre sur la géométrie de X à partir de la seule donnée de ses équations.

Exemple : La cubique gauche vs. une conique dans $\mathbf{P}^3$.

Soit $[x : y : z : w]$ des coordonnées homogènes sur l'espace projectif $\mathbf{P}^3$, et considérons le plongement

$$[\lambda : \mu] \in \mathbf{P}^1 \mapsto [\lambda^3 : \lambda^2\mu : \lambda\mu^2 : \mu^3] \in \mathbf{P}^3.$$

L'image de ce plongement est appelée cubique gauche. Elle est isomorphe à $\mathbf{P}^1$, en somme, une courbe simple, mais ce n'est pas une intersection complète car elle a codimension 2 tout en étant le lieu des zéros des trois mineurs maximaux de la matrice suivante :

$$\begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}.$$

Si l'on note ainsi les trois mineurs :

$$\mathtt{m}_{12} = xz - y^2,\ \mathtt{m}_{13} = xw - yz,\ \mathtt{m}_{23} = yw - z^2,$$

alors toute intersection complète donnée par l'annulation de deux d'entre eux (ou de deux combinaisons linéaires indépendantes de ces trois mineurs) est une courbe réductible dont les composantes sont la cubique gauche et une droite "résiduelle". Les relations algébriques entre les trois mineurs sont engendrées par ces deux-ci:

$$x\,\mathtt{m}_{23} - y\,\mathtt{m}_{13} + z\,\mathtt{m}_{12} = y\,\mathtt{m}_{23} - z\,\mathtt{m}_{13} + w\,\mathtt{m}_{12} = 0.$$

D'un autre côté, considérons le plongement

$$[\lambda : \mu] \in \mathbf{P}^1 \mapsto [\lambda^2 : \lambda\mu : \mu^2 : 0] \in \mathbf{P}^3,$$

dont l'image (une conique) est aussi isomorphe à $\mathbf{P}^1$, mais est donnée dans $\mathbf{P}^3$ par les équations

$$w = xz - y^2 = 0.$$

Contrairement aux trois mineurs de la cubique gauche, ces deux équations ci-dessus ne satisfont aucune relation algébrique outre la relation triviale

$$(xz - y^2)w - w(wz - y^2) = 0,$$

et la conique est en effet une intersection complète dans $\mathbf{P}^3$. Il est par example possible de déterminer à la main par le calcul son fibré tangent et son fibré normal grâce à la Jacobienne du système $(w, xz - y^2)$.

Plus généralement, lorsque X est une intersection complète dans $\mathbf{P}^N$, les seules relations algébriques entre ses équations polynomiales sont les relations triviales. Par exemple, si l'on a trois polynômes homogènes f, g et h sur $\mathbf{P}^N$ tels que

$$X = \{f = g = h = 0\}$$

avec $\dim X = N - 3$, alors les trois équations bénéficient d'une sorte d'indépendance algébrique, ce qui signifie que les seules relations qui les lient sont engendrées par

$$fg - gf = fh - hf = gh - hg = 0.$$

En particulier, il n'y a aucune relation lorsque X est une hypersurface, c'est à dire le lieu des zéros d'un seul polynôme homogène.

Globalement, la même philosophie s'applique quand X est plongée comme sous-variété d'une autre variété Y, qui pourrait ne pas être un espace projectif. Si Y est elle-même une variété projective et X est une intersection complète dans Y, dont la géométrie et l'anneau de coordonnées $R_Y = {}^{R}/_{I(Y)}$ sont déjà connus, alors il est facile d'obtenir des informations sur X et sur son anneau de coordonnées à partir de Y. Par exemple, si X est donnée comme le lieu d'annulation d'un unique élément $f \in R_Y$, alors on a $R_X \simeq {}^{R_Y}/_{(f)}$, et le fibré tangent de X peut être retrouvé à la main à partir de celui de Y. De plus, les syzygies de X ont une forme similaire à celle des syzygies de Y. Pour une reférence extensive sur les syzygies et leur lien avec la géométrie, sur les intersections complètes et incomplètes, nous suggérons [Ei05].

Donnons maintenant une définition qui est au centre de notre premier chapitre.

Définition 0.0.1. *Soit $X \subset \mathbf{P}^N$ une variété projective non-dégénérée, c'est à dire, contenue dans aucun hyperplan. On dit que X est* extensible *s'il existe $Y \subset \mathbf{P}^{N+1}$ qui n'est pas un cône et qui contient X comme section hyperplane.*

Remarque. Si X est une intersection complète dans $\mathbf{P}^N$, alors la construction d'une extension est relativement facile. Pour illustrer notre propos, considérons une hypersurface $X \subset \mathbf{P}^N$ donnée en coordonnées homogènes $[x_0 : \cdots x_N]$ par une équation $f(\mathbf{x}) = 0$, avec f un polynôme homogène. Il est alors possible d'introduire une nouvelle coordonnée x_{N+1} et de noter $g = f + x_{N+1}^{\deg f}$, puis de considérer l'hypersurface $Y \subset \mathbf{P}^{N+1}$ donnée par l'équation $g = 0$. Cette variété Y n'est alors par un cône et elle contient X comme section par l'hyperplan $\{x_{N+1} = 0\}$. Notons que Y est elle-même une hypersurface dans son espace ambient $\mathbf{P}^{N+1}$, et qu'elle peut donc être étendue par le même argument, et qu'une telle extension de Y peut aussi être étendue, ainsi de suite.

Inversement, l'existence d'une extension pour X est une question compliquée lorsque X n'est pas une intersection complète (ce qui est, en un sens, la situation la plus courante en géométrie projective). La difficulté vient des syzygies de X, puisque construire une extension requiert de trouver une variété de dimension plus grande avec des syzygies similaires. Plus précisément, supposons que $Y \subset \mathbf{P}^{N+1}$ est une extension de X, où l'espace projectif $\mathbf{P}^{N+1}$ est doté de coordonnées homogènes $[x_0 : \cdots x_{N+1}]$ de telle sorte que $X = Y \cap \{x_{N+1} = 0\}$. Soit S l'anneau $\mathbf{C}[x_0, ..., x_{N+1}]$ et $I(Y) \subset S$ l'idéal de Y. Alors les syzygies de Y prennent la forme d'une suite exacte

$$0 \to N_r \to \cdots \to N_1 \to N_0 \to S \to {}^{S}/_{I(Y)} \to 0,$$

où chaque N_i est isomorphe à $S^{\oplus k_i}$ pour un certain $k_i \in \mathbf{N}$. Les syzygies de X sont obtenues en spécifiant $x_{N+1} = 0$ dans les relations entre les équations qui définissent Y, et les relations entre lesdites relations, et ainsi de suite. En d'autres termes, les syzygies de X sont entièrement déterminées par la suite exacte ci-dessus.

Naturellement, une variété projective X peut toujours être réalisée comme section hyperplane d'un cône sur elle-même. Toutefois, il n'est pas possible d'obtenir des informations supplémentaires sur la géométrie de X à partir de la géométrie du cône en question. Dans la définition d'une extension ci-dessus, nous imposons spécifiquement que Y ne doit pas être un cône, et par conséquent, il n'existe pas d'extension de X dans le cas général.

Quand une extension Y de X existe, il s'agit d'un espace naturel dans lequel déformer X en bougeant simplement l'hyperplan H tel que $Y \cap H = X$ vers d'autres hyperplans H'. Bien sûr, la même question de l'existence d'une extension se pose pour Y. Si l'on parvient à étendre Y, alors on a une variété dans laquelle X a codimension 2, donnant lieu à un espaces de déformations pour X d'autant plus grand.

Terminologie. Si l'on a une chaîne d'extensions

$$X \subset Y = X_1 \subset X_2 \subset \cdots$$

où X_k est une extension de X_{k-1}, avec $\dim X_k = 1 + \dim X_{k-1}$, on dit que X_k est une k-extension de X.

D'après la remarque précédente, une variété $X \subset \mathbf{P}^N$ qui est une intersection complète admet une telle chaîne infinie d'extensions. Dans [Bal21], E. Ballico a donné des exemples d'intersections incomplètes admettant également des extensions de dimension arbitrairement grande. Cependant, le théorème dit de la tour Babylonienne (*Babylonian tower theorem*), dû à A. N. Tyurin, W. Barth et A. Van de Ven, énonce que les intersections complètes sont les seules variétés localement intersections complètes qui admettent des chaînes infinies d'extensions localement intersections complètes. Pour une preuve du théorème de la tour Babylonienne, nous renvoyons à [Co12].

L'extensibilité des variétés projectives a intéressé bien des mathématiciens dans les années récentes, mais également tout au long du XXe siècle. Mentionnons [LD23], un article de A. Lopez qui donne une idée globale de la question et mentionne des travaux de S. Lvovski [Lvo92] et J. Wahl [Wa90]. Citons de plus anciennes références qui remontent au début du XXe siècle, au moment où des mathématiciens tels que G. Scorza et A. Terracini ont étudié la notion d'extensibilité, prouvant par exemple que ni une variété de Veronese de dimension 2 ou plus, ni une variété de Segre de dimension 3 ou plus n'est intégrable ([Sc10], [Te13]).

Dans le chapitre 1, nous nous intéresserons à l'extensibilité de *courbes canoniques*. Il s'agit de courbes projectives lisses non-hyperelliptiques $\Gamma \subset \mathbf{P}^{g-1}$ plongées par le système linéaire de leur fibré cotangent. Un tel plongement est dit canonique car le fibré cotangent d'une courbe lisse est intrinsèque et constitue un sujet d'intérêt pour les géomètres complexes en général. L'extensibilité d'une courbe canonique fait intervenir beaucoup d'arguments sur les surfaces K3, les 3-variétés de Fano et l'application dite de *Gauß-Wahl* de la courbe en question (parfois simplement appelée application de Gauß, ou application gaußienne, ou application de Wahl, voir par exemple [Wa90] ou [CHM88]). Plus précisément, c'est un fait standart que toute surface projective avec des singularités canoniques qui est une extension de courbe canonique est une surface K3, et réciproquement, toute section hyperplane lisse d'une surface K3 projective est une courbe canonique (pour une preuve, se référer au Lemme 1.2.8). En outre, la section anticanonique générale d'une 3-variété de Fano à singularités canoniques est une surface K3, et réciproquement, toute 3-variété à singularités canoniques qui est extension d'une surface K3 projective est de Fano.

Dès 1987, J. Wahl prouva dans [Wa87] que si une courbe lisse est hypersurface d'une surface K3, alors son application de Gauß-Wahl n'est pas surjective. Ceci fut rapidement suivi d'une preuve alternative du même résultat par A. Beauville et J.-Y. Mérindol dans [BM87]. Plus tard, E. Arbarello, A. Bruno et E. Sernesi prouvèrent dans [ABS17] qu'une courbe canonique de genre au moins 11 et d'indice de Clifford au moins 3 est extensible *si et seulement si* son application de Gauß-Wahl n'est pas surjective. Le lien entre les applications de Gauß-Wahl et les extensions de dimension supérieures de courbes canoniques fut étudié plus avant par C. Ciliberto, T. Dedieu et E. Sernesi dans [CDS20].

Donnons maintenant la définition d'un espace projectif à poids.

Définition 0.0.2. *Étant donné un ensemble d'entiers positifs non nuls premiers entre eux $a_0, ..., a_N$, l'espace projectif à poids associé $\mathbf{P}(a_0, ..., a_N)$ est défini comme le quotient de $\mathbf{C}^{N+1} - \{0\}$ par l'action suivante de $\mathbf{C}^*$:*

$$\lambda \cdot (x_0, ..., x_N) = (\lambda^{a_0} x_0, ..., \lambda^{a_N} x_N).$$

L'extensibilité de courbes canoniques et de surfaces K3 est le sujet du chapitre 1, dans lequel nous examinons la liste des 14 espaces projectifs à poids de dimension 3 qui peuvent être réalisés comme extensions de surface K3 et de courbes canoniques. La liste de ces extensions de dimensions 3 peut être trouvée dans [DS23]. Plus précisément, étant donné un espace $\mathbf{P}$ parmi ces 14, il existe un plongement $\mathbf{P} \hookrightarrow \mathbf{P}^N$ qui contient des surfaces K3 et des courbes canoniques comme sections par des hyperplans. De plus, il n'existe pas d'autre espace projectif à poids de dimension 3 avec cette propriété. Pour chacun de ces 14 espaces projectifs à poids, la dimension d'une extension de $\mathbf{P} \subset \mathbf{P}^N$ est bornée supérieurement, et pour 8 d'entre eux, la construction d'une extension de dimension maximale était connue grâce à [DS23]. Le chapitre 1 se concentre sur les 6 cas restants.

Les espaces projectifs à poids forment une classe de variétés sur lesquelles les calculs et l'étude des intersections complètes sont intelligibles grâce à l'existence de coordonnées homogènes. Nous avons alors une liste de courbes canoniques $\Gamma \subset \mathbf{P}^{g-1}$ qui sont sections linéaires de modèles projectifs d'espaces projectifs à poids ; ceci rend possible d'obtenir des informations sur lesdites courbes, car la géométrie d'un espace projectif à poids est bien connue.

Les résultats principaux du chapitre 1 sont listés à la fin de cette introduction.

⋆ ⋆

Qu'est-ce que le MMP ? Qu'est-ce que la théorie de Mori ?

L'idée générale du Programme du Modèle Minimal — ou MMP, abréviation de l'anglais Minimal Model Program — est de trouver un "bon" représentant d'une variété donnée X à équivalence birationnelle près. Originellement pour les surfaces, il s'agissait de trouver un représentant sans courbes d'auto-intersection -1, c'est à dire un modèle birationnel X' de X sans courbe que l'on pourrait contracter via un morphisme birationnel vers une surface lisse. Le but plus moderne à partir d'une variété X de dimension quelconque est de trouver un représentant X' qui a soit son diviseur canonique nef, soit un morphisme $X' \to Y$, vers une variété Y de dimension plus petite, tel que le diviseur anticanonique de la fibre générale est ample. Un tel morphisme de dimension relative strictement positive avec fibre générale de Fano est appelé une *fibration de Fano*. Plus spécifiquement, celles obtenues par une méthode particulière (appelée Programme de Mori) sont appelées *fibrations de Mori*.

Comme références sur le MMP en général, citons [KMM87] (dans laquelle Y. Kawamata, K. Matsuda et K. Matsuki consacrent un chapitre à la conjecture du flip), [Kol96], [KM98] et [Mor82] (dans laquelle S. Mori se concentre sur le cas de dimension 3).

Le Programme de Mori se base sur l'idée suivante : si une variété X contient une courbe projective C, il est possible de donner sens à l'intersection de n'importe quel diviseur de Cartier $D \in \mathrm{Pic}(X)$ (de façon équivalente, un fibré en droites) avec C, que l'on définit comme le degré de la restriction $D|_C$. Considérant au départ une variété X dont le diviseur canonique est Cartier (ou simplement $\mathbf{Q}$-Cartier), le Programme de Mori vise à se débarrasser du lieu dans X qui est couvert par les courbes K_X-négatives, c'est à dire les courbes projectives C telles que $K_X \cdot C < 0$. Pour une belle vue d'ensemble de ce sujet, nous renvoyons à [Deb01], un livre culte par O. Debarre. Nous souhaitons également citer [Deb16], un cours de M2 du même auteur sur la théorie de Mori. Une référence plus ancienne mais fondatrice est le livre de J. Kollár et S. Mori, [KM98], paru en 1998.

Il peut arriver qu'une variété lisse X ne soit pas recouverte par des courbes C telles que $K_X \cdot C < 0$. Sous cette condition, le lieu que l'on vise à modifier n'est pas la variété totale X ; la théorie développée par Y. Kawamata, J. Kollár, S. Mori et V. Shokurov dans les années 80 donne l'existence de morphismes birationnels $X \to Y$, chacun vers une variété normale Y, qui contractent certaines des courbes K_X-négatives. Un tel morphisme n'est pas unique, et en général il en existe une quantité dénombrable d'*élémentaires*. L'élémentarité est à comprendre au sens de la Définition 0.0.3 ci-dessous.

Définition 0.0.3. *Étant donnée une variété X dont le diviseur canonique est $\mathbf{Q}$-Cartier, un morphisme à fibres connexes depuis X qui contracte des courbes K_X-négatives et pour lequel $-K_X$ est relativement ample est ce qu'on appelle une* contraction de Mori. *De plus, une contraction de Mori $X \to Y$ est* élémentaire *si elle ne peut pas être factorisée par une autre contraction de Mori $X \to Y'$ dans un diagramme commutatif*

$$\begin{array}{ccc} X & \longrightarrow & Y' \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

où $Y' \to Y$ n'est pas un isomorphisme.

Il existe aussi des cas de variétés lisses X pour lesquelles la théorie de Mori donne l'existence de contractions de Mori élémentaires $X \to Y$ avec $\dim Y < \dim X$. Dans cette situation, on dit que $X \to Y$ est une *fibration de Mori*, comme mentionné précédemment, et il s'agit d'un objet final du MMP.

Notons que certains mathématiciens appellent les contractions de Mori *contractions de Fano-Mori*, puisque la fibre générale est Fano.

Définition 0.0.4 (Longueur)**.** *Soit X une variété de diviseur canonique $\mathbf{Q}$-Cartier et $f : X \to Y$ une contraction de Mori élémentaire. La* longueur *de f est*

$$l(f) = \min \{-K_X \cdot \Gamma \mid \Gamma \subset X \text{ une courbe rationnelle contractée}\}.$$

Grâce à un résultat classique de P. Ionescu et J. Wiśniewski (voir [Io86] et [Wi91]), la longueur d'une contraction de Mori élémentaire $X \to Y$, pour X une variété lisse, admet une borne supérieure qui fait intervenir la dimension de son lieu exceptionnel et celle d'une fibre dans son lieu exceptionnel.

Remarque. Si $f : X \to Y$ est birationnelle à fibres connexes, alors le lieu exceptionnel de f désigne habituellement le lieu dans X où f n'est pas un isomorphisme. Si en revanche f est à fibres connexes, mais pas birationnelle, auquel cas $\dim X > \dim Y$, alors nous nous autorisons à dire que le lieu exceptionnel de f intervenant dans la borne supérieure de sa longueur est la variété totale X, bien que ce ne soit pas un choix consensuel de vocabulaire.

Théorème 0.0.1 ([Io86] Theorem 0.4, [Wi91] Theorem 1.1)**.** *Soit $f : X \to Y$ une contraction de Mori élémentaire d'une variété lisse X. Soit $E \subset X$ une composante irréductible du lieu f-exceptionnel et $F \subset E$ une composante irréductible d'une fibre. Alors*

$$\dim E + \dim F \geq \dim X + l(f) - 1.$$

Lorsque la borne est atteinte, autrement dit lorsque qu'il existe une composante irréductible E de $\mathrm{Exc}(f)$ et une fibre $F \subset E$ telles que

$$l(f) = \dim E + \dim F + 1 - \dim X,$$

la contraction est dite de *longueur maximale*. Dans le cas d'une contraction de Mori élémentaire de longueur maximale d'une variété lisse projective, A. Höring et C. Novelli ont mis en évidence dans [HN13] la structure d'un fibré projectif pour le lieu exceptionnel sur son image, à modification birationnelle près :

Théorème 0.0.2 ([HN13], Theorems 1.3 & 1.4)**.** *Soit X une variété projective lisse.*

- *Soit $f : X \to Y$ une contraction de Mori élémentaire telle que* $\dim X - \dim Y > 0$ *et $l(f)$ est maximale.*

  Si f est équidimensionnelle, alors il s'agit d'un fibré projectif.

  Si f n'est pas équidimensionnelle, il existe des morphismes birationnels $X' \to X$ et $Y' \to Y$ avec X' et Y' lisses, et un fibré projectif $X' \to Y'$ tel que le diagramme suivant commute :

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

- *Soit $f : X \to Y$ une contraction de Mori élémentaire birationnelle de longueur maximale, et E le lieu exceptionnel de f. Soit E' la normalization de E, $Z = f(E)$ et $E' \to Z'$ la fibration obtenue par la factorisation de Stein de $E' \to Z$. Alors $E' \to Z'$ est un fibré projectif en codimension 1 ; de plus, il existe des morphismes birationnels $\overline{E} \to E'$ et $\overline{Z} \to Z'$ avec $\overline{E}$ et $\overline{Z}$ lisses, et un fibré projectif $\overline{E} \to \overline{Z}$ tel que le diagramme suivant est commutatif :*

$$\begin{array}{ccccc} \overline{E} & \longrightarrow & E' & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow f \\ \overline{Z} & \longrightarrow & Z' & \longrightarrow & Z \end{array}$$

Dans le chapitre 2 de cette thèse, nous assouplissons l'hypothèse de longueur, nous concentrant sur le cas de longueur sous-maximale, c'est à dire, le cas d'une contraction de Mori élémentaire $f : X \to Y$ avec X lisse, dont la longueur n'est pas maximale, mais telle qu'il existe E une composante irréductible de $\mathrm{Exc}(f)$ et $F \subset E$ une fibre vérifiant

$$l(f) = \dim E + \dim F - \dim X.$$

Considérons le résultat suivant sur les variétés de Fano lisses dont le diviseur anticanonique est de grand degré par rapport à la dimension :

Théorème 0.0.3 ([CMSB02], [Ke02], [Mi04], [DH17])**.** *Soit X une variété de Fano lisse de dimension n.*

- *[CMSB02], [Ke02]: Si pour toute courbe rationnelle $\Gamma \subset X$, on a $-K_X \cdot \Gamma \geq n+1$, alors $X \simeq \mathbf{P}^n$.*
- *[Mi04], [DH17]: Si pour toute courbe rationnelle $\Gamma \subset X$, on a $-K_X \cdot \Gamma \geq n$, alors X est isomorphe soit à une quadrique, soit à un espace projectif.*

L'étude des contractions de Mori élémentaires de longueur maximale ou sous-maximale peut être perçue comme une version relative du théorème ci-dessus. En effet, si par exemple $X \to Y$ est une contraction de Mori élémentaire vers un point, la condition de longueur porte sur le degré minimal de $-K_X$ sur les courbes rationnelles qui sont dans X. Réciproquement, toute fibre lisse F d'une contraction de Mori élémentaire $X \to Y$ avec $\dim X > \dim Y$ est de Fano, et par adjonction on a $K_F = K_X|_F$. L'inégalité du Théorème 0.0.1 porte donc sur le degré minimal que le diviseur $-K_F$ peut avoir sur les courbe rationnelles dans F.

L'étude des contractions de Mori a été un domaine de recherche actif pour de nombreux mathématiciens au cours des trois dernières décennies, parmi lesquels nous souhaitons citer J. Wiśniewski, M. Andreatta, G. Occhetta et L. Tasin.

Dans [Wi91] J. Wiśniewski donna un critère d'amplitude pour un fibré en droite sur une variété de Fano lisse X sous certaines conditions qui font intervenir les contractions de Mori de X.

Dans [AW93] M. Andreatta et J. Wiśniewski donnèrent des critères pour qu'un fibré en droites soit sans point base relativement à une contraction, et pour la structure locale d'une contraction autour d'une fibre satisfaisant une condition de dimension ; ceci implique la notion de *diviseur support* et celle de *valeur nef* d'une contraction.

Les deux articles conjointement écrits par M. Andreatta et G. Occhetta que nous souhaitons citer sont [AO02] et [AO05]. Dans le premier, ils fournirent un critère numérique pour qu'une contraction de Mori d'une variété lisse soit un éclatement d'une autre variété lisse en un point. Le deuxième est un critère numérique pour qu'une variété de Fano lisse soit un produit d'espaces projectifs, ou bien l'éclatement d'un espace projectif le long d'un sous-espace linéaire, ledit critère portant sur l'existence d'un type particulier de contraction de Mori.

M. Andreatta et L. Tasin contribuèrent à la théorie avec les deux articles relativement récents [AT14] et [AT16]. Ces deux articles portent également sur la notion de valeur nef d'une contraction et sa relation avec la dimension de l'espace total, ou bien celle des fibres.

De façon générale, la valeur nef d'une contraction $X \to Y$ peut nous permettre d'en savoir beaucoup sur la structure de la contraction, puisqu'il s'agit (dit succinctement) de la mesure d'à quel point le diviseur canonique K_X est divisible le long des fibres du morphisme $X \to Y$, et puisqu'elle est intimement liée à la longueur de la contraction dont il est question, autrement dit le plus petit degré que le diviseur $-K_X$ peut avoir sur les courbes rationnelles contractées.

⋆ ⋆

Plan de thèse : résultats principaux.

Ce texte est séparé en deux chapitres principaux. Dans le premier chapitre, nous nous concentrons sur la liste des 14 espaces projectifs à poids $\mathbf{P}$ de dimension 3 admettant chacun un modèle projectif $\mathbf{P} \hookrightarrow \mathbf{P}^N$ qui soit extension de surfaces K3 et de courbes canoniques. Les 6 espaces parmi ces 14 pour lesquels la construction d'une extension maximale n'était pas connue par [DS23] sont listés ci-dessous.

$\mathbf{P}(1,4,5,10)$
$\mathbf{P}(1,2,6,9)$
$\mathbf{P}(1,2,3,6)$
$\mathbf{P}(1,3,8,12)$
$\mathbf{P}(1,6,14,21)$
$\mathbf{P}(2,3,10,15)$

Le résultat central du chapitre 1 est le suivant.

Théorème. *Soit $\mathbf{P} = \mathbf{P}(a_0, a_1, a_2, a_3)$ l'un des espaces projectifs à poids de la liste ci-dessus. Alors son diviseur anticanonique $-K_{\mathbf{P}}$ est très ample et induit un modèle projectif $\mathbf{P} \hookrightarrow \mathbf{P}^N$ dont la section hyperplane générale est une surface K3. En outre, une construction d'une extension maximale Y de ce modèle est fournie dans le tableau ci-dessous :*

$\mathbf{P}$	Y	$\dim(Y)$
$\mathbf{P}(1,4,5,10)$	*quintique spéciale de* $\mathbf{P}(1^3,2,4^3)$	5
$\mathbf{P}(1,2,6,9)$	10-*ique spéciale de* $\mathbf{P}(1^2,3,5,9^2)$	4
$\mathbf{P}(1,2,3,6)$	$\mathbf{P}(1,2,3,6)$	3
$\mathbf{P}(1,3,8,12)$	9-*ique spéciale de* $\mathbf{P}(1^2,3,4,8^2)$	4
$\mathbf{P}(1,6,14,21)$	*heptique spéciale de* $\mathbf{P}(1^2,2,3,6^2)$	4
$\mathbf{P}(2,3,10,15)$	*intersection complète de codim.* 2 *dans un fibré en* $\mathbf{P}(1^2,2,3,5^3)$ *sur* $\mathbf{P}^1$	5

Pour davantage de détails, voir le Théorème 1.4.5. La méthode que nous allons utiliser pour construire des extensions de ces plongements $\mathbf{P} \hookrightarrow \mathbf{P}^N$ est similaire à celle développée par B. Totaro (non publiée) qui fut utilisée dans [CD21, §3.2] et [CD24', 4.8].

Les polarisations hyperplanes H sur les surfaces K3 obtenues comme sections linéaires de ces espaces projectifs à poids sont parfois non-primitives. Sur chaque telle surface K3 S, lorsque la polarisation n'est pas primitive, nous considérons la polarisation primitive $H' = \frac{1}{i_S}H$, où i_S est le plus grand entier positif r tel que $\frac{1}{r}H$ est Cartier, puis nous étudions la géométrie de la courbe générale du système linéaire ainsi obtenu $|H'|$ dans le Théorème 1.4.20. Au cours du premier chapitre, nous donnons également une classification rapide des espaces projectifs à poids de dimension 4 ou plus qui admettent des modèles projectifs qui sont extensions de surfaces K3.

Dans le second chapitre, nous examinons les contractions de Mori élémentaires de longueur sous-maximale, c'est à dire, dont la longueur est égale à sa borne supérieur moins 1 (voir le Théorème 0.0.1). Lorsque la contraction est de dimension relative 1, le résultat [Sa82, Theorem 1.13] s'applique, assurant l'existence d'un modèle birationnel qui est un fibré en coniques. Sous l'hypothèse que la longueur est sous-maximale, nous donnons la construction explicite d'un modèle birationnel (voir la Sous-section 2.2.1) et nous prouvons qu'il s'agit d'un fibré en coniques en codimension 1. Dans le cas birationnel, nous supposons que la contraction de Mori est divisorielle. Notre résultat central est :

Théorème. • *Soit $f : X \to Y$ une fibration équidimensionnelle d'une variété projective normale X vers une variété projective lisse Y avec $\dim X = \dim Y + 1$. Supposons l'existence d'un diviseur de Cartier relativement ample A sur X tel que $K_X + A$ est trivial sur les fibres lisses. Alors f est un fibré en coniques en codimension* 1.

• *Soit $f : X \to Y$ une contraction de Mori élémentaire de longueur sous-maximale d'une variété X lisse. Si f est divisorielle, c'est à dire que le lieu f-exceptionnel E est une hypersurface de X, alors le lieu équidimensionnel de $E \to f(E)$ est birationnel soit à un fibré en quadriques, soit à un fibré projectif.*

Pour plus de détails et des arguments au cas par cas, voir les Théorèmes 2.2.3 et 2.2.10 avec le Corollaire 2.2.11, ainsi que le Théorème 2.3.3.

Introduction in English

This Ph.D. thesis is centered around the notion of complex projective variety. Such a geometric object is by definition a ringed space which admits locally the structure of a complex affine algebraic variety, and globally admits an embedding into a projective space over $\mathbf{C}$, inside which it is the zero locus of an ideal of homogeneous polynomials. One of the main points of complex algebraic geometry is the classification of such objects from a geometric point of view. More precisely, a principal concern is the relations between the geometry of a given variety and its homogeneous polynomial equations, as well as the identification of its subvarieties (e.g., some "special" curves inside a variety of dimension 2 or more) which one may contract via a morphism.

Without further specifications, all the varieties we will consider in this thesis are over the field of complex numbers and endowed with their Zariski topology. This topology is a lot less fine than that to which other geometers might be used to, but is a good framework for doing algebraic geometry, as it is the simplest one for which polynomials are continuous.

⋆ ⋆

What is a (non)complete intersection? What is an extension of a projective variety?

Assume X is a variety embedded in the projective space $\mathbf{P}^N$. Then it is described by a set of equations, i.e., the vanishing of homogeneous polynomials in the homogeneous coordinates $\mathbf{x} = [x_0 : \cdots : x_N]$ on $\mathbf{P}^N$. The philosophy of projective algebraic geometry is that the datum of the image of the embedding $X \hookrightarrow \mathbf{P}^N$ and the data of its polynomial equations are equivalent. Namely, it is possible to recover all geometric information about X from the ideal of polynomials by which it is defined. However, when the codimension of X is large, this process is rather difficult.

The most common situation is that of a noncomplete intersection, i.e., a variety $X \subset \mathbf{P}^N$ which is cut out by a larger amount of polynomial equations than its actual codimension. One aspect of that is the algebraic dependence between the equations for X: if a set of polynomial equations for X in this embedding is

$$f_1(\mathbf{x}) = \cdots = f_r(\mathbf{x}) = 0,$$

and I denotes the ideal generated by the f_i's in the ring $R = \mathbf{C}[x_0, ..., x_N]$, then the homogeneous polynomials f_i satisfy "nontrivial" algebraic relations (naturally, the trivial relations $f_i f_j - f_j f_i = 0$ always hold). The entirety of the relations between the f_i's is encoded by the kernel of the ring morphism $R^{\oplus r} \to R$ which maps each $(h_1, ..., h_r)$ to $\sum_{i=1}^r f_i h_i$, and has image I. This takes the form of an exact sequence

$$M_0 \to R^{\oplus r} \xrightarrow{(f_1 \cdots f_r)} R \to {}^R/_I \to 0,$$

where M_0 is a free R-module of finite type, the kernel of $R^{\oplus r} \to R$ being finitely generated by noetherianity. One may also want to identify the kernel of $M_0 \to R^{\oplus r}$, which encodes "relations between the relations" of the f_i's. This kernel is the image of another map $M_1 \to M_0$, with M_1 another free R-module of finite type. We may repeat the process: in the end, by Hilbert's syzygy theorem, we get an exact sequence of finite length

$$0 \to M_s \to \cdots \to M_0 \to R^{\oplus r} \xrightarrow{(f_1 \cdots f_r)} R \to {}^R/_I \to 0,$$

where the M_i's are free R-modules of finite type. The *syzygies* of I (or equivalently, the syzygies of X), as defined by D. Eisenbud in [Ei05], refer to the elements of the kernels of the arrows involved in this exact sequence.

The condition that X is not a complete intersection in $\mathbf{P}^N$ translates into the fact that the f_i's are not a regular sequence in R, in other words the fact that one of the f_i's is a zero divisor in $R/(f_1, \ldots, f_{i-1})$, and the syzygies are nontrivial. In this situation, the ideal generated by the f_i's is, in a sense, more complicated than it is for a complete intersection. In other words, it is not an easy process to learn about the geometry of X only from the data of its equations.

Example: The twisted cubic vs. a conic in $\mathbf{P}^3$.

Let $[x : y : z : w]$ be homogeneous coordinates on the projective space $\mathbf{P}^3$, and consider the embedding

$$[\lambda : \mu] \in \mathbf{P}^1 \mapsto [\lambda^3 : \lambda^2\mu : \lambda\mu^2 : \mu^3] \in \mathbf{P}^3.$$

We refer to the image of this embedding as a twisted cubic. It is isomorphic to $\mathbf{P}^1$, in other words a fairly simple curve, but it is not a complete intersection, as it has codimension 2 while being the vanishing locus of the three maximal minors of the following matrix:

$$\begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}.$$

If we denote the three minors as follows,

$$\mathtt{m}_{12} = xz - y^2,\ \mathtt{m}_{13} = xw - yz,\ \mathtt{m}_{23} = yw - z^2,$$

then any complete intersection given by the vanishing of any two of these minors (or two linear combinations of them) is a reducible curve whose components are the twisted cubic and a "residual" line. The algebraic relations between the three minors are generated by the following:

$$x\,\mathtt{m}_{23} - y\,\mathtt{m}_{13} + z\,\mathtt{m}_{12} = y\,\mathtt{m}_{23} - z\,\mathtt{m}_{13} + w\,\mathtt{m}_{12} = 0.$$

On the other hand, consider the embedding

$$[\lambda : \mu] \in \mathbf{P}^1 \mapsto [\lambda^2 : \lambda\mu : \mu^2 : 0] \in \mathbf{P}^3.$$

Then the image (a plane conic) is also isomorphic to $\mathbf{P}^1$, but it is given in $\mathbf{P}^3$ by the equations

$$w = xz - y^2 = 0.$$

Unlike the three minors of the twisted cubic, these two equations above don't satisfy any algebraic relations besides the trivial one:

$$(xz - y^2)w - w(xz - y^2) = 0,$$

and the conic is indeed a complete intersection in $\mathbf{P}^3$. It is for example possible to compute by hand its tangent and normal bundle thanks to the Jacobian of the system $(w, xz - y^2)$.

More generally, when X is a complete intersection in $\mathbf{P}^N$, the only algebraic relations between its polynomial equations are the trivial ones. For example, if we have three homogeneous polynomials f, g and h on $\mathbf{P}^N$ such that

$$X = \{f = g = h = 0\}$$

with $\dim X = N - 3$, then the three equations benefit from a form of algebraic independence, meaning that the only relations between them are generated by

$$fg - gf = fh - hf = gh - hg = 0.$$

In particular, there is no relation when X is a hypersurface, i.e., the zero locus of a single homogeneous polynomial.

Overall, the same philosophy applies when X is embedded as a subvariety of another variety Y which may not be a projective space. If Y is itself a projective variety and

X is realized as a complete intersection inside Y, the geometry and the coordinate ring $R_Y = {}^{R}/_{I(Y)}$ of which are already known, then it is easy to obtain information on X and its coordinate ring from Y. Namely, if X is given as the vanishing locus of a single element $f \in R_Y$, then we have $R_X \simeq {}^{R_Y}/_{(f)}$, and the tangent bundle of X can be recovered by hand from that of Y. In addition, the syzygies of X have a similar form as those of Y. As an extensive reference about syzygies and their link with geometry, complete and noncomplete intersections, we suggest [Ei05].

We now make a definition which is at the core of our first chapter.

Definition 0.0.5. *Let $X \subset \mathbf{P}^N$ be a projective variety which is nondegenerate, i.e., not contained in any hyperplane. We say that X is* extendable *if there exists $Y \subset \mathbf{P}^{N+1}$ which is not a cone and which contains X as a hyperplane section.*

Remark. If X is a complete intersection in $\mathbf{P}^N$, then the construction of an extension is fairly easy. To illustrate our point, let us consider a hypersurface X in $\mathbf{P}^N$ given in homogeneous coordinates $[x_0 : \cdots : x_N]$ by an equation $f(\mathbf{x}) = 0$, with f a homogeneous polynomial. One may introduce an additional coordinate x_{N+1}, denote $g = f + x_{N+1}^{\deg f}$ and consider the hypersurface $Y \subset \mathbf{P}^{N+1}$ given by the equation $g = 0$. Then Y is not a cone and contains X as its intersection with the hyperplane $\{x_{N+1} = 0\}$. Note that Y is itself a hypersurface in its ambient space $\mathbf{P}^{N+1}$, so it can be extended in turn, and any extension of Y can be extended as well, etc.

Conversely, the existence of an extension of X is a relatively complicated question when X is not a complete intersection (which is the most common situation in projective geometry). The difficulty comes from the syzygies of X, as constructing an extension requires to find a variety of larger dimension with similar syzygies. To be more precise, assume $Y \subset \mathbf{P}^{N+1}$ is an extension of X, where the projective space $\mathbf{P}^{N+1}$ is endowed with homogeneous coordinates $[x_0 : \cdots : x_{N+1}]$ so that $X = Y \cap \{x_{N+1} = 0\}$. Let S denote the ring $\mathbf{C}[x_0, ..., x_{N+1}]$ and $I(Y) \subset S$ the ideal of Y. Then the syzygies of Y take the form of an exact sequence

$$0 \to N_r \to \cdots \to N_1 \to N_0 \to S \to {}^{S}/_{I(Y)} \to 0,$$

where each N_i is isomorphic to $S^{\oplus k_i}$ for some $k_i \in \mathbf{N}$. The szygyies or X are obtained by implementing $x_{N+1} = 0$ in the relations between the equations which define Y, and the relations between said relations, and so on. In other words, the syzygies of X are fully determined by the exact sequence above.

Of course, any projective variety X can always be realized as a hyperplane section of a cone over itself. However, it is not possible to gain additional information on the geometry of X from the geometry of said cone. In the definition of an extension above, we specifically ask that Y is not a cone; as a result, there exists no extension of X in the general case.

When an extension Y of X exists, it is a natural space in which it is possible to deform X by simply moving the hyperplane H for which $Y \cap H = X$ to other hyperplanes H'. Of course, the same question of the existence of an extension holds about Y. If we manage to extend Y, then we have a variety inside which X has codimension 2, yielding an even larger space of deformations of X.

Terminology. If we have a chain of extensions

$$X \subset Y = X_1 \subset X_2 \subset \cdots$$

in which X_k is an extension of X_{k-1}, with $\dim X_k = 1 + \dim X_{k-1}$, we say that X_k is a *k-extension* of X.

By the previous remark, a variety $X \subset \mathbf{P}^N$ which is a complete intersection admits such an infinite chain of extensions. In [Bal21], E. Ballico gave examples of noncomplete intersections which also admit extensions of arbitrarily large dimension. However, the so-called Babylonian tower theorem, due to A. N. Tyurin, W. Barth and A. Van de Ven, states that the only l.c.i. varieties which admit infinite chains of l.c.i. extensions are complete intersections. For a proof of the Babylonian tower theorem, see [Co12].

The extendability of projective varieties has interested many mathematicians in the recent years, but also for a long time during the twentieth century. We refer to [LD23], an article by A. Lopez which gives a tour of the question and mentions works by S. Lvovski [Lvo92] and J. Wahl [Wa90]. We cite older references from as early as the beginning of the twentieth century, when mathematicians such as G. Scorza and A. Terracini studied the extendability question, proving for instance that neither a Veronese variety of dimension ≥ 2 nor a Segre variety of dimension ≥ 3 are extendable ([Sc10], [Te13]).

In chapter 1, we will be interested in the extendability of *canonical curves*. These are smooth nonhyperelliptic projective curves $\Gamma \subset \mathbf{P}^{g-1}$ embedded by the linear system of their cotangent bundle. Such an embedding is said canonical since the cotangent line bundle of a smooth curve is intrinsic and a main research interest to complex geometers in general. The extendability of a canonical curve involves a great deal about K3 surfaces, Fano threefolds and the so-called *Gauß-Wahl map* of Γ (sometimes simply called the Gauß map, or the Wahl map, see for instance [Wa90] or [CHM88]). To be more precise, this is a standard fact that any projective surface with canonical singularities which is an extension of a canonical curve is K3, and conversely, any smooth hyperplane section of a projective K3 surface is a canonical curve (for a proof, see Lemma 1.2.8). Furthermore, the general anticanonical section of a Fano threefold with canonical singularities is a K3 surface, and conversely, any threefold extension with canonical singularities of a projective K3 surface is Fano.

As early as 1987, J. Wahl proved in [Wa87] that if a smooth curve is a hypersurface of a smooth K3 surface, then its Gauß-Wahl map is not surjective. This was shortly followed by an alternative proof by A. Beauville and J.-Y. Mérindol in [BM87]. Later, E. Arbarello, A. Bruno and E. Sernesi proved in [ABS17] that a canonical curve of genus at least 11 and Clifford index at least 3 is extendable *if and only if* its Gauß-Wahl map is not surjective. The link between Gauß-Wahl maps and the higher-dimensional extensions of canonical curves was further studied by C. Ciliberto, T. Dedieu and E. Sernesi in [CDS20].

Let us now introduce the definition of a weighted projective space:

Definition 0.0.6. *Given a set of coprime positive integers $a_0, ..., a_N$, the associated weighted projective space $\mathbf{P}(a_0, ..., a_N)$ is defined as the quotient $\mathbf{C}^{N+1} - \{0\}$ by the following action of $\mathbf{C}^*$:*

$$\lambda \cdot (x_0, ..., x_N) = (\lambda^{a_0} x_0, ..., \lambda^{a_N} x_N).$$

The extendability of canonical curves and K3 surfaces is the topic of chapter 1, in which we examine the list of the 14 weighted projective spaces of dimension 3 which can be realized as extensions of K3 surfaces and canonical curves. The list of these threefold extensions of canonical curves can be found in [DS23]. More precisely, given any weighted projective space $\mathbf{P}$ among these 14, there exists an embedding $\mathbf{P} \hookrightarrow \mathbf{P}^N$ which contains K3 surfaces and canonical curves as linear sections; moreover, there exist no other weighted projective space of dimension 3 satisfying this property. For each of the 14 spaces on the list, the dimension of an extension of $\mathbf{P} \subset \mathbf{P}^N$ is bounded from above, and for 8 spaces among these 14, the construction of an extension of maximal dimension was known from [DS23]. Chapter 1 focuses on the 6 other items.

Weighted projective spaces are a class of varieties on which the computations and the study of complete intersections are manageable thanks to the existence of homogeneous coordinates. We then have a list of canonical curves $\Gamma \subset \mathbf{P}^{g-1}$ which are linear sections of projective models of weighted projective spaces; this makes it possible to recover information on these curves since the geometry of a weighted projective space is well-known.

The main results of chapter 1 are listed at the end of this introduction.

⋆ ⋆

What is the MMP? What is Mori theory?

The rough idea of the Minimal Model Program — or MMP — is to find a "good" representative of a given variety X under birational equivalence. Originally for surfaces, it meant finding a representative without (-1)-curves, i.e., a birational model X' of X such that X' contains no curve which may be contracted via a birational morphism to a smooth surface. Today's understanding of the objective of the MMP is the following: for a given variety X of any dimension, find a representative X' which has either nef canonical divisor, or admits a morphism $X' \to Y$ onto a variety of smaller dimension such that the anticanonical divisor of a general fibre is ample. Such morphisms of positive relative dimension with Fano general fibres are called *Fano fibre spaces.* More specifically, the ones obtained via a specific method (called the Mori Program) are called *Mori fibre spaces.*

As references about the MMP in general, we would like to cite [KMM87] (in which Y. Kawamata, K. Matsuda and K. Matsuki dedicate a chapter to the flip conjecture), [Kol96], [KM98] and [Mor88] (in which S. Mori focuses on the case of dimension 3).

The Mori Program consists in the following idea: if a variety X contains a projective curve C, we can make sense of the intersection of any Cartier divisor $D \in \mathrm{Pic}(X)$ (equivalently, a line bundle) with C, which can be defined as the degree of the restriction $D|_C$. Starting from a variety X with Cartier (or just $\mathbf{Q}$-Cartier) canonical class, the Mori Program aims to get rid of the locus inside X which is covered by the K_X-negative curves, i.e., the projective curves $C \subset X$ for which $K_X \cdot C = \deg(K_X|_C) < 0$. For a good tour of this subject, generalities and notations, we refer to [Deb01], a classical book by O. Debarre. We would also like to mention [Deb16], an M2 course by the same author about Mori theory. An older, but more extensive reference about the Mori Program is the book from 1998 by J. Kollár and S. Mori, [KM98].

More precisely, it may happen that a smooth variety X is not covered by curves C with $K_X \cdot C < 0$. Under this condition, the locus we want to modify is not the whole variety X; the theory developped by Y. Kawamata, J. Kollár, S. Mori and V. Shokurov in the 80s tells us that in that case, we may consider a birational morphism $X \to Y$ onto a normal variety Y which contracts some of the K_X-negative curves. Such a morphism is not unique, and in general there are countably many such elementary ways of contracting negative curves (for more details, see Definition 0.0.7 below). Such an elementary contraction is an "elementary step" in the Mori Program.

Definition 0.0.7. *Given a variety X with $\mathbf{Q}$-Cartier canonical divisor, a morphism with connected fibres from X which contracts K_X-negative curves and for which $-K_X$ is relatively ample is what we call a* Mori contraction. *Moreover, a Mori contraction $X \to Y$ is* elementary *if it cannot be factored via another Mori contraction $X \to Y'$, in a commutative diagram*

$$\begin{array}{ccc} X & \longrightarrow & Y' \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

where $Y' \to Y$ is not an isomorphism.

There also exist cases of smooth varieties X for which Mori theory gives the existence of an elementary contraction $X \to Y$ of K_X-negative curves with $\dim Y < \dim X$. In this case, we say that $X \to Y$ is a *Mori fibre space*, as mentioned previously, and it is a final object for the MMP.

Note that some mathematicians refer to Mori contractions as *Fano-Mori contractions*, since the general fibre is Fano.

Definition 0.0.8 (Length)**.** *Let X be a variety with $\mathbf{Q}$-Cartier canonical class and $f : X \to Y$ an elementary Mori contraction. The* length *of f is*

$$l(f) = \min\left\{-K_X \cdot \Gamma \mid \Gamma \subset X \text{ a contracted rational curve}\right\}.$$

Thanks to a classical result by P. Ionescu and J. Wiśniewski (see [Io86] and [Wi91]), the length of an elementary Mori contraction $X \to Y$ from a smooth variety X has an upper bound which involves the dimension of its exceptional locus and that of a fibre in its exceptional locus.

Remark. If $f : X \to Y$ is birational with connected fibres, then the exceptional locus of f ususally refers to the locus where f is not an isomorphism. If f has connected fibres but is not birational, in which case $\dim X > \dim Y$, then we allow ourselves to say that the exceptional locus involved in the upper bound is the whole variety X, although it may usually not be an appropriate choice of vocabulary.

Theorem 0.0.9 ([Io86] Theorem 0.4, [Wi91] Theorem 1.1)**.** *Assume $f : X \to Y$ is an elementary Mori contraction from a smooth variety X. Let $E \subset X$ be an irreducible component of the f-exceptional locus and $F \subset E$ an irreducible component of a fibre. Then*

$$\dim E + \dim F \geq \dim X + l(f) - 1.$$

When the bound is reached, in other words when there exists an irreducible component E of $\mathrm{Exc}(f)$ and a fibre $F \subset E$ such that

$$l(f) = \dim E + \dim F + 1 - \dim X,$$

the contraction is said to be of *maximal length.* In the case of an elementary Mori contraction of maximal length from a smooth projective variety, A. Höring and C. Novelli have exhibited in [HN13] the structure of a projective bundle for the exceptional locus over its image, up to some birational modification:

Theorem 0.0.10 ([HN13], Theorems 1.3 & 1.4)**.** *Assume that X is a smooth projective variety.*

- *Let $f : X \to Y$ be an elementary Mori contraction such that* $\dim X - \dim Y > 0$ *and $l(f)$ is maximal.*

  If f is equidimensional, then it is a projective bundle.

  If f is not equidimensional, then there exist birational morphisms $X' \to X$ and $Y' \to Y$ such that X' and Y' are smooth, and a projective bundle $X' \to Y'$ such that the following diagram is commutative:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

- *Let $f : X \to Y$ be a birational elementary Mori contraction of maximal length, and E the exceptional locus of f. Let E' be the normalization of E, $Z = f(E)$ and $E' \to Z'$ the fibration obtained by the Stein factorization of $E' \to Z$. Then $E' \to Z'$ is a projective bundle in codimension one; moreover there exist birational morphisms $\overline{E} \to E'$ and $\overline{Z} \to Z'$ such that $\overline{E}$ and $\overline{Z}$ are smooth, and $\overline{E} \to \overline{Z}$ a projective bundle such that the following diagram is commutative:*

$$\begin{array}{ccccc} \overline{E} & \longrightarrow & E' & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow f \\ \overline{Z} & \longrightarrow & Z' & \longrightarrow & Z \end{array}$$

In chapter 2 of this thesis, we relax the length hypothesis, moving to the submaximal case, i.e., the case of elementary Mori contractions $f : X \to Y$ with X smooth, the length of which is not maximal, but for which there exists an irreducible component E of $\mathrm{Exc}(f)$ and a fibre $F \subset E$ such that

$$l(f) = \dim E + \dim F - \dim X.$$

Consider the following result about smooth Fano varieties with large anticanonical degree on rational curves with respect to their dimension:

Theorem 0.0.11 ([CMSB02], [Ke02], [Mi04], [DH17])**.** *Let X be a smooth Fano variety of dimension n.*

• *[CMSB02], [Ke02]: If for every rational curve $\Gamma \subset X$, one has $-K_X \cdot \Gamma \geq n+1$, then $X \simeq \mathbf{P}^n$.*

• *[Mi04], [DH17]: If for every rational curve $\Gamma \subset X$, one has $-K_X \cdot \Gamma \geq n$, then X is isomorphic either to a quadric, or to a projective space.*

The study of elementary Mori contractions of maximal or submaximal length can be understood as a relative version of the above theorem. Indeed, if for instance $X \to Y$ is an elementary Mori contraction to a point, the length condition tells us about the minimal degree of $-K_X$ on the rational curves inside X. Conversely, any smooth fibre F of an elementary Mori contraction $X \to Y$ with $\dim X > \dim Y$ is Fano, and by adjunction we have $K_F = K_X|_F$. Then the inequality in Theorem 0.0.9 gives information on the minimal degree the Cartier divisor $-K_F$ can have on the rational curves in F.

The study of Mori contractions has been an active research domain for many mathematicians in the past 3 decades, among which we would like to cite J. Wiśniewski, M. Andreatta, G. Occhetta and L. Tasin.

In [Wi91] J. Wiśniewski gave an ampleness criterion for a line bundle on a smooth Fano variety X under some conditions which involve the Mori contractions from X.

In [AW93] M. Andreatta and J. Wiśniewski gave criteria for the relative basepoint-freeness of a line bundle with respect to a contraction, and for the local structure of a contraction around a fibre satisfying a dimension condition; this involves the notion of a *supporting divisor* and that of the *nef value* of a contraction.

The two joint papers by M. Andreatta and G. Occhetta we would like to refer to are [AO02] and [AO05]. In the former, they provided a numerical criterion for a Mori contraction from a smooth variety to be a blow-up of another smooth variety at a point. The latter is a numerical criterion for a Fano smooth variety to be either a product of projective spaces, or a blow-up of a projective space along a linear subspace, which has to do with the existence of a particular type of Mori contraction.

M. Andreatta and L. Tasin contributed to the theory with the two fairly recent papers [AT14] and [AT16]. Both these articles also deal with the notion of the nef value of a contraction and its relation with the dimension of the total space, or that of the fibres.

All in all, the nef value of a contraction $X \to Y$ can allow us to know a lot about the structure of the contraction, as it is (briefly speaking) the measure of how much the canonical divisor K_X is divisible along the fibres of the morphism $X \to Y$, and it is closely related to the length of the contraction, in other words the smallest degree the Cartier divisor $-K_X$ can have on the contracted rational curves.

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Outline of this thesis: main results

The text is split into two main chapters. In chapter 1 we focus on the list of the 14 weighted projective spaces $\mathbf{P}$ of dimension 3 admitting projective models $\mathbf{P} \hookrightarrow \mathbf{P}^N$ which are extensions of K3 surfaces and canonical curves. The 6 spaces among these 14 for which the construction of a maximal extension was not known from [DS23] are listed below.

$\mathbf{P}(1,4,5,10)$
$\mathbf{P}(1,2,6,9)$
$\mathbf{P}(1,2,3,6)$
$\mathbf{P}(1,3,8,12)$
$\mathbf{P}(1,6,14,21)$
$\mathbf{P}(2,3,10,15)$

The main result of chapter 1 is the following.

Theorem. *Let $\mathbf{P} = \mathbf{P}(a_0, a_1, a_2, a_3)$ be one of the weighted projective spaces from the list above. Then the anticanonical divisor $-K_{\mathbf{P}}$ is very ample and it induces a projective* *model* $\mathbf{P} \hookrightarrow \mathbf{P}^N$ *whose general linear section is a K3 surface. Moreover, a construction of an extension of maximal extension* Y *for this model is given in the table below:*

$\mathbf{P}$	Y	*dim*(Y)
$\mathbf{P}(1,4,5,10)$	*nongeneral quintic of* $\mathbf{P}(1^3,2,4^3)$	5
$\mathbf{P}(1,2,6,9)$	*nongeneral* 10*-ic of* $\mathbf{P}(1^2,3,5,9^2)$	4
$\mathbf{P}(1,2,3,6)$	$\mathbf{P}(1,2,3,6)$	3
$\mathbf{P}(1,3,8,12)$	*nongeneral* 9*-ic of* $\mathbf{P}(1^2,3,4,8^2)$	4
$\mathbf{P}(1,6,14,21)$	*nongeneral heptic of* $\mathbf{P}(1^2,2,3,6^2)$	4
$\mathbf{P}(2,3,10,15)$	*codim.* 2 *complete intersection in a* $\mathbf{P}(1^2,2,3,5^3)$*-bundle over* $\mathbf{P}^1$	5

For more details, see Theorem 1.4.5. The method that we will use to construct extensions of the embeddings $\mathbf{P} \hookrightarrow \mathbf{P}^N$ is similar to that developed by B. Totaro (unpublished), which was used in [CD21, §3.2] and [CD24', 4.8].

The hyperplane polarizations H on the K3 surfaces which are hyperplane sections of those weighted projective spaces $\mathbf{P} \hookrightarrow \mathbf{P}^N$ are sometimes nonprimitive. On each such K3 surface S, when the polarization is not primitive, we consider the primitive polarization $H' = \frac{1}{i_S}H$, where i_S is the largest positive integer r such that $\frac{1}{r}H$ is Cartier, and then we study the geometry of the general curve in the newly obtained linear system $|H'|$ in Theorem 1.4.20. In chapter 1, we also give a quick classification of all the weighted projective spaces of dimension 4 or more which admit projective models in which they are extensions of K3 surfaces.

In chapter 2, we investigate elementary Mori contractions of submaximal length, i.e., whose length is equal to its upper bound minus one (see Theorem 0.0.9). When the contraction is of relative dimension 1, a result by V. G. Sarkisov [Sa82, Theorem 1.13] applies, ensuring the existence of a birational model which is a conic bundle. Under the condition that the length is submaximal, we provide in the case of relative dimension 1 the construction of an explicit birational model of our elementary Mori contraction (see Subsection 2.2.1) and prove that this model is a conic bundle in codimension 1. In the birational case, we make the assumption that the contraction is divisorial. Our main result is:

Theorem. • *Let* $f : X \to Y$ *be an equidimensional fibration from a normal projective variety* X *to a smooth projective variety* Y *with* $\dim X = \dim Y + 1$*. Assume that there exists a relatively ample Cartier divisor* A *on* X *such that* $K_X + A$ *is trivial on the smooth fibres. Then* f *is a conic bundle in codimension* 1*.*

• *Let* $f : X \to Y$ *be an elementary Mori contraction of submaximal length from a smooth variety* X*. If* f *is divisorial, i.e., birational with the* f*-exceptional locus* E *a hypersurface of* X*, then the equidimensional locus of* $E \to f(E)$ *is birational either to a quadric bundle, or to a projective bundle.*

The proof of this theorem is broken down into intermediate results, namely: Theorems 2.2.3 and 2.2.10 (with Corollary 2.2.11) for the case where f has relative dimension 1, and Theorem 2.3.3 for the case where f is a divisorial Mori contraction.

For the case $\dim X - \dim Y = 1$, the result Theorem 2.2.3 (respectively, Theorem 2.2.10) is based on the assumptions of Setup A.1 (respectively, of Setup A.2). These assumptions are obtained via an application to Setup A of the results of Subsection 2.2.1.

Chapter 1

K3 surfaces in weighted projective spaces

1.1 General facts about weighted projective spaces

1.1.1 Definition and notations

We cite [Ia00, §5 & §6] as a reference about weighted projective spaces. Here, we give the definition as well as some properties of weighted projective spaces, and we set some notations that will be of use later on.

We recall down below the definition of a weighted projective space, which was first given in Definition 0.0.6.

Definition 1.1.1. *Given a $(d+1)$-tuple of positive integers $(a_0, ..., a_d)$, consider the action $\mathbf{C}^* \curvearrowright (\mathbf{C}^{d+1} - \{0\})$:*

$$\lambda \cdot (x_0, ..., x_d) = (\lambda^{a_0} x_0, ..., \lambda^{a_d} x_d).$$

The weighted projective space $\mathbf{P}(a_0, ..., a_d)$ is the quotient space of said action.

A weighted projective space is a natural generalization of a projective space; namely, it follows by the definition that $\mathbf{P}^d = \mathbf{P}(1, ..., 1)$ with the weight 1 occurring $d+1$ times
Terminology. The integers a_i's are called the weights. Classically, they are written in increasing order. Note that a weighted projective space is invariant (up to isomorphism) by a reordering of its weights.

Notations. We will sometimes write WPS, which is short for weighted projective space. The points of the weighted projective space $\mathbf{P}(a_0, ..., a_d)$ are denoted

$$[x_0 : \cdots : x_d].$$

Furthermore, the point $[1 : 0 : \cdots : 0]$ which corresponds to the vanishing of $x_1, ..., x_d$ is denoted p_{x_0}, or simply p_0 when there is no ambiguity. The point $[0 : 1 : 0 : \cdots : 0]$ is denoted p_1, and so on. Given a sub-family of weights $J \subset \{0, ..., d\}$, then the vanishing locus

$$\{x_j = 0 \mid j \in J\}$$

is denoted $\Lambda_{(p_i \mid i \notin J)}$. This is the weighted projective subspace spanned by the points p_i for $i \notin J$, and we have

$$\Lambda_{(p_i \mid i \notin J)} \simeq \mathbf{P}(a_i \mid i \notin J).$$

We also adopt the following notation for brevity: when a given weight appears multiple times, say

$$a_i = a_{i+1} = \cdots = a_{i+s} = a$$

then we denote

$$\mathbf{P}(a_0, ..., a_d) = \mathbf{P}(a_0, ..., a_{i-1}, a^{s+1}, a_{i+s+1}, ..., a_d).$$

As an example, $\mathbf{P}(1^3, 3^3)$ is $\mathbf{P}(1, 1, 1, 3, 3, 3)$. In such a case, when some weights appear multiple times, it is also convenient to denote the homogeneous coordinates by repeating some letters. For the particular example of $\mathbf{P}(1^3, 3^3)$, homogeneous coordinates can be chosen to be $[u_0 : u_1 : u_2 : v_0 : v_1 : v_2]$, where each u_i has weight 1 and each v_i has weight 3. We usually use the notation $\mathbf{P}(1^3, 3^3)_{[u_0:u_1:u_2:v_0:v_1:v_2]}$, with a given choice of coordinates written as a subscript. To lighten the notation, when there is not risk of confusion, we allow ourselves to write $[\mathbf{u} : \mathbf{v}]$ instead of $[u_0 : u_1 : u_2 : v_0 : v_1 : v_2]$.

From the scheme-theoretic point of view, the weighted projective space $\mathbf{P}(a_0, ..., a_d)$ consists of all the closed points of the scheme

$$\mathrm{Proj}(\mathbf{C}[x_0, ..., x_d]),$$

where the ring $R = \mathbf{C}[x_0, ..., x_d]$ is endowed with the grading $\deg x_i = a_i$.

Now, we give a list of facts about weighted projective spaces.

Remark. • If the weights are not coprime, for instance

$$a_i = b_i q$$

for some common divisor $q > 1$, then the action $\mathbf{C}^* \curvearrowright (\mathbf{C}^{d+1} - \{0\})$ mentioned in Definition 1.1.1 such that

$$\mathbf{P}(a_0, ..., a_d) = {}^{(\mathbf{C}^{d+1} - \{0\})}/_{\mathbf{C}^*}$$

is not faithful: if λ is a q-th root of unity, then it acts trivially. We may precompose this action by the morphism $\lambda \mapsto \lambda^q$ and denote $\mu = \lambda^q$. Then the action becomes

$$\mu \cdot (x_0, ..., x_d) = (\mu^{b_0} x_0, ..., \mu^{b_d} x_d),$$

yielding the isomorphism

$$\mathbf{P}(b_0, ..., b_d) \simeq \mathbf{P}(a_0, ..., a_d).$$

As a consequence, in Definition 1.1.1, we may assume that the weights are coprime.

• Let $f \in R$ be a polynomial in the variables $x_0, ..., x_d$. Then its vanishing locus is well defined in $\mathbf{P}(a_0, ..., a_d)$ if and only if f is homogeneous with respect to the grading of R, in other words for all $\lambda \in \mathbf{C}^*$ there is an integer $\deg f$ such that

$$f(\lambda^{a_0} x_0, ..., \lambda^{a_d} x_d) = \lambda^{\deg f} f(x_0, ..., x_d).$$

Lemma 1.1.2 ([Ia00], 5.10.a). *Any WPS of dimension one is isomorphic to* $\mathbf{P}^1$*:*

$$\mathbf{P}(a, b) \simeq \mathbf{P}^1.$$

Lemma 1.1.3 ([Ia00], 5.15). *The WPS* $\mathbf{P}(a_0, ..., a_d)$ *has at worst quotient singularities.*
• *With respect to the coordinates*

$$[x_0 : \cdots : x_d]$$

the point p_{x_i} *is a quotient singularity of the form*

$$\frac{1}{a_i}(a_0, ..., \widehat{a_i}, ..., a_d).$$

That is to say, if μ_{a_i} *denotes the group of* a_i*-th roots of unity in* $\mathbf{C}$*, the singularity at* p_{x_i} *is isomorphic to the image of the origin in the quotient of* $\mathbf{C}^d$ *by the following action of* μ_{a_i}*:*

$$\zeta \cdot (z_0, ..., \widehat{z_i}, ..., z_d) = (\zeta^{a_0} z_0, ..., \zeta^{a_d} z_d).$$

• *More generally, the WPS* $\mathbf{P}(a_0, ..., a_d)$ *might be singular along weighted projective subspaces. Namely, for* $J \subset \{0, ..., d\}$*, the subspace*

$$\{x_j = 0 \mid j \in J\} = \Lambda_{(p_i \mid i \notin J)} \simeq \mathbf{P}(a_i \mid i \notin J)$$

contains a copy of $(\mathbf{C}^*)^{d-|J|}$ *whose points are singularities of the ambient space* $\mathbf{P}(a_0, ..., a_d)$*. We say that* $\Lambda_{(p_i \mid i \notin J)}$ *is generically a*

$$(\mathbf{C}^*)^{d-|J|} \times \frac{1}{\gcd(a_i \mid i \notin J)}(a_j \mid j \in J),$$

meaning that for a subvariety $X \subset \mathbf{P}(a_0, ..., a_d)$ *of dimension* $|J|$ *meeting* $\Lambda_{(p_i \mid i \notin J)}$ *transversally along the copy of* $(\mathbf{C}^*)^{d-|J|}$*, the intersection locus*

$$X \cap \Lambda_{(p_i \mid i \notin J)}$$

consists of singular points of X *of the form*

$$\frac{1}{\gcd(a_i \mid i \notin J)}(a_j \mid j \in J).$$

In the WPS $\mathbf{P}(a_0, ..., a_d)$, consider the point p_{a_0}. The singularity at p_{a_0} is of the form

$$\frac{1}{a_0}(a_1, ..., a_d),$$

which is a smooth point if and only if a_0 is a divisor of a_i for all $i \neq 0$. This quotient $\mathbf{C}^d/\mu_{a_0}$ plays the role of an "étale chart" for the affine open subset $\{x_0 \neq 0\}$ in $\mathbf{P}(a_0, ..., a_d)$. Naturally, if all the weights are 1, then we recover from this a standard affine chart of $\mathbf{P}^d$.

Example. The last point of Lemma 1.1.3 deserves an example. Consider the WPS

$$\mathbf{P} = \mathbf{P}(1, 2, 2, 3)$$

endowed with its homogeneous coordinates $[x : y : z : w]$. Then the locus

$$\Lambda_{p_y, p_z} = \{x = w = 0\}$$

is isomorphic to $\mathbf{P}(2, 2) \simeq \mathbf{P}^1$. It consists of the two points p_y and p_z, which are threefold singularities of $\mathbf{P}$ of the form

$$\frac{1}{2}(1, 2, 3) = \frac{1}{2}(1, 0, 1),$$

and $U = \Lambda_{p_y, p_z} - \{p_y, p_z\}$, which is isomorphic to $\mathbf{C}^*$. For a general surface $S \subset \mathbf{P}$ meeting U transversally, the points $S \cap U$ are surface singularities in S of the form

$$\frac{1}{\gcd(2, 2)}(1, 3) = \frac{1}{2}(1, 1).$$

Given $k \in \mathbf{Z}$, on the weighted projective space

$$\mathbf{P} = \mathbf{P}(a_0, ..., a_d)$$

we introduce the sheaf $\mathcal{O}_{\mathbf{P}}(k)$ whose sections over any open subset $U \subset \mathbf{P}$ are fractions of polynomials of the form $\frac{P}{Q}$, where P and Q are homogeneous with respect to the grading of $\mathbf{P}$, $\deg P - \deg Q = k$ and Q never vanishes on U.

Definition 1.1.4 (Well-formedness)**.** *The weights* $(a_0, ..., a_d)$ *are well formed if for all* $i \in \{0, ..., d\}$*, we have*

$$\gcd(a_0, ..., \widehat{a_i}, ..., a_d) = 1.$$

Equivalently, a weighted projective space $\mathbf{P}(a_0, ..., a_d)$ *is well formed if its weights are well formed.*

Lemma 1.1.5 (Picard group — Canonical sheaf: [Kol96] V.1.3, [Do81] Theorem 3.3.4)**.** *Let* $\mathbf{P}$ *be a well formed weighted projective space. Then the divisor class group of* $\mathbf{P}$ *is isomorphic to* $\mathbf{Z}$*:*

$$Cl(\mathbf{P}) = \mathbf{Z}[\mathcal{O}_{\mathbf{P}}(1)],$$

while the Picard group is generated by $\mathcal{O}_{\mathbf{P}}(\gcd(a_0,...,a_d))$. *In particular,* $\mathbf{P}$ *is* $\mathbf{Q}$-*factorial.*

The intersection form on $Cl(\mathbf{P}) = \mathrm{Pic}(\mathbf{P}) \otimes \mathbf{Q}$ *is given by*

$$\mathcal{O}_{\mathbf{P}}(1)^d = \frac{1}{a_0 a_1 \cdots a_d}.$$

Moreover, the canonical sheaf of $\mathbf{P}$ *is* $K_{\mathbf{P}} = \mathcal{O}_{\mathbf{P}}(-\sum_{i=0}^d a_i)$.

We also mention the following result about the local freeness of a sheaf on a general Cartier divisor on a WPS:

Lemma 1.1.6. *Let* D *be a general member of a linear system* $|\mathcal{O}_{\mathbf{P}}(n)|$ *on a well formed weighted projective space* $\mathbf{P} = \mathbf{P}(a_0,...,a_d)$, *where* $\mathcal{O}_{\mathbf{P}}(n)$ *is any line bundle on* $\mathbf{P}$. *For all* $k \in \mathbf{Z}$, *the restriction* $\mathcal{O}_{\mathbf{P}}(k)|_D$ *is locally free if and only if*

$$\forall i \neq j \; : \; \gcd(a_i, a_j) | k.$$

The proof is an adaptation of [DS23, Lemma 2.2] to the case of general dimension.

Proof: The local freeness needs only to be verified at the singular points. By generality, D can only be singular along the singular locus of $\mathbf{P}$, i.e., only along codimension 2 or more weighted projective subspaces $\Lambda_{p_{i_1} \cdots p_{i_s}}$ spanned by families of coordinate points. Since $\mathcal{O}_{\mathbf{P}}(n)$ is locally free, D avoids all the coordinate points by generality.

Let p be a point on a weighted projective subspace spanned by coordinate points $p_{i_0},...,p_{i_s}$. Assume that p is not a coordinate point of $\mathbf{P}$: in the local ring $\mathcal{O}_{\mathbf{P},p}$, the invertible monomials are the nonvanishing ones, i.e., those of the form $x_{i_0}^{n_0} \cdots x_{i_s}^{n_s}$. Their degrees generate the ideal $(\gcd(a_{i_0},...,a_{i_s})) \subset \mathbf{Z}$. It follows that $\mathcal{O}_{\mathbf{P}}(k)$ is locally free at p if there exists such a monomial of degree k, i.e., if $\gcd(a_{i_0},...,a_{i_s})$ divides k.

Conversely, if $s = 2$, the space spanned by p_{i_1} and p_{i_2} is a line which meets D finitely many times, away frow the coordinate points p_{i_1} and p_{i_2}. By the above, at a given point p of this line (except the twoo coordinate points p_{i_1} and p_{i_2}) the condition for $\mathcal{O}_{\mathbf{P}}(k)$ to be locally free at p is

$$\gcd(a_{i_1}, a_{i_2}) | k.$$

■

The notation $\mathcal{O}_{\mathbf{P}}(k)$ is consistent with that of the sheaves $\mathcal{O}_{\mathbf{P}^d}(k)$ on a regular projective space $\mathbf{P}^d$. The group law is naturally

$$(\mathcal{O}_{\mathbf{P}}(k) \otimes \mathcal{O}_{\mathbf{P}}(h))^{\vee\vee} = \mathcal{O}_{\mathbf{P}}(k+h), \, \mathcal{O}_{\mathbf{P}}(0) = \mathcal{O}_{\mathbf{P}},$$

so the intersection form is

$$\mathcal{O}_{\mathbf{P}}(k_1) \cdot \mathcal{O}_{\mathbf{P}}(k_2) \cdots \mathcal{O}_{\mathbf{P}}(k_d) = \frac{k_1 k_2 \cdots k_d}{a_0 a_1 \cdots a_d}.$$

The members of the linear system $|\mathcal{O}_{\mathbf{P}}(k)|$ are the hypersurfaces $X \subset \mathbf{P}$ which are given as vanishing loci of homogeneous polynomials (with respect to the weights of $\mathbf{P}$) of degree k. In particular, $|\mathcal{O}_{\mathbf{P}}(k)|$ is empty as soon as $k < 0$. It follows that $-K_{\mathbf{P}}$ is effective, and it is Cartier (in other words, $\mathbf{P}$ is Gorenstein) if and only if each weight a_i is a divisor of $\sum_{i=0}^d a_i$.

Terminology. In a well formed weighted projective space $\mathbf{P}$, for a fixed k and $X \in |\mathcal{O}_{\mathbf{P}}(k)|$, we say that X is a hypersurface of degree k in $\mathbf{P}$. If X and Y are two hypersurfaces of respective degrees k and h such that

$$\dim X \cap Y = d - 2,$$

then we say that $X \cap Y$ is a complete intersection of type (k, h) in $\mathbf{P}$, or sometimes a (k, h)-complete intersection, and so on.

We emphasize that the notion of degree of a hypersurface $X \subset \mathbf{P}$ is closely related to the weights of the ambient space $\mathbf{P}$. Consider as an example the embedding $\mathbf{P}(1,1,2) \hookrightarrow \mathbf{P}^3$ induced by the linear system $|\mathcal{O}_{\mathbf{P}(1,1,2)}(2)|$, which is given in coordinates by

$$[x : y : z] \mapsto [u_0 : u_1 : u_2 : u_3] = [x^2 : xy : y^2 : z],$$

whose image is the quadric cone $\{u_0 u_2 = u_1^2\}$ in $\mathbf{P}^3$. Then a *cubic section* of the quadric cone is a *sextic* hypersurface of $\mathbf{P}(1,1,2)$.

Lemma 1.1.7 (Veronese embeddings: [Do81] 1.3.1)**.** *Let* $\mathbf{P} = \mathbf{P}(a_0, ..., a_d)$ *be a weighted projective space, and* n *a positive integer. Denote* $R = \mathbf{C}[x_0, ..., x_d]$ *with the suitable grading such that* $\mathbf{P} = \mathrm{Proj}(R)$*. Then the following isomorphism holds:*

$$\mathbf{P} \simeq \mathrm{Proj}(R^{(n)}),$$

where $R^{(n)}$ *is the graded ring whose degree* d *part is* $(R^{(n)})_d = R_{nd}$*.*

Moreover, $R^{(n)}$ *is isomorphic to a quotient of the form* $\mathbf{C}[u_0, ..., u_m]/I$ *where* $\mathbf{C}[u_0, ..., u_m]$ *is endowed with a grading* $\deg u_i = b_i$ *and* I *is a homogeneous ideal. This gives rise to an embedding*

$$v_n : \mathbf{P} \hookrightarrow \mathrm{Proj}(\mathbf{C}[u_0, ..., u_m]) = \mathbf{P}(b_0, ..., b_m),$$

which we refer to as the n*-Veronese map. This makes* $\mathbf{P}$ *a subvariety of* $\mathbf{P}(b_0, ..., b_m)$*, more precisely*

$$\mathbf{P} \simeq V(I) = \{\mathbf{u} = [u_0 : \cdots : u_m] \mid f(\mathbf{u}) = 0 \textit{ for all } f \in I\}$$

in $\mathbf{P}(d_0, ..., d_m)$*.*

Example. In the case of $\mathbf{P} = \mathbf{P}(1, 4, 5, 10)$, with coordinates $[x : y : z : w]$ with the following grading:

$$\deg x = 1, \deg y = 4, \deg z = 5, \deg w = 10,$$

the 5-Veronese embedding of $\mathbf{P}$ is

$$v_5 : \mathbf{P} \hookrightarrow \mathbf{P}(1^3, 2, 4),$$

given in coordinates by

$$[x : y : z : w] \in \mathbf{P} \mapsto [u_0 : u_1 : u_2 : v : s] = [x^5 : xy : z : w : y^5] \in \mathbf{P}(1^3, 2, 4).$$

Note that we have $\deg u_i = 1, \deg v = 2$ and $\deg s = 4$ in $\mathbf{P}(1^3, 2, 4)$. The image of v_5 is the quintic hypersurface of equation $u_0 s = u_1^5$, encoding the relation $x^5 y^5 = (xy)^5$.

The embedding v_5 admits such an expression in coordinates since the monomials

$$x^5,\ xy,\ z,\ w \text{ and } y^5$$

are generators of $\mathbf{C}[x, y, z, w]^{(5)}$, i.e., they generate the subalgebra of polynomials

$$\{f(x, y, z, w) \mid \deg f \text{ is divisible by } 5\},$$

and they satisfy only one relation, namely $x^5 y^5 = (xy)^5$, so that we have indeed

$$\mathbf{C}[x, y, z, w]^{(5)} \simeq \mathbf{C}[u_0, u_1, u_2, v, s]/(u_0 s - u_1^5).$$

Remark. Be aware of the fact that, although it is the case for standard weighted projective spaces, the n-Veronese map of a weighted projective space does not only involve monomials of degree n in general.

Corollary 1.1.8. *If* $\mathbf{P} = \mathbf{P}(a_0, ..., a_d)$ *is not well formed, then there exists a composition of Veronese embeddings which isomorphically identifies it with a well formed WPS.*

Proof: Assume that $\mathbf{P}$ is not well formed. Without loss of generality, we have

$$gcd(a_1, ..., a_d) = \delta \neq 1,$$

and $a_i = b_i \delta$ for some integers b_i, $i > 0$. In this situation, the δ-Veronese map of $\mathbf{P}$ is

$$v_\delta : \mathbf{P} \xrightarrow{\simeq} \mathbf{P}(a_0, b_1, ..., b_d)$$

which maps the point $[x_0 : x_1 : \cdots : x_d]$ to $[x_1^\delta : x_1 : \cdots : x_d]$. Moreover the b_i's are coprime.

If we assume that $\mathbf{P}(a_0, b_1, ..., b_d)$ is not well formed, then there exists an $i > 0$ such that

$$\gcd(a_0, b_1, ..., \widehat{b_i}, ..., b_d) = \gamma \neq 1,$$

$a_0 = c_0\gamma$, $b_j = c_j\gamma$ for $j > 0$, $j \neq i$. In this case, we consider the γ-Veronese map of $\mathbf{P}(a_0, b_1, ..., b_d)$ and the composition

$$\mathbf{P} \xrightarrow{v_\delta} \mathbf{P}(a_0, b_1, ..., b_d) \xrightarrow{v_\gamma} \mathbf{P}(c_0, ..., c_{i-1}, b_i, c_{i+1}, ..., c_d).$$

In addition, we have

$$\gcd(c_1, ..., c_{i-1}, b_i, c_{i+1}, ..., c_d) = 1$$

and

$$\gcd(c_0, ..., c_d) = 1.$$

If the WPS $\mathbf{P}(c_0, ..., c_{i-1}, b_i, c_{i+1}, ..., c_d)$ is not well formed, we repeat the process. This stops after at most d steps. ■

Example. We have the isomorphism $\mathbf{P}(1, 2, 4) \simeq \mathbf{P}(1, 1, 2)$ via

$$v_2 : [x : y : z] \mapsto [x^2 : y : z].$$

Without further notice, any weighted projective space we consider from now on is well formed.

1.1.2 The toric point of view

We cite [CLS11] for a reference about toric varieties. In general, a toric variety is a normal variety X containing a dense open subset $U \subset X$ such that $U \simeq (\mathbf{C}^*)^n$ and the action by multiplication of U on itself extends to a regular action $U \curvearrowright X$. Such varieties are characterized by fans, i.e., collections of cones in an $\mathbf{R}$-vector space satisfying some combinatorial conditions ([CLS11, §3.1]).

If X is affine, it is characterized by a cone, and if it is projective, it is characterized by the fan of a polytope, in other words there exists a polytope P in a $\mathbf{R}$-vector space V of dimension n, such that the vertices of P are in a lattice $\Lambda \subset V$, and for each $(n-1)$-dimensional face $P_i \subset P$ we consider the *primitive inwards normal* $u_{P_i} \in \Lambda^\vee$ and the one dimensional cone $\mathbf{R}_+ u_{P_i}$. Then the fan of P consists of all the cones indexed by families of codimension 1 faces with non-empty intersection:

$$\left\{ \Sigma_{i \in I} \mathbf{R}_+ u_{P_i} \text{ such that } \cap_{i \in I} P_i \neq \varnothing \right\}.$$

As an example, the toric points, which correspond to the vertices of the polytope, are the zero-dimensional orbits of the action, and they are parametrized by the n-dimensional cones of the fan.

The correspondence between projective toric varieties and polytopes is explained in further detail in [CLS11, §2.3].

Lemma 1.1.9. *Weighted projective spaces are toric.*

Proof: Consider the weighted projective space $\mathbf{P} = \mathbf{P}(a_0, ..., a_d)$ and the $(d+1)$-dimensional torus $(\mathbf{C}^*)^{d+1}$ with the action $(\mathbf{C}^*)^{d+1} \curvearrowright \mathbf{P}$:

$$(t_0, ..., t_d) \cdot [x_0 : \cdots : x_d] = [t_0^{a_0} x_0 : \cdots : t_d^{a_d} x_d].$$

By construction, the diagonal $\Delta \simeq \mathbf{C}^*$ acts trivially on $\mathbf{P}$. The quotient $(\mathbf{C}^*)^{d+1}/\Delta \simeq (\mathbf{C}^*)^d$ acts faithfully on $\mathbf{P}$. This endows $\mathbf{P}$ with the structure of a toric variety: the dense torus is the orbit of the point $[1 : 1 : \cdots : 1]$. The action $(\mathbf{C}^*)^{d+1} \curvearrowright \mathbf{P}$ is indeed an extension of the action of $(\mathbf{C}^*)^d$ on itself by multiplication, as for all $(t_i), (s_i) \in (\mathbf{C}^*)^{d+1}$ we have

$$(s_0, ..., s_d) \cdot [t_0^{a_0} : \cdots : t_d^{a_d}] = (s_0 t_0, ..., s_d t_d) \cdot [1 : \cdots : 1].$$

■

An algorithm for the construction of a polytope for the weighted projective space $\mathbf{P}(a_0, ..., a_d)$ is given in [RT13, Proposition 2.8].

⋆ ⋆

We now introduce a specific example of a birational map between two weighted projective spaces which we describe via polytopes. This example will be useful in §1.4.1, where we will give a list of birational maps. More precisely, the example below is the first item of the list displayed in Table 9, which can be found in §1.4.1.

Example. We endow $\mathbf{P} = \mathbf{P}(1, 4, 5, 10)$ with homogeneous coordinates $[x : y : z : w]$, and $X = \mathbf{P}(1^3, 2, 4)$ with homogeneous coordinates $[u_0 : u_1 : u_2 : v : s]$. The 5-Veronese map of $\mathbf{P}$ is an embedding $\mathbf{P} \hookrightarrow XX$ as follows:

$$[x : y : z : w] \in \mathbf{P} \mapsto [x^5 : xy : z : w : y^5] \in X.$$

Now consider the embedding of X by the very ample line bundle $\mathcal{O}_X(4)$:

$$X \xhookrightarrow{|\mathcal{O}_X(4)|} \mathbf{P}^{22}.$$

Let $\mathbf{P}' = \mathbf{P}(1^3, 2)$ be the weighted projective subspace of X given by the vanishing of s. It is embedded in $\mathbf{P}^{21}$ by $\mathcal{O}_X(4)|_{\mathbf{P}'}$, making X a cone over $\mathbf{P}'$ with the point p_s as vertex. As the image of $\mathbf{P}$ in X passes through the point p_s, the projection from the vertex p_s onto $\mathbf{P}'$ induces a rational map $\varphi : \mathbf{P} \dashrightarrow \mathbf{P}'$. In homogeneous coordinates, we can express φ as

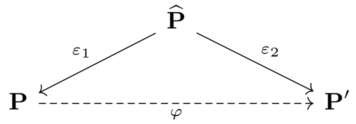
$$\varphi : [x : y : z : w] \in \mathbf{P} \mapsto [u_0 : u_1 : u_2 : v] = [x^5 : xy : z : w] \in \mathbf{P}'.$$

This is the 5-Veronese map without its last component y^5. Using toric methods, we can prove that φ is birational.

Lemma 1.1.10. *The map φ is birational.*

Proof: The map φ is toric, i.e., equivariant under the torus actions on $\mathbf{P}$ and $\mathbf{P}'$. It is encoded by a transformation of the polytopes of $\mathbf{P}$ and $\mathbf{P}'$.

We prove that φ is birational by exhibiting a commutative diagram



in which ε_1 and ε_2 are weighted blow-ups, and $\widehat{P}$ is a projective toric variety encoded by a fan which involves a subdivision of the fans of $\mathbf{P}$ and $\mathbf{P}'$.

Let us first construct ε_1. The map φ is regular outside the point $p_y = \{x = z = w = 0\}$. Since this indeterminacy point is a toric point, i.e., it is fixed by action of the torus on $\mathbf{P}$ (see Lemma 1.1.9), we may resolve the indeterminacy of φ via a toric birational modification given by a weighted blow-up of $\mathbf{P}$ at p_y. This corresponds to a subdivision of the cone of the affine chart $\{y \neq 0\}$.

We introduce the following elements of the lattice $\mathbf{Z}^3$ in $\mathbf{R}^3$:

$$\mathsf{e}_x = \begin{bmatrix} -4 \\ -5 \\ -10 \end{bmatrix}, \mathsf{e}_y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathsf{e}_z = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathsf{e}_w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The algorithm given in [RT13, Proposition 2.8] gives rise to the following fan in $\mathbf{Z}^3$ for the toric variety $\mathbf{P}$.

$$\Sigma_{\mathbf{P}} = \mathrm{Fan}(\mathsf{e}_x, \mathsf{e}_y, \mathsf{e}_z, \mathsf{e}_w) = \mathrm{Fan}\left(\begin{bmatrix} -4 \\ -5 \\ -10 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

This notation means that the cones of the fan $\Sigma_{\mathbf{P}}$ are the ones generated by all possible proper subsets of the family $\{\mathsf{e}_x, \mathsf{e}_y, \mathsf{e}_z, \mathsf{e}_w\}$.

There is a one-to-one correspondence between the cones of $\Sigma_{\mathbf{P}}$ and the toric subsets of $\mathbf{P}$ (in other words, the subsets which are stable under the action of the torus). For instance, the toric hypersurface $\{x = 0\}$ is encoded by the cone $\mathbf{R}_+\mathsf{e}_x$, and likewise, the toric point p_x is encoded by the cone $\mathbf{R}_+\mathsf{e}_y + \mathbf{R}_+\mathsf{e}_z + \mathbf{R}_+\mathsf{e}_w$.

In particular, the indeterminacy point p_y of φ is the origin of the affine chart $\{y \neq 0\}$ and a weighted blow-up of $\mathbf{P}$ at this point results in a subdivision of the cone $\mathbf{R}_+\mathsf{e}_x +$

$\mathbf{R}_+\mathtt{e}_z+\mathbf{R}_+\mathtt{e}_w$ by adding to the fan a new ray $\mathbf{R}_+\mathtt{e}_\zeta$ with $\mathtt{e}_\zeta \in \mathbf{Z}^3$ a $\mathbf{Q}$-linear combination with positive coefficients of $\mathtt{e}_x, \mathtt{e}_z$ and $\mathtt{e}_w$, as well as all the cones spanned by $\mathtt{e}_\zeta$ and the faces of $\mathbf{R}_+\mathtt{e}_x + \mathbf{R}_+\mathtt{e}_z + \mathbf{R}_+\mathtt{e}_w$ of dimension 1 and 2.

We introduce as a generator of the new ray:

$$\mathtt{e}_\zeta = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}.$$

Then $4\mathtt{e}_\zeta$ is a linear combination of $\mathtt{e}_x, \mathtt{e}_z$ and $\mathtt{e}_w$ with coefficients $1, 1$ and 2:

$$4\begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ -10 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This gives rise to a variety $\widehat{\mathbf{P}}$ which is the weighted blow-up of $\mathbf{P}$ at p_y with weights $1, 1$ and 2 whose exceptional divisor is isomorphic to $\mathbf{P}(1,1,2)$. This yields a fan $\Sigma_{\widehat{\mathbf{P}}}$ for $\widehat{\mathbf{P}}$ which contains a subdivision of the cone $\mathbf{R}_+\mathtt{e}_x + \mathbf{R}_+\mathtt{e}_z + \mathbf{R}_+\mathtt{e}_w$ as above. The other cones of maximal dimension in the fan $\Sigma_{\mathbf{P}}$ are left unchanged and are also cones of the fan $\Sigma_{\widehat{\mathbf{P}}}$.

We now refer to the construction of homogeneous coordinates on toric varieties which is explained in [CLS11, §5.1]. This allows us to introduce five toric coordinates on $\widehat{\mathbf{P}}$ which we denote by $(\mathtt{x}, \zeta, \mathtt{y}, \mathtt{z}, \mathtt{w})$ with the following grading in $\mathbf{Z}^2$:

	x	ζ	y	z	w
degree	1	0	4	5	10
in $\mathbf{Z}^2$:	0	1	1	1	2

The blow-up map ε_1 from $\widehat{\mathbf{P}}$ to $\mathbf{P}$ in homogeneous coordinates is

$$\varepsilon_1 : [\mathtt{x} : \zeta : \mathtt{y} : \mathtt{z} : \mathtt{w}] \in \widehat{\mathbf{P}} \mapsto [\mathtt{x}\zeta : \mathtt{y}\zeta^3 : \mathtt{z}\zeta^4 : \mathtt{w}\zeta^8] \in \mathbf{P}.$$

It is well defined everywhere and contracts the exceptional divisor $\{\zeta = 0\}$ to the point p_y. Indeed, if we fix a point with representative $(\mathtt{x}, \zeta, \mathtt{y}, \mathtt{z}, \mathtt{w})$, $\zeta^{1/4}$ a fourth root of ζ and $\mu = (\zeta^{1/4})^{-3}$, then the image of this point in $\mathbf{P}$ is

$$[\mu\mathtt{x}\zeta : \mu^4\mathtt{y}\zeta^3 : \mu^5\mathtt{z}\zeta^4 : \mu^{10}\mathtt{w}\zeta^8] = [\mathtt{x}\zeta^{1/4} : \mathtt{y} : \mathtt{z}\zeta^{1/4} : \mathtt{w}\zeta^{1/2}].$$

Let us now construct the weighted blow-up ε_2 and check that $\varphi \circ \varepsilon_1 = \varepsilon_2$. The following is a fan for the WPS $\mathbf{P}' = \mathbf{P}(1,1,1,2)$.

$$\Sigma_{\mathbf{P}'} = \mathrm{Fan}(\mathtt{e}_\zeta, \mathtt{e}_y, \mathtt{e}_z, \mathtt{e}_w) = \mathrm{Fan}\left(\begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right).$$

Besides, $\widehat{\mathbf{P}}$ is also the weighted blow-up of $\mathbf{P}'$ along a toric curve. Indeed, we have

$$\begin{bmatrix} -4 \\ -5 \\ -10 \end{bmatrix} = 5\begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

in other words, $\mathtt{e}_x = 5\mathtt{e}_\zeta + \mathtt{e}_y$, where $\mathtt{e}_\zeta$ and $\mathtt{e}_y$ are rays of the fan $\Sigma_{\mathbf{P}'}$. This subdivision of the cone $\mathbf{R}_+\mathtt{e}_\zeta + \mathbf{R}_+\mathtt{e}_y$ yields the same fan as $\Sigma_{\widehat{\mathbf{P}}}$.

The blow-up map ε_2 from $\widehat{\mathbf{P}}$ to $\mathbf{P}'$ is the following:

$$\varepsilon_2 : [\mathtt{x} : \zeta : \mathtt{y} : \mathtt{z} : \mathtt{w}] \in \widehat{\mathbf{P}} \mapsto [\mathtt{x}^5\zeta : \mathtt{x}\mathtt{y} : \mathtt{z} : \mathtt{w}] \in \mathbf{P}',$$

contracting the exceptional locus $\{\mathtt{x} = 0\}$ to a curve.

Moreover, given any point $[\mathtt{x} : \zeta : \mathtt{y} : \mathtt{z} : \mathtt{w}] \in \widehat{\mathbf{P}}$, we have

$$\varphi \circ \varepsilon_1([\mathtt{x} : \zeta : \mathtt{y} : \mathtt{z} : \mathtt{w}]) = [\mathtt{x}^5\zeta^5 : \mathtt{x}\mathtt{y}\zeta^4 : z\zeta^4 : \mathtt{w}\zeta^8] = \varepsilon_2([\mathtt{x} : \zeta : \mathtt{y} : \mathtt{z} : \mathtt{w}]).$$

So $\varphi \circ \varepsilon_1 = \varepsilon_2$, as required. ∎

1.2 Extendability of K3 surfaces and canonical curves

1.2.1 Extendability of projective varieties

Let us recall the definition of an extension of a projective variety (which can be found in Definition 0.0.5):

Definition 1.2.1. *Let $X \subset \mathbf{P}^n$ be a nondegenerate projective variety (nondegenerate means not contained in any hyperplane). We say that X is* extendable *if there exist a positive integer r and a nondegenerate variety $Y \subset \mathbf{P}^{n+r}$ which is not a cone and such that X is a linear section of Y, in other words, an intersection of Y with several hyperplanes.*

If such an extension Y of X exists with $\dim Y - \dim X = r$, we sometimes say to be more specific that Y is an r-extension of X.

Let $C(X) \subset \mathbf{P}^{n+r}$ be the cone over X with an $(r-1)$-dimensional linear subspace as vertex. Then for the general n-dimensional linear subspace Λ of $\mathbf{P}^{n+r}$ we have $C(X) \cap \Lambda \simeq X$. However, in the definition above Y is required not to be a cone. As a consequence, the existence of an extension is not obvious.

Definition 1.2.2 (Maximal extension)**.** *Given $X \subset \mathbf{P}^n$ and Y an extension of X, we say that Y is* maximal *if there exists no extension of X of dimension larger than* $\dim Y$.

Remark. Assume that a projective variety $X \subset \mathbf{P}^n$ admits a chain of extensions

$$X \subset X_1 \subset \cdots \subset X_s,$$

where X_s is not extendable. Then in general we may not conclude that X_s is a maximal extension of X in the sense of Definition 1.2.2, as there may exist other longer chains of extensions of X not involving X_s. An example of such a situation is given in [Tev03, Example 1.33]; see also [SL24, Example 2.2].

We make the comment that it might be more appropriate to refer to a maximal extension as in Definition 1.2.2 as an *extension of maximal dimension*, but we will simply say *maximal extension* to lighten the notation.

Definition 1.2.3 (Extendability)**.** *Given $X \subset \mathbf{P}^n$, we define its extendability to be*

$$\epsilon_{\mathbf{P}^n}(X) = \sup\left(\{0\} \cup \left\{r \in \mathbf{N} \mid \exists Y \subset \mathbf{P}^{n+r} \text{ an } r\text{-extension of } X\right\}\right).$$

For terminology, when $\epsilon_{\mathbf{P}^n}(X) = +\infty$, we say that X is *infinitely extendable*, sometimes denoted ∞-extendable. Conversely, if no extension of X exists, then we have $\epsilon_{\mathbf{P}^n}(X) = 0$ and in this situation, if X is not a cone, we say that it is its own maximal extension.

Remark. Let $X \subset \mathbf{P}^n$ be a smooth extendable variety. Then in general there exist many non-pairwise isomorphic smooth extensions of X in $\mathbf{P}^{n+1}$. Consider for instance $\mathcal{S}$ the moduli space of smooth cubic surfaces and $\mathcal{C}$ that of smooth cubic curves. Then we know that $\mathcal{C}$ has dimension 1, while

$$\dim \mathcal{S} = h^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3)) - \dim \mathrm{PGL}_4(\mathbf{C}) = \binom{6}{3} - 15 = 5.$$

We now introduce the moduli space

$$\mathcal{SC} = \{(\Sigma, \Gamma) \mid \Sigma \in \mathcal{S},\, \Gamma \text{ a smooth hyperplane section of } \Sigma\},$$

and the two projections onto $\mathcal{S}$ and $\mathcal{C}$:

$$\begin{array}{ccc} & \mathcal{SC} & \\ {}^{\pi_{\mathcal{S}}}\swarrow & & \searrow^{\pi_{\mathcal{C}}} \\ \mathcal{S} & & \mathcal{C} \end{array}$$

Each fibre of $\pi_{\mathcal{S}}$ has dimension $h^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1)) - 1 = 3$, yielding $\dim \mathcal{SC} = 8$. It follows that the $\pi_{\mathcal{C}}$-fibre over the general $\Gamma \in \mathcal{C}$ has dimension 7. In other words the smooth surface extensions of Γ up to isomorphism form a family of dimension 7.

Examples.

- There exist varieties which are infinitely extendable. Any hypersurface $X \subset \mathbf{P}^n$ is ∞-extendable: as an example, consider the Fermat hypersurface F_d^n of degree d in $\mathbf{P}^{n+1}$ given by the equation

$$x_0^d + x_1^d + \cdots + x_n^d = 0.$$

  Then F_d^n is not a cone, and its intersection with the hyperplane $\{x_n = 0\}$ is F_d^{n-1}. We have the following infinite chain of extensions

$$\cdots \subset F_d^{n-1} \subset F_d^n \subset F_d^{n+1} \subset \cdots$$

- There exist nonextendable varieties. See for instance Table 6: the WPS $\mathbf{P}(1,2,3,6)$ admits an embedding in $\mathbf{P}^{26}$ whose image is not extendable.

- There exist varieties which are extendable, but not infinitely many times. It is for instance the case of canonical curves of genus ≥ 11 and Clifford index ≥ 3 (this is the topic of Theorem 1.2.12).

More specific examples will be treated in §1.4, where we realize weighted projective spaces as extensions of K3 surfaces.

Definition 1.2.4. *Let $X \subset \mathbf{P}^n$ be a projective variety, with $L = \mathcal{O}_X(1)$. We introduce*

$$\alpha(X, L) = h^0(X, N_{X/\mathbf{P}^n} \otimes L^{-1}) - n - 1.$$

Now we cite the following theorem, which is an argument ensuring the existence of varieties which are not infinitely extendable.

Theorem 1.2.5 ([Lvo92])**.** *If $X \subset \mathbf{P}^n$ is smooth, nondegenerate, not a quadric, and $\alpha(X, L) < n$, then it is at most $\alpha(X, L)$-extendable:*

$$\epsilon_{\mathbf{P}^n}(X) \leq \alpha(X, L).$$

1.2.2 The Gauß-Wahl maps and extendability of canonical curves

We start by recalling the definition of a canonical curve.

Definition 1.2.6. *An irreducible projective curve Γ embedded in $\mathbf{P}^n$ is canonical if it is nondegenerate, i.e., not contained in any hyperplane, and*

$$\mathcal{O}_{\mathbf{P}^n}(1)|_\Gamma \simeq \mathcal{O}_\Gamma(K_\Gamma).$$

We recall that the canonical divisor K_Γ of a curve Γ of genus 2 or more is very ample if and only if Γ is nonhyperelliptic (see [Har77, Proposition IV.5.2]). In this case, the curve Γ is embedded as a canonical curve by K_Γ:

$$\Gamma \xhookrightarrow{|K_\Gamma|} \mathbf{P}^{g(\Gamma)-1}.$$

We also recall the definition of a K3 surface:

Definition 1.2.7. *A K3 surface is a surface S with canonical singularities such that $h^1(S, \mathcal{O}_S) = 0$ and the canonical divisor K_S is Cartier with $\mathcal{O}_S(K_S) \simeq \mathcal{O}_S$.*

Classically, the surface S is asked to be smooth to be considered K3, but we extend the definition to surfaces with canonical singularities. K3 surfaces play a significant role in the extendability of canonical curves, according to the following lemma:

Lemma 1.2.8. *If $\Gamma \subset \mathbf{P}^{g-1}$ is a canonical curve, and S is a projective surface with canonical singularities and a projective model $S \subset \mathbf{P}^g$ which is an extension of Γ, then S is a K3 surface.*

Conversely, if $S \subset \mathbf{P}^g$ is a projective K3 surface, and Γ is a general hyperplane section of S, then $\Gamma \subset \mathbf{P}^{g-1}$ is a canonical curve.

Proof: Since Γ is canonical, we have

$$\mathcal{O}_S(\Gamma)|_\Gamma = \mathcal{O}_\Gamma(1) \simeq \mathcal{O}_\Gamma(K_\Gamma),$$

and by adjunction we have the equality of Cartier divisors

$$(K_S + \Gamma)|_\Gamma = K_\Gamma = \Gamma|_\Gamma$$

hence $K_S|_\Gamma = 0$. It follows from [AP23, Proposition 1.1] that $K_S = 0$.

Consider the restriction exact sequence

$$0 \to \mathcal{O}_S(-\Gamma) = \mathcal{O}_S(-1) \to \mathcal{O}_S \to \mathcal{O}_\Gamma \to 0$$

and its cohomology long exact sequence

$$\begin{array}{ccccc} 0 \longrightarrow H^0(S, \mathcal{O}_S(-1)) & \longrightarrow & H^0(S, \mathcal{O}_S) & \longrightarrow & H^0(\Gamma, \mathcal{O}_\Gamma) \\ & & & & \downarrow \\ H^1(\Gamma, \mathcal{O}_\Gamma) & \longleftarrow & H^1(S, \mathcal{O}_S) & \longleftarrow & H^1(S, \mathcal{O}_S(-1)) \\ \downarrow & & & & \\ H^2(S, \mathcal{O}_S(-1)) & \longrightarrow & H^2(S, \mathcal{O}_S) & \longrightarrow & 0, \end{array}$$

in which $h^0(S, \mathcal{O}_S) = h^2(S, \mathcal{O}_S) = 1$ and $h^2(S, \mathcal{O}_S(-1)) = h^0(S, \mathcal{O}_S(1)) = g+1$ by Serre duality, and $h^0(S, \mathcal{O}_S(-1)) = h^1(S, \mathcal{O}_S(-1)) = 0$. Since the genus of Γ is g we have $h^1(\Gamma, \mathcal{O}_\Gamma) = g$ and by a count of dimensions we obtain

$$h^1(S, \mathcal{O}_S) = 0,$$

so S is a K3 surface.

Conversely, if S is a K3 surface embedded in $\mathbf{P}^g$ and Γ is an irreducible hyperplane section of S, then by adjunction we have the equality of Cartier divisors

$$K_\Gamma = (K_S + \Gamma)|_\Gamma = \Gamma|_\Gamma,$$

so $\mathcal{O}_\Gamma(1) \simeq \mathcal{O}_\Gamma(K_\Gamma)$, and Γ is canonical. ∎

⋆ ⋆

Let us now fix a canonical curve $\Gamma \subset \mathbf{P}^{g-1}$ of genus $g \geq 3$.

Definition 1.2.9. *The* Gauß-Wahl map *of* Γ *is the linear map between vector spaces*

$$\Phi_{K_\Gamma} : \wedge^2 H^0(\Gamma, K_\Gamma) \to H^0(\Gamma, 3K_\Gamma)$$

which maps any pure wedge product $s \wedge t$ *to* $sdt - tds$.

This map can be surjective only if the starting dimension is larger than, or equal to, the dimension of the target space. Namely, we compute

$$\dim \wedge^2 H^0(\Gamma, K_\Gamma) = \binom{g}{2} = \frac{g(g-1)}{2},$$

and by Riemann-Roch

$$\dim H^0(\Gamma, 3K_\Gamma) = 5(g-1).$$

So the surjectivity of the Gauß-Wahl requires $\frac{1}{2}g(g-1) \geq 5(g-1)$, in other words $g \geq 10$.

More precisely, by [CHM88, Theorem 8.7], for $g \geq 10$ (except $g = 11$) the surjectivity of Φ_{K_Γ} holds for the general $\Gamma \in \mathcal{M}_g$, where $\mathcal{M}_g$ denotes the moduli space of projective curves of genus g.

Definition 1.2.10. *Let* $D \in \operatorname{Pic}(\Gamma)$*. The* Clifford index *of* D *is*

$$\operatorname{Cliff}(D) = \deg(D) - 2(h^0(\Gamma, D) - 1).$$

The Clifford index of Γ *is*

$$\operatorname{Cliff}(\Gamma) = \min \{\operatorname{Cliff}(D) \mid D \in \operatorname{Pic}(\Gamma), D \nsim 0, D \nsim K_\Gamma\}.$$

Definition 1.2.11. *Let Γ be an irreducible projective curve. The gonality of Γ is the smallest d such that there exists a g^1_d on Γ, i.e., a pencil of divisors of degree d.*

In other words the gonality is equal to the lowest possible degree of a quasi-finite rational map $\Gamma \dashrightarrow \mathbf{P}^1$.

In the above, we know by Clifford's theorem (see [Cl78]) that $\mathrm{Cliff}(D) \geq 0$ for any Cartier divisor D on Γ, and that $\mathrm{Cliff}(D) = 0$ if and only if D is linearly equivalent to either the trivial divisor or the canonical divisor, or Γ is hyperelliptic with the linear system $|D|$ containing a g^1_2. Then the Clifford index of Γ is well defined as a nonnegative integer, and in that sense it measures how far the curve Γ is from being hyperelliptic.

The Clifford index is known for a general curve of genus g. In other words, in the moduli space $\mathcal{M}_g$ of smooth curves of genus g, there exists a dense open subset $\mathcal{U}_g$ such that for every member Γ of $\mathcal{U}_g$ we have

$$\mathrm{Cliff}(\Gamma) = \left\lfloor \frac{g-1}{2} \right\rfloor,$$

which is equal to the gonality of Γ minus 2.

For some results about the role of the Clifford index in the extendability of canonical curves, see [ABS17, Theorems 1 & 2, Corollary 4]. The following theorem, very important to us, deals with the maximal extendability of smooth canonical curves, under some conditions on the genus and the Clifford index:

Theorem 1.2.12 ([CDS20], Theorem 2.1 & Corollary 5.5)**.** *Let $\Gamma \subset \mathbf{P}^{g-1}$ be a smooth canonical curve with $g \geq 11$ and such that $\mathrm{Cliff}(\Gamma) > 2$. Then the maximal extendability of Γ is equal to $\alpha := \mathrm{corank}(\Phi_{K_\Gamma})$, in other words there exists a maximal extension $Y \subset \mathbf{P}^{g-1+\alpha}$ of Γ with $\dim(Y) = 1 + \alpha$.*

In addition, there exists a maximal extension Y of Γ which is universal*, meaning that it contains each surface extension of Γ as a unique linear section.*

Furthermore, by [Lvo92] the invariant $\alpha(\Gamma, K_\Gamma)$ that we defined in Definition 1.2.4 is equal to the corank of Φ_{K_Γ} under the assumption that $\Gamma \subset \mathbf{P}^{g-1}$ is a smooth canonical curve. Then Theorem 1.2.12 says that the upper bound for the extendability given in Theorem 1.2.5

$$\alpha(\Gamma, K_\Gamma) \geq \epsilon_{\mathbf{P}^{g-1}}(\Gamma)$$

is an equality whenever $g \geq 11$ and $\mathrm{Cliff}(\Gamma) > 2$.

We make the comment that the universal extension of a canonical curve satisfying these two conditions is unique up to isomorphism, by the construction provided in [CDS20, §5].

⋆ ⋆

Consider a smooth canonical curve $\Gamma \subset \mathbf{P}^{g-1}$ of genus $g \geq 11$ and Clifford index $\mathrm{Cliff}(\Gamma) > 2$. Let $Y \subset \mathbf{P}^{g-1+\alpha(\Gamma,K_\Gamma)}$ be the universal extension of Γ. Then the family of surface extensions of Γ is parametrized by $\mathbf{P}(\mathrm{coker}(\Phi_{K_\Gamma})) \simeq \mathbf{P}^{\alpha(\Gamma,K_\Gamma)-1}$. Specifically, each surface extension of Γ is the intersection of Y with a g-dimensional linear subspace $\Lambda \subset \mathbf{P}^{g-1+\alpha(\Gamma,K_\Gamma)}$ containing Γ, and all such Λ form a family which is isomorphic to $\mathbf{P}^{\alpha(\Gamma,K_\Gamma)-1}$.

In [CDS20, 4.8], a correspondence is made explicit between the points of $\mathbf{P}(\mathrm{coker}(\Phi_{K_\Gamma}))$ and the surface extension of Γ. The choice of a line in $\mathbf{P}(\mathrm{coker}(\Phi_{K_\Gamma}))$ yields a pencil of surfaces in Y which are all extensions of Γ, and this pencil of surfaces spans a threefold extension of Γ which is a linear section of Y. Likewise, the choice of a s-dimensional linear subspace of $\mathbf{P}(\mathrm{coker}(\Phi_{K_\Gamma}))$ yields an s-dimensional family of surfaces extension of Γ, and this family spans an $(s+2)$-fold extension of Γ which is a linear section of Y.

Lemma 1.2.13. *Let $\Gamma \subset \mathbf{P}^{g-1}$ be a smooth canonical curve of genus $g \geq 11$ such that $\mathrm{Cliff}(\Gamma) \geq 3$. Let $\alpha = \mathrm{coker}(\Phi_{K_\Gamma})$ and $Y \subset \mathbf{P}^{g-1+\alpha}$ be a maximal extension of Γ with $\dim Y \geq 2$. Then Y is the universal extension of Γ if and only if it contains no cone over Γ as a linear surface section.*

Proof: Let us denote by $\mathcal{H}$ the family of g-planes $\Lambda \subset \mathbf{P}^{g-1+\alpha}$ such that $\Gamma \subset \Lambda$, and $\mathcal{S} = \mathbf{P}(\operatorname{coker}(\Phi_{K_\Gamma}))$ the space which parametrizes the surface extensions of Γ. Then we have $\mathcal{H} \simeq \mathcal{S} \simeq \mathbf{P}^{\alpha-1}$, and the map $\mathcal{H} \to \mathcal{S}$ which maps Λ to $Y \cap \Lambda$ is well-defined since no linear surface section of Y is a cone. Moreover, it lifts to a linear map $\mathbf{C}^\alpha \to \mathbf{C}^\alpha$ which has trivial kernel, i.e., it is an isomorphism (see [CD24, 6.6]).

It follows that Y contains all the surface extensions of Γ as unique linear sections. By Theorem 1.2.12, it is the universal extension of Γ. ■

By [CHM88, Theorem 8.7], starting from $g = 10$ the Gauß-Wahl map of a general member of $\mathcal{M}_g$ is onto. But Theorem 1.2.12 only applies for $g \geq 11$.

⋆ ⋆

We make now a short digression about the extendability of Brill-Noether-Petri general curves.

Definition 1.2.14 (Brill-Noether-Petri generality)**.** *A curve* Γ *is* Brill-Noether-Petri general *if the multiplication map (called the Petri map)*

$$H^0(\Gamma, L) \otimes H^0(\Gamma, K_\Gamma \otimes L^{-1}) \to H^0(\Gamma, K_\Gamma)$$

is injective for all $L \in \operatorname{Pic}(L)$*.*

Many authors write *BNP generality*, which is short for Brill-Noether-Petri generality. Then we have the following result about the extendability of BNP general canonical curves. It implicitly imposes a condition on the Clifford index, since for BNP general curves Γ we have $\operatorname{Cliff}(\Gamma) = \lfloor \frac{g-1}{2} \rfloor$.

Theorem 1.2.15 ([ABS17], Theorem 1)**.** *Let* Γ *be a BNP general canonical curve of genus* 12 *or more. Then* Γ *is a hyperplane section of a K3 surface, or a degeneration of K3 surfaces, if and only if* $\alpha(\Gamma, K_\Gamma) > 0$*.*

This theorem can now be seen as a particular case of Theorem 1.2.12.

1.3 Weighted projective spaces as extensions of canonical curves

Assume that $\Gamma \subset \mathbf{P}^{g-1}$ is an extendable canonical curve with an extension Y of dimension $d \geq 3$. We know from [CDS20, Theorem 5.1] that such an extension Y is arithmetically Gorenstein, hence Gorenstein, and it satisfies the equality of Cartier divisors

$$K_Y = (d-2)H,$$

where H is the class of the hyperplane sections of Y. A proof in the case $d = 3$ can be found in [CM85].

The aim of this section is to study extensions of canonical curves which are isomorphic to weighted projective spaces. In particular, such a space $\mathbf{P}$ satisfies $-K_\mathbf{P} = (d-2)H$ by the above, where $d = \dim \mathbf{P}$. We will study the existence of weighted projective spaces $\mathbf{P}$ for which $-\frac{1}{d-2}K_\mathbf{P}$ is Cartier, and in each case for which the image of the induced morphism $\mathbf{P} \to \mathbf{P}^N$ is not a cone, we will prove that $-\frac{1}{d-2}K_\mathbf{P}$ is very ample. In each of these situations, we will examine the linear sections of the image of the associated embedding.

By the condition $-K_\mathbf{P} = (d-2)H$ we know that the general hyperplane section of $\mathbf{P}$, i.e., the general member of $|H|$, has at worst canonical singularities by [Mel99, Theorem 1]. Note that there exist extensions of canonical curves with worse than canonical singularities, but here we will only focus on the general members of $|H|$.

Lemma 1.3.1. *There is a one-to-one correspondence between:*

- *The weighted projective spaces* $\mathbf{P}$ *of dimension* d *for which* $K_\mathbf{P}$ *is Cartier and divisible by* $d-2$ *in* $\operatorname{Pic}(\mathbf{P})$*,*

- *The decompositions of* $d-2$ *as* Egyptian fractions *of* $d+1$ *terms, i.e., sums of the form*
$$\frac{1}{k_0}+\frac{1}{k_1}+\cdots+\frac{1}{k_d}=d-2,$$
where $k_0 \geq k_1 \geq \cdots \geq k_d$ *are positive integers.*

Proof: If $\mathbf{P}=\mathbf{P}(a_0,...,a_d)$ is a Gorenstein weighted projective space with $\frac{1}{d-2}K_{\mathbf{P}}$ Cartier, and we denote
$$\sigma = a_0+\cdots+a_d$$
then by Lemma 1.1.5, for each weight a_i we have $a_ik_i=\frac{\sigma}{d-2}$ for some nonzero $k_i \in \mathbf{N}$. Moreover we have the equality
$$(d-2)\left(\frac{a_0}{\sigma}+\cdots+\frac{a_d}{\sigma}\right)=d-2$$
which can be rewritten as the Egyptian fraction
$$\frac{1}{k_0}+\cdots+\frac{1}{k_d}=d-2.$$
Conversely, given such an Egyptian fraction, we set $\sigma=(d-2)\mathrm{lcm}(k_0,...,k_d)$ and $a_i=\frac{\sigma}{k_i(d-2)}$ for $i=0,...,d$. Then the weighted projective space
$$\mathbf{P}(a_0,...,a_d)$$
is well formed with $\frac{1}{d-2}K_{\mathbf{P}}$ Cartier. ■

In the following, through the use of the correspondence given in Lemma 1.3.1, we classify the weighted projective spaces $\mathbf{P}$ of dimension d for which $-\frac{1}{d-2}K_{\mathbf{P}}$ is Cartier, starting with $d=3$.

1.3.1 Dimension 3

By Lemma 1.3.1, the list of the Gorenstein weighted projective spaces of dimension 3 corresponds to the list of Egyptian fractions of the form
$$\frac{1}{k_0}+\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}=1,$$
in which we assume without loss of generality that the largest summand is $\frac{1}{k_3}$. It follows that $\frac{1}{k_3}\geq\frac{1}{4}$ and thus $k_3 \leq 4$. Naturally $k_3 \neq 1$ since the other summands are nonzero, hence the integer k_3 can only take three different values, and in each case there is a method to compute all the Egyptian fractions by induction:

- The case $k_3=4$, where all the k_i's are equal to 4.
- The case $k_i=3$, in which we have the sum of three terms
$$\frac{1}{k_0}+\frac{1}{k_1}+\frac{1}{k_2}=\frac{2}{3},$$
where, without loss of generality, $\frac{1}{k_2}$ is the largest summand, so that $\frac{1}{k_2}\geq\frac{2}{9}$, i.e., $2k_2 \leq 9$, and so on.
- Lastly, the case $k_3=2$, in which we have
$$\frac{1}{k_0}+\frac{1}{k_1}+\frac{1}{k_2}=\frac{1}{2}$$
and $\frac{1}{k_2}\geq\frac{1}{6}$, i.e., $k_2 \leq 6$, etc.

From the correspondence given in Lemma 1.3.1, we deduce the list of all the Gorenstein weighted projective spaces of dimension 3, i.e., all the WPS $\mathbf{P}$ of dimension 3 with $K_{\mathbf{P}}$ Cartier, and we display them in the table below. We denote by σ the sum of the weights in each case, so that $\mathcal{O}_{\mathbf{P}}(-K_{\mathbf{P}})=\mathcal{O}_{\mathbf{P}}(\sigma)$ by Lemma 1.1.5.

$\frac{1}{k_0}+\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}=1$	$\mathbf{P}$	σ
$\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$	$\mathbf{P}(1,1,1,1)=\mathbf{P}^3$	4
$\frac{1}{6}+\frac{1}{4}+\frac{1}{4}+\frac{1}{3}$	$\mathbf{P}(2,3,3,4)$	12
$\frac{1}{6}+\frac{1}{6}+\frac{1}{3}+\frac{1}{3}$	$\mathbf{P}(1,1,2,2)$	6
$\frac{1}{12}+\frac{1}{4}+\frac{1}{3}+\frac{1}{3}$	$\mathbf{P}(1,3,4,4)$	12
$\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{2}$	$\mathbf{P}(1,1,1,3)$	6
$\frac{1}{10}+\frac{1}{5}+\frac{1}{5}+\frac{1}{2}$	$\mathbf{P}(1,2,2,5)$	10
$\frac{1}{8}+\frac{1}{8}+\frac{1}{4}+\frac{1}{2}$	$\mathbf{P}(1,1,2,4)$	8
$\frac{1}{12}+\frac{1}{6}+\frac{1}{4}+\frac{1}{2}$	$\mathbf{P}(1,2,3,6)$	12
$\frac{1}{20}+\frac{1}{5}+\frac{1}{4}+\frac{1}{2}$	$\mathbf{P}(1,4,5,10)$	20
$\frac{1}{15}+\frac{1}{10}+\frac{1}{3}+\frac{1}{2}$	$\mathbf{P}(2,3,10,15)$	30
$\frac{1}{12}+\frac{1}{12}+\frac{1}{3}+\frac{1}{2}$	$\mathbf{P}(1,1,4,6)$	12
$\frac{1}{18}+\frac{1}{9}+\frac{1}{3}+\frac{1}{2}$	$\mathbf{P}(1,2,6,9)$	18
$\frac{1}{24}+\frac{1}{8}+\frac{1}{3}+\frac{1}{2}$	$\mathbf{P}(1,3,8,12)$	24
$\frac{1}{42}+\frac{1}{7}+\frac{1}{3}+\frac{1}{2}$	$\mathbf{P}(1,6,14,21)$	42

Table 1: The Gorenstein WPS of dimension 3.

In particular, if $\mathbf{P}$ is a Gorenstein weighted projective space of dimension 3 and S is a general member of the linear system $|-K_{\mathbf{P}}|$, then it is a K3 surface (possibly with ADE singularities). If $\Gamma \subset \mathbf{P}$ is the intersection of two general members of the linear system $|-K_{\mathbf{P}}|$, then we have $K_\Gamma = -K_{\mathbf{P}}|_\Gamma$. Each of these weighted projective spaces $\mathbf{P}$ admits a projective model in which it is an extension of K3 surfaces and canonical curves under the condition that $-K_{\mathbf{P}}$ is very ample. This was proven for 6 items out of the 14 of the list above:

Theorem 1.3.2 ([De23], Theorem 2.5)**.** *Let* $\mathbf{P}$ *be one of the following Gorenstein weighted projective* 3*-space:*

$\mathbf{P}(1,4,5,10)$
$\mathbf{P}(1,2,6,9)$
$\mathbf{P}(1,2,3,6)$
$\mathbf{P}(1,3,8,12)$
$\mathbf{P}(1,6,14,21)$
$\mathbf{P}(2,3,10,15)$

Then $-K_{\mathbf{P}}$ *is very ample.*

Comment. There are other items of the list Table 1 for which this condition holds. As an example, it holds for $\mathbf{P}=\mathbf{P}^3$. However, the 6 cases of Theorem 1.3.2 are discriminated by the following criterion.

In [DS23] T. Dedieu and E. Sernesi showed that $\mathbf{P}$ deforms to a threefold extension of a smooth deformation (S', L') of $(S, -K_{\mathbf{P}}|_S)$ under the condition $\alpha(S, -K_{\mathbf{P}}|_S) = \alpha(S', L')$. The term *smooth deformation* is to be understood as in Definition 1.3.3 below.

However, as we will see in Table 7, there are cases for which any smooth deformation (S', L) of $(S, -K_{\mathbf{P}}|_S)$ satisfies $\alpha(S, -K_{\mathbf{P}}|_S) > \alpha(S', L) > 0$. In this situation, it is not known whether $\mathbf{P}$ deforms to a threefold extension of a smooth deformation (S', L). These items are the ones mentioned in Theorem 1.3.2 above.

Definition 1.3.3 (Smoothing)**.** *Let* (S, L) *be a polarized K3 surface with canonical singularities. A* smoothing *of* (S, L) *is the data of a polarized threefold* $(\mathcal{S}, \mathcal{L})$ *and a morphism to a pointed smooth affine curve* $\mathcal{S} \to (\Delta, 0)$ *whose general fibre is smooth, and* S *is isomorphic to the fibre* $\mathcal{S}_0$ *over* 0 *with* $L \simeq \mathcal{L}|_{\mathcal{S}_0}$*.*

Let $\mathcal{S}_t$ *be a smooth fibre of* $\mathcal{S} \to \Delta$*. Then we say that* $(\mathcal{S}_t, \mathcal{L}|_{\mathcal{S}_t})$ *is a* smooth deformation *of* (S, L)*.*

More results about the weighted projective 3-spaces as extensions of K3 surfaces and canonical curves will follow in §1.4. Namely, our main result is Theorem 1.4.5, in which we construct maximal extensions of the items of Table 1 for which a construction was not known before. For now, we make a digression about the weighted projective spaces of dimension ≥ 4 which are extensions of canonical curves.

1.3.2 Dimension 4

By [CDS20, Theorem 5.1], the condition for a WPS $\mathbf{P}$ of dimension 4 to have a projective model which is an extension of canonical curves requires

$$\frac{1}{2}K_{\mathbf{P}} \in \operatorname{Pic}(\mathbf{P}).$$

Such a space $\mathbf{P}$ is what we call a Gorenstein weighted projective 4-space of even index.

A weighted projective 4-space $\mathbf{P} = \mathbf{P}(a_0, a_1, a_2, a_3, a_4)$ satisfies $\frac{1}{2}K_{\mathbf{P}} \in \operatorname{Pic}(\mathbf{P})$ iff. each a_i is a divisor of

$$\frac{\sigma}{2} = \frac{a_0 + a_1 + a_2 + a_3 + a_4}{2}.$$

By Lemma 1.3.1 these weighted projective spaces are classified by Egyptian fractions of the form

$$\frac{1}{k_0} + \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4} = 2.$$

The largest of the five summands being $\frac{1}{k_4}$, we have $2k_4 \leq 5$, leaving only two possibilities:

- $k_4 = 1$, in which case we have the equality

  $$\frac{1}{k_0} + \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = 1.$$

  But this corresponds to a Gorenstein weighted projective 3-space by Lemma 1.3.1. In this case, the 4-fold $\mathbf{P} = \mathbf{P}(a_0, a_1, a_2, a_3, a_4)$ verifies

  $$a_4 = a_0 + a_1 + a_2 + a_3,$$

  and the subspace $\mathbf{P}' = \mathbf{P}(a_0, a_1, a_2, a_3) \subset \mathbf{P}$ is one of the 14 spaces listed in Table 1. Moreover, we have

  $$-\frac{1}{2}K_{\mathbf{P}}|_{\mathbf{P}'} = -K_{\mathbf{P}'}.$$

  The divisor $-\frac{1}{2}K_{\mathbf{P}}$ is Cartier and basepoint-free, and the image of the associated morphism $\mathbf{P} \to \mathbf{P}^N$ is a cone over the isomorphic image of $\mathbf{P}'$ in $\mathbf{P}^{N-1}$. Naturally, as we are looking for extensions of canonical curves, these cases will not be useful to us.

- $k_4 = 2$, in which case we have the following equality involving k_0, k_1, k_2 and k_3:

  $$\frac{1}{k_0} + \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = \frac{3}{2}.$$

  The largest summand being $\frac{1}{k_3}$, we have $3k_3 \leq 8$ and $k_3 \geq k_4 = 2$, so necessarily k_3 must be equal to 2. Hence

  $$\frac{1}{k_0} + \frac{1}{k_1} + \frac{1}{k_2} = 1$$

  with $k_0 \geq k_1 \geq k_2$, for which there are only 3 solutions:

  $$(k_0, k_1, k_2) = (3, 3, 3) \text{ or } (4, 4, 2) \text{ or } (6, 3, 2).$$

Lemma 1.3.4. *Let* $\mathbf{P}$ *be a weighted projective space of dimension* 4 *with* $-\frac{1}{2}K_{\mathbf{P}}$ *Cartier and such that the induced projective model*

$$\mathbf{P} \xrightarrow{|-\frac{1}{2}K_{\mathbf{P}}|} \mathbf{P}^N$$

is not a cone. Then it is one of the items of Table 2 below.

$\frac{1}{k_0}+\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}+\frac{1}{k_4}=2$	$\mathbf{P}$	$\frac{\sigma}{2}$
$\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{2}+\frac{1}{2}$	$\mathbf{P}(2,2,2,3,3)$	6
$\frac{1}{4}+\frac{1}{4}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$	$\mathbf{P}(1,1,2,2,2)$	4
$\frac{1}{6}+\frac{1}{3}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$	$\mathbf{P}(1,2,3,3,3)$	6

Table 2: The Gorenstein WPS of dimension 4 which are extensions of K3 surfaces.

Lemma 1.3.5. *For each Gorenstein weighted projective* 4*-space* $\mathbf{P}$ *given in Table 2 above, the line bundle* $\mathcal{O}_{\mathbf{P}}(-\frac{1}{2}K_{\mathbf{P}}) = \mathcal{O}_{\mathbf{P}}(\frac{\sigma}{2})$ *is very ample.*

We give a case-by-case proof. We recall that we use the following notation for brevity: the WPS $\mathbf{P}(1,1,1,3,3,3)$, for example, is denoted by $\mathbf{P}(1^3,3^3)$ since it involves the weights 1 and 3 three times each.

Proof: Recall from Lemma 1.1.7 the definition of the Veronese maps on weighted projective spaces. Each case of Table 2 above involves a Veronese embedding of $\mathbf{P}$ as a hypersurface in a weighted projective 5-space over which a very ample divisor is clearly identified.

- If $\mathbf{P} = \mathbf{P}(2^3,3^2)$, then the 2-Veronese map (as in Lemma 1.1.7) is an embedding

  $$v_2 : \mathbf{P} \hookrightarrow \mathbf{P}(1^3,3^3) =: W.$$

  If we set weighted coordinates $[\mathbf{u}:\mathbf{v}] = [u_0:u_1:u_2:\mathrm{v}_0:\mathrm{v}_1:\mathrm{v}_2]$ on $\mathbf{P}(1^3,3^3)$, then the image of this embedding is the hypersurface of equation

  $$\mathrm{v}_0\mathrm{v}_2 = \mathrm{v}_1^2.$$

  Moreover, the morphism induced by the linear system $|-\frac{1}{2}K_{\mathbf{P}}| = |\mathcal{O}_{\mathbf{P}}(6)|$ is the composition

  $$\mathbf{P} \xrightarrow{v_2} W \xrightarrow{|\mathcal{O}_W(3)|} \mathbf{P}^{12}.$$

  It follows that $-\frac{1}{2}K_{\mathbf{P}}$ is very ample from the very ampleness of $\mathcal{O}_W(3)$. Indeed, the morphism induced by the linear system $|\mathcal{O}_W(3)|$ is the 3-Veronese of W, and the isomorphic image of W in $\mathbf{P}^{12}$ is a cone over $v_3(\mathbf{P}^2) \subset \mathbf{P}^9$ with vertex a plane.
- If $\mathbf{P} = \mathbf{P}(1^2,2^3)$, the 2-Veronese map embeds $\mathbf{P}$ as a hypersurface in $\mathbf{P}^5$ of equation

  $$u_0u_2 = u_1^2$$

  with respect to the homogeneous coordinates $[u_0:u_1:u_2:u_3:u_4:u_5]$ on $\mathbf{P}^5$. The morphism induced by the linear system $|-\frac{1}{2}K_{\mathbf{P}}| = |\mathcal{O}_{\mathbf{P}}(4)|$ is the composition

  $$\mathbf{P} \xrightarrow{v_2} \mathbf{P}^5 \xrightarrow{|\mathcal{O}_{\mathbf{P}^5}(2)|} \mathbf{P}^{20},$$

  and the very ampleness of $-\frac{1}{2}K_{\mathbf{P}}$ comes from the fact that $\mathcal{O}_{\mathbf{P}^5}(2)$ is very ample.
- If $\mathbf{P} = \mathbf{P}(1,2,3^3)$, then the 3-Veronese map embeds $\mathbf{P}$ as a hypersurface in $\mathbf{P}(1^5,2) =: V$ whose equation is

  $$u_0v = u_1^3$$

  in the choice of coordinates $[\mathbf{u}:v] = [u_0:u_1:u_2:u_3:u_4:v]$ on V. The map induced by the linear system $|-\frac{1}{2}K_{\mathbf{P}}| = |\mathcal{O}_{\mathbf{P}}(6)|$ is the composition

  $$\mathbf{P} \xrightarrow{v_3} V \xrightarrow{|\mathcal{O}_V(2)|} \mathbf{P}^{15},$$

  where $\mathcal{O}_V(2)$ is very ample. Indeed, the morphism induced by $|\mathcal{O}_V(2)|$ is the 2-Veronese of V, whose image in $\mathbf{P}^{15}$ is a cone over $v_2(\mathbf{P}^4) \subset \mathbf{P}^{14}$ with vertex a point.

■

Now, for each space $\mathbf{P}$ of Table 2, let us identify the general linear section T of the projective model $\mathbf{P} \subset \mathbf{P}^{g+2}$ given by the linear system $|-\frac{1}{2}K_{\mathbf{P}}|$. By adjunction we have

$$K_T = (K_{\mathbf{P}} + T)|_T = \frac{1}{2}K_{\mathbf{P}}|_T,$$

so $T \subset \mathbf{P}^{g+1}$ is a Fano threefold embedded by its anticanonical divisor, hence an extension of K3 surfaces and of canonical curves of genus g.

Lemma 1.3.6. *Let $\mathbf{P}$ be one of the weighted projective spaces from Table 2, and T a general hyperplane section of the image of the embedding*

$$\mathbf{P} \xhookrightarrow{|-\frac{1}{2}K_{\mathbf{P}}|} \mathbf{P}^{g+2}$$

The nature of the threefold T and the value of g are as indicated in Table 3 below.

$\mathbf{P}$	T	g
$\mathbf{P}(2,2,2,3,3)$	quadric section of the cone over $v_3(\mathbf{P}^2) \subset \mathbf{P}^9$ with a line as vertex	10
$\mathbf{P}(1,1,2,2,2)$	$(2,2)$-complete intersection in $\mathbf{P}^5$	18
$\mathbf{P}(1,2,3,3,3)$	cubic of $\mathbf{P}^4$	13

Table 3: Projective models for T.

Proof: The values of g are known from Lemma 1.3.5. Indeed, in the proof we have made explicit the embedding induced by $-\frac{1}{2}K_{\mathbf{P}}$ and the dimension $g+2$ of the ambient space.

- If $\mathbf{P} = \mathbf{P}(2^3, 3^2)$, then T is the complete intersection in $\mathbf{P}(1^3, 3^3)_{[\mathbf{u}:\mathbf{v}]}$ given by the equation $\mathtt{v}_0\mathtt{v}_2 = \mathtt{v}_1^2$ and a general cubic. By generality, this cubic can be chosen to be $\mathtt{v}_2 = \phi_3(\mathbf{u}, \mathtt{v}_0, \mathtt{v}_1)$, where ϕ_3 has degree 3. Hence T is a sextic hypersurface of

$$\{\mathtt{v}_2 = \phi_3(\mathbf{u}, \mathtt{v}_0, \mathtt{v}_1)\} \simeq \mathbf{P}(1^3, 3, 3)_{[\mathbf{u}:\mathtt{v}_0:\mathtt{v}_1]}$$

i.e., a quadric section of the cone over $v_3(\mathbf{P}^2) \subset \mathbf{P}^9$ with a line as vertex.

- $\mathbf{P} = \mathbf{P}(1^2, 2^3)$ is isomorphic to a quadric of $\mathbf{P}^5$, according to Lemma 1.3.5. The threefold T is thus a complete intersection of two quadrics in in $\mathbf{P}^5$.
- $\mathbf{P} = \mathbf{P}(1, 2, 3^3)$ is isomorphic to a cubic hypersurface of $\mathbf{P}(1^5, 2)_{[\mathbf{u}:v]}$ according to Lemma 1.3.5. The threefold T is cut out on $\mathbf{P}$ by a quadric of $\mathbf{P}(1^5, 2)$, whose equation can be chosen to be

$$v = \phi_2(u_0, u_1, u_2, u_3)$$

so T is a cubic hypersurface of

$$\{v = \phi_2(u_0, u_1, u_2, u_3)\} \simeq \mathbf{P}^4_{[u_0:u_1:u_2:u_3]}.$$

■

Remark. The first item in Table 3 is a Fano threefold of genus 10, for which Theorem 1.2.12 fails and the maximal extendability of the general canonical curve in $\mathbf{P}$ is not known. In that case, T admits the structure of a conic bundle over $\mathbf{P}^2$ with two sections, so it is rational. The third item, a cubic threefold, is not rational.

Let now S be a general hyperplane section of $T \subset \mathbf{P}^{g+1}$. It follows from the above that S admits the following model. Moreover, the general hyperplane section Γ of $S \subset \mathbf{P}^g$ is a canonical curve of genus g.

$\mathbf{P}$	S
$\mathbf{P}(2,2,2,3,3)$	quadric section of the cone over $v_3(\mathbf{P}^2) \subset \mathbf{P}^9$ with a point as vertex i.e., double cover of $\mathbf{P}^2$ ramified over a sextic curve
$\mathbf{P}(1,1,2,2,2)$	$(2,2,2)$-complete intersection in $\mathbf{P}^5$
$\mathbf{P}(1,2,3,3,3)$	$(3,2)$-complete intersection in $\mathbf{P}^4$

Table 4: Projective models for S.

Remark. The two last items are K3 complete intersections in projective spaces. The only type of K3 complete intersections $S \subset \mathbf{P}^N$ which we do not obtain as linear sections of weighted projective 4-spaces is that of quartic hypersurfaces.

Lemma 1.3.7. *In each case of Table 4, let Γ denote a general member of $-\frac{1}{2}K_{\mathbf{P}}|_S$, in other words a general hyperplane section of $S \subset \mathbf{P}^g$.*

For $\mathbf{P} = \mathbf{P}(2^3, 3^2)$ the curve Γ is isomorphic to a smooth plane sextic.

In the last two cases, the divisor $-\frac{1}{4}K_{\mathbf{P}}|_S$ is Cartier on S. If C is a general member of $-\frac{1}{4}K_{\mathbf{P}}|_S$, then in S we have $\Gamma = 2C$ as Cartier divisors, and C is as follows.

$\mathbf{P}$	C
$\mathbf{P}(1, 1, 2, 2, 2)$	$(2, 2, 2)$-complete intersection in $\mathbf{P}^4$
$\mathbf{P}(1, 2, 3, 3, 3)$	$(3, 2)$-complete intersection in $\mathbf{P}^3$

Table 5: Projective models for C.

Proof: We consider first $\mathbf{P} = \mathbf{P}(2^3, 3^2)$ embedded in $\mathbf{P}^{12}$ by $-\frac{1}{2}K_{\mathbf{P}}$. The canonical curve $\Gamma \subset \mathbf{P}^9$ is a general hyperplane section of S. We know thanks to Table 4 that S is a quadric section of the cone over $v_3(\mathbf{P}^2)$. Hence Γ is a quadric section of $v_3(\mathbf{P}^2)$, in other words a plane sextic. Moreover Γ is smooth thanks to Bertini's theorem since it is a general hyperplane section of a K3 surface with canonical singularities.

Now we consider the two items from Table 5.

- For $\mathbf{P} = \mathbf{P}(1^2, 2^3)$, we have
$$-\frac{1}{4}K_{\mathbf{P}} = \mathcal{O}_{\mathbf{P}}(2),$$
which is Cartier.
We recall that S is a $(2, 2, 2)$-complete intersection in $\mathbf{P}^5$ and that Γ can be obtained as the intersection of S with a quadric of $\mathbf{P}^5$. Therefore C is the intersection of S with a hyperplane of $\mathbf{P}^5$, in other words a complete intersection of type $(2, 2, 2)$ in $\mathbf{P}^4$.
- For $\mathbf{P} = \mathbf{P}(1, 2, 3^3)$, we have
$$-\frac{1}{4}K_{\mathbf{P}} = \mathcal{O}_{\mathbf{P}}(3),$$
which is Cartier except at the point $[0 : 1 : 0 : 0 : 0]$. But S avoids this point by generality.
We recall that S is the intersection of a cubic and a general quadric of $\mathbf{P}^4$, and that Γ is the intersection of S with a quadric of $\mathbf{P}^4$. Therefore C is the intersection of S with a hyperplane of $\mathbf{P}^4$, in other words the intersection of a cubic with a general quadric of $\mathbf{P}^3$. ■

Remark. Let S be a K3 linear section of $\mathbf{P} = \mathbf{P}(2^3, 3^2)$. Then S is a double cover of $\mathbf{P}^2$ branched over a sextic curve. By the lemma above, the general hyperplane section of $S \subset \mathbf{P}^{10}$ is isomorphic to a sextic curve; a study of such double covers of the plane as extensions of canonical models of plane sextic curves can be found in [CD24].

Let Γ' be a smooth sextic curve given by the equation $\phi_6(\mathbf{u}) = 0$ on $\mathbf{P}^2_{[u_0:u_1:u_2]}$, with ϕ_6 a homogeneous sextic. One might wonder under which conditions Γ' is isomorphic to a hyperplane section of a K3 linear section S' of $\mathbf{P}$. Since S' is a sextic hypersurface of $\mathbf{P}(1^3, 3)_{[\mathbf{u}:v]}$ given by an equation of the form
$$v^2 + \psi_3(\mathbf{u})v + \psi_6(\mathbf{u}) = 0,$$
where ψ_3 and ψ_6 are fixed, respectively a cubic and a sextic of $\mathbf{P}^2$, the curve Γ' is a hyperplane section of S' iff. there exists a homogeneous cubic $\phi_3(\mathbf{u})$ such that
$$(v^2 + \psi_3(\mathbf{u})v + \psi_6(\mathbf{u}))|_{v=\phi_3(\mathbf{u})} = \phi_6(\mathbf{u}).$$
Hence ϕ_6 must be a sextic such that $\phi_6 - \psi_6 = (\phi_3 + \psi_3)\phi_3$ is reducible. More precisely, the plane sextic curve of equation $\phi_6 = \psi_6$ must have two cubic curves as irreducible components (one of which is the cubic $\phi_3 = 0$).

1.3.3 Dimension 5

We are now looking for Gorenstein weighted projective 5-spaces which are extensions of canonical curves. For such a space $\mathbf{P}$, by [CDS20, Theorem 5.1] we need to have $\frac{1}{3}K_{\mathbf{P}} \in \mathrm{Pic}(\mathbf{P})$. If the image of the morphism

$$\mathbf{P} \xrightarrow{|-\frac{1}{3}K_{\mathbf{P}}|} \mathbf{P}^N$$

is not a cone and admits a nonhyperelliptic curve $\Gamma \subset \mathbf{P}^{N-4}$ as a general linear section, by adjunction Γ is a canonical curve of genus $g = N - 3$.

A weighted projective 5-space $\mathbf{P} = \mathbf{P}(a_0, a_1, a_2, a_3, a_4, a_5)$ satisfies $\frac{1}{3}K_{\mathbf{P}} \in \mathrm{Pic}(\mathbf{P})$ iff. each weight a_i is a divisor of

$$\frac{\sigma}{3} = \frac{a_0 + a_1 + a_2 + a_3 + a_4 + a_5}{3}.$$

Thanks to Lemma 1.3.1 we are able to establish the full list of such spaces $\mathbf{P}$ through the use of Egyptian fractions. Namely, for all weights a_i we have $3a_ik_i = \sigma$ for some positive integer k_i, so we have an Egyptian fraction of the form

$$\frac{1}{k_0} + \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4} + \frac{1}{k_5} = 3.$$

We may assume that the largest summand is $\frac{1}{k_5}$, yielding $\frac{1}{k_5} \geq \frac{1}{2}$. Hence the only two possibilities are $k_5 = 1$ and $k_5 = 2$. If $k_5 = 1$, then we have

$$\frac{1}{k_0} + \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4} = 2$$

which encodes a Gorenstein weighted projective 4-space

$$\mathbf{P}' := \mathbf{P}(a_0, a_1, a_2, a_3, a_4) \subset \mathbf{P}$$

with $-\frac{1}{3}K_{\mathbf{P}}|_{\mathbf{P}'} = -\frac{1}{2}K_{\mathbf{P}'} \in \mathrm{Pic}(\mathbf{P}')$ by Lemma 1.3.1. In that case, $\mathbf{P}$ is of the form $\mathbf{P}(a_0, a_1, a_2, a_3, a_4, a_5)$ with $3a_5 = \sigma$ and $\frac{1}{3}K_{\mathbf{P}}$ is basepoint-free, and the image of

$$\mathbf{P} \xrightarrow{|-\frac{1}{3}K_{\mathbf{P}}|} \mathbf{P}^N$$

is a cone over the image of the embedding

$$\mathbf{P}' \xhookrightarrow{|-\frac{1}{2}K_{\mathbf{P}'}|} \mathbf{P}^{N-1}.$$

Since we are looking for extensions of canonical curves, these cases will not be useful to us, so we will only consider the cases where $k_5 = 2$.

Lemma 1.3.8. *Let $\mathbf{P}$ be a Gorenstein weighted projective 5-space such that $-\frac{1}{3}K_{\mathbf{P}}$ is Cartier. Under the condition that the image of the induced map*

$$\mathbf{P} \xrightarrow{|-\frac{1}{3}K_{\mathbf{P}}|} \mathbf{P}^N$$

is not a cone, then the only possibility is $\mathbf{P} = \mathbf{P}^5$. In particular, $-\frac{1}{3}K_{\mathbf{P}} = \mathcal{O}_{\mathbf{P}^5}(2)$ is very ample.

Proof: In the Egyptian fraction

$$\frac{1}{k_0} + \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4} + \frac{1}{k_5} = 3,$$

since

$$\frac{1}{k_5} = \frac{1}{2} = \frac{1}{6}\sum_{i=0}^{5} \frac{1}{k_i}$$

and $\frac{1}{k_5}$ is the largest of the six summands, then for all i we have $k_i = 2$, yielding $\sigma = 6$ and $a_i = 1$. ■

The associated projective model is the 2-Veronese of $\mathbf{P}^5$:

$$v_2 : \mathbf{P}^5 \to \mathbf{P}^{20}$$

whose general linear curve section is a canonical curve of genus 17, which can be seen as a complete intersection of type $(2, 2, 2, 2)$ in $\mathbf{P}^5$.

1.3.4 Dimension 6 or larger

Lastly, we are looking for weighted projective spaces $\mathbf{P}$ of dimension $d \geq 6$ such that $-\frac{1}{d-2}K_{\mathbf{P}}$ is Cartier. By Lemma 1.1.5, this amounts to the equality $(d-2)k_i a_i = \sigma$ with $k_i \in \mathbf{N}$, for each weight a_i. By Lemma 1.3.1 these are in a one-to-one correspondence with the Egyptian fractions of the form:

$$\frac{1}{k_0} + \cdots + \frac{1}{k_d} = d - 2.$$

Where $\frac{1}{k_d}$ is the largest term, so that $k_d \leq \frac{d+1}{d-2}$ with $d \geq 6$. Since $\frac{d+1}{d-2}$ is smaller than 2, we necessarily have $k_d = 1$ and $\mathbf{P}$ is a cone over a weighted projective space of dimension $d-1$.

Lemma 1.3.9. *Let $\mathbf{P}$ be a weighted projective space of dimension $d \geq 6$ for which $-\frac{1}{d-2}K_{\mathbf{P}}$ is Cartier. Then*

$$\mathbf{P} = \mathbf{P}(a_0, ..., a_d)$$

with $a_d = \frac{1}{d-2}\sigma$. In particular $-\frac{1}{d-2}K_{\mathbf{P}}$ is basepoint-free and the image of the morphism $\mathbf{P} \xrightarrow{|-\frac{1}{d-2}K_{\mathbf{P}}|} \mathbf{P}^N$ is a cone.

Proof: Let $\mathbf{P}' = \mathbf{P}(a_0, ..., a_{d-1})$ be the $(d-1)$-dimensional subspace of $\mathbf{P}$ given by the vanishing of the last coordinate of $\mathbf{P}$. Since in the Egyptian fraction

$$\frac{1}{k_0} + \cdots + \frac{1}{k_d} = d - 2,$$

the equality $k_d = 1$ holds (hence $a_d = \frac{\sigma}{(d-2)}$),

$$\frac{1}{k_0} + \cdots + \frac{1}{k_{d-1}} = d - 3,$$

which is the Egyptian fraction of d summands encoding $\mathbf{P}'$ via the correspondence given in Lemma 1.3.1.

Moreover, it follows by adjunction that $-\frac{1}{d-3}K_{\mathbf{P}'} = -\frac{1}{d-2}K_{\mathbf{P}}|_{\mathbf{P}'}$, and the morphism induced by the linear system $|-\frac{1}{2}K_{\mathbf{P}}| = |\mathcal{O}_{\mathbf{P}}(\frac{\sigma}{d-2})| = |\mathcal{O}_{\mathbf{P}}(a_d)|$ can be written as

$$[x_0 : \cdots x_d] \in \mathbf{P} \mapsto [\mathtt{f}_0 : \cdots : \mathtt{f}_s : x_d] \in \mathbf{P}^N,$$

where the $\mathtt{f}_i'$s form a basis of degree $\frac{\sigma}{d-2}$ polynomials in the variables $x_0, ..., x_{d-1}$. Hence the equations of the image are the algebraic relations between the $\mathtt{f}_i'$s and do not involve the linear form x_d, which means that the image is a cone. ■

1.4 K3 surfaces in weighted projective 3-spaces

Now we go back to the study of the weighted projective spaces of dimension 3 which are extensions of K3 surfaces. We know the full list of the Gorenstein weighted projective 3-spaces by Table 1. In each case, we know by Theorem 1.3.2 that $-K_{\mathbf{P}}$ is very ample. For each of those spaces $\mathbf{P}$, we investigate the anticanonical model

$$\mathbf{P} \xhookrightarrow{|-K_{\mathbf{P}}|} \mathbf{P}^N$$

with $g = N - 1$ the genus of the polarization $(\mathbf{P}, -K_{\mathbf{P}})$. The general hyperplane section S of $\mathbf{P}$ is a K3 surface, and the general hyperplane section Γ of S is a smooth canonical curve of genus g. Any extension of $\mathbf{P}$ is an extension of S and Γ by construction.

The extendability of $\mathbf{P}$ is bounded from above by the value of $\alpha(\Gamma, K_\Gamma)$. Indeed, in each case the genus of Γ is larger than 11 (see for instance [DS23, Table 2]), and by [De23, Corollary 2.8] we know that its Clifford index is larger than 2. Then Theorem 1.2.12 applies to the canonical curve $\Gamma \subset \mathbf{P}^{g-1}$, ensuring that Γ is $\alpha(\Gamma, K_\Gamma)$-extendable, and the extendability of the anticanonical model of $\mathbf{P}$ is at most $\alpha(\Gamma, K_\Gamma) - 2$:

$$\epsilon_{\mathbf{P}^{g+1}}(\mathbf{P}) \leq \alpha(\Gamma, K_\Gamma) - 2.$$

So $\mathbf{P} \subset \mathbf{P}^{g+1}$ can only be extended finitely many times, and any extension of dimension $1 + \alpha(\Gamma, K_\Gamma)$ is maximal. The Gorenstein weighted projective 3-spaces for which $\alpha(\Gamma, K_\Gamma) = 2$ are nonextendable.

T. Dedieu and E. Sernesi have computed the value of $\alpha(\Gamma, K_\Gamma)$ for each case: see [DS23, Proposition 3.2 & Table 3]. However, the construction of a maximal extension for the anticaonical model of $\mathbf{P}$ was not provided in all the cases.

In each case of Table 1 except the 5 items for which it was not known, we exhibit in the table below a maximal extension which has dimension $1 + \alpha(\Gamma, K_\Gamma)$.

$\mathbf{P}$	$g = g(\mathbf{P}, K_\mathbf{P})$	extendable?	maximal extension
$\mathbf{P}(1,1,1,1)$	33	no	itself
$\mathbf{P}(1,1,1,3)$	37	no	itself
$\mathbf{P}(1,1,4,6)$	37	no	itself
$\mathbf{P}(1,2,2,5)$	26	yes	sextic hypersurface of $\mathbf{P}(1^3, 3, 5^3)$
$\mathbf{P}(1,1,2,4)$	33	no	itself
$\mathbf{P}(1,3,4,4)$	19	yes	quartic hypersurface of $\mathbf{P}(1^4, 3^4)$
$\mathbf{P}(1,1,2,2)$	28	no	itself
$\mathbf{P}(2,3,3,4)$	13	yes	cubic hypersurface of $\mathbf{P}(1^5, 2^5)$
$\mathbf{P}(1,4,5,10)$	21	yes	was not known
$\mathbf{P}(1,2,6,9)$	28	yes	was not known
$\mathbf{P}(1,2,3,6)$	25	no	itself
$\mathbf{P}(1,3,8,12)$	25	yes	was not known
$\mathbf{P}(1,6,14,21)$	22	yes	was not known
$\mathbf{P}(2,3,10,15)$	16	yes	was not known

Table 6

The topic of §1.4.2 is to exhibit a maximal extension for the 5 remaining cases.

For each of them, we endow the general anticanonical divisor $S \subset \mathbf{P}$ with the polarization $-K_\mathbf{P}|_S$. Then $(S, -K_\mathbf{P}|_S)$ is a polarized K3 surface of genus g and index i_S, i.e.,

$$i_S = \max\left\{ r \in \mathbf{N} \text{ such that } -\frac{1}{r} K_\mathbf{P}|_S \in \mathrm{Pic}(S) \right\}.$$

Definition 1.4.1. *Let $\mathcal{K}_g^i$ be the moduli stack of the polarized surfaces (S, L) with S a K3 surface and L an ample Cartier divisor class on S such that the general member of $|L|$ is a curve of genus g, and the index of L in $\mathrm{Pic}(S)$ equals i.*

For any member (S, L) of $\mathcal{K}_g^i$ we consider the integer $\alpha(S, L)$ given in Definition 1.2.4. This is a map $\alpha : \mathcal{K}_g^i \to \mathbf{Z}$.

If $\mathbf{P}$ is a Gorenstein weighted projective space of dimension 3 and S is a general anticanonical divisor of $\mathbf{P}$, then the polarized surface $(S, -K_\mathbf{P}|_S)$ is a member of the moduli space $\mathcal{K}_g^{i_S}$. In each case of Table 6, T. Dedieu and E. Sernesi have computed in [DS23, Proposition 6.2] a constant $\alpha_g^{i_S}$ such that α takes the value $\alpha_g^{i_S}$ on a dense open subset of $\mathcal{K}_g^{i_S}$. The first 8 cases of Table 6 are those for which $\alpha(S, -K_\mathbf{P}|_S) = \alpha_g^{i_S}$, and we are going to examine the ones for which this equality doesn't hold.

Lemma 1.4.2 ([DS23], Proposition 6.2)**.** *Let $\mathbf{P}$ be one of the spaces from Table 6 and S a general hyperplane section of $\mathbf{P} \subset \mathbf{P}^{g+1}$, with $(S, -K_\mathbf{P}|_S) \in \mathcal{K}_g^{i_S}$. Assume that*

$$\alpha(S, -K_\mathbf{P}|_S) \neq \alpha_g^{i_S}.$$

Then $\mathbf{P}$ is one of the last six items of the list Table 6. Moreover, in these cases, for Γ a general hyperplane section of S, we have $\alpha(S, -K_\mathbf{P}|_S) = \alpha(\Gamma, K_\Gamma) - 1$.

We give the list of these spaces in the table below, together with the values of $\alpha_g^{i_S}$ and $\alpha(S, -K_\mathbf{P}|_S)$ for $S \in |-K_\mathbf{P}|$ general (known from [DS23, Proposition 6.2]).

$\mathbf{P}$	$\alpha(S, -K_{\mathbf{P}}\|_S)$	α_g^{is}
$\mathbf{P}(1,4,5,10)$	3	1
$\mathbf{P}(1,2,6,9)$	2	1
$\mathbf{P}(1,2,3,6)$	1	0
$\mathbf{P}(1,3,8,12)$	2	0
$\mathbf{P}(1,6,14,21)$	2	0
$\mathbf{P}(2,3,10,15)$	3	0

Table 7

Remark. In the last four cases, we have $\alpha(S, -K_{\mathbf{P}}|_S) \geq 1$ and $\alpha(S', L') = 0$ for the general member (S', L') of $\mathcal{K}_g^{is}$. These are polarized K3 hypersurfaces $(S, -K_{\mathbf{P}}|_S)$ which are extendable and can be deformed to nonextendable polarized K3 surfaces.

1.4.1 The birational models

Notice that 5 of the 6 weighted projective spaces listed in Table 7 are the items of Table 6 for which the construction of a maximal extension was not known (namely, all of them except $\mathbf{P}(1,2,3,6)$). Before constructing maximal extensions for these spaces, which is the topic of Theorem 1.4.5, we introduce for each of them a birational map $\varphi : \mathbf{P} \dashrightarrow \mathbf{P}'$ to another weighted projective 3-space $\mathbf{P}'$ such that φ restricts to an isomorphism on the general $S \in |-K_{\mathbf{P}}|$. This will allow us to express the general $\Gamma \in |-K_{\mathbf{P}}|_S|$ as a complete intersection of two surfaces of different degrees in $\mathbf{P}'$ and to construct extensions of Γ.

⋆ ⋆

The first step consists in the introduction of a suitable Veronese map $v_n : \mathbf{P} \hookrightarrow X$, where X is a weighted projective space of dimension 4, so that the anticanonical embedding

$$\mathbf{P} \xhookrightarrow{|-K_{\mathbf{P}}|} \mathbf{P}^{g+1}$$

factors as a composition of two maps

$$\mathbf{P} \xhookrightarrow{v_n} X \dashrightarrow \mathbf{P}^{g+1}$$

where $X \dashrightarrow \mathbf{P}^{g+1}$ is a rational map that we will specify and whose indeterminacy locus does not meet the image of v_n. Recall that σ denotes the sum of the weights of $\mathbf{P}$. Then the number n is chosen among the divisors of σ, yielding

$$\mathcal{O}_{\mathbf{P}}(-K_{\mathbf{P}}) = \mathcal{O}_{\mathbf{P}}(\sigma) = v_n^* \mathcal{O}_X\left(\frac{\sigma}{n}\right).$$

For the items of Table 7 a suitable choice of n is given in Table 8 below, realizing each time $\mathbf{P}$ as a hypersurface in a weighted projective space X of dimension 4. A choice of homogeneous coordinates on X is fixed each time, as well as a defining equation of the image $v_n(\mathbf{P}) \subset X$.

Remark. We also provide a Veronese embedding for $\mathbf{P} = \mathbf{P}(1,2,3,6)$, although this space is nonextendable in its anticanonical model. This embedding will be of use in Theorem 1.4.20 where we will study the geometry of curves inside the general $S \in |-K_{\mathbf{P}}|$.

$\mathbf{P}$	σ	n	X	$v_n(\mathbf{P}) \subset X$	$-K_{\mathbf{P}}$
$\mathbf{P}(1,4,5,10)$	20	5	$\mathbf{P}(1,1,1,2,4)_{[u_0:u_1:u_2:v:s]}$	quintic $(u_0 s = u_1^5)$	$v_5^*\mathcal{O}_X(4)$
$\mathbf{P}(1,2,6,9)$	18	2	$\mathbf{P}(1,1,3,5,9)_{[u_0:u_1:v:s:t]}$	10-ic $(u_0 t = s^2)$	$v_2^*\mathcal{O}_X(9)$
$\mathbf{P}(1,2,3,6)$	12	2	$\mathbf{P}(1,1,2,3,3)_{[u_0:u_1:v:s_0:s_1]}$	quartic $(u_0 s_0 = v^2)$	$v_2^*\mathcal{O}_X(6)$
$\mathbf{P}(1,3,8,12)$	24	3	$\mathbf{P}(1,1,3,4,8)_{[u_0:u_1:v:s:t]}$	9-ic $(u_0 t = v^3)$	$v_3^*\mathcal{O}_X(8)$
$\mathbf{P}(1,6,14,21)$	42	7	$\mathbf{P}(1,1,2,3,6)_{[u_0:u_1:v:s:t]}$	heptic $(u_0 t = u_1^7)$	$v_7^*\mathcal{O}_X(6)$
$\mathbf{P}(2,3,10,15)$	30	3	$\mathbf{P}(1,2,4,5,10)_{[u:v:s:t:r]}$	12-ic $(vr = s^3)$	$v_3^*\mathcal{O}_X(10)$

Table 8

By a divisibility criterion, we can check fairly easily that $\mathcal{O}_X(\frac{\sigma}{n})$ is not always basepoint-free. This criterion is purely numerical: $\mathcal{O}_X(\frac{\sigma}{n})$ is basepoint-free if and only if $\frac{\sigma}{n}$ is divisible by all the weights of X. Namely, it is not the case for $\mathbf{P}(1,2,6,9)$, $\mathbf{P}(1,3,8,12)$ and $\mathbf{P}(2,3,10,15)$, for which the induced map $X \dashrightarrow \mathbf{P}^{g+1}$ is nonregular.

To be exhaustive, let us list down below an expression of each Veronese embedding $\mathbf{P} \hookrightarrow X$ from Table 8.

- For $\mathbf{P} = \mathbf{P}(1,4,5,10)_{[x:y:z:w]}$, the 5-Veronese is
$$[x:y:z:w] \in \mathbf{P} \mapsto [x^5 : xy : z : w : y^5] \in X = \mathbf{P}(1^3, 2, 4).$$
- For $\mathbf{P} = \mathbf{P}(1,2,6,9)_{[x:y:z:w]}$, the 2-Veronese is
$$[x:y:z:w] \in \mathbf{P} \mapsto [x^2 : y : z : xw : w^2] \in X = \mathbf{P}(1,1,3,5,9).$$
- For $\mathbf{P} = \mathbf{P}(1,2,3,6)_{[x:y:z:w]}$, the 2-Veronese is
$$[x:y:z:w] \in \mathbf{P} \mapsto [x^2 : y : xz : z^2 : w] \in X = \mathbf{P}(1,1,2,3,3).$$
- For $\mathbf{P} = \mathbf{P}(1,3,8,12)_{[x:y:z:w]}$, the 3-Veronese is
$$[x:y:z:w] \in \mathbf{P} \mapsto [x^3 : y : xz : w : z^3] \in X = \mathbf{P}(1,1,3,4,8).$$
- For $\mathbf{P} = \mathbf{P}(1,6,14,21)_{[x:y:z:w]}$, the 7-Veronese is
$$[x:y:z:w] \in \mathbf{P} \mapsto [x^7 : xy : z : w : y^7] \in X = \mathbf{P}(1,1,2,3,6).$$
- For $\mathbf{P} = \mathbf{P}(2,3,10,15)_{[x:y:z:w]}$, the 3-Veronese is
$$[x:y:z:w] \in \mathbf{P} \mapsto [y : x^3 : xz : w : z^3] \in X = \mathbf{P}(1,2,4,5,10).$$

⋆ ⋆

Given an extendable Gorenstein weighted projective 3-space $\mathbf{P}$ from Table 7, the method that we will be using consists in exhibiting a birational map from $\mathbf{P}$ to another 3-dimensional weighted projective space $\mathbf{P}'$ which realizes S as a nongeneral anticanonical divisor of $\mathbf{P}'$, and the general linear curve section Γ of S as a complete intersection of two surfaces of different degrees in $\mathbf{P}'$.

In each case, we have specified in Table 8 a Veronese embedding $\mathbf{P} \hookrightarrow X$, where X is a weighted projective space of dimension 4. Each time, except for $\mathbf{P}(1,2,3,6)$ (which is the only nonextendable item of Table 7), the image of X in $\mathbf{P}^{g+1}$ is a cone with a point as vertex. This follows from the fact that $\frac{\sigma}{n}$ equals the largest weight of X; say $X = \mathbf{P}(d_0, d_1, d_2, d_3, d_4)$ with $\frac{\sigma}{n} = d_4$. Then the map given by the linear system $|\mathcal{O}_X(\frac{\sigma}{n})|$ is the following.

$$[x_0 : x_1 : x_2 : x_3 : x_4] \in X \mapsto [\mathtt{f}_0 : \mathtt{f}_1 : \cdots : \mathtt{f}_s : x_4]$$

where the $\mathtt{f}_i$'s form a basis of the degree $\frac{\sigma}{n}$ homogeneous polynomials in the variables x_0, x_1, x_2 and x_3. Hence the equations for the image of X encode the algebraic relations between the $\mathtt{f}_i$'s and do not involve x_4.

This suggests projecting from the vertex point of this cone to $\mathbf{P}^g$. This projection restricts to the identity on S by generality, since $S \subset \mathbf{P}^g$ is a hyperplane section of $\mathbf{P}$ which does not contain the vertex point.

This projection restricted to $\mathbf{P}$ is a birational map $\varphi : \mathbf{P} \dashrightarrow \mathbf{P}'$, where $\mathbf{P}'$ is the weighted projective 3-space whose weights are those of X but the last one. In other words, if $X = \mathbf{P}(d_0, d_1, d_2, d_3, d_4)$ with $d_4 = \frac{\sigma}{n}$, then $\mathbf{P}' = \mathbf{P}(d_0, d_1, d_2, d_3)$. The restriction of this map to the general $S \in |-K_{\mathbf{P}}|$ is an isomorphism; the image of S is K3 so it is a (nongeneral) anticanonical divisor of $\mathbf{P}'$. We denote by $\mathcal{L}$ the (noncomplete) linear system whose members are the anticanonical divisors of $\mathbf{P}'$ which are the direct images of all $D \in |-K_{\mathbf{P}}|$,
$$\mathcal{L} = \varphi(|-K_{\mathbf{P}}|) \subset |-K_{\mathbf{P}'}|,$$

so that the surface $\varphi(S)$ is a general member of $\mathcal{L}$.

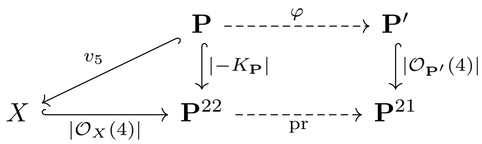
Since $S \simeq \varphi(S)$ we will drop the notation $\varphi(S)$ for the sake of brevity and write S instead. Likewise, we will refer to $\varphi(\Gamma)$ as Γ. As our computations will show, the restriction of $\mathcal{L}$ to S has $|\mathcal{O}_{\mathbf{P}'}(\frac{\sigma}{n})|_S|$ as its mobile part. That way, Γ is cut out on $\mathbf{P}'$ by two equations of different degrees.

Example. Recall from Lemma 1.1.10 that the map on $\mathbf{P} = \mathbf{P}(1, 4, 5, 10)$ given by the following expression

$$\varphi : [x : y : z : w] \in \mathbf{P} \mapsto [x^5 : xy : z : w] \in \mathbf{P}' = \mathbf{P}(1, 1, 1, 2)$$

is birational. The map φ is the projection from the vertex point of $X \subset \mathbf{P}^{22}$, where $X = \mathbf{P}(1, 1, 1, 2, 4)$ is embedded by the linear system of its quartics, and $\mathbf{P}$ is embedded in X via its 5-Veronese map. Let S be a general anticanonical divisor of $\mathbf{P}$, i.e., given by an equation of degree 20 on $\mathbf{P}$.

In this case, the birational map φ is given in coordinates by all the monomials of the 5-Veronese map of $\mathbf{P}$, except y^5. We recall that the 5-Veronese map $v_5 : \mathbf{P} \hookrightarrow X$ satisfies $-K_{\mathbf{P}} = v_5^* \mathcal{O}_X(4)$, and φ is in the following commutative diagram



(1.1)

in which pr is the projection map from the vertex point of the cone X onto $\mathbf{P}'$. However, in $\mathbf{P}'$, the surface S (which is a hyperplane section of $\mathbf{P} \subset \mathbf{P}^{22}$) can not be described by an equation of degree 4, since the monomial y^5 is not the pullback of a quartic of $\mathbf{P}'$ by φ.

The surface S is general in the basepoint-free linear system $|-K_{\mathbf{P}}|$, so it avoids the indeterminacy point of φ. Its image being a K3 surface, it is an anticanonical divisor of $\mathbf{P}'$, i.e., a quintic surface in $\mathbf{P}(1, 1, 1, 2)$. Using the description of φ in homogeneous coordinates, we see that S in $\mathbf{P}(1, 1, 1, 2)$ has equation

$$u_0 f_4(\mathbf{u}, v) + u_1^5 = 0 \tag{1.2}$$

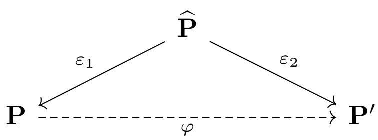
with f_4 a general homogeneous polynomial of degree 4 in the variables $\mathbf{u} = (u_0, u_1, u_2)$ and v. Indeed, such an equation pulls back to an equation on $\mathbf{P}$ of the form

$$x^5 f_{20}(x, y, z, w) = 0,$$

where f_{20} is a general 20-ic on $\mathbf{P}$. The fact that f_{20} is general follows from the fact that the only missing monomial of degree 20 on $\mathbf{P}$ which is missing from $\varphi^* H^0(\mathbf{P}', \mathcal{O}_{\mathbf{P}'}(4))$ is y^5, which can be recovered from the equality

$$\varphi^* u_1^5 = x^5 y^5.$$

In other words, the pullback to $\mathbf{P}$ of $S \subset \mathbf{P}'$ via φ is $S + (x^5)$ where the locus $x = 0$ is contracted by φ. We have a commutative diagram



where ε_1 and ε_2 are weighted blow-ups, and $\varepsilon^{-1}(\{x = 0\})$ is the exceptional divisor of ε_2 (see Lemma 1.1.10).

Therefore, $\mathcal{L} = \varphi(|-K_{\mathbf{P}}|) \subset |-K_{\mathbf{P}'}|$ consists of the quintic surfaces of the form

$$u_0 f_4(\mathbf{u}, v) + \lambda u_1^5 = 0,$$

with $\deg(f_4) = 4$ and $\lambda \in \mathbf{C}$. The surface S being general in $\mathcal{L}$, λ is non zero and up to scaling, we may assume $\lambda = 1$ as in (1.2). The base locus of $\mathcal{L}$ is the curve $\Delta := \{u_0 = u_1 = 0\}$.

Lemma 1.4.3. *In the case of $\mathbf{P} = \mathbf{P}(1,4,5,10)$, given a general $S \in |-K_{\mathbf{P}}|$, the general $\Gamma \in |-K_{\mathbf{P}}|_S|$ is cut out on S in $\mathbf{P}'$ by a general quartic.*

Proof: Let S' be another general member of $\mathcal{L}$, i.e., the image via φ of a general anticanonical divisor of $\mathbf{P}$, which is the zero locus

$$u_0 f_4'(\mathbf{u}, v) + u_1^5 = 0,$$

with f_4' a homogeneous quartic. Then

$$S \cap S' = \{u_0 f_4 + u_1^5 = u_0 f_4' + u_1^5 = 0\} = S \cap \{u_0(f_4 - f_4') = 0\},$$

and $f_4 - f_4'$ is a general quartic of $\mathbf{P}'$. This shows that the restriction $\mathcal{L}|_S$ has

$$S \cap \{u_0 = 0\} = \Delta$$

as its fixed part, and its mobile part is $|\mathcal{O}_{\mathbf{P}'}(4)|_S|$. Thus the map from S given by the restriction of $\mathcal{L}$ is the same map as the one induced by the quartics of $\mathbf{P}'$,

$$S \xrightarrow{|\mathcal{O}_{\mathbf{P}'}(4)|_S|} \mathbf{P}^{21}$$

so that the curve Γ is the pullback to S of a hyperplane of $\mathbf{P}^{21}$, as indicated in the diagram (1.1). Hence, Γ is cut out on S by a general quartic of $\mathbf{P}'$. ■

In conclusion, the curve Γ is a $(5,4)$-complete intersection in $\mathbf{P}'$ given by two equations of the form $u_0 f_4(\mathbf{u}, v) + u_1^5 = g_4(\mathbf{u}, v) = 0$, with g_4 a general quartic on $\mathbf{P}'$.

⋆ ⋆

In each case from Table 7 except $\mathbf{P}(1,2,3,6)$, for which there exists no extension, we can apply a similar method as in the example above, yielding a birational map $\varphi : \mathbf{P} \dashrightarrow \mathbf{P}'$ to a weighted projective 3-space, and a description of Γ as a complete intersection of two different degrees in $\mathbf{P}'$. This description will allow us to construct a maximal extension of Γ which is also an extension of the anticanonical model of $\mathbf{P}$. All the needed pieces of information are listed in Table 9 and Table 10 below. The proof that φ is birational in each case follows the toric method, involving the decomposition of $\mathbf{P}$ via two weighted blow-ups $\widehat{\mathbf{P}} \to \mathbf{P}$ and $\widehat{\mathbf{P}} \to \mathbf{P}'$:

$$\begin{array}{ccc} & \widehat{\mathbf{P}} & \\ \swarrow & & \searrow \\ \mathbf{P} & \overset{\varphi}{\dashrightarrow} & \mathbf{P}' \end{array}$$

as does the proof of Lemma 1.1.10.

Lemma 1.4.4. *In each case, the birational map $\varphi : \mathbf{P} \dashrightarrow \mathbf{P}'$ is an isomorphism from the general anticanonical surface $S \subset \mathbf{P}$ onto its image.*

Proof: The image of $\mathbf{P}$ in $\mathbf{P}^{g+1}$ lies inside X, which is a cone over $\mathbf{P}'$ with vertex a point. The birational map $\varphi : \mathbf{P} \to \mathbf{P}'$ is the restriction to $\mathbf{P}$ of the projection $X \dashrightarrow \mathbf{P}'$ from the vertex point; this is an isomorphism on the hyperplane section $S \subset \mathbf{P}$ as soon as the vertex point of the cone X does not belong to S. ■

Now, we list information about the birational maps $\varphi : \mathbf{P} \to \mathbf{P}'$, including the expression of φ in coordinates for each case and its indeterminacy point p.

$\mathbf{P}$	$\mathbf{P}'$	expression of φ	p
$\mathbf{P}(1,4,5,10)$	$\mathbf{P}(1,1,1,2)_{[u_0:u_1:u_2:v]}$	$[x^5 : xy : z : w]$	p_y
$\mathbf{P}(1,2,6,9)$	$\mathbf{P}(1,1,3,5)_{[u_0:u_1:v:s]}$	$[x^2 : y : z : xw]$	p_w
$\mathbf{P}(1,3,8,12)$	$\mathbf{P}(1,1,3,4)_{[u_0:u_1:v:s]}$	$[x^3 : y : xz : w]$	p_z
$\mathbf{P}(1,6,14,21)$	$\mathbf{P}(1,1,2,3)_{[u_0:u_1:v:s]}$	$[x^7 : xy : z : w]$	p_y
$\mathbf{P}(2,3,10,15)$	$\mathbf{P}(1,2,4,5)_{[u:v:s:t]}$	$[y : x^3 : xz : w]$	p_z

Table 9

Each time, we denote by $\mathcal{L} = \varphi(|-K_{\mathbf{P}}|) \subset |-K_{\mathbf{P}'}|$ the noncomplete linear system whose members are the direct images $\varphi(D)$ of all the anticanonical divisors $D \in |-K_{\mathbf{P}}|$. For the general member $S \in |-K_{\mathbf{P}}|$, we display a defining equation for its isomorphic image in $\mathbf{P}'$ (denoting by f_d a general degree d homogeneous polynomial according to the grading of $\mathbf{P}'$). The base locus of $\mathcal{L}$ is denoted by Δ: it is also the fixed component of $\mathcal{L}|_S$.

$\mathbf{P}$	equation for S in $\mathbf{P}'$	Δ	Γ
$\mathbf{P}(1,4,5,10)$	$u_0 f_4(\mathbf{u}, v) + u_1^5 = 0$	$u_0 = u_1 = 0$	$S \cap$ quartic
$\mathbf{P}(1,2,6,9)$	$u_0 f_9(\mathbf{u}, v, s) + s^2 = 0$	$u_0 = s = 0$	$S \cap$ 9-ic
$\mathbf{P}(1,3,8,12)$	$u_0 f_8(\mathbf{u}, v, s) + v^3 = 0$	$u_0 = v = 0$	$S \cap$ 8-ic
$\mathbf{P}(1,6,14,21)$	$u_0 f_6(\mathbf{u}, v, s) + u_1^7 = 0$	$u_0 = u_1 = 0$	$S \cap$ sextic
$\mathbf{P}(2,3,10,15)$	$v f_{10}(u, v, s, t) + s^3 = 0$	$v = s = 0$	$S \cap$ 10-ic

Table 10

Comment. In each case, the expression of φ in coordinates $[x : y : z : w]$ is the same as that of the n-Veronese map v_n from $\mathbf{P}$ (as listed in Table 8), minus the last monomial of v_n. We have an inclusion

$$\varphi^* H^0\left(\mathbf{P}', \mathcal{O}_{\mathbf{P}'}\left(\frac{\sigma}{n}\right)\right) \subset H^0(\mathbf{P}, -K_{\mathbf{P}})$$

which is strict, so the image of the general anticanonical surface $S \subset \mathbf{P}$ via φ cannot be described by an equation of degree $\frac{\sigma}{n}$ on $\mathbf{P}'$. The linear system $\varphi^*|\mathcal{O}_{\mathbf{P}'}(\frac{\sigma}{n})|$ is a proper subsystem of $[-K_{\mathbf{P}}|$ whose members are all the anticanonical divisors of $\mathbf{P}$ through the point p. Moreover, note that S is a K3 surface in $\mathbf{P}'$, hence an anticanonical surface of $\mathbf{P}'$, whereas

$$\mathcal{O}_{\mathbf{P}'}\left(\frac{\sigma}{n}\right) \neq -K_{\mathbf{P}'}.$$

Each equation indicated above for the hypersurface $S \subset \mathbf{P}'$ is indeed an equation of the same degree as $\deg(-K_{\mathbf{P}'})$ with respect to the grading of $\mathbf{P}'$.

Focusing again on the case $\mathbf{P} = \mathbf{P}(1,4,5,10)$, with the birational map

$$\varphi : [x : y : z : w] \in \mathbf{P} \mapsto [x^5 : xy : z : w] \in \mathbf{P}' = \mathbf{P}(1^3, 2),$$

we have $\frac{\sigma}{n} = \frac{20}{5} = 4$, while the anticanonical surfaces in $\mathbf{P}'$ have degree 5.

- On the one hand, if we consider a general member of $|\mathcal{O}_{\mathbf{P}'}(4)|$, then its proper transform via φ is an anticanonical divisor of $\mathbf{P}$ passing through the point p_y.
- On the other hand, the general anticanonical surface $S \in |-K_{\mathbf{P}}|$ is isomorphic to its image via φ (and we allow ourselves to write S instead of $\varphi(S)$ for said image), so that S is a member of $|\mathcal{O}_{\mathbf{P}'}(5)|$. It cannot be *any* member of the linear linear system $|\mathcal{O}_{\mathbf{P}'}(5)|$ though, since it contains the curve Δ (which is the indeterminacy locus of the rational inverse φ^{-1}), and $\varphi^* S$ contains S as an irreducible component.
- The general $\Sigma \in |\mathcal{O}_{\mathbf{P}'}(5)|$ meets the indeterminacy locus Δ of φ^{-1} in 3 points. The proper transform of Σ via φ is the blow-up of Σ at these 3 points.

The motivation for considering φ is that the weighted projective space $\mathbf{P}'$ contains more K3 surfaces which may extend the curve Γ than $\mathbf{P}$ does, hence it is a necessary framework for the construction of a maximal extension of Γ.

As a side note, for this particular example, the general anticanonical surface $S \subset \mathbf{P}$ is realized as a *special* anticanonical surface of the weighted projective space $\mathbf{P}' = \mathbf{P}(1^3, 2)$. This space is isomorphic to the cone over the 2-Veronese of $\mathbf{P}^2$, through the following embedding:

$$\mathbf{P}' \xrightarrow{|\mathcal{O}_{\mathbf{P}'}(2)|} \mathbf{P}^6.$$

Since S is a quintic surface of $\mathbf{P}'$, one may consider a sextic of $\mathbf{P}'$, in other words a cubic section of the cone over $v_2(\mathbf{P}^2)$, containing S. This cubic section contains two irreducible components in $\mathbf{P}^6$, which are S and a quadric cone.

We didn't come up with a similar interpretation of $S \subset \mathbf{P}'$ as above for the other items of Tables 9 and 10.

1.4.2 Maximal extensions

For each case of Table 7, the goal is now to provide the construction of a maximal extension of $\mathbf{P}$ in its anticanonical model. This is done thanks to the datum of the birational model

$$\varphi : \mathbf{P} \dashrightarrow \mathbf{P}'$$

which is specified in Tables 9 and 10.

Theorem 1.4.5. *Each Gorenstein weighted projective 3-space of Table 7 in its anticanonical model admits a maximal extension Y which is described in the table below. Moreover, for a general linear curve section Γ of $\mathbf{P}$, we indicate whenever Y has been identified as the universal extension of Γ, in the sense of Theorem 1.2.12.*

$\mathbf{P}$	Y	$\dim(Y)$	*universal extension of* Γ?
$\mathbf{P}(1,4,5,10)$	*nongeneral quintic of* $\mathbf{P}(1^3,2,4^3)$	5	yes
$\mathbf{P}(1,2,6,9)$	*nongeneral* 10*-ic of* $\mathbf{P}(1^2,3,5,9^2)$	4	yes
$\mathbf{P}(1,2,3,6)$	$\mathbf{P}(1,2,3,6)$	3	yes
$\mathbf{P}(1,3,8,12)$	*nongeneral* 9*-ic of* $\mathbf{P}(1^2,3,4,8^2)$	4	yes
$\mathbf{P}(1,6,14,21)$	*nongeneral heptic of* $\mathbf{P}(1^2,2,3,6^2)$	4	unknown
$\mathbf{P}(2,3,10,15)$	*codim.* 2 *complete intersection in a* $\mathbf{P}(1^2,2,3,5^3)-$ *bundle over* $\mathbf{P}^1$	5	unknown

Table 11: Maximal extensions.

We know from Table 7 that the Gorenstein WPS $\mathbf{P} = \mathbf{P}(1,2,3,6)$ is not extendable in its anticanonical model by Theorem 1.2.12, as the corank of the Gauß-Wahl map of its general linear curve section Γ equals 2.

We provide a case-by-case proof of Theorem 1.4.5, constructing the maximal extensions Y for all the extendable spaces. In the last two cases of the list, we didn't manage to check whether Y is the universal extension of Γ.

⋆ ⋆

$\mathbf{P}(1,2,3,6)$ is the universal extension of its general linear curve section

The anticanonical model of $\mathbf{P} = \mathbf{P}(1,2,3,6)$ is embedded by the linear system of its hypersurfaces of degree 12 (or as we call them, 12-ics):

$$\mathbf{P} \xhookrightarrow{|\mathcal{O}_{\mathbf{P}}(12)|} \mathbf{P}^{26}.$$

In this model, the curve Γ is given by two equations of degree 12 in $\mathbf{P}$:

$$f_{12}(x,y,z,w) = g_{12}(x,y,z,w) = 0,$$

where f_{12} and g_{12} are general homogeneous polynomials of degree 12. The pencil of hyperplane sections $S \subset \mathbf{P}$ containing Γ consists of members of the form

$$S_{[\lambda:\mu]} = \{\lambda f_{12}(x,y,z,w) + \mu g_{12}(x,y,z,w) = 0\}$$

for $[\lambda : \mu] \in \mathbf{P}^1$.

Lemma 1.4.6. *The anticanonical model of* $\mathbf{P}$ *contains no cone over* Γ *as a hyperplane section.*

Proof: Assume by contradiction that there exists a member of the pencil $S_{[\lambda:\mu]}$ which is a cone in $\mathbf{P}^{25}$. Then it contains a line, i.e., a curve $\ell \subset S_{[\lambda:\mu]}$ such that $\ell \cdot \mathcal{O}_{\mathbf{P}}(12) = 1$. Hence we have $\ell \cdot \mathcal{O}_{\mathbf{P}}(6) = \frac{1}{2}$, which is not possible, since $\mathcal{O}_{\mathbf{P}}(6)$ is Cartier. ■

Corollary 1.4.7. *The anticanonical model of* $\mathbf{P}$ *is the universal extension of* Γ.

Proof: Since $\mathbf{P} \subset \mathbf{P}^{26}$ is a maximal extension of the canonical curve Γ, and since it contains no cone of Γ as a hyperplane section, we may conclude that it is the universal extension of Γ by Lemma 1.2.13. ■

A maximal extension of $\mathbf{P}(1,4,5,10)$

According to Table 9 and Table 10, the general linear curve section Γ of $\mathbf{P} = \mathbf{P}(1,4,5,10)$ is cut out on $\mathbf{P}(1,1,1,2)$ with homogeneous coordinates $[u_0 : u_1 : u_2 : v]$ by the equations

$$u_0 f_4(\mathbf{u}, v) + u_1^5 = g_4(\mathbf{u}, v) = 0,$$

where f_4 and g_4 are general homogeneous quartic polynomials. Consider then the equation

$$u_0 s_0 + u_1 s_1 + u_2 s_2 = u_1^5,$$

where s_0, s_1 and s_2 are coordinates of weight 4. This defines a quintic hypersurface Y in $\mathbf{X} = \mathbf{P}(1^3, 2, 4^3)$.

Lemma 1.4.8. *The variety Y has a model in $\mathbf{P}^{24}$ which is a maximal extension of $\mathbf{P}$, i.e., it has dimension $5 = 1 + \alpha(\Gamma, K_\Gamma)$ (according to Lemma 1.4.2), contains $\mathbf{P}$ as a linear section, and is not a cone.*

Proof: Consider the linear system $|\mathcal{O}_{\mathbf{X}}(4)|$, which is very ample (the induced morphism is the 4-Veronese of $\mathbf{X}$) and realizes Y in $\mathbf{P}^{24}$ as a variety of degree $\mathcal{O}_{\mathbf{X}}(5) \cdot \mathcal{O}_{\mathbf{X}}(4)^5 = \frac{4^5 \times 5}{4^3 \times 2} = 40 = 2g - 2$. This model is an extension of Γ, as

$$Y \cap \{s_0 = -f_4(\mathbf{u}, v),\ s_1 = s_2 = g_4(\mathbf{u}, v) = 0\} = \Gamma.$$

Moreover, we have by adjunction $K_\Gamma = \mathcal{O}_{\mathbf{X}}(4)|_\Gamma$, so the linear section $\Gamma \subset Y$ is the canonical model of Γ, as required.

The fivefold Y is a maximal extension of Γ by Lemma 1.4.2 since it has dimension $1 + \alpha(\Gamma, K_\Gamma)$ and it contains $\mathbf{P} = \mathbf{P}(1,4,5,10)$ as a 3-fold linear section. Indeed, as indicated in Table 8, $\mathbf{P}$ embeds into $\mathbf{P}(1,1,1,2,4)$ as the quintic hypersurface $u_0 s_0 = u_1^5$. It follows that Y is an extension of $\mathbf{P}$, as

$$\mathbf{P} = Y \cap \{s_1 = s_2 = 0\}.$$

Now we assume by contradiction that Y is a cone in $\mathbf{P}^{24}$. Then it admits a linear surface section which is a cone over Γ. But such a cone in Y does not exist, by Lemma 1.4.9 below. ■

Letting $(\lambda_0, \lambda_1, \lambda_2)$ move in $\mathbf{C}^3$, we get a family of K3 surfaces

$$Y \cap \{s_0 = \lambda_0 g_4(\mathbf{u}, v) - f_4(\mathbf{u}, v),\ s_1 = \lambda_1 g_4(\mathbf{u}, v),\ s_2 = \lambda_2 g_4(\mathbf{u}, v)\}$$

which are all linear sections of Y and contain Γ as a hyperplane section. Indeed, the curve Γ is cut out on all of them by $\{g_4(\mathbf{u}, v) = 0\}$. Among them, those that are members of the linear system $\mathcal{L} = \varphi(|-K_{\mathbf{P}}|)$ are those parametrized by $\lambda_1 = \lambda_2 = 0$ (see Tables 9 and 10).

We make the remark that the surface extensions of Γ which are not members of $\mathcal{L}$ could not be obtained from $\mathbf{P}$, but are hyperplane sections of other threefold extensions of Γ. Moreover the family above is only parametrized by the affine space $\mathbf{C}^3$, whereas the family of surface extensions of Γ in its universal extension is parametrized by $\mathbf{P}^3$.

Lemma 1.4.9. *The variety Y in $\mathbf{P}^{24}$ does not contain any cone over Γ as a surface linear section. Therefore, it is the universal extension of Γ.*

Proof: We identify the total family of surface extensions of Γ inside Y via the following method: the equations which cut out Γ from Y being

$$s_0 + f_4(\mathbf{u}, v) = s_1 = s_2 = g_4(\mathbf{u}, v) = 0,$$

any surface extension of Γ which is a linear section of $Y \subset \mathbf{P}^{24}$ is given by three linear combinations of the equations above. In other words, any surface extension of Γ inside Y is given by a set of four equations in $\mathbf{X} = \mathbf{P}(1^3, 2, 4^3)_{[u_0:u_1:u_2:v:s_0:s_1:s_2]}$ as follows:

$$\lambda_0(s_0 + f_4(\mathbf{u}, v)) + \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 g_4(\mathbf{u}, v) = 0, \tag{1.3}$$

$$\mu_0(s_0 + f_4(\mathbf{u}, v)) + \mu_1 s_1 + \mu_2 s_2 + \mu_3 g_4(\mathbf{u}, v) = 0, \tag{1.4}$$

$$\eta_0(s_0 + f_4(\mathbf{u}, v)) + \eta_1 s_1 + \eta_2 s_2 + \eta_3 g_4(\mathbf{u}, v) = 0, \tag{1.5}$$

$$u_0s_0 + u_1s_1 + u_2s_2 = u_1^5, \tag{1.6}$$

where the three vectors $(\lambda_0, \lambda_1, \lambda_2, \lambda_3), (\mu_0, \mu_1, \mu_2, \mu_3)$ and $(\eta_0, \eta_1, \eta_2, \eta_3)$ are linearly independent.

Assume by contradiction that there exists such a linear surface section S which is a cone over Γ, given by the four equations (1.3), (1.4), (1.5) and (1.6). In particular, S contains a line $\ell \subset \mathbf{X}$ such that $\ell \cdot \mathcal{O}_{\mathbf{X}}(4) = 1$ and $\ell \cdot \mathcal{O}_{\mathbf{X}}(2) = \frac{1}{2}$. Let $U \subset \mathbf{X}$ denote the locus where $\mathcal{O}_{\mathbf{X}}(2)$ is locally free: it is the complement in $\mathbf{X}$ of the vanishing locus

$$u_0 = u_1 = u_2 = v = 0,$$

i.e., U is the complement of the weighted projective subspace spanned by the three points p_{s_0}, p_{s_1} and p_{s_2}.

- If there exists a line $\ell \subset S$ such that $\ell \subset U$, then $\mathcal{O}_{\mathbf{X}}(2)$ is Cartier on ℓ, which yields a contradiction with $\ell \cdot \mathcal{O}_{\mathbf{X}}(2) = \frac{1}{2}$.
- If all the lines $\ell \subset S$ meet $\mathbf{X} - U$, then the vertex point of the cone S belongs to $\mathbf{X} - U$. Such a point is of the form

  $$[0 : 0 : 0 : 0 : s_0 : s_1 : s_2]$$

  with respect to the coordinates $[u_0 : u_1 : u_2 : v : s_0 : s_1 : s_2]$ on $\mathbf{X}$. Up to a change of the coordinates s_0, s_1, s_2, we may assume that the vertex point is

  $$p_{s_2} = [0 : 0 : 0 : 0 : 0 : 0 : 1].$$

  In other words, there exists a change of variables of the form

  $$[s_0 : s_1 : s_2] \mapsto A \cdot [s_0 : s_1 : s_2],$$

  with

  $$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \in \mathrm{GL}_3(\mathbf{C}),$$

  which moves the vertex point of S to p_{s_2}. Such a linear transformation of the coordinates s_i applied to the equations (1.3), (1.4), (1.5) and (1.6) yields

  $$\begin{array}{rcl} 0 & = & \lambda_0(a_{00}s_0 + a_{01}s_1 + a_{02}s_2) \\ & + & \lambda_1(a_{10}s_0 + a_{11}s_1 + a_{12}s_2) \\ & + & \lambda_2(a_{20}s_0 + a_{21}s_1 + a_{22}s_2) \\ & + & \lambda_0 f_4 + \lambda_3 g_4 \end{array} \tag{1.7}$$

  $$\begin{array}{rcl} 0 & = & \mu_0(a_{00}s_0 + a_{01}s_1 + a_{02}s_2) \\ & + & \mu_1(a_{10}s_0 + a_{11}s_1 + a_{12}s_2) \\ & + & \mu_2(a_{20}s_0 + a_{21}s_1 + a_{22}s_2) \\ & + & \mu_0 f_4 + \mu_3 g_4 \end{array} \tag{1.8}$$

  $$\begin{array}{rcl} 0 & = & \eta_0(a_{00}s_0 + a_{01}s_1 + a_{02}s_2) \\ & + & \eta_1(a_{10}s_0 + a_{11}s_1 + a_{12}s_2) \\ & + & \eta_2(a_{20}s_0 + a_{21}s_1 + a_{22}s_2) \\ & + & \eta_0 f_4 + \eta_3 g_4 \end{array} \tag{1.9}$$

  $$\begin{array}{rcl} 0 & = & u_0(a_{00}s_0 + a_{01}s_1 + a_{02}s_2) \\ & + & u_1(a_{10}s_0 + a_{11}s_1 + a_{12}s_2) \\ & + & u_2(a_{20}s_0 + a_{21}s_1 + a_{22}s_2) \\ & - & u_1^5. \end{array} \tag{1.10}$$

  Since $S|_{s_2=1}$ is an affine cone, the ideal $I(S)$ of S in the ring $\mathbf{C}[u_0, u_1, u_2, v, s_0, s_1, s_2]$ admits four generators (namely, three of degree 4 and one of degree 5) whose restrictions to $s_2 = 1$ are homogeneous. Note that in the above, the equation (1.10) has degree 5, while (1.7), (1.8), (1.9) have degree 4. As a consequence, as generators of the ideal $I(S)$ not involving the variable s_2, there are

  (i) three homogeneous quartics which are linear combinations of (1.7), (1.8) and (1.9) with constant coefficients,

  (ii) one homogeneous quintic which is a combination of (1.10) with (1.7), (1.8) and (1.9).

  Taking a linear combination as (i) suggests, up to a change of notation for coefficients λ_i, μ_i, η_i, we obtain three equations of the same form as (1.7), (1.8) and (1.9) which do not involve s_2.

  Then the condition (ii) implies $a_{02} = a_{12} = a_{22} = 0$, which is not possible.

The conclusion follows that Y is the universal extension of the curve Γ, by Lemma 1.2.13. ■

A maximal extension of $\mathbf{P}(1,2,6,9)$

By Table 9 and Table 10, the general linear curve section Γ of $\mathbf{P} = \mathbf{P}(1,2,6,9)$ is given in $\mathbf{P}(1,1,3,5)$ by the following two equations

$$u_0 f_9(\mathbf{u}, v, s) + s^2 = g_9(\mathbf{u}, v, s) = 0,$$

where f_9 and g_9 are general homogeneous polynomials of degree 9. Adding two coordinates t_0 and t_1 of weight 9, we consider the 10-ic hypersurface Y in $\mathbf{X} = \mathbf{P}(1^2, 3, 5, 9^2)$ given by the equation

$$u_0 t_0 + u_1 t_1 = s^2.$$

Lemma 1.4.10. *The variety Y has a model in $\mathbf{P}^{30}$ which is a maximal extension of $\mathbf{P}$.*

Proof: The linear system $|\mathcal{O}_{\mathbf{X}}(9)|$ has one base point in $\mathbf{X}$ but its restriction to Y defines an embedding which realizes Y as a projective variety in $\mathbf{P}^{30}$ of degree $\mathcal{O}_{\mathbf{X}}(10)\cdot\mathcal{O}_{\mathbf{X}}(9)^4 = \frac{9^4\times 10}{9^2\times 5\times 3} = 54 = 2g-2$. It has dimension $4 = 1+\alpha(\Gamma, K_\Gamma)$ by Lemma 1.4.2 and contains Γ as a linear section in $\mathbf{P}^{27}$:

$$Y \cap \{t_0 = -f_9(\mathbf{u}, v, s),\ t_1 = g_9(\mathbf{u}, v, s) = 0\} = \Gamma.$$

Besides, Y is not a cone, by the same arguments as those mentioned in the proof of Lemma 1.4.8. Hence the fourfold Y is a maximal extension of Γ.

Recall from Table 8 that $\mathbf{P}$ is the hypersurface of $\mathbf{P}(1,1,3,5,9)$ given by the equation $u_0 t_0 = s^2$. It follows that $\mathbf{P}$ is a linear section of Y, namely:

$$\mathbf{P} = Y \cap \{t_1 = 0\}.$$

■

In particular,

$$Y \cap \{t_0 = \lambda_0 g_9(\mathbf{u}, v, s) - f_9(\mathbf{u}, v, s),\ t_1 = \lambda_1 g_9(\mathbf{u}, v, s)\}$$

describes a family of K3 surfaces in Y indexed by $(\lambda_0, \lambda_1) \in \mathbf{C}^2$ which all contain Γ as a hyperplane section. Among them, the members of $\mathcal{L} = \varphi(|-K_{\mathbf{P}}|)$ are the ones for which $\lambda_1 = 0$ (see Tables 9 and 10).

Lemma 1.4.11. *The variety Y in $\mathbf{P}^{30}$ is the universal extension of Γ.*

The proof of this lemma follows a similar argument as the proof of Lemma 1.4.9. From the equations of Γ in $\mathbf{X} = \mathbf{P}(1^2, 3, 5, 9^2)_{[\mathbf{u}:v:s:\mathbf{t}]}$:

$$u_0 t_0 + u_1 t_1 - s^2 = t_0 + f_9(\mathbf{u}, v, s) = t_1 = g_9(\mathbf{u}, v, s) = 0,$$

the first one of which being the equation of Y in $\mathbf{X}$, we identify the whole family of surface extensions of Γ in Y. This family is parametrized by $\mathbf{P}^2$ and it consists of all the intersections of Y with the vanishing locus of any two independent linear combinations of

$$t_0 + f_9(\mathbf{u}, v, s),\ t_1 \text{ and } g_9(\mathbf{u}, v, s).$$

One shows by the same argument as Lemma 1.4.9 that this family contains no cone, and we may conclude that Y is the universal extension of Γ by Lemma 1.2.13.

A maximal extension of $\mathbf{P}(1,3,8,12)$

By Table 9 and Table 10, the general linear curve section Γ of $\mathbf{P} = \mathbf{P}(1,3,8,12)$ is given in $\mathbf{P}(1,1,3,4)$ by the following equations

$$u_0 f_8(\mathbf{u}, v, s) + v^3 = g_8(\mathbf{u}, v, s) = 0,$$

where f_8 and g_8 are general homogeneous polynomials of degree 8. After adding two coordinates t_0 and t_1 of weight 8, we consider the 9-ic hypersurface Y in $\mathbf{X} = \mathbf{P}(1^2, 3, 4, 8^2)$ of equation

$$u_0 t_0 + u_1 t_1 = v^3.$$

Lemma 1.4.12. *The variety Y has a model in $\mathbf{P}^{27}$ which is a maximal extension of $\mathbf{P}$.*

Proof: It is embedded in $\mathbf{P}^{27}$ by the restriction of the linear system $|\mathcal{O}_{\mathbf{X}}(8)|$. This model has degree $\mathcal{O}_{\mathbf{X}}(9)\cdot\mathcal{O}_{\mathbf{X}}(8)^4 = \frac{8^4\times 9}{8^2\times 4\times 3} = 46 = 2g-2$ and dimension $4 = 1+\alpha(\Gamma, K_\Gamma)$ by Lemma 1.4.2 and contains Γ as a linear section:

$$Y \cap \{t_0 = -f_8(\mathbf{u}, v, s),\ t_1 = g_8(\mathbf{u}, v, s) = 0\} = \Gamma.$$

Hence it is a maximal extension of Γ. It is also an extension of $\mathbf{P}$; indeed, we know from Table 8 that $\mathbf{P}$ is the hypersurface $u_0t_0 = v^3$ in $\mathbf{P}(1,1,3,4,8)$. This exhibits $\mathbf{P}$ as a hyperplane section of Y, namely

$$\mathbf{P} = Y \cap \{t_1 = 0\}.$$

The fact that Y is not a cone can be proven in the same way as in Lemma 1.4.8. ■

Letting (λ_0, λ_1) move in $\mathbf{C}^2$, we get a family

$$Y \cap \{t_0 = \lambda_0 g_8(\mathbf{u}, v, s) - f_8(\mathbf{u}, v, s),\ t_1 = \lambda_1 g_8(\mathbf{u}, v, s)\}$$

of K3 surfaces in Y which contain Γ as a hyperplane section. The surfaces in this family that are members of $\mathcal{L} = \varphi(|-K_{\mathbf{P}}|)$ are the ones for which $\lambda_1 = 0$ (see Tables 9 and 10).

Lemma 1.4.13. *The variety Y in $\mathbf{P}^{27}$ is the universal extension of Γ.*

The proof of this lemma follows a similar argument as the proof of Lemma 1.4.9. From the equations of Γ in $\mathbf{X} = \mathbf{P}(1^2, 3, 4, 8^2)_{[\mathbf{u}:v:s:\mathbf{t}]}$:

$$u_0t_0 + u_1t_1 - v^3 = t_0 + f_8(\mathbf{u}, v, s) = t_1 = g_8(\mathbf{u}, v, s) = 0,$$

the first one of which being the equation of Y in $\mathbf{X}$, we identify the whole family of surface extensions of Γ in Y. This family is parametrized by $\mathbf{P}^2$ and it consists of all the intersections of Y with the vanishing locus of any two independent linear combinations of

$$t_0 + f_8(\mathbf{u}, v, s),\ t_1 \text{ and } g_8(\mathbf{u}, v, s).$$

One shows by the same argument as Lemma 1.4.9 that this family contains no cone, and we may conclude that Y is the universal extension of Γ by Lemma 1.2.13.

A maximal extension of $\mathbf{P}(1, 6, 14, 21)$

By Table 9 and Table 10, the general linear curve section Γ of $\mathbf{P} = \mathbf{P}(1, 6, 14, 21)$ is given in $\mathbf{P}(1, 1, 2, 3)$ by the following equations

$$u_0 f_6(\mathbf{u}, v, s) + u_1^7 = g_6(\mathbf{u}, v, s) = 0,$$

with f_6 and g_6 general homogeneous polynomials of degree 6. Adding two coordinates t_0 and t_1 of weight 6, we consider the heptic hypersurface Y in $\mathbf{X} = \mathbf{P}(1^2, 2, 3, 6^2)$ given by

$$u_0t_0 + u_1t_1 = u_1^7.$$

Lemma 1.4.14. *The variety Y has a model in $\mathbf{P}^{24}$ which is a maximal extension of $\mathbf{P}$.*

Proof: It is embedded in $\mathbf{P}^{24}$ by restriction of the linear system $|\mathcal{O}_{\mathbf{X}}(6)|$. This model has degree $\mathcal{O}_{\mathbf{X}}(7)\cdot\mathcal{O}_{\mathbf{X}}(6)^4 = \frac{6^4\times 7}{6^2\times 3\times 2} = 42 = 2g-2$ and contains Γ as a linear section:

$$Y \cap \{t_0 = -f_6(\mathbf{u}, v, s),\ t_1 = g_6(\mathbf{u}, v, s) = 0\} = \Gamma.$$

Besides, Y also contains $\mathbf{P}$ as a hyperplane section: recall from Table 8 that $\mathbf{P}$ is the hypersurface $u_0t_0 = u_1^7$ in $\mathbf{P}(1, 1, 2, 3, 6)$. As a consequence, we have the hyperplane section

$$Y \cap \{t_1 = 0\} = \mathbf{P}.$$

Now we assume by contradiction that Y is a cone over $\mathbf{P}$. By the equality above, we may assume up to a change of variables that the vertex of this cone is the point p_{t_1}. In other words, there must exist a change of variables on $\mathbf{X} = \mathbf{P}(1^2, 2, 3, 6^2)_{[\mathbf{u}:v:s:\mathbf{t}]}$ of the form

$$[\mathbf{u} : v : s : \mathbf{t}] \mapsto [\mathbf{u} : v : s : at_0 + bt_1 + \alpha(\mathbf{u}, v, s) : ct_0 + dt_1 + \beta(\mathbf{u}, v, s)],$$

where α and β are homogeneous sextics in $(\mathbf{u}, v, s)$ and $ad \neq bc$, which eliminates the coordinate t_1 from the equation of Y:

$$u_0t_0 + u_1t_1 = u_1^7,$$

i.e., makes the affine chart $Y|_{t_1=1}$ an affine cone. But such a change of variables does not exist. We may thus conclude that Y is a maximal extension of $\mathbf{P}$ and Γ. ■

Letting (λ_0, λ_1) move in $\mathbf{C}^2$, we have a family

$$Y \cap \{t_0 = \lambda_0 g_6(\mathbf{u}, v, s) - f_6(\mathbf{u}, v, s),\ t_1 = \lambda_1 g_6(\mathbf{u}, v, s)\}$$

of K3 surfaces in Y which are extensions of Γ. Those surfaces members of $\mathcal{L} = \varphi(|-K_{\mathbf{P}}|)$ are the ones for which $\lambda_1 = 0$. Let $\lambda = (\lambda_0, \lambda_1)$, then the surface given by the intersection above is the following hypersurface in $\mathbf{P}' = \mathbf{P}(1, 1, 2, 3)$:

$$S_\lambda = \left\{u_1^7 + u_0 f_6(\mathbf{u}, v, s) = (\lambda_0 u_0 + \lambda_1 u_1) g_6(\mathbf{u}, v, s)\right\},$$

so that $\Gamma = S_\lambda \cap \{g_6(\mathbf{u}, v, s) = 0\}$.

The question arises whether there exists a cone over Γ as a linear surface section of Y. If the answer is no, then Y is the universal extension of Γ by (Lemma 1.2.13).

However, the argument used in Lemma 1.4.9 doesn't apply here, since a curve $\ell \subset S_\lambda$ such that $\ell \cdot \mathcal{O}_{\mathbf{P}'}(6) = 1$ could pass through the base points of $\mathcal{O}_{\mathbf{P}'}(2)$ and $\mathcal{O}_{\mathbf{P}'}(3)$, allowing $\ell \cdot \mathcal{O}_{\mathbf{P}'}(2) = \frac{1}{3}$ and $\ell \cdot \mathcal{O}_{\mathbf{P}'}(3) = \frac{1}{2}$.

An alternative birational model for $\mathbf{P}(2, 3, 10, 15)$ and two nonmaximal extensions

Lastly, we want to construct a maximal extension for the anticanonical model of $\mathbf{P} = \mathbf{P}(2, 3, 10, 15)$. Unlike the previous cases, the birational model indicated in Tables 9 and 10 will only allow us to construct a nonmaximal extension of $\mathbf{P}$. We haven't come up with an explanation of why the method which worked in all other cases (namely, add coordinates to $\mathbf{P}' = \mathbf{P}(1, 2, 4, 5)$ to extend the defining equations of Γ in $\mathbf{P}'$) wasn't sufficient here to obtain a maximal extension.

We will introduce another birational model for $\mathbf{P}$ than the model indicated in Tables 9 and 10. We will then have two birational models, which will allow us to construct two nonmaximal extensions Y_1, Y_2 of the anticanonical model $\mathbf{P} \subset \mathbf{P}^{17}$. Each of these two extensions is a fourfold in $\mathbf{P}^{18}$. The data of the Y_i's will make it possible to construct a maximal extension Y, which is a fivefold in $\mathbf{P}^{19}$.

Idea. Let us consider a projective space $\mathbf{P}^{19}$ with a fixed subspace $\Lambda = \mathbf{P}^{17}$ such that Λ contains the anticanonical model of $\mathbf{P}$. Then we may consider two hyperplanes H_1, H_2 of $\mathbf{P}^{19}$ both containing Λ and such that H_1 contains Y_1 and H_2 contains Y_2, with Y_1 and Y_2 the two nonmaximal extensions of $\mathbf{P}$ mentioned above. The fourfolds Y_1 and Y_2 generate a pencil of fourfolds sections in a fivefold Y which we will exhibit as a maximal extension of $\mathbf{P}$.

According to the information displayed in Tables 9 and 10, the general linear curve section Γ of the anticanonical model $\mathbf{P} \subset \mathbf{P}^{17}$ is given in $\mathbf{P}(1, 2, 4, 5)$ by the equations

$$v f_{10}(u, v, s, t) + s^3 = g_{10}(u, v, s, t) = 0,$$

with f_{10} and g_{10} general homogeneous polynomials of degree 10. After adding two coordinates r_0 and r_1 of weight 10 we construct an extension of Γ as the hypersurface

$$u^2 r_0 + v r_1 = s^3$$

which we denote by Y_1, in $\mathbf{X} = \mathbf{P}(1, 2, 4, 5, 10^2)$.

Lemma 1.4.15. *The variety Y_1 has a model in $\mathbf{P}^{18}$ which is an extension of $\mathbf{P}$.*

Proof: It is embedded in $\mathbf{P}^{18}$ by the restriction of the linear system $|\mathcal{O}_{\mathbf{X}}(10)|$. This model has degree $\mathcal{O}_{\mathbf{X}}(12) \cdot \mathcal{O}_{\mathbf{X}}(10)^4 = \frac{10^4 \times 12}{10^2 \times 5 \times 4 \times 2} = 30 = 2g - 2$ and contains Γ as a linear section:

$$Y_1 \cap \{r_1 = -f_{10}(u,v,s,t),\ r_0 = g_{10}(u,v,s,t) = 0\} = \Gamma.$$

In accordance with the equation $vr_1 = s^3$ which is given for $\mathbf{P}$ in Table 8 as a hypersurface in $\mathbf{P}(1,2,4,5,10)$, one checks that $Y_1 \cap \{r_0 = 0\} = \mathbf{P}$. We also check that Y_1 is not a cone over $\mathbf{P}$, by a similar argument as in Lemma 1.4.14. ■

However, Y_1 has dimension 4, while $1 + \alpha(\Gamma, K_\Gamma) = 5$, so this extension of $\mathbf{P}$ is a nonmaximal extension of the curve Γ.

⋆ ⋆

Now we introduce an alternative birational model for $\mathbf{P}$. This will allow us to construct another four dimensional nonmaximal extension Y_2 of $\mathbf{P}$. The data of these two nonmaximal extensions Y_1, Y_2 will make it possible to construct a maximal extension of Γ and $\mathbf{P}$.

First, we construct the nonmaximal extension Y_2. Introducing homogeneous coordinates $[u' : v' : s' : t']$ on the weighted projective space $\mathbf{P}(1,3,5,9)$, consider the following rational map ψ from $\mathbf{P}$ to $\mathbf{P}(1,3,5,9)$

$$\psi : [x : y : z : w] \in \mathbf{P} \mapsto [u' : v' : s' : t'] = [x : y^2 : z : yw] \in \mathbf{P}(1,3,5,9).$$

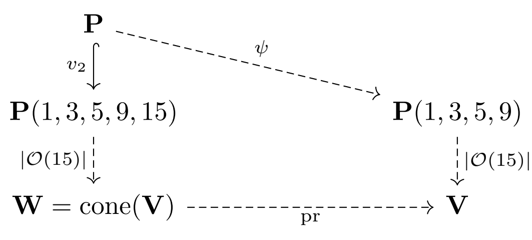
The expression of ψ in homogeneous coordinates is obtained from the 2-Veronese map v_2 on $\mathbf{P}$,

$$[x : y : z : w] \mapsto [x : y^2 : z : yw : w^2],$$

by removing the last monomial w^2. This construction is similar to that of φ displayed in Table 9, which was obtained from the 3-Veronese map v_3. For the general linear curve section of the anticanonical model of $\mathbf{P}$, the datum of this birational map will provide surface extensions which are neither anticanonical surfaces of $\mathbf{P}$ anticanonical surfaces of $\mathbf{P}' = \mathbf{P}(1,2,4,5)$. These surfaces which we could not recover from $\mathbf{P}$ and $\varphi : \mathbf{P} \dashrightarrow \mathbf{P}'$ are necessary in the construction of a maximal extension.

Lemma 1.4.16. *The map ψ is birational and it restricts to an isomorphism on the general anticanonical divisor of $\mathbf{P}$.*

The proof that ψ is birational consists in a resolution of the indeterminacy point of ψ, as was done in the proof of Lemma 1.1.10. A similar argument applies to ψ as the one which was used to show that $\varphi : \mathbf{P} \dashrightarrow \mathbf{P}(1,2,4,5)$ induces an isomorphism on the general $S \in |-K_{\mathbf{P}}|$. It revolves around the following commutative diagram.



Here, v_2 is the 2-Veronese map from $\mathbf{P}$ to $\mathbf{P}(1,3,5,9,15)$, $-K_{\mathbf{P}} = v_2^* \mathcal{O}_{\mathbf{P}(1,3,5,9,15)}(15)$, $\mathbf{W}$ is a cone over $\mathbf{V}$ with vertex a point and pr is the projection map from the vertex point of $\mathbf{W}$ onto $\mathbf{V}$.

A consequence of this is that S can be realized as a nongeneral anticanonical divisor of $\mathbf{P}(1,3,5,9)$ with equation

$$v' f_{15}(u', v', s', t') + t'^2 = 0,$$

where f_{15} is a homogeneous polynomial of degree 15. One checks that the pullback to $\mathbf{P}$ via ψ of such a hypersurface is $S + (y^2)$, and the locus $y = 0$ is contracted by ψ.

In $\mathbf{P}(1,3,5,9)$, the curve Γ is cut out on S by a general hypersurface of degree 15, as the diagram above shows. Hence Γ is given in $\mathbf{P}(1,3,5,9)$ by the following equations

$$v'f_{15}(u',v',s',t') + t'^2 = g_{15}(u',v',s',t') = 0$$

where g_{15} is a general homogeneous polynomial of degree 15. Let's add two coordinates r_0' and r_1' of weight 15 and examine the hypersurface Y_2 in $\mathbf{X}' = \mathbf{P}(1,3,5,9,15^2)$ given by the equation

$$u'^3 r_0' + v' r_1' = t'^2.$$

Lemma 1.4.17. *The variety Y_2 has a model in $\mathbf{P}^{18}$ which is also an extension of $\mathbf{P}$.*

Proof: It has dimension 4 and is embedded in $\mathbf{P}^{18}$ by restriction of $|\mathcal{O}_{\mathbf{X}'}(15)|$. This model contains Γ as a linear section; indeed, given two constants λ_0 and λ_1 :

$$Y_2 \cap \left\{ \begin{array}{l} r_0' = \lambda_0 g_{15}(u',v',s',t') \\ r_1' = \lambda_1 g_{15}(u',v',s',t') - f_{15}(u',v',s',t') \\ g_{15}(u',v',s',t') = 0 \end{array} \right\} = \Gamma.$$

This is an extension of $\mathbf{P}$ as well. Indeed, we have $Y_2 \cap \{r_0' = 0\} = \left\{v'r_1' = t'^2\right\} = \mathbf{P}$ in $\mathbf{P}(1,3,5,9,15)$. We also check that Y_2 is not a cone over $\mathbf{P}$, by a similar argument as in Lemma 1.4.14. ∎

However, this extension of Γ has dimension 4, so it is nonmaximal.

A maximal extension of $\mathbf{P}(2,3,10,15)$

Now that we have constructed in Lemma 1.4.15 and Lemma 1.4.17 two fourfold extensions Y_1 and Y_2 of Γ, we construct a fivefold extension Y of $\mathbf{P}$, such that Y contains both Y_1 and Y_2 as hyperplane sections in $\mathbf{P}^{19}$. This construction involves a weighted projective bundle over $\mathbf{P}^1$, i.e., a quotient of a vector bundle with a weighted projective space as fibre.

The construction of Y will require a realization of Y_1 and Y_2 as complete intersections in $\mathbf{P}(1^2,2,3,5^3)$. Note that the image of the 6-Veronese map v_6 of $\mathbf{P}(2,3,10,15)$ lies in $\mathbf{P}(1^2,2,3,5^2)$, so one might think that it could be possible to recover Y_1 and Y_2 from v_6. However, all our attempts in trying so have been unsuccessful.

On the one hand, Y_1 is the 12-ic hypersurface in $\mathbf{X} = \mathbf{P}(1,2,4,5,10^2)_{[u:v:s:t:r_0:r_1]}$ of equation

$$u^2 r_0 + v r_1 = s^3.$$

On the other hand, Y_2 is the 18-ic hypersurface in $\mathbf{X}' = \mathbf{P}(1,3,5,9,15^2)_{[u':v':s':t':r_0':r_1']}$ given by the equation

$$u'^3 r_0' + v' r_1' = t'^2.$$

Both $\mathbf{X}$ and $\mathbf{X}'$ can be embedded in $\mathbf{P}(1^2,2,3,5^3)$ by the following Veronese maps.

$$(v_2)_{\mathbf{X}} : \begin{cases} \mathbf{X} = \mathbf{P}(1,2,4,5,10^2) & \longrightarrow \mathbf{P}(1^2,2,3,5^3) \\ [u:v:s:t:r_0:r_1] & \longmapsto [u^2 : v : s : ut : t^2 : r_1 : r_0]. \end{cases}$$

$$(v_3)_{\mathbf{X}'} : \begin{cases} \mathbf{X}' = \mathbf{P}(1,3,5,9,15^2) & \longrightarrow \mathbf{P}(1^2,2,3,5^3) \\ [u':v':s':t':r_0':r_1'] & \longmapsto [v' : u'^3 : u's' : t' : r_1' : s'^3 : r_0']. \end{cases}$$

We may choose $[U_0 : U_1 : V : W : X_0 : X_1 : X_2]$ as homogeneous coordinates on $\mathbf{P}(1^2,2,3,5^3)$, whose pullbacks by the Veronese maps are

	U_0	U_1	V	W	X_0	X_1	X_2
pullback to $\mathbf{X}$	u^2	v	s	ut	t^2	r_1	r_0
pullback to $\mathbf{X}'$	v'	u'^3	$u's'$	t'	r_1'	s'^3	r_0'

Hence the above realizes $\mathbf{X}$ (respectively $\mathbf{X}'$) as the hypersurface of equation $U_0X_0 = W^2$ (respectively $U_1X_1 = V^3$). The descriptions we know for Y_1 and Y_2 in $\mathbf{X}$ and $\mathbf{X}'$ yield

$$Y_1 = \left\{ \begin{array}{l} U_0X_0 = W^2 \\ U_1X_1 + U_0X_2 = V^3 \end{array} \right\}$$

and

$$Y_2 = \left\{ \begin{array}{l} U_0X_0 + U_1X_2 = W^2 \\ U_1X_1 = V^3 \end{array} \right\}.$$

Besides, we know from Lemma 1.4.15 and Lemma 1.4.17 that $\mathbf{P}$ is cut out on Y_1 and Y_2 by the same equation, namely: $Y_1 \cap \{X_2 = 0\} = Y_2 \cap \{X_2 = 0\} = \mathbf{P}$. In particular, $\mathbf{P}$ in $\{X_2 = 0\}$ is given by the equations

$$U_0X_0 = W^2, \ \ U_1X_1 = V^3. \tag{1.11}$$

We introduce now two coordinates λ, μ and consider $F = \mathrm{Proj}(R)$ with

$$R = \mathbf{C}[\lambda, \mu, U_0, U_1, V, W, X_0, X_1, X_2]$$

endowed with the following grading in $\mathbf{Z}^2$:

	λ	μ	U_0	U_1	V	W	X_0	X_1	X_2
degree	1	1	0	0	0	0	0	0	-1
in $\mathbf{Z}^2$:	0	0	1	1	2	3	5	5	5

It is a bundle over $\mathbf{P}^1$ with fiber $\mathbf{P}(1^2, 2, 3, 5^3)$, with coordinates $[\lambda : \mu]$ on the base $\mathbf{P}^1$, and the locus $X_2 = 0$ is the trivial subbundle $\mathbf{P}^1 \times \mathbf{P}(1^2, 2, 3, 5^2)$.

There is a morphism $\phi : F \to \mathbf{P}(1^2, 2, 3, 5^4)$ which is given by the folllowing expression: for any point $p = F$ of coordinates $[\lambda : \mu : U_0 : U_1 : V : W : X_0 : X_1 : X_2]$, one has

$$\phi(p) = [U_0 : U_1 : V : W : X_0 : X_1 : \lambda X_2 : \mu X_2].$$

The map $F \dashrightarrow \mathbf{P}^{19}$ induced by the linear system $|\mathcal{O}_F(0, 5)|$ is the composition

$$F \xrightarrow{\ \phi\ } \mathbf{P}(1^2, 2, 3, 5^4) \overset{|\mathcal{O}(5)|}{\dashrightarrow} \mathbf{P}^{19}.$$

Notice that ϕ contracts the trivial bundle $\mathbf{P}^1 \times \mathbf{P}(1^2, 2, 3, 5^2)$ given by the equation $X_2 = 0$ onto $\mathbf{P}(1^2, 2, 3, 5^2)$. Hence the image of $\{X_2 = 0\}$ by $|\mathcal{O}_F(0, 5)|$ is the image of $\mathbf{P}(1^2, 2, 3, 5^2)$ in $\mathbf{P}^{17}$.

Consider the complete intersection Z in F given by the two homogeneous equations

$$U_0X_0 + \lambda U_1X_2 = W^2,$$
$$U_1X_1 + \mu U_0X_2 = V^3.$$

Lemma 1.4.18. *The image Y of Z in $\mathbf{P}^{19}$ is not a cone and contains Y_1 and Y_2 as hyperplane sections. By Lemma 1.4.2, it has dimension $1 + \alpha(\Gamma, K_\Gamma)$, and thus it is a maximal extension of $\mathbf{P}$.*

Proof: The restriction of Z to $\{X_2 = 0\}$ is the complete intersection in $\mathbf{P}^1 \times \mathbf{P}(1^2, 2, 3, 5^2)$ defined by the equations $U_0X_0 = W^2$ and $U_1X_1 = V^3$. These are the defining equations for $\mathbf{P}$ in $\mathbf{P}(1^2, 2, 3, 5^2)$ as mentioned in (1.2), hence

$$Z \cap \{X_2 = 0\} = \mathbf{P}^1 \times \mathbf{P}$$

and it is contracted by ϕ to $\mathbf{P}$.

Let Y be the image of Z in $\mathbf{P}^{19}$. Let us show that it contains Y_1 and Y_2 as hyperplane sections. On the one hand, $\{\lambda X_2 = 0\}$ is the pullback to F of a hyperplane in $\mathbf{P}^{19}$, such that

$$Z \cap \{\lambda X_2 = 0\} = Z|_{\lambda=0} + Z|_{X_2=0}.$$

In the above, $Z|_{X_2=0}$ is contracted onto $\mathbf{P}$, and $Z|_{\lambda=0}$ has image Y_1.

On the other hand,

$$Z \cap \{\mu X_2 = 0\} = Z|_{\mu=0} + Z|_{X_2=0},$$

where once again, $Z|_{X_2=0}$ is contracted onto $\mathbf{P}$, and $Z|_{\mu=0}$ has image Y_2.

It remains to be proven that Y is not a cone. The pencil of fourfold extensions of $\mathbf{P}$ contained in Y consists of all the $Y \cap H$, where $H \subset \mathbf{P}^{19}$ is a hyperplane such that

$\mathbf{P} \subset H$. These fourfolds are each cut out on Y by $\ell(\lambda,\mu)X_2 = 0$, with ℓ a linear form. Hence, they are complete intersections in $\mathbf{P}(1^2, 2, 3, 5^3)$ of the form

$$U_0X_0 + \lambda U_1X_2 = W^2,$$
$$U_1X_1 + \mu U_0X_2 = V^3,$$

where λ and μ are fixed constant coefficients (to be precise, solutions to $\ell(\lambda,\mu) = 0$). Let $Y_{(\lambda,\mu)}$ be the fourfold section of Y given by the equations above, so that $Y_1 = Y_{(0,1)}$ and $Y_2 = Y_{(1,0)}$. We first notice that $Y_{(\lambda,\mu)} \simeq Y_{(\alpha\lambda,\beta\mu)}$ for all $\alpha, \beta \in \mathbf{C}^*$; indeed, the automorphism which consists in the change of variables

$$U_1 \mapsto \alpha U_1,\ U_0 \mapsto \beta U_0,\ X_1 \mapsto \frac{1}{\alpha}X_1 \text{ and } X_0 \mapsto \frac{1}{\beta}X_0$$

identifies $Y_{(\alpha\lambda,\beta\mu)}$ with $Y_{(\lambda,\mu)}$. Therefore, among the $Y_{(\lambda,\mu)}$ there are at most three isomorphism classes: $Y_{(1,0)}, Y_{(0,1)}$ and $Y_{(1,1)}$. In particular, the class represented by $Y_{(1,1)}$ is dense in the pencil

$$\left\{Y \cap H \mid H \text{ a hyperplane of } \mathbf{P}^{19} \text{ such that } \mathbf{P} \subset H\right\}.$$

Assume now by contradiction that Y is a cone. It contains $\mathbf{P}$ as a linear section of codimension 2, and $\mathbf{P}$ is not a cone, so there are only two possible cases: either the vertex of Y is a point, or it is a line. In the latter case, all the $Y_{(\lambda,\mu)}$'s are cones over $\mathbf{P}$ with each time a point as vertex; in the former case, there is a unique member $Y_{(\lambda,\mu)}$ which is a cone over $\mathbf{P}$. This unique member is either $Y_{(1,0)}$ or $Y_{(0,1)}$ since the class of $Y_{(1,1)}$ is dense in the pencil, so without loss of generality we may assume that $Y_{(1,0)}$ is a cone over $\mathbf{P}$ with a point as vertex (the rest of the proof is analogous if the cone is $Y_{(0,1)}$).

Let us recall the equations for $Y_{(1,0)}$ in $\mathbf{P}(1^2, 2, 3, 5^3)$ with respect to the coordinates $[U_0 : U_1 : V : W : X_0 : X_1 : X_2]$.

$$U_0X_0 + U_1X_2 = W^2,$$
$$U_1X_1 = V^3.$$

We recall as well the fact that $\mathbf{P}$ is the hyperplane section $Y_{(1,0)} \cap \{X_2 = 0\}$ in $\mathbf{P}^{18}$. There is a change of variable which fixes the hyperplane $\{X_2 = 0\}$ and moves the vertex point to p_{X_2}. This change of variables makes the affine chart $Y_{(1,0)}|_{X_2=1}$ an affine cone, i.e., it eliminates the variable X_2 from the equations above.

Indeed, let $F = G = 0$ be the defining equations of a cone whose vertex point is p_{X_2}, such that F and G are two homogeneous sextics on $\mathbf{P}(1^2, 2, 3, 5^3)$, not divisible by X_2, and set $f = F|_{X_2=1}, g = G|_{X_2=1}$. If one of them is not homogeneous, say f (which means that the polynomial F involves the coordinate X_2), then

$$f = f_6 + \mathtt{f},$$

where $f_6 = f_6(U_0, U_1, V, W, X_0, X_1)$ is homogeneous of degree 6 and $\mathtt{f}$ has degree 5 or less. By the fact that $\deg(\mathtt{f}) < 6$ and g and f are sextics, we have $\mathtt{f} \notin (f, g)$ and thus there exists a point $q = (U_0, U_1, V, W, X_0, X_1)$ in the affine cone $\{f = g = 0\}$ such that $\mathtt{f}(q) \neq 0$. From the condition $f(q) = 0$, we have the equality $f_6(q) = -\mathtt{f}(q)$, and by the fact that $\{f = g = 0\}$ is an affine cone, then for all $\lambda \in \mathbf{C}^*$ the point

$$\lambda \cdot q = (\lambda U_0, \lambda U_1, \lambda^2 V, \lambda^3 W, \lambda^5 X_0, \lambda^5 X_1)$$

also belongs to $\{f = g = 0\}$. If λ is general, we have

$$f_6(\lambda \cdot q) = \lambda^6 f_6(q) = -\lambda^6 \mathtt{f}(q) \neq -\mathtt{f}(\lambda \cdot q)$$

since the equality $\lambda^6 \mathtt{f}(q) = \mathtt{f}(\lambda \cdot q)$ is a polynomial condition of degree 6 on λ. This leads to the contradiction that $\lambda \cdot q \notin \{f = g = 0\}$ and the conclusion that f and g are homogeneous.

As a consequence, there exists a change of variables on $\mathbf{P}(1^2, 2, 3, 5^2)$ which eliminates the variables X_2 from the equations:

$$U_0X_0 + U_1X_2 = W^2,$$
$$U_1X_1 = V^3.$$

To eliminate X_2 from these equations, an automorphism of $\mathbf{P}(1^2, 2, 3, 5^2)$ must be of the form

$$[U_0 : U_1 : V : W : X_0 : X_1 : X_2] \mapsto [A\mathbf{U} : V : W : M\mathbf{X}],$$

where $A \in GL_2(\mathbf{C})$ and $M \in GL_3(\mathbf{C})$. Let us denote

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } M = \begin{pmatrix} \alpha_0 & \beta_0 & \gamma_0 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{pmatrix}.$$

This change of variables applied to the equations of $Y_{(1,0)}$ yields

$$(aU_0 + bU_1)(\alpha_0 X_0 + \beta_0 X_1 + \gamma_0 X_2) + (cU_0 + dU_1)(\alpha_2 X_0 + \beta_2 X_1 + \gamma_2 X_2) = W^2,$$
$$(cU_0 + dU_1)(\alpha_1 X_0 + \beta_1 X_1 + \gamma_1 X_2) = V^3.$$

By the fact that this does not involve the variable X_2, we have

$$a\gamma_0 + c\gamma_2 = c\gamma_1 = b\gamma_0 + d\gamma_2 = d\gamma_1 = 0$$

in other words,

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} \gamma_0 \\ \gamma_2 \end{pmatrix} = 0 \text{ and } \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 \\ \gamma_1 \end{pmatrix} = 0.$$

This means that we have either $\det(A) \neq 0$ or $\det(M) \neq 0$, which is a contradiction. The conclusion follows that $Y_{(1,0)}$ is not a cone over $\mathbf{P}$ and therefore, Y is not a cone ■

This maximal extension of $\mathbf{P} \subset \mathbf{P}^{17}$ also hasn't been identified as the universal extension of the general linear curve section $\Gamma \subset \mathbf{P}$.

1.4.3 The primitive polarizations

We go back to Table 7. For each of those spaces $\mathbf{P}$, we consider a general hyperplane section S of the anticanonical model of $\mathbf{P}$, which is a K3 surface.

We recall that the index of the polarized K3 surface $(S, -K_{\mathbf{P}}|_S)$, which is denoted by i_s, is the divisibility of $-K_{\mathbf{P}}|_S$ in the Picard group of S, in other words the largest integer r such that $-\frac{1}{r}K_{\mathbf{P}}|_S$ is a Cartier divisor on S. Here, we denote as usual by Γ a general member of $|-K_{\mathbf{P}}|_S|$, and we introduce C a general member of $|-\frac{1}{i_S}K_{\mathbf{P}}|_S|$, so that $\Gamma = i_S C$ in $\mathrm{Pic}(S)$.

Definition 1.4.19 (Primitive curve). *Let $S \subset \mathbf{P}^N$ be a projective K3 surface and Γ a general hyperplane section of S. Let i_S be the index of $(S, \mathcal{O}_{\mathbf{P}^N}(1)|_S)$, i.e., the largest integer such that $\frac{1}{r}\Gamma$ is Cartier on S. Assume that $C \subset S$ is a curve such that we have the linear equivalence of Cartier divisors $i_S C \sim \Gamma$. Then C is called* primitive.

We list in Table 12 below the values of i_S for the general anticanonical surfaces S of the weighted projective 3-spaces from Table 7, coupled with the genera of Γ and C, where Γ is a general member of $-K_{\mathbf{P}}|_S$ and C is a general member of $-\frac{1}{i_S}K_{\mathbf{P}}|_S$. The nature of the singularities of S is also provided. We mention [DS23, Tables 1 & 2] as a reference for such data.

$\mathbf{P}$	i_S	$g = g(\Gamma)$	$g(C)$	$\mathrm{Sing}(S)$
$\mathbf{P}(1,4,5,10)$	2	21	6	$A_1, 2A_4$
$\mathbf{P}(1,2,6,9)$	3	28	4	$3A_1, A_2$
$\mathbf{P}(1,2,3,6)$	2	25	7	$2A_1, 2A_2$
$\mathbf{P}(1,3,8,12)$	2	25	7	$2A_2, A_3$
$\mathbf{P}(1,6,14,21)$	1	22	22	A_1, A_2, A_6
$\mathbf{P}(2,3,10,15)$	1	16	16	$3A_1, 2A2, A_4$

Table 12: indices of the polarizations $(S, -K_{\mathbf{P}}|_S)$

Now, we examine all those cases and give a geometric description of the curve C:

Theorem 1.4.20. *Let* $\mathbf{P}$ *be a Gorenstein weighted projective space from the list Table 12 and* S *a general anticanonical divisor of* $\mathbf{P}$*. If* C *is a general member of* $-|\frac{1}{i_S}K_{\mathbf{P}}|_S|$ *then it can be described as follows.*

$\mathbf{P}$	C
$\mathbf{P}(1,4,5,10)$	*plane quintic with a total inflection point*
$\mathbf{P}(1,2,6,9)$	*smooth hyperelliptic curve of genus 4*
$\mathbf{P}(1,2,3,6)$	*normalization of a plane oscnodal sextic curve*
$\mathbf{P}(1,3,8,12)$	*trigonal curve of genus 7 with a total ramification point*
$\mathbf{P}(1,6,14,21)$	*member of the linear system* $\lvert -7K_{DP_1}\rvert$ *such that there exists* $\gamma \in \lvert -K_{DP_1}\rvert$ *with* $C\lvert_\gamma = 7p,$ *where* p *is the base point of* $\lvert -K_{DP_1}\rvert$
$\mathbf{P}(2,3,10,15)$	*sextic section of the cone over a conic with two smooth local branches meeting at a singular point* p*, a line with contact order 6 at the point* p*, and a tri-tangent line*

Table 13: Primitive curves in the general K3 surfaces $S \subset \mathbf{P}$

Conversely, for all items on this list, except $\mathbf{P}(1,2,3,6)$*, any curve with the given description is isomorphic to a member of* $-\frac{1}{i_{S'}}K_{\mathbf{P}}|_{S'}$ *for some K3 surface* $S' \subset |-K_{\mathbf{P}}|$*.*

Comment. • The general primitive curve in $S \subset \mathbf{P}(2,3,10,15)$ is a 6-gonal curve of geometric genus 16 such that one member of the g^1_6 is a sextuple point, and another member consists of three double points.

• The description of the general primitive curve C in $S \subset \mathbf{P}(1,6,14,21)$ is equivalent to the following: C is the blowup of a plane curve of degree 21 at 8 septuple points $p_1,...,p_8$, and for p the ninth base point of the pencil $\mathcal{P}$ of cubics through the points p_i's, there exists a cubic curve $\gamma \in \mathcal{P}$ such that

$$C|_\gamma = 7p + 7p_1 + \cdots + 7p_8.$$

Let us now provide a case-by-case proof of Theorem 1.4.20. It is broken down into Lemmas 1.4.21, 1.4.22, 1.4.24, 1.4.25, 1.4.26 and 1.4.29.

The general primitive curve in $S \subset \mathbf{P}(1,4,5,10)$

Let S be a general anticanonical divisor of $\mathbf{P} = \mathbf{P}(1,4,5,10)$.

According to Table 12, the index i_S is equal to 2 and the genus of C is 6. In Table 10, S is explicitly given as the quintic hypersurface $u_0 f_4(\mathbf{u}, v) + u_1^5 = 0$ in $\mathbf{P}(1,1,1,2)$, with $\deg f_4 = 4$, and Γ is cut out on S by a quartic. Therefore $C \sim \frac{1}{2}\Gamma$ is cut out by a quadric, so its defining equations are

$$u_0 f_4(\mathbf{u}, v) + u_1^5 = g_2(\mathbf{u}, v) = 0$$

with g_2 a general homogeneous quadric polynomial.

Lemma 1.4.21. *The curve* C *is isomorphic to a plane quintic with a total inflection point, i.e., there is a line* Δ *which is tangent to* C *in* $\mathbf{P}^2$ *and* $C|_\Delta$ *is a quintuple point.*

Conversely, any such plane quintic can be realized as a member of $|-\frac{1}{2}K_{\mathbf{P}}|_{S'}|$ *for a K3 surface* $S' \in |-K_{\mathbf{P}}|$*.*

Proof: Up to scaling, we may choose by generality $g_2(\mathbf{u}, v) = v - \alpha(u_0, u_1, u_2)$ where α is a conic. Hence C is cut out by

$$u_0 f_4(u_0, u_1, u_2, v) + u_1^5 = 0,\ v = \alpha(u_0, u_1, u_2).$$

Substituting $\alpha(u_0, u_1, u_2)$ for v in the first equation naturally realizes C as a quintic in $\mathbf{P}^2$ with coordinates $[u_0 : u_1 : u_2]$. Moreover, the restriction of C to the line $u_0 = 0$ is a quintuple point. Indeed, for $\Delta = \{u_0 = 0\}$ we have

$$C|_\Delta = 5p$$

where $p = \{u_0 = u_1 = 0\}$ in $\mathbf{P}^2$. This is an inflection point of order 5 of the curve C. The tangent cone of C at this point is the reduced line Δ by generality of f_4 (the curve

C is indeed smooth by Bertini's theorem, since it is a general hyperplane section of S in $\mathbf{P}^6$ and S has isolated singularities).

Conversely, let C' be such a plane quintic. Up to a choice of coordinates, C' is given by an equation of the form

$$u_0 g_4(u_0, u_1, u_2) + u_1^5 = 0$$

with $\deg g_4 = 4$. The curve C' in its canonical model can be extended by a quintic surface in $\mathbf{P}(1, 1, 1, 2)$, as follows: according to the construction that was done in [LD23, Appendix A.2], there exists a quintic polynomial $f_5(\mathbf{u}, v)$ and a quadric $\alpha(\mathbf{u}) = \alpha(u_0, u_1, u_2)$ such that

$$C' = \{f_5(\mathbf{u}, v) = 0, v = \alpha(\mathbf{u})\}$$

in $\mathbf{P}(1, 1, 1, 2)$. Hence the quintic surface $\{f_5(\mathbf{u}, v) = 0\}$ in $\mathbf{P}(1, 1, 1, 2)$ is an extension of C'.

Here f_5 and α are so that

$$f_5|_{v=\alpha(\mathbf{u})} = u_0 g_4(u_0, u_1, u_2) + u_1^5.$$

Thus $f_5 = u_1^5 + \lambda\beta(\mathbf{u}, v)(v - \alpha(\mathbf{u})) \pmod{u_0}$ in $\mathbf{C}[u_0, u_1, u_2, v]$ for some constant λ and $\deg \beta = 3$. Picking $\lambda = 0$ doesn't change C', and yields $f_5 = u_0 f_4(\mathbf{u}, v) + u_1^5$ for some homogeneous quartic f_4 on $\mathbf{P}(1, 1, 1, 2)$.

Hence C' is cut out on S' by a quadric, where S' is the quintic $u_0 f_4(\mathbf{u}, v) + u_1^5 = 0$. Recall from Lemma 1.4.4 that $\varphi : \mathbf{P} \dashrightarrow \mathbf{P}(1, 1, 1, 2)$ restricts to an isomorphism on the general member of $|-K_{\mathbf{P}}|$; here S' is a member of $\mathcal{L} = \varphi(|-K_{\mathbf{P}}|)$ so it is isomorphic to a general anticanonical divisor of $\mathbf{P}$. Moreover, $C' \sim -\frac{1}{2}K_{\mathbf{P}}|_{S'}$ in $\mathrm{Pic}(S')$. ■

The general primitive curve in $S \subset \mathbf{P}(1, 2, 6, 9)$

Let S be a general anticanonical divisor of $\mathbf{P} = \mathbf{P}(1, 2, 6, 9)$.

According to Table 12, the index i_S of the polarization $(S, -K_{\mathbf{P}}|_S)$ equals 3 and the curve C has genus 4. We know by Table 10 that S is a degree 10 hypersurface in $\mathbf{P}(1, 1, 3, 5)$, of equation $u_0 f_9(\mathbf{u}, v, s) + s^2 = 0$ and Γ is the intersection of S with a general 9-ic. Hence C is cut out on S by a general cubic of $\mathbf{P}(1, 1, 3, 5)$, i.e., it is defined by two equations as follows:

$$u_0 f_9(\mathbf{u}, v, s) + s^2 = 0, \ v = \alpha(u_0, u_1),$$

where α is a homogeneous cubic polynomial on $\mathbf{P}^1$.

Lemma 1.4.22. *The curve C is isomorphic to a 10-ic curve in $\mathbf{P}(1, 1, 5)$, i.e., a quadric section of the cone over a rational quintic curve. Equivalently, C is a smooth hyperelliptic curve of genus 4.*

Conversely, any such curve can be realized as a member of $|-\frac{1}{3}K_{\mathbf{P}}|_{S'}|$ for a K3 surface $S' \in |-K_{\mathbf{P}}|$.

Proof: By the equations

$$u_0 f_9(\mathbf{u}, v, s) + s^2 = 0, \ v = \alpha(u_0, u_1)$$

in $\mathbf{P}(1, 1, 3, 5)$ with coordinates $[u_0 : u_1 : v : s]$, C is naturally realized as a degree 10 curve on $\mathbf{P}(1, 1, 5)$ with coordinates $[u_0 : u_1 : s]$ of the form $u_0 h_9(\mathbf{u}, s) + s^2 = 0$. Hence the linear system $|\mathcal{O}_{\mathbf{P}(1,1,5)}(1)|$, whose base locus $\{u_0 = u_1 = 0\}$ does not meet C, restricts to a g^1_2 on C.

Conversely, let C' be a curve in $\mathbf{P}(1, 1, 5)$ of degree 10. In a suitable choice of coordinates, the point $[u_0 : u_1] = [0 : 1]$ belongs to the branch locus of the double cover $C \to \mathbf{P}^1$. Hence the line $\Delta = \{u_0 = 0\}$ in $\mathbf{P}(1, 1, 5)$ is tangent to C'. As a result, the restriction $C'|_{u_0=1}$ is a double point, which yields the following equation for C':

$$u_0 h'_9(\mathbf{u}, s) + s^2 = 0,$$

with $\deg h'_9 = 9$. Introducing v a coordinate of weight 3 and $f'_9(\mathbf{u}, v, s)$ a degree 9 homogeneous polynomial on $\mathbf{P}(1, 1, 3, 5)$ such that $f'_9(\mathbf{u}, v, s)|_{v=\alpha(u_0,u_1)} = h'_9(\mathbf{u}, v)$, we realize C' as a complete intersection in $\mathbf{P}(1, 1, 3, 5)$ of equations

$$u_0 f'_9(\mathbf{u}, v, s) + s^2 = 0, \ v = \alpha(u_0, u_1).$$

This makes C' a curve in the surface $S' = \{u_0 f_9'(\mathbf{u}, v, s) + s^2 = 0\}$, which is a member of $\mathcal{L} = \varphi(|-K_{\mathbf{P}}|)$, meaning that S' is isomorphic to a general anticanonical divisor of $\mathbf{P}$. Moreover, the moving part of $\mathcal{L}|_{S'}$ is the restriction to S' of the 9-ics, and therefore $3C$ is a member of the moving part of $\mathcal{L}|_S$. This makes $3C$ the class of the hyperplane sections of S in $\mathbf{P}^{28}$, in other words $3C = -K_{\mathbf{P}}|_S$ in $\mathrm{Pic}(S)$. This yields $C \sim -\frac{1}{3}K_{\mathbf{P}}|_{S'}$ as Cartier divisors on S'. ■

The general primitive curve in $S \subset \mathbf{P}(1,2,3,6)$

Let S be a general anticanonical divisor of $\mathbf{P} = \mathbf{P}(1,2,3,6)$.

Remark. This is the only example of our list for which $-\frac{1}{i_S}K_{\mathbf{P}}$ is Cartier on $\mathbf{P}$. The index i_S is equal to 2 and $-\frac{1}{2}K_{\mathbf{P}}$ is the class of sextic surfaces. Hence, S is a general surface of degree 12, and we consider a curve C cut out on S by a general sextic of $\mathbf{P}$. The projective model associated to $-\frac{1}{2}K_{\mathbf{P}}$ in which C is a hyperplane section of S is a realization of $\mathbf{P}$ as a variety of degree $(-\frac{1}{2}K_{\mathbf{P}})^3 = 6$ in $\mathbf{P}^7$. It factors as the composite map

$$\mathbf{P} \xhookrightarrow{\;v_2\;} \mathbf{P}(1,1,2,3,3) \overset{|\mathcal{O}(3)|}{\dashrightarrow} \mathbf{P}^7$$

where v_2 is the 2-Veronese embedding.

Let $[u_0 : u_1 : v : s_0 : s_1]$ be coordinates on $\mathbf{P}(1,1,2,3,3)$, then v_2 is given by

$$[u_0 : u_1 : v : s_0 : s_1] = [x^2 : y : xz : z^2 : w]$$

and it realizes $\mathbf{P}$ as the hypersurface $u_0 s_0 = v^2$, S as the intersection of $\mathbf{P}$ with a general sextic, and C as the intersection of S with a general cubic.

Consider now $\mathbf{P'} := \mathbf{P}(1,1,1,3)$ with coordinates $[a_0 : a_1 : a_2 : b]$ and the rational map $\psi : \mathbf{P'} \dashrightarrow \mathbf{P}(1,1,2,3,3)$, birational onto its image, given by the expression

$$[u_0 : u_1 : v : s_0 : s_1] = [a_0 : a_1 : a_0 a_2 : a_0 a_2^2 : b].$$

Its image satisfies the same equation as the image of v_2, hence $\psi(\mathbf{P'}) = v_2(\mathbf{P})$. There is a birational map φ which makes the following diagram commute.

$$\begin{array}{ccc} \mathbf{P} = \mathbf{P}(1,2,3,6) & \overset{\varphi}{\dashrightarrow} & \mathbf{P}(1,1,1,3) = \mathbf{P'} \\ & {\scriptstyle v_2} \searrow \quad \swarrow {\scriptstyle \psi} & \\ & \mathbf{P}(1,1,2,3,3) & \end{array}$$

To study the geometry of C, we will examine its image via φ. In the end, since $\mathbf{P'} = \mathbf{P}(1^3, 3)$ is isomorphic to a cone over the 3-Veronese of $\mathbf{P}^2$, the diagram above will allow us to realize $\varphi(C)$ as a plane curve in Lemma 1.4.24. The idea which motivated the choice of ψ is the following: the data of ψ and its expression in coordinates make it possible to give an expression for φ in coordinates.

One checks from the expressions of v_2 and ψ that φ has the following expression with regard to the weighted coordinates:

$$\varphi : [x : y : z : w] \in \mathbf{P} \mapsto [a_0 : a_1 : a_2 : b] = [x^3 : xy : z : x^3 w] \in \mathbf{P'}. \tag{1.12}$$

The rational map φ admits indeed a rational inverse φ^{-1}, which is:

$$\varphi^{-1} : [a_0 : a_1 : a_2 : b] \in \mathbf{P'} \mapsto [x : y : z : w] = [a_0 : a_0 a_1 : a_0^2 a_2 : a_0^3 b].$$

Indeed, a computation gives

$$\varphi \circ \varphi^{-1} : [a_0 : a_1 : a_2 : b] \mapsto [a_0^3 : a_0^2 a_1 : a_0^2 a_2 : a_0^6 b] = [a_0 : a_1 : a_2 : b],$$

for any point $[a_0 : a_1 : a_2 : b]$ in the regular locus of φ^{-1}. Likewise, one can compute $\varphi^{-1} \circ \varphi = \mathrm{id}_{\mathbf{P}}$.

As a sanity check, let us consider $q = [a_0 : a_1 : a_2 : b]$ a fixed point in $\mathbf{P}'$ with a chosen representative $(a_0, a_1, a_2, b) \in \mathbf{C}^4$, and $\sqrt{a_0}$ a square root of a_0. Then the image of q by φ^{-1} is

$$\varphi^{-1}(q) = [\sqrt{a_0} : a_1 : \sqrt{a_0}a_2 : b] \in \mathbf{P}, \tag{1.13}$$

and a computation yields

$$v_2 \circ \varphi^{-1}(q) = \psi(q),$$

as required from the commutative diagram.

The birational map φ contracts the vanishing locus of x to a point p. The curve C is cut out on $\mathbf{P}$ by two general equations of respective degree 12 and 6 with regard to the grading of $\mathbf{P}$, and $D = \{x = 0\} \in |\mathcal{O}_{\mathbf{P}}(1)|$, so the degree of the restriction $D|_C$ is

$$\deg D|_C = D \cdot C = \mathcal{O}_{\mathbf{P}}(1) \cdot \mathcal{O}_{\mathbf{P}}(12) \cdot \mathcal{O}_{\mathbf{P}}(6) = 2.$$

Therefore, on $D \simeq \mathbf{P}(2, 3, 6)$, the curve C cuts two general points. The indeterminacy locus $x = z = 0$ does not meet C by the generality assumption, so the map φ induces by restriction to C a morphism $C \to \mathbf{P}(1, 1, 1, 3)$ which is birational onto its image and makes C the normalization of its image. Let us denote $C_0 = \varphi(C)$.

Lemma 1.4.23. *The curve C_0 has a unique singularity at which two smooth local branches meet. The morphism $C \to C_0$ is an isomorphism over the complement of this singular point.*

Proof: From the expression (1.12), we identify that the exceptional locus of φ is $D = \{x = 0\} \subset \mathbf{P}$ and $\varphi(D)$ is the point $p = [0 : 0 : 1 : 0] \in \mathbf{P}' = \mathbf{P}(1^3, 3)$. Moreover, C is smooth and the transverse intersection $D \cap C$ consists of two points. From the expression (1.13), we identify that the only indeterminacy point of the rational inverse φ^{-1} of φ is p. Since D is the preimage of p by φ and $\varphi \circ \varphi^{-1} = \mathrm{id}$ on the regular locus of φ^{-1}, this ensures that $\mathrm{Exc}(\varphi) = D$, and moreover φ is an isomorphism on its regular locus minus D, i.e., φ induces by restriction an isomorphism from $\mathbf{P} - (D \cup \{x = z = 0\})$ onto its image.

As a result, the morphism $\varphi|_C : C \to C_0$ is an isomorphism outside of $D \cap C$, and C_0 has only one singular point which is the intersection of two local branches. To prove that these two local branches are smooth, it suffices to show that the kernel of the differential $d\varphi$ coincides with the tangent of D at any of the two points $D \cap C$.

Consider the open subset $U = \{z \neq 0\} \subset \mathbf{P}(1, 2, 3, 6)$. By generality of C, this opens subset contains the two points $D \cap C$. On the one hand, by Lemma 1.1.3, U is isomorphic to the quotient of $\mathbf{C}^3_{(\mathtt{x},\mathtt{y},\mathtt{w})}$ under the following action of the group of cubic roots of unity μ_3:

$$\zeta \cdot (\mathtt{x}, \mathtt{y}, \mathtt{w}) = (\zeta \mathtt{x}, \zeta^2 \mathtt{y}, \mathtt{w}).$$

On the other hand, the indeterminacy locus $\{x = z = 0\}$ of φ is disjoint from U, and the image of U via φ is the open subset $\{a_2 \neq 0\}$ of $\mathbf{P}(1^3, 3)_{[a_0:a_1:a_2:b]}$, which is isomorphic to $\mathbf{C}^3_{(\mathtt{a}_0,\mathtt{a}_1,\mathtt{b})}$.

Consider the map $\overline{\varphi} : \mathbf{C}^3_{(\mathtt{x},\mathtt{y},\mathtt{w})} \to \mathbf{C}^3_{(\mathtt{a}_0,\mathtt{a}_1,\mathtt{b})}$ given by the expression

$$\overline{\varphi} : (\mathtt{x}, \mathtt{y}, \mathtt{w}) \mapsto (\mathtt{x}^3, \mathtt{x}y, \mathtt{x}^3\mathtt{w}).$$

This map is invariant under the action of μ_3 and it fits in the following commutative diagram, where the left vertical arrow is the quotient map onto $U \simeq \mathbf{C}^3/\mu_3$, which is generically a local diffeomorphism:

$$\begin{array}{ccc} \mathbf{C}^3 & \xrightarrow{\ \overline{\varphi}\ } & \mathbf{C}^3 \\ {\scriptstyle 3:1}\downarrow & & \downarrow{\scriptstyle \simeq} \\ U & \xrightarrow[\ \varphi\ ]{} & \varphi(U) \end{array}$$

Let us denote by $C', D' \subset \mathbf{C}^3$ the respective preimages of $C \cap U$ and $D \cap U$ by the quotient map $\mathbf{C}^3 \to U$. Let p_1, p_2 be the two intersection points of D and C; by generality, each point p_i has three distinct preimages in $D' \cap C'$, and above the point p_i the quotient map $\mathbf{C}^3 \to U$ is a local diffeomorphism. Hence at the general point $\mathtt{p} \in D' \cap C'$, it is enough to show that

$$\ker(d_{\mathtt{p}}\overline{\varphi}) = T_{\mathtt{p}}D'.$$

The equality above follows from a straightforward computation, as D' is the vanishing locus of $\mathtt{x}$ and $\overline{\varphi}$ is the map $(\mathtt{x}, \mathtt{y}, \mathtt{w}) \mapsto (\mathtt{x}^3, \mathtt{x}\mathtt{y}, \mathtt{x}^3\mathtt{w})$. ■

Lemma 1.4.24. *The curve C is the normalization of its image $C_0 = \varphi(C)$, which is isomorphic to a plane sextic with an oscnode (i.e., C_0 has two smooth local branches meeting with contact order 3 at its singular point p).*

Moreover, there is a line $\Delta \subset \mathbf{P}^2$ through p such that $C_0|_\Delta = 6p$.

Proof: Let Σ be the direct image of S via $\varphi : \mathbf{P} \to \mathbf{P}'$; it is the proper transform of S via φ^{-1}. From the expression

$$\varphi^{-1} : [a_0 : a_1 : a_2 : b] \in \mathbf{P}' \mapsto [x : y : z : w] = [\sqrt{a_0} : a_1 : \sqrt{a_0}a_2 : b],$$

where $\sqrt{a_0}$ is any square root of a_0, we identify the exceptional locus of φ^{-1} as $\{a_0 = 0\}$ (indeed, $\varphi^{-1}(\{a_0 = 0\})$ is a curve in $\mathbf{P}$). The pullback to $\mathbf{P}'$ of the general 12-ic surface S is

$$(\varphi^{-1})^* S = \left\{a_0^6\Big(a_0 f_5(\mathbf{a}, b) + \lambda a_1^6 + \mu a_1^3 b + \gamma b^2\Big) = 0\right\}$$

where $f_5(\mathbf{a}, b)$ is a quintic on $\mathbf{P}'$ and λ, μ, γ are constant. Hence the proper transform Σ of S is a nongeneral sextic of $\mathbf{P}' = \mathbf{P}(1, 1, 1, 3)$ of equation

$$a_0 f_5(\mathbf{a}, b) + \lambda a_1^6 + \mu a_1^3 b + \gamma b^2 = 0.$$

On the one hand, we have $h^0(\mathbf{P}, \mathcal{O}_\mathbf{P}(-K_\mathbf{P})) = 27$, while $h^0(\mathbf{P}', \mathcal{O}_{\mathbf{P}'}(5))+3 = 30$, so even the quintic f_5 can't be general, and it must belong to a subspace $V \subset H^0(\mathbf{P}', \mathcal{O}_{\mathbf{P}'}(5))$ with $\dim V = \dim H^0(\mathbf{P}', \mathcal{O}_{\mathbf{P}'}(5)) - 3 = 24$. Namely, it follows from the expression of φ^{-1} that f_5 does not involve the monomials $a_2^5, a_2^4 a_1$ and $a_2^3 a_1^2$.

As C is a hyperplane section of S in $\mathbf{P}^7$, it is cut out on S by a general cubic $\alpha(u_0, u_1, v, s_0, s_1) = 0$ in $\mathbf{P}(1, 1, 2, 3, 3)$, with

$$[u_0 : u_1 : v : s_0 : s_1] = [\sqrt{a_0} : a_1 : \sqrt{a_0}a_2 : b],$$

and thus its image C_0 in $\mathbf{P}'$ is cut out on Σ by a nongeneral cubic of the form

$$b = \tau a_1^3 + a_0 q(\mathbf{a})$$

with τ a constant and q a general quadric in the variables $[a_0 : a_1 : a_2]$. It follows

$$C_0 = \Sigma \cap \left\{b = \tau a_1^3 + a_0 q(\mathbf{a})\right\} = \left\{ \begin{array}{l} a_0 f_5(\mathbf{a}, b) + \lambda a_1^6 + \mu a_1^3 b + \gamma b^2 = 0 \\ b = \tau a_1^3 + a_0 q(\mathbf{a}) \end{array} \right\}.$$

This makes C_0 a sextic curve in $\mathbf{P}^2$ with coordinates $[a_0 : a_1 : a_2]$ such that $C_0|_{a_0=0} = (a_1^6)$. Hence the restriction of C_0 to the line $\Delta = \{a_0 = 0\}$ in $\mathbf{P}^2_{[a_0:a_1:a_2]}$, which is set-theoretically the point $p = \{a_0 = a_1 = 0\}$, has multiplicity six.

By Lemma 1.4.23, the curve C_0 has a unique singular point which is the intersection of two local branches, and the morphism from C to C_0 is finite and birational. Since it is smooth, we deduce that C is the normalization of C_0. Moreover, the curve C_0 has two smooth local branches at p, say B_1 and B_2, such that

$$6p = C_0|_\Delta = B_1|_\Delta + B_2|_\Delta.$$

This implies $B_i|_\Delta = \beta_i p$ with $\beta_i \in \mathbf{N}$ for $i = 1, 2$, and $\beta_1 + \beta_2 = 6$. There are three cases to distinguish.

(*i*) If $(\beta_1, \beta_2) = (1, 5)$, then C_0 is a sextic plane curve with a node at p, and $g(C) = \frac{(6-1)(6-2)}{2} - 1 = 9$. But $g(C) = 7$ by Table 12.

(*ii*) If $(\beta_1, \beta_2) = (2, 4)$, then C_0 has a tacnode at p, and $g(C) = \frac{(6-1)(6-2)}{2} - 2 = 8$, which is also a contradiction.

(*iii*) If $(\beta_1, \beta_2) = (3, 3)$, then C_0 has an oscnode at p, and we indeed have $g(C) = \frac{(6-1)(6-2)}{2} - 3 = 7$.

The conclusion follows that p is an oscnodal point of C_0, as required. ■

The general primitive curve in $S \subset \mathbf{P}(1,3,8,12)$

Let S be a general anticanonical divisor of $\mathbf{P} = \mathbf{P}(1,3,8,12)$.

As stated in Table 12, the general primitive curve C in S has genus 7. It follows from Table 10 that S is isomorphic to the 9-ic hypersurface $u_0 f_8(\mathbf{u}, v, s) + v^3 = 0$ in $\mathbf{P}(1,1,3,4)$ with coordinates $[u_0 : u_1 : v : s]$ and Γ is cut out on S by a degree 8 hypersurface of $\mathbf{P}(1,1,3,4)$. The index i_S is equal to 2, therefore $C \sim \frac{1}{2}\Gamma$ in $\mathrm{Pic}(S)$ and C is the intersection of S with a general quartic. Such a quartic has equation $s = \alpha(\mathbf{u}, v)$, where $\deg \alpha = 4$. Hence C is cut out by the equations

$$u_0 f_8(\mathbf{u}, v, s) + v^3 = 0,\ s = \alpha(\mathbf{u}, v).$$

Lemma 1.4.25. *The curve C is isomorphic to a degree 9 curve in $\mathbf{P}(1,1,3)$, i.e., a cubic section of the cone over a rational normal cubic curve, with an inflection point of order 3 along a line of the ruling, i.e., there is a line $\Delta \in |\mathcal{O}_{\mathbf{P}(1,1,3)}(1)|$ which is tangent to C at a point p, and $C|_\Delta = 3p$. In particular, C is a trigonal curve of genus 7 with a total ramification point.*

Conversely, any such curve in $\mathbf{P}(1,1,3)$ is isomorphic to a member of $|-\frac{1}{2}K_\mathbf{P}|_{S'}|$ for a K3 surface $S' \in |-K_\mathbf{P}|$.

Proof: By the equations

$$u_0 f_8(\mathbf{u}, v, s) + v^3 = 0,\ s = \alpha(\mathbf{u}, v)$$

in $\mathbf{P}(1,1,3,4)$ with coordinates $[u_0 : u_1 : v : s]$, C is naturally realized as the curve of degree 9 in $\mathbf{P}(1,1,3)$ given by the following

$$u_0 h_8(\mathbf{u}, v) + v^3 = 0$$

where $h_8 = f_8|_{s=\alpha(\mathbf{u},v)}$, $\deg h_8 = 8$. Let Δ be the line $\{u_0 = 0\}$, then $C|_\Delta = 3p$ where p is the point $\{u_0 = v = 0\}$. The tangent cone of C at p is the reduced line Δ, hence C is smooth and has an inflection point of order 3 at p.

Conversely, let $C' \subset \mathbf{P}(1,1,3)$ be such a curve. Then there exists a choice of coordinates such that C' is cut out by an equation of the form

$$u_0 h'_8(\mathbf{u}, v) + v^3 = 0$$

with $\deg h'_8 = 8$. We introduce a coordinate s of weight 4 and a homogeneous degree 8 polynomial $f'_8(\mathbf{u}, v, s)$ on $\mathbf{P}(1,1,3,4)$ such that $f'_8(\mathbf{u}, v, s)|_{s=\alpha(\mathbf{u},v)} = h'_8(\mathbf{u}, v)$. In this setting, C' is the complete intersection in $\mathbf{P}(1,1,3,4)$ given by

$$u_0 f'_8(\mathbf{u}, v, s) + v^3 = 0,\ s = \alpha(\mathbf{u}, v),$$

meaning it is cut out by a general quartic on the surface $S' = \{u_0 f'_8(\mathbf{u}, v, s) + v^3 = 0\}$. Recall from Lemma 1.4.4 that the birational map φ from $\mathbf{P}$ to $\mathbf{P}(1,1,3,4)$ restricts to an isomorphism on the general anticanonical divisors of $\mathbf{P}$. Here S' is a member of $\mathcal{L} = \varphi(|-K_\mathbf{P}|)$ whose equation involves the monomial v^3, so it is isomorphic to a general member of $|-K_\mathbf{P}|$, and furthermore the moving part of $\mathcal{L}|_{S'}$ is the restriction to S' of the 8-ics of $\mathbf{P}(1,1,3,4)$. As a result, $2C$ is a member of the moving part of $\mathcal{L}|_S$, i.e., it is a hyperplane section of S in $\mathbf{P}^{25}$. Hence $2C = -K_\mathbf{P}|_S$ in $\mathrm{Pic}(S)$, and the conclusion follows that $C \sim -\frac{1}{2}K_\mathbf{P}|_S$. ∎

The general primitive curve in $S \subset \mathbf{P}(1,6,14,21)$

Let S be a general anticanonical divisor of $\mathbf{P} = \mathbf{P}(1,6,14,21)$.

We know from Table 10 that S is a heptic hypersurface in $\mathbf{P}(1,1,2,3)$ with coordinates $[u_0 : u_1 : v : s]$ of equation $u_0 f_6(\mathbf{u}, v, s) + u_1^7 = 0$. In this case, the index i_S is equal to 1, hence C and Γ are two curves of genus 22 which represent the same Cartier divisor on S, which is cut out by a general sextic of $\mathbf{P}(1,1,2,3)$. Such a sextic is smooth by generality, since $\mathbf{P}(1,1,2,3)$ has only two isolated singularities and the linear system of its sextics doesn't have base points. Moreover, the general sextic is a double cover of $\mathbf{P}(1,1,2)$ ramified over a general curve of degree 6. It is indeed given by an equation of the form $s^2 = h_6(\mathbf{u}, v)$ with $\deg(h_6) = 6$, and the ramification locus in $\mathbf{P}(1,1,2)$ is the curve $h_6(\mathbf{u}, v) = 0$. This sextic of $\mathbf{P}(1,1,2,3)$ is then a Del Pezzo surface of degree 1 and we shall denote it by DP_1. In particular, it can be obtained from $\mathbf{P}^2$ by blowing up 8 general points.

Lemma 1.4.26. *The curve C is the blow-up of a plane 21-ic curve C_0 at 8 septuple points $p_1, ..., p_8$. Moreover, if p is the ninth base points of the pencil $\mathcal{P}$ whose members are the plane cubics through the p_i's, then there exists γ a member of $\mathcal{P}$ such that*

$$C_0|_\gamma = 7p + 7p_1 + \cdots + 7p_8.$$

Conversely, the proper transform in DP_1 by the blow-up map $DP_1 \to \mathbf{P}^2$ of any such plane curve C_0' of degree 21 is isomorphic to a member of $|-K_{\mathbf{P}}|_{S'}|$ for a K3 surface $S' \in |-K_{\mathbf{P}}|$.

Proof: Let $\varepsilon : DP_1 \to \mathbf{P}^2$ be the blow-up map, $H = \varepsilon^* \mathcal{O}_{\mathbf{P}^2}(1)$ the pullback of the lines and E_i the exceptional curve over p_i, $i \in \{1, ..., 8\}$. On the one hand, we have $-K_{DP_1} = 3H - \sum_{i=1}^8 E_i$. On the other hand, the adjunction formula yields $-K_{DP_1} = [\mathcal{O}_{\mathbf{P}(1,1,2,3)}(1)|_{DP_1}]$. Since $C = DP_1 \cap S$, where S is a heptic in $\mathbf{P}(1,1,2,3)$, it holds that

$$C = -7K_{DP_1} = 21H - \sum_{i=1}^{8} 7E_i$$

and thus it is the proper transform of a degree 21 curve C_0 in $\mathbf{P}^2$ which passes through the points p_i, each with multiplicity 7.

The curve C is given by the two following equations.

$$u_0 f_6(\mathbf{u}, v, s) + u_1^7 = g_6(\mathbf{u}, v, s) = 0$$

for g_6 a general homogeneous sextic polynomial, so that $g_6 = 0$ is the defining equation of DP_1. The base point of $-K_{DP_1}$ is the intersection point of DP_1 with the locus $\{u_0 = u_1 = 0\}$, which we denote by p. Let B be the curve $DP_1 \cap \{u_0 = 0\}$. It is an anticanonical curve of DP_1 and by the equations above we have

$$C|_B = (DP_1 \cap \{u_0 f_6(\mathbf{u}, v, s) +_1^7 = 0\})|_{u_0=0} = DP_1|_{u_0=0} \cap (u_1^7)|_{u_0=0} = 7p.$$

If $\gamma = \varepsilon(B)$, which is a plane cubic through $p, p_1, ..., p_8$, then the above implies that $C_0|_\gamma = 7p + 7p_1 + \cdots + 7p_8$.

Conversely, if C_0' is such a plane 21-ic curve, then its blow-up $C' \subset DP_1$ at the points p_i is in the Cartier class $-7K_{DP_1}$, and by the surjectivity of

$$H^0(\mathbf{P}(1,1,2,3), \mathcal{O}_{\mathbf{P}(1,1,2,3)}(7)) \to H^0(DP_1, \mathcal{O}_{DP_1}(-7K_{DP_1}))$$

which follows from the restriction short exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}(1,1,2,3)} \to \mathcal{O}_{\mathbf{P}(1,1,2,3)}(7) \to \mathcal{O}_{DP_1}(-7K_{DP_1}) \to 0$$

and the vanishing $h^1(\mathbf{P}(1,1,2,3), \mathcal{O}_{\mathbf{P}(1,1,2,3)}) = 0$, we have $C' = S' \cap DP_1$ where S' is a heptic surface in $\mathbf{P}(1,1,2,3)$. It follows that C' in $\mathbf{P}(1,1,2,3)$ has equations

$$f_7'(\mathbf{u}, v, s) = g_6(\mathbf{u}, v, s) = 0.$$

Besides, there exists B' an anticanonical curve of DP_1 such that $C'|_{B'} = 7p$. We may choose the coordinates $[u_0 : u_1 : v : s]$ on $\mathbf{P}(1,1,2,3)$ such that $B' = DP_1 \cap \{u_0 = 0\}$. This yields

$$f_7'(\mathbf{u}, v, s)|_{u_0 = g_6(\mathbf{u}, v, s) = 0} = u_1^7.$$

In other words, $f_7' = u_1^7 + \lambda\alpha(u_0, u_1) g_6(\mathbf{u}, v, s) (\text{mod } u_0)$ for some constant λ and $\deg \alpha = 1$. We may choose $\lambda = 0$, which does not change C' and realizes it as a complete intersection in $\mathbf{P}(1,1,2,3)$ of the form

$$u_0 f_6'(\mathbf{u}, v, s) + u_1^7 = g_6(\mathbf{u}, v, s) = 0$$

with $\deg f_6' = 0$. Thus C' lies on the surface $S' = \{u_0 f_6'(\mathbf{u}, v, s) + u_1^7 = 0\}$. It is a member of $\mathcal{L} = \varphi(|-K_{\mathbf{P}}|)$, for φ the birational map displayed in Table 9. Therefore S' is isomorphic to a general member of $|-K_{\mathbf{P}}|$ and $C' \sim -K_{\mathbf{P}}|_{S'}$. ∎

The general primitive curve in $S \subset \mathbf{P}(2,3,10,15)$

Let S be a general anticanonical divisor of $\mathbf{P} = \mathbf{P}(2,3,10,15)$.

The index i_S is equal to 1, meaning that both C and Γ represent the same Cartier divisor on S. The curve C is then the intersection of $\mathbf{P}$ in $\mathbf{P}^{17}$ with two general hyperplanes. Recall from (1.11) that $\mathbf{P}$ is realized as a complete intersection in $\mathbf{P}(1^2,2,3,5^2)$ with coordinates $[U_0 : U_1 : V : W : X_0 : X_1]$ of equations

$$U_0X_0 = W^2,\ U_1X_1 = V^3,$$

and that its hyperplane sections in $\mathbf{P}^{17}$ are its sections by the quintics of $\mathbf{P}(1^2,2,3,5^2)$. Let us use lower case letters instead of upper case ones to denote the coordinates, as there is no risk of confusion here.

The equations that cut out the curve C in $\mathbf{P}$ are thus general quintics of $\mathbf{P}(1^2,2,3,5^2)$, and by the generality assumption we may choose them to be

$$x_0 = f_5(\mathbf{u}, v, w) \text{ and } x_1 = h_5(\mathbf{u}, v, w),$$

where f_5 and h_5 are homogeneous of degree 5 in the coordinates $(\mathbf{u}, v, w)$.

This realizes C as the curve in $\mathbf{P}(1,1,2,3)$ given by the two following sextic equations:

$$w^2 = u_0 f_5(\mathbf{u}, v, w),\ v^3 = u_1 h_5(\mathbf{u}, v, w).$$

In other words, C is the intersection of the two sextic surfaces $\Sigma = \left\{v^3 = u_1 h_5(\mathbf{u}, v, w)\right\}$ and $\Theta = \left\{w^2 = u_0 f_5(\mathbf{u}, v, w)\right\}$. By adjunction, we have $K_C = \mathcal{O}_{\mathbf{P}(1,1,2,3)}(5)|_C$ and we have indeed

$$g(C) = h^0(C, K_C) = h^0(\mathbf{P}(1,1,2,3), \mathcal{O}_{\mathbf{P}(1,1,2,3)}(5)) = 16,$$

as expected from Table 12.

Lemma 1.4.27. *Consider the map $\varepsilon : \mathbf{P}(1,1,2,3) \dashrightarrow \mathbf{P}(1,1,2)$ given by*

$$[u_0 : u_1 : v : w] \mapsto [u_0 : u_1 : v].$$

The restriction of ε to Σ yields a birational map $\Sigma \dashrightarrow \mathbf{P}(1,1,2)$ which contracts the rational curve $\mathfrak{f} = \{u_1 = v = 0\}$ to the point $p_{u_0} = [1 : 0 : 0] \in \mathbf{P}(1,1,2)$. In addition, the restriction of ε to $\Sigma - \mathfrak{f}$ is an isomorphism onto its image.

Proof: First of all, we identify the unique indeterminacy point of ε as $p_w = [0 : 0 : 0 : 0 : 1]$.

The fiber of ε over a smooth point of $\mathbf{P}(1,1,2)$ (say, over $[u_0 : u_1 : v] \neq [0 : 0 : 1]$) is a $\mathbf{P}(1,3)$ in $\mathbf{P}(1,1,2,3)$ parametrized by

$$[\mathtt{u} : \mathtt{w}] \in \mathbf{P}(1,3) \to [\mathtt{u}u_0 : \mathtt{u}u_1 : \mathtt{u}^2 v : \mathtt{w}] \in \varepsilon^{-1}([u_0 : u_1 : v]).$$

In addition, in this parametrization, the indeterminacy point of ε along this fibre is $[0 : 1]$.

It follows from the defining equation of Σ in $\mathbf{P}(1,1,2,3)$ that its restriction to such a fiber is the locus in $\mathbf{P}(1,3)$ given by an equation of the form $\mathtt{u}^3(\mu\mathtt{u}^3 + \mathtt{w}) = 0$, with μ a constant. Via the isomorphism

$$[\mathtt{u} : \mathtt{w}] \in \mathbf{P}(1,3) \mapsto [\mathtt{u}^3 : \mathtt{w}] \in \mathbf{P}^1,$$

we see that Σ has degree 2 on this fibre and it cuts out (transversally) two disctinct points, one of which is the indeterminacy point p_w of ε (which is parametrized by $[\mathtt{u} : w] = [0 : 1]$). The only exception is the particular fiber $\mathfrak{f} := \{u_1 = v = 0\}$, which is a $\mathbf{P}(1,3)$ with coordinates $[u_0 : w]$, contained in Σ and contracted to the smooth point $p_{u_0} \in \mathbf{P}(1,1,2)$.

Meanwhile, the fiber over the singular point $p_v \in \mathbf{P}(1,1,2)$, consists of all the points $[u_0 : u_1 : v : w]$ for which $(u_0, u_1) = (0,0)$ and thus is a $\mathbf{P}(2,3)$ with coordinates $[v : w]$, and the restriction of Σ to this particular fiber is given by the equation $v^3 = 0$. Hence Σ meets this particular fiber only at the indeterminacy point of ε.

Therefore, $\varepsilon|_\Sigma$ is a birational map from Σ to $\mathbf{P}(1,1,2)$ with indeterminacy point $p_w = \{u_0 = u_1 = v = 0\}$ and its restriction to $\Sigma - \{p_w\}$ is a birational morphism

$$\Sigma - \{p_w\} \to \mathbf{P}(1,1,2) - \{p_v\}$$

which contracts the curve $\mathfrak{f}$ to the point p_{u_0} and is an isomorphism on $\Sigma - \mathfrak{f}$. ■

Lemma 1.4.28. *At the general point q of $\mathfrak{f}$, we have*

$$\ker(d_q\varepsilon) = T_q\mathfrak{f}.$$

Proof: Consider the open subset $U = \{u_0 \neq 0\}$ of $\mathbf{P}(1,1,2,3)$. It is isomorphic to $\mathbf{C}^3$ with coordinates $(\mathtt{u}_1, \mathtt{v}, \mathtt{w})$ and its image via ε is the open subset $\{u_0 \neq 0\}$ of $\mathbf{P}(1,1,2)_{[u_0:u_1:v]}$, which is isomorphic to $\mathbf{C}^2_{(\mathtt{u}_1,\mathtt{v})}$. Moreover, the intersection of U with $\mathfrak{f} = \{u_1 = v = 0\}$ is $\mathfrak{f} - \{p_w\}$.

The map $\bar{\varepsilon} : \mathbf{C}^3 \to \mathbf{C}^3$ given by $(\mathtt{u}_1, \mathtt{v}, \mathtt{w}) \mapsto (\mathtt{u}_1, \mathtt{v})$ lifts ε in the commutaitive diagram:

$$\begin{array}{ccc} \mathbf{C}^3 & \xrightarrow{\ \bar{\varepsilon}\ } & \mathbf{C}^3 \\ \simeq\downarrow & & \downarrow\simeq \\ U & \xrightarrow[\ \varepsilon\ ]{} & \varepsilon(U) \end{array}$$

Since the preimage $\bar{\mathfrak{f}}$ of $\mathfrak{f}$ in $\mathbf{C}^3_{(\mathtt{u}_1,\mathtt{v},\mathtt{w})}$ is the vanishing locus of $\mathtt{u}_1$ and $\mathtt{v}$, we check by a straightforward computation that for the general point $\mathtt{q} \in \bar{\mathfrak{f}}$, we have

$$\ker(d_{\mathtt{q}}\bar{\varepsilon}) = T_{\mathtt{q}}\bar{\mathfrak{f}}.$$

This ends the proof. ■

Lemma 1.4.29. *The curve C is the normalization of a singular curve C_0 of degree 12 in $\mathbf{P}(1,1,2)$, i.e., a sextic section of the cone over a conic. The curve C_0 admits a singular point p at which two smooth local branches meet. Moreover, p is a smooth point of $\mathbf{P}(1,1,2)$ and the line $\Delta \in |\mathcal{O}_{\mathbf{P}(1,1,2)}(1)|$ through p is such that $C_0|_\Delta = 6p$. Furthermore, there is another line $\Delta' \in |\mathcal{O}_{\mathbf{P}(1,1,2)}(1)|$ such that C_0 is tri-tangent to Δ', meaning*

$$C_0|_{\Delta'} = 2p_1 + 2p_2 + 2p_3,$$

where p_1, p_2 and p_3 are general points of Δ'.

Conversely, the normalization of any such 12-ic curve $C_0' \subset \mathbf{P}(1,1,2)$ can be realized as a member of $|-K_{\mathbf{P}}|_{S'}|$ for a K3 surface $S' \in |-K_{\mathbf{P}}|$.

Proof: Using the parametrization of $\mathfrak{f} = \operatorname{Exc}(\varepsilon|_\Sigma)$ as a $\mathbf{P}(1,3)$ with coordinates $[u_0 : w]$, we know from the equations for C in $\mathbf{P}(1,1,2,3)$:

$$v^3 = u_1 h_5(\mathbf{u}, v, w),\ w^2 = u_0 f_5(\mathbf{u}, v, w),$$

that the restriction $C|_\mathfrak{f}$ is cut out by an equation of the form $\tau u_0^6 + \lambda u_0^3 w + w^2 = 0$, where λ and τ are constants. Hence, the curve C meets the fibre $\mathfrak{f}$ of ε at two points, but it does not contain the indeterminacy point p_w of ε. This implies in particular that the restriction $\varepsilon|_C$ is a regular map with image a curve $C_0 \subset \mathbf{P}(1,1,2)$. Since $\varepsilon|_\Sigma$ is birational and an isomorphism on $\Sigma - \mathfrak{f}$ (Lemma 1.4.27), the morphism from C to C_0 is birational; besides, it maps the two points $C \cap \mathfrak{f}$ to a single point p. By Lemma 1.4.28, the curve C_0 has two smooth local branches at p, and C is the normalization of C_0, since it is smooth.

Furthermore, C_0 is a member of $|\mathcal{O}_{\mathbf{P}(1,1,2)}(d)|$ for some d such that

$$\frac{d}{2} = C_0 \cdot \mathcal{O}_{\mathbf{P}(1,1,2)}(1) = C \cdot \mathcal{O}_{\mathbf{P}(1,1,2,3)}(1) = \mathcal{O}_{\mathbf{P}(1,1,2,3)}(6)^2 \cdot \mathcal{O}_{\mathbf{P}(1,1,2,3)}(1) = 6.$$

The curve C_0 is thus given by a degree $d = 12$ equation on $\mathbf{P}(1,1,2)$; in other words, it is a sextic section of the cone over a conic.

In particular, let $\Delta = \{u_1 = 0\}$ be the line through p in $\mathbf{P}(1,1,2)$. By the above, $C_0|_\Delta$ has degree 6. But the intersection of C_0 with Δ is the image via ε of $C \cap \{u_1 = 0\}$, and since we have $C \subset \Sigma$ and $\Sigma \cap \{u_1 = 0\} = \{u_1 = v = 0\} = \mathfrak{f}$, the curve C_0 has only one contact point with Δ:

$$C_0 \cap \Delta = \varepsilon(C|_{u_1=0}) \subset \varepsilon(\mathfrak{f}) = \{p\}.$$

Hence $C_0|_\Delta = 6p$.

Now let Δ' be the line $\{u_0 = 0\}$ in $\mathbf{P}(1,1,2)$. The restriction of C_0 to Δ' is the image via ε of $C|_{u_0=0}$. Consider the surface

$$\Theta = \{w^2 = u_0 f_5(\mathbf{u}, v, w)\}$$

in $\mathbf{P}(1,1,2,3)$; by the fact that $C = \Sigma \cap \Theta$ and

$$\Theta|_{u_0=0} = (w^2) = 2\ell$$

where ℓ is the curve $u_0 = w = 0$, which is a $\mathbf{P}(1,2)$ with coordinates $[u_1 : v]$, and $\Sigma \cdot \ell = 3$ in $\mathbf{P}(1,1,2,3)$, we have

$$C|_{u_0=0} = \Theta|_{u_0=0} \cap \Sigma|_{u_0=0} = 2\ell \cap \Sigma|_{u_0=0}$$

which consists of three double points. Therefore, $C_0|_{\Delta'} = 2p_1+2p_2+2p_3$ where p_1, p_1, p_3 are general points of Δ'. Since $C \to C_0$ is an isomorphism over $C_0 - \{p\}$, the three contact points of C_0 with Δ' are tangency points.

Conversely, let C_0' be such a curve in $\mathbf{P}(1,1,2)$ and C' its proper transform in Σ via the birational map

$$\varepsilon|_\Sigma : \Sigma \dashrightarrow \mathbf{P}(1,1,2).$$

As C_0' is given by an equation of degree 12, we have

$$6 = C_0' \cdot \mathcal{O}_{\mathbf{P}(1,1,2)}(1) = C' \cdot \mathcal{O}_{\mathbf{P}(1,1,2,3)}(1) = \Sigma \cdot \mathcal{O}_{\mathbf{P}(1,1,2,3)}(6) \cdot \mathcal{O}_{\mathbf{P}(1,1,2,3)}(1),$$

i.e., $C' = \mathcal{O}_{\mathbf{P}(1,1,2,3)}(6)|_\Sigma$ in $\mathrm{Pic}(\Sigma)$.

The restriction exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}(1,1,2,3)} \to \mathcal{O}_{\mathbf{P}(1,1,2,3)}(6) \to \mathcal{O}_{\mathbf{P}(1,1,2,3)}(6)|_\Sigma \to 0$$

and the vanishing of $h^1(\mathbf{P}(1,1,2,3), \mathcal{O}_{\mathbf{P}(1,1,2,3)})$ imply that the map

$$H^0(\mathbf{P}(1,1,2,3), \mathcal{O}_{\mathbf{P}(1,1,2,3)}(6)) \to H^0(\Sigma, \mathcal{O}_{\mathbf{P}(1,1,2,3)}(6)|_\Sigma)$$

is surjective, hence there exists a sextic Θ' of $\mathbf{P}(1,1,2,3)$ such that $C' = \Sigma \cap \Theta'$. Let $f_6(\mathbf{u}, v, w) = 0$ be an equation for Θ'. As C_0' is tri-tangent to the line Δ', the set $C_0' \cap \{u_0 = 0\}$ has cardinality 3, whereas

$$\deg C'|_{u_0=0} = \Theta'|_{u_0=0} \cdot \Sigma|_{u_0=0} = \mathcal{O}_{\mathbf{P}(1,1,2,3)}(6)^2 \cdot \mathcal{O}_{\mathbf{P}(1,1,2,3)}(1) = 6.$$

Hence $\Theta'|_{u_0=0}$ is a nonreduced curve $2\ell'$ of $\mathbf{P}(1,2,3)_{[u_1:v:w]}$ such that $\ell' \cdot \Sigma|_{u_0=0} = 3$. This yields $f_6(\mathbf{u}, v, w)|_{u_0=0} = h_3(u_1, v, w)^2$ with h_3 a homogeneous cubic. Up to scaling, we have $h_3 = w + \alpha(u_1, v)$ with $\deg \alpha = 3$, and the change of variables $w \mapsto w + \alpha(u_1, v)$, which is an automorphism of $\mathbf{P}(1,1,2,3)$, yields

$$f_6(\mathbf{u}, v, w)|_{u_0=0} = w^2,$$

thus

$$f_6(\mathbf{u}, v, w) = u_0 f_5'(\mathbf{u}, v, w) + w^2$$

for some homogeneous quintic f_5'. Hence C' lies on the surface Θ' with equation

$$u_0 f_5'(\mathbf{u}, v, w) + w^2 = 0.$$

As a consequence, C' is the complete intersection in $\mathbf{P}(1,1,2,3)$ given by the following equations:

$$u_0 f_5'(\mathbf{u}, v, w) + w^2 = 0, v^3 = u_1 h_5(\mathbf{u}, v, w).$$

As $\mathbf{P}$ is cut out in $\mathbf{P}(1^2, 2, 3, 5^2)_{[u_0:u_1:v:w:x_0:x_1]}$ by the equations $u_0 x_0 = w^2$ and $u_1 x_1 = v^3$ (see (1.11)) and the pullbacks of the hyperplanes of $\mathbf{P}^{17}$ are the quintic hypersurfaces of $\mathbf{P}(1^2, 2, 3, 5^2)$, we see that C' is a linear section of $\mathbf{P}$ in $\mathbf{P}^{17}$:

$$C' = \mathbf{P} \cap \{x_0 = -f_5'(\mathbf{u}, v, w), x_1 = h_5(\mathbf{u}, v, w)\}.$$

In other words, there exists a hyperplane section $S' \in |-K_\mathbf{P}|$ of $\mathbf{P}$ in $\mathbf{P}^{17}$ such that C' is a hyperplane section of S', i.e., $C' \in |-K_\mathbf{P}|_{S'}|$. ■

1.5 Future projects

This chapter has been dedicated to the study of weighted projective spaces as extensions of canonical curves and K3 surfaces. Weighted projective spaces make a good class of varieties as they are a natural generalization of projective spaces and they are endowed with homogeneous weighted coordinates making the computations manageable.

However, as we have seen in Lemmas 1.3.4, 1.3.8 and 1.3.9, the condition for a weighted projective space to be an extension of a K3 surface is very restrictive. Two new questions arise naturally.

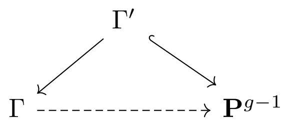
In the first place, one could look at the list of the K3 hypersurfaces which are listed in [Bel97]. These K3 surfaces are given as hypersurfaces of weighted projective 3-spaces, some of which are not Gorenstein. By a classification from Reid and Fletcher (see for instance [Ia00, 13.3]), there are 95 such weighted projective 3-spaces containing K3 surfaces as hypersurfaces.

Given such a WPS $\mathbf{P}$ and $S \in |-K_{\mathbf{P}}|$, if $K_{\mathbf{P}}$ is not basepoint-free, then it is only $\mathbf{Q}$-Cartier. However, it may happen that the induced map $\mathbf{P} \dashrightarrow \mathbf{P}^N$ is birational onto its image, and that the image of S is K3 as well. A first step would be to identify such items for which $-K_{\mathbf{P}}$ induces a birational map onto an extension of a K3 surface, identify their image and then investigate the existence of further extensions.

Example. $\mathbf{P} = \mathbf{P}(3, 6, 7, 8)$ is not Gorenstein, as the canonical divisor admits a basepoint, namely $p_z = [0 : 0 : 1 : 0]$. Resolving the indeterminacy of the induced map $\mathbf{P} \dashrightarrow \mathbf{P}^{12}$ would require to do a blow-up (possibly weighted) at the point p_z. Likewise, any anticanonical K3 surface $S \subset \mathbf{P}$ passes through the point p_z, and it is possible that its image is not K3. However, the image of $\mathbf{P}$ is by construction an extension of the image of S. Note that given a curve $\Gamma = S \cap S'$ given as the intersection of two general anticanonical divisors of $\mathbf{P}$, we have by adjunction

$$K_\Gamma = -K_{\mathbf{P}}|_\Gamma$$

and the image of Γ via the linear system $|-K_{\mathbf{P}}|$ is a birational modification of Γ at the point p_z. It is thus a curve of the same geometric genus; by generality we may assume that Γ is smooth outside p_z, resolve its singularities to get the smooth curve Γ' and consider the canonical model $\Gamma' \hookrightarrow \mathbf{P}^{g-1}$, with the commutative diagram



It follows that the image X of $\mathbf{P}$ via $|-K_{\mathbf{P}}|$ is a threefold extension of Γ'. It would be interesting to identify X and look for extensions of larger dimension.

In the second place, instead of looking for extensions of K3s among weighted projective spaces, one could try to find extensions among complete intersections in WPS. This is promising, as a lot of interesting varieties can be realized as complete intersections in weighted projective spaces, and these form a class of varieties for which adjunction and computations on the ambient space make the situation manageable.

Chapter 2

Mori contractions of submaximal length

2.1 Mori theory and Mori contractions

2.1.1 Mori's cone theorem

When considering a quasi-projective variety X with Cartier (or even just $\mathbf{Q}$-Cartier) canonical divisor K_X, the question arises whether there exist projective curves $C \subset X$ which are negative against K_X, i.e.

$$K_X \cdot C < 0.$$

Definition 2.1.1. *Two projective curves $C, C' \subset X$ are numerically equivalent if for every Cartier divisor $D \in \mathrm{Pic}(X)$, we have $D \cdot C = D \cdot C'$. In this case, we write $C \sim_{\mathrm{num}} C'$.*

The quotient of the group of 1-cycles on X under numerical equivalence is denoted by $N_1(X)$:

$$N_1(X) = \frac{\{\sum_{\mathrm{finite}} a_C C \mid a_C \in \mathbf{Z},\ C \textit{ curve } \subset X\}}{\sim_{\mathrm{num}}}.$$

Two Cartier divisors $D, D' \in \mathrm{Pic}(X)$ are numerically equivalent if for every projective curve $C \subset X$, we have $D \cdot C = D' \cdot C$. In this case, we write $D \sim_{\mathrm{num}} D'$.

The quotient of the Picard group of X under numerical equivalence is denoted by $N^1(X)$:

$$N^1(X) = \frac{\mathrm{Pic}(X)}{\sim_{\mathrm{num}}}.$$

We set

$$N^1(X)_{\mathbf{R}} = N^1(X) \otimes_{\mathbf{Z}} \mathbf{R} \textit{ and } N_1(X)_{\mathbf{R}} = N_1(X) \otimes_{\mathbf{Z}} \mathbf{R}.$$

Notation. We usually denote by $[C]$ the numerical equivalence class of a curve C, and by $[D]$ the numerical equivalence class of a Cartier divisor D.

Remark. Since X is quasi-projective, note that there exists no projective curve $C \subset X$ such that $C \sim_{\mathrm{num}} 0$.

⋆ ⋆

It follows from the definition that the intersection pairing

$$N_1(X)_{\mathbf{R}} \otimes N^1(X)_{\mathbf{R}} \to \mathbf{R}$$

is bilinear and nondegenerate, i.e., it is a duality pairing. The dimension of $N_1(X)_{\mathbf{R}}$ and $N^1(X)_{\mathbf{R}}$ is finite, called the Picard rank of X and denoted by $\rho(X)$.

A similar notion can be defined in the relative setting. If $f : X \to Y$ is a surjective morphism onto an algebraic variety Y, then there is an inclusion $f^*\mathrm{Pic}(Y) \hookrightarrow \mathrm{Pic}(X)$.

If Δ is a numerically trivial Cartier divisor on Y, then for every irreducible curve $C \subset X$ we have $(f^*\Delta) \cdot C = 0$ since the restriction of f to C is a morphism

$$C \to f(C)$$

with either $f(C)$ a point, or $f(C)$ a curve such that $\deg \Delta|_{f(C)} = 0$. This yields by $\mathbf{R}$-linearity an inclusion

$$f^* N^1(Y)_{\mathbf{R}} \hookrightarrow N^1(X)_{\mathbf{R}}.$$

Definition 2.1.2. *We define the relative Picard group of f as the quotient*

$$\frac{\mathrm{Pic}(X)}{f^*\mathrm{Pic}(Y)}.$$

We define the relative Picard rank of f, which we denote by $\rho(X/Y)$, as the dimension of the quotient

$$\frac{N^1(X)_{\mathbf{R}}}{f^* N^1(Y)_{\mathbf{R}}}.$$

Lemma 2.1.3. *Given any surjective morphism $f : X \to Y$, the relative Picard rank $\rho(X/Y)$ is equal to $\rho(X) - \rho(Y)$.*

This lemma is a straightforward consequence of the definition of $\rho(X/Y)$.

Definition 2.1.4 (Contraction — Elementary contraction)**.** *A contraction from a variety X is a projective morphism $f : X \to Y$ with connected fibres for which there exists at least one irreducible curve $C \subset X$ such that $f(C)$ is a point. We distinguish three cases:*

- $\dim Y < \dim X$*; we say that f is a fibration (or sometimes, of fibre type),*
- *the exceptional locus of f is a hypersurface of X; we say that f is a divisorial contraction,*
- *the exceptional locus of f has codimension more than 1 in X; we say that f is a small contraction.*

Moreover, we say that f is elementary *if it satisfies*

$$\rho(X/Y) = 1.$$

If $f' : X \to Y'$ is a proper morphism which contracts a curve but admits a disconnected fibre, then we may consider its Stein factorization: $f' = g \circ f$, in which g has finite fibres and f has connected fibres, hence f is a contraction:

$$X \xrightarrow{f} Y \xrightarrow[\text{finite}]{g} Y'.$$

Besides, it follows from the definition that a contraction is either a fibration or birational.

If moreover f is elementary, then $f^* N^1(Y)_{\mathbf{R}}$ is a hyperplane of $N^1(X)_{\mathbf{R}}$. In this situation, consider C, C' two curves which are contracted by f. Then for every Cartier divisor Δ on Y, we have

$$f^*\Delta \cdot C = f^*\Delta \cdot C' = 0,$$

so according to the duality pairing

$$N_1(X)_{\mathbf{R}} = N^1(X)_{\mathbf{R}}^{\vee},$$

the numerical equivalence classes $[C]$ and $[C']$ both have the same kernel $f^* N^1(Y)_{\mathbf{R}}$ as linear forms on $N^1(X)_{\mathbf{R}}$, hence they are proportional:

$$\exists \lambda \in \mathbf{R} \, : \, [C] = \lambda [C'].$$

⋆ ⋆

We now make a very important definition.

Definition 2.1.5 (Mori cone)**.** *The cone of curves is the subset of $N_1(X)_{\mathbf{R}}$ which consists of effective 1-cycles, i.e., linear combinations of curves with nonnegative coefficients.*

$$NE(X) = \{\sum_{\text{finite}} a_C[C] \mid a_C \in \mathbf{R}, a_C \geq 0\}.$$

The Mori cone $\overline{NE(X)}$ *of X is the closure of $NE(X)$ inside $N_1(X)_{\mathbf{R}}$.*

We cite [Deb16] as a reference about Mori theory, and in particular Theorem 4.10 (Kleiman's criterion), Proposition 4.21 about the relative cone of curves, and §4.8 for topological properties about cones.

Now come the two most important theorems of this section.

Theorem 2.1.6 (Mori's cone theorem)**.** *Let X be a smooth projective variety. Up to numerical equivalence, there exists a countable set of K_X-negative rational curves C_i such that*

$$0 < -K_X \cdot C_i \leq \dim X + 1$$

for every i, and the extremal rays of the K_X-negative part of the Mori cone $\overline{NE(X)}$ are $\mathbf{R}_+[C_i]$, in other words

$$\overline{NE(X)} = \overline{NE(X)}_{K_X \geq 0} + \sum_i \mathbf{R}_+[C_i].$$

Moreover, an accumulation of such extremal rays $\mathbf{R}_+[C_i]$ can only happen towards the K_X-trivial part of $\overline{NE(X)}$.

We may refer to [Deb16, §8.1] for the proof of this theorem, although the original proof by Mori can be found in [KM98, §1.3] or in [Mor82].

Theorem 2.1.7. *Let X be a smooth projective variety. Given $R \subset \overline{NE(X)}$ an extremal K_X-negative extremal ray, there exists an elementary contraction $f_R : X \to Y$ to a normal variety such that a curve $C \subset X$ is contracted to a point by f if and only if its numerical equivalence class $[C]$ belongs to R. Moreover, $-K_X$ is f_R-ample.*

Proof: By [Deb16, Corollary 8.2] and [KMM87, Theorem 3.1.1], there exists a nef divisor M_R which is semiample (i.e., a high enough multiple is basepoint-free) such that

$$R = \left\{\zeta \in \overline{NE(X)} \mid M_R \cdot \zeta = 0\right\}.$$

For $m \gg 0$, we may consider the morphism induced by the linear system $|mM_R|$. It follows from the above that the only contracted curves are those whose numerical equivalence classes lie in the ray R. Up to taking the Stein factorization of this morphism, we end up with a contraction $f_R : X \to Y$ such that $\rho(X/Y) = 1$. Indeed, by [Deb16, Corollary 8.4], given any numerical equivalence class of a curve $[C] \in R$, we have the exact sequence

$$0 \to \mathrm{Pic}(Y) \xrightarrow{f_R^*} \mathrm{Pic}(X) \xrightarrow{\cdot C} \mathbf{Z},$$

which ensures that $f_R^* N^1(Y)_{\mathbf{R}}$ is a hyperplane of $N^1(X)_{\mathbf{R}}$.

The fact that $-K_X$ is relatively ample is a consequence of the relative version of Kleiman's criterion ([Deb16, Exercise 4.18]). ■

Definition 2.1.8 (Supporting divisor)**.** *Given a K_X-negative extremal ray R on X, a divisor M_R such that*

$$R = \left\{\zeta \in \overline{NE(X)} \mid M_R \cdot \zeta = 0\right\}$$

is called a supporting divisor *of R.*

Corollary 2.1.9. *Let $f_R : X \to Y$ be an elementary contraction associated with an extremal K_X-negative ray R of the Mori cone of X. If C is a curve such that $[C] \in R$ and L and M are two line bundles on X having the same degree on C, then they are relatively linearly equivalent, meaning $L \otimes M^\vee \simeq f^*J$ for some $J \in \mathrm{Pic}(Y)$.*

This is a straightforward consequence of the exact sequence

$$0 \to f^*\mathrm{Pic}(Y) \to \mathrm{Pic}(X) \xrightarrow{\cdot C} \mathbf{Z}$$

which is exact under the condition that f_R is an elementary contraction and $[C]$ lies in the associated K_X-negative extremal ray.

Terminology. A contraction f_R of a K_X-negative extremal ray R is called an *elementary Mori contraction*. When there is no ambiguity, we may denote f_R as f. This definition agrees with Definition 0.0.7, and such an elementary Mori contraction has indeed relative Picard rank 1, as required in Definition 2.1.4.

The locus on which f is not an isomorphism is the locus covered by the curves $C \subset X$ for which $[C] \in R$.

Remark. When the relative dimension of f is positive, meaning that X is covered by curves whose numerical equivalence classes lie in R, it is an unusual choice of vocabulary to say that X is the exceptional locus of f. However, to lighten the notation, we allow ourselves to write $\mathrm{Exc}(f) = X$ for a fibration f, and to say that X is the exceptional locus.

2.1.2 Length

Let us recall the notion of length for an elementary Mori contraction, which we initially gave in Definition 0.0.8:

Definition 2.1.10 (Length)**.** *Let X be a smooth projective variety and $f : X \to Y$ an elementary Mori contraction associated with a K_X-negative extremal ray R. The length of f (or equivalently, the length of R) is defined as the minimal degree of $-K_X$ on the rational curves which are contracted by f:*

$$l(f) = l(R) = \min \{-K_X \cdot \Gamma \mid [\Gamma] \in R, \Gamma \textit{ a rational curve}\}.$$

Definition 2.1.11 (Minimal curve)**.** *Let $\Gamma \subset X$ be a rational curve such that $[\Gamma] \in R$ and $-K_X \cdot \Gamma = l(f)$. We say that Γ is an f−minimal curve, or a $(-K_X)$-minimal curve, or more simply a* minimal curve *when there is no risk of confusion.*

⋆ ⋆

We know thanks to P. Ionescu and J. Wiśniewski that the length of f satisfies the inequality of Theorem 0.0.9: see [Io86, Theorem 0.4] and [Wi91, Theorem 1.1]. Let us recall here the said inequality: under the condition that $f : X \to Y$ is an elementary Mori contraction from a smooth projective variety, $E \subset X$ is an irreducible component of the f-exceptional locus and $F \subset E$ is an irreducible component of a fibre, we have

$$\dim E + \dim F \geq \dim X + l(f) - 1.$$

Remark. If f is a fibration, then $E = X$ is irreducible. If f is divisorial, then $E = \mathrm{Exc}(f)$ is irreducible thanks to [Deb16, Proposition 8.7.b]. However, if f is a small contraction, then it may happen that $\mathrm{Exc}(f)$ is disconnected (see [Deb16, Example 8.22]).

Terminology. Assume that E is an irreducible component of the f-exceptional locus such that $\dim E$ is smallest among the irreducible components of $\mathrm{Exc}(f)$. Let moreover F be an irreducible component of a general fibre in E, so that F has the expected dimension:

$$\dim F = \dim E - \dim f(E).$$

Then the optimal upper bound for $l(f)$ is

$$l(f) \leq \dim E - \dim X + \dim F + 1.$$

If f is an elementary Mori contraction such that the upper bound is met, we say that its length is *maximal*. If $l(f) = \dim E - \dim X + \dim F$, we say that the length of f is *submaximal*.

We recall Theorem 0.0.10, a result by A. Höring and C. Novelli about the classification of elementary Mori contractions of maximal length. The rest of this chapter is dedicated to the study of elementary Mori contractions of submaximal length, both of fibre type and divisorial.

Concerning small Mori contractions, see [AT16, Theorem 1.2.B], in which M. Andreatta and L. Tasin exhibited the structure of a projective space for the f-exceptional locus when $\dim \mathrm{Exc}(f) = \dim X - 2$, under a condition on a supporting divisor of f (in the sense of Definition 2.1.8).

Theorem 2.1.12 ([AT16], Theorem 1.2.B)**.** *Let $f : X \to Y$ be a contraction from a $\mathbf{Q}$-factorial terminal variety X such that $-K_X$ is f-ample. Assume that X has only points of index 1 and 2 and that f is supported by the nef divisor $K_X + \tau L$ with L an f-ample line bundle and τ a positive rational number such that $\tau > \dim X - 3 \geq 0$. Assume furthermore that every irreducible component of $\mathrm{Exc}(f)$ has codimension 2.*

Then $\mathrm{Exc}(f)$ is irreducible, $f(\mathrm{Exc}(f))$ is a point, $(E, L|_E) \simeq (\mathbf{P}^{\dim X - 2}, \mathcal{O}_{\mathbf{P}^{\dim X-2}}(1))$ and $\tau = \frac{2\dim X - 5}{2}$.

Definition 2.1.13. *Let X, L and f be as above. The positive rational number τ such that $K_X + \tau L$ is a nef supporting divisor of f is called the* nef value *of the pair (X, L).*

The nef value of the pair (X, L) is the smallest positive rational number r such that $K_X + rL$ is nef, hence the terminology. This notion is often used in the works of Andreatta, Tasin and Wiśniewski. When f is an elementary Mori contraction supported by $K_X + \tau L$ with L ample, the length of f is a multiple of the nef value τ of (X, L).

⋆ ⋆

We introduce now a tool for the deformation of rational curves on a projective variety X whose notation follows [Kol96, §II.2].

Notation. Given a quasi-projective variety X, we denote by $\mathrm{RatCurves}^n(X)$ the space parametrizing rational curves in X.

If $f : X \to Y$ is a projective morphism, then we denote by $\mathrm{RatCurves}^n(X/Y)$ the space parametrizing rational curves in X which are contracted by f.

This notation is the same as the one used in [HN13]. The existence of these spaces follows from [Kol96, Proposition II.2.11.1]. Briefly speaking, the space $\mathrm{RatCurves}^n(X)$ is constructed as the normalization of the quotient $\mathrm{Hom}_{\mathrm{bir}}(\mathbf{P}^1, X)/\mathrm{Aut}(\mathbf{P}^1)$, where $\mathrm{Hom}_{\mathrm{bir}}(\mathbf{P}^1, X)$ parametrizes morphisms $\mathbf{P}^1 \to X$ which are birational onto their image.

In [Kol96, I.6.9], J. Kollár constructed a morphism $\theta : \mathrm{Hom}(\mathbf{P}^1, X) \to \mathrm{Chow}(X)$ which identifies any morphism

$$\eta : \mathbf{P}^1 \to X$$

with any other representative of its class under the action of $\mathrm{Aut}(\mathbf{P}^1)$ by precomposition:

$$\forall h \in \mathrm{Aut}(\mathbf{P}^1),\ h \cdot \eta = \eta \circ h.$$

Let us denote by W an irreducible component of the open subset $\mathrm{Hom}_{\mathrm{bir}}(\mathbf{P}^1, X) \subset \mathrm{Hom}(\mathbf{P}^1, X)$, and consider $\overline{V}$ the closure of its image $\theta(W)$. By [Kol96, II.2.2], the points of $\overline{V}$ represent some 1-cycles in X whose components are rational. We consider $V \subset \overline{V}$ the open subset which consists in irreducible and reduced such 1-cycles, i.e., rational curves in X. Then by the construction of Kollár all the irreducible components of $\mathrm{RatCurves}^n(X)$ are normalizations of such irreducible subspaces $V \subset \mathrm{Chow}(X)$.

The construction of $\mathrm{RatCurves}^n(X/Y)$ follows a similar method in the relative case.

Definition 2.1.14 (Deformation family)**.** *Let $\Gamma \subset X$ be a rational curve. By construction, it is represented by a point of $\mathrm{RatCurves}^n(X)$, and there exists an irreducible component $\mathcal{H} \subset \mathrm{RatCurves}^n(X)$ containing this point. We say that $\mathcal{H}$ is a* deformation family *of Γ in X.*

If $X \to Y$ is a projective morphism and $\Gamma \subset X$ is a rational curve contained in a fibre, then it is represented by a point of an irreducible component $\mathcal{H} \subset \mathrm{RatCurves}^n(X/Y)$. We say that $\mathcal{H}$ is a deformation family *of Γ in X over Y.*

Remark. By construction, the points of a deformation family $\mathcal{H}$ represent only irreducible rational curves. This family $\mathcal{H}$ is the normalization of a subspace V of $\mathrm{Chow}(X)$ whose closure is $\overline{V}$. Then $\mathcal{H}$ is dense in the normalization $\overline{\mathcal{H}}$ of $\overline{V}$. The points of $\overline{\mathcal{H}}$ represent 1-cycles in X which are degenerations of rational curves.

Terminology. Let a rational curve $\Gamma \subset X$, a deformation family $\mathcal{H}$, and $\overline{\mathcal{H}}$ be as above. We say that $\overline{\mathcal{H}}$ is a *closed deformation family* of Γ in X.

In the relative case, given a projective morphism $X \to Y$ and a rational curve Γ contained in a fibre and $\mathcal{H}$ a deformation family of Γ in X over Y, it also follows by the construction of [Kol96] that $\mathcal{H}$ is contained in a closed subspace $\overline{\mathcal{H}}$, which is the normalization of a closed irreducible component of $\mathrm{Chow}(X/Y)$ containing the point representing Γ.

Definition 2.1.15 (Unsplit deformation family)**.** *Let $X \to Y$ be a projective morphism and $\Gamma \subset X$ a contracted rational curve, with $\mathcal{H}$ a deformation family of Γ in X over Y and $\overline{\mathcal{H}}$ the associated closed deformation family. We say that $\overline{\mathcal{H}}$ is unsplit if all its members are reduced and irreducible, i.e., $\mathcal{H} = \overline{\mathcal{H}}$.*

Moreover, given a closed deformation family $\overline{\mathcal{H}}$ of a rational curve in a variety X, there exists a *universal family* over $\overline{\mathcal{H}}$

$$\mathcal{U} = \left\{(p, Z) \in X \times \overline{\mathcal{H}} \mid Z \in \overline{\mathcal{H}},\ p \in \mathrm{Supp}(Z)\right\},$$

with the two projections

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\ ev\ } & X \\ {\scriptstyle q}\downarrow & & \\ \overline{\mathcal{H}} & & \end{array}$$

In the above, the notation ev stands for "evaluation morphism". The existence of $\mathcal{U}$ is explained in [Kol96, II.2.8, II.2.12].

⋆ ⋆

Lemma 2.1.16. *Assume $X \to Y$ is an equidimensional projective fibration between quasi-projective varieties. If Y is normal and L is a line bundle on X, then the degree of L is constant across the fibres of $X \to Y$. In other words, for any two fibres X_p, X_t of dimension $n = \dim X - \dim Y$ we have*

$$(L|_{X_p})^n = (L|_{X_t})^n.$$

This is an application of [Kol96, I.3.12]. This holds if $X \to Y$ is a so-called *well-defined* family of cycles, which is true under the condition that Y is normal (see [Kol96, I.3.10.5]). In general, the well-definedness of the family relies on [Kol96, I.3.10.4], which is a rather difficult condition to verify.

The following result about closed deformation families of rational curves follows from Lemma 2.1.16:

Lemma 2.1.17. *Let $f : X \to Y$ be a projective morphism, Γ a rational curve contained in a fibre of f and $\overline{\mathcal{H}}$ a closed deformation family of Γ in X over Y. Let L be a line bundle on X. Then L has the same degree on all the members of $\overline{\mathcal{H}}$.*

Proof: Let $\mathcal{U}$ be the universal family over $\overline{\mathcal{H}}$, with $ev : \mathcal{U} \to X$ the evaluation map. The fibration $q : \mathcal{U} \to \overline{\mathcal{H}}$ is equidimensional onto a normal variety, and the line bundle ev^*L has the same degree on all the fibres of q by Lemma 2.1.16. By construction, the fibres of q are the members of the family $\overline{\mathcal{H}}$. More precisely, given such a member $Z = q^{-1}(z)$ with $z \in \overline{\mathcal{H}}$, we may write

$$Z = \sum_{i=1}^{s} a_i \Gamma_i$$

where the a_i's are integers and the Γ_i's are rational curves in X. Then the degree

$$ev^*L \cdot Z = \sum_{i=1}^{r} a_i L \cdot \Gamma_i$$

is independent of the choice of the point $z \in \overline{\mathcal{H}}$. ∎

Corollary 2.1.18. *Let $f : X \to Y$ be an elementary Mori contraction and $\Gamma \subset X$ a contracted rational curve such that $-K_X \cdot \Gamma = l(f)$. Then any closed deformation family of Γ in X over Y is unsplit.*

Proof: Assume by contraction that there exists $\overline{\mathcal{H}}$ a closed deformation family of Γ in X over Y which contains a reducible or nonreduced member Z. By Lemma 2.1.17 the line bundle $-K_X$ has the same degree on all the members of $\overline{\mathcal{H}}$, hence $-K_X \cdot Z = -K_X \cdot \Gamma = l(f)$. Since Z is reducible or nonreduced, we may write

$$Z = a\Gamma' + Z',$$

where Γ' is a rational curve and Z' is a 1-cycle with only rational components such that $f(\Gamma') = f(Z') = f(Z)$ is a point, and $a > 1$ or $Z' \neq 0$. But $-K_X$ is ample, which yields $-K_X \cdot \Gamma' < -K_X \cdot Z = l(f)$, which is not possible. ■

2.1.3 Some general facts about singularities, contractions and normalizations

Here, we make a list of notions and results about contractions and normalizations in general which will be useful in the remaining part of the chapter.

We would like to first recall the definition of some types of **Q**-Gorenstein singularities that are essential in the MMP, as it will be of use later on:

Definition 2.1.19. *Let Y be a normal variety whose canonical divisor K_Y is **Q**-Cartier. Let $\varepsilon : X \to Y$ be any resolution of its singularities. Then the ε-exceptional divisors E_i on X are involved in the following equality:*

$$K_X = \varepsilon^* K_Y + \sum_i a_i E_i.$$

The singularities of Y are called

$$\begin{cases} \textit{terminal} & \textit{if} \quad a_i > 0 \textit{ for all } i \\ \textit{canonical} & \textit{if} \quad a_i \geq 0 \textit{ for all } i \\ \textit{klt} & \textit{if} \quad a_i > -1 \textit{ for all } i \\ \textit{log-canonical} & \textit{if} \quad a_i \geq -1 \textit{ for all } i \end{cases}$$

It is a nontrivial result that the above properties of singularities on Y does not depend on the choice of a resolution ε. We cite [Kol13] as a well-rounded reference about singularities of projective varieties and the role they play in the MMP. It emphasizes the importance of the classification above, as there are very fine results about the existence of certain types of birational modifications under conditions on the singular points, among which we may mention **Q**-factorial modifications ([Kol13, Corollary 1.37]). An even more general result is [Kol13, Theorem 1.31], ensuring the existence of a birational modification $g : X^{\text{can}} \to X$ if X is normal, with X^{can} a variety with canonical singularities such that $K_{X^{\text{can}}}$ is g-ample. Moreover X^{can} is unique up to isomorphism.

Theorem 2.1.20 (Zariski's main theorem, [Zar43]). *Let $f : X \to Y$ be a finite birational morphism to a normal variety. Then f is an isomorphism.*

Lemma 2.1.21 (Conductor). *Let X be a reduced variety and $\nu : X' \to X$ its normalization. Assume that K_X is Cartier and that X satisfies the S_2 condition. Then there exists an effective Weil divisor $\mathcal{D}$ on X' such that $K_{X'} + \mathcal{D}$ is Cartier, and*

$$\nu^* K_X \simeq K_{X'} + \mathcal{D}.$$

Moreover, $\nu(\mathcal{D})$ is the nonnormal locus of X.

In particular, if X is a projective, irreducible, reduced, Gorenstein curve, then the degree of the divisor $\mathcal{D}$ is even.

Terminology. The divisor $\mathcal{D}$ is called the *conductor* of the normalization $X' \to X$.

Proof: A construction of the conductor is provided in [Kol13, 5.2.2]. In general, this is just a subscheme of X', but when X is S_2 $\mathcal{D}$ is a hypersurface of X'. By construction, the support of the conductor is the locus where ν is not an isomorphism, i.e., the locus where X is nonnormal. The formula

$$\nu^* K_X \simeq K_{X'} + \mathcal{D}$$

follows from [Kol13, 5.7.1], which holds under the condition that K_X is Cartier and X is S_2.

Now consider a projective reduced and irreducible curve C with K_C Cartier and its normalization $C' \to C$. By the Riemann-Roch formula for singular curves (see for instance [Har77, Exercise IV.1.9]), we have $\deg K_C = 2p_a(C) - 2$. Hence the degree of the conductor divisor is $\deg K_C - \deg K_{C'} = 2(p_a(C) - g(C))$. ■

Lemma 2.1.22 (Evaluation map)**.** *Let $f : X \to Y$ be a contraction with projective fibres and L a line bundle on X. Then the $\mathcal{O}_X$-module $f^* f_* L$ admits a morphism of $\mathcal{O}_X$-modules*

$$f^* f_* L \to L$$

which we call the relative evaluation map*, or simply the* evaluation map *to simplify the terminology.*

If $p \in Y$ is a point such that $f_* L$ is locally free of rank r at p, and the fibre $X_p = f^{-1}(p)$ is such that

$$\dim H^0(X_p, L|_{X_p}) = r$$

then the restriction of the evaluation map $f^* f_* L \to L$ to X_p is a morphism of vector bundles

$$H^0(X_p, L|_{X_p}) \otimes \mathcal{O}_{X_p} \to L|_{X_p}.$$

Proof: By [Har77, II.5], there is a canonical isomorphism of groups

$$\mathrm{Hom}_{\mathcal{O}_X}(f^* f_* L, L) \simeq \mathrm{Hom}_{\mathcal{O}_Y}(f_* L, f_* L),$$

and the evaluation map $f^* f_* L \to L$ is the element of $\mathrm{Hom}_{\mathcal{O}_X}(f^* f_* L, L)$ which corresponds to the identity $id : f_* L \to f_* L$.

Now let $p \in Y$ be a point such that $f_* L$ is locally free at p of rank $r = h^0(X_p, L)$. In this situation $(f_* L)_p$ is identified with the vector space $H^0(X_p, L)$ by [Har77, Corollary III.12.9]. Moreover let us denote by $\iota : X_p \hookrightarrow X$ the inclusion map. Then the restriction of the evaluation map

$$f^* f_* L \to L$$

to X_p is its pullback via ι, which is

$$\iota^* f^* f_* L \to L|_{X_p}.$$

But $\iota^* f^* f_* L = (f \circ \iota)^* f_* L$ where $f \circ \iota$ is the constant morphism $X_p \to p$. This yields

$$\iota^* f^* f_* L = H^0(X_p, L) \otimes \mathcal{O}_{X_p}.$$

■

To be more precise, the restriction of the evaluation map to X_p

$$H^0(X_p, L) \otimes \mathcal{O}_{X_p} \to L|_{X_p}$$

maps any $\sigma \otimes \tau$ to $\tau\sigma|_U$, for U any open subset of X_p and $\sigma \in H^0(X_p, L|_{X_p})$, $\tau \in H^0(U, \mathcal{O}_{X_p})$.

Theorem 2.1.23 ([AW93], Theorem 5.1)**.** *Let $f : X \to Y$ be a contraction from a projective normal variety with at worst klt singularities. Assume that K_X is $\mathbf{Q}$-Cartier and f is supported by $K_X + rL$ for some line bundle L and r a rational number (see Definition 2.1.8). Assume that L is f-ample, i.e., its restriction to every f-fibre is ample.*

Consider X_p a fibre of f. Assume moreover that

- *f is a fibration and* $\dim X_p < r + 1$*, or*

- *f is birational and* $\dim X_p \leq r+1$.

Then the evaluation map $f^*f_*L \to L$ is surjective along X_p.

Moreover, when $f^*f_*L \to L$ is surjective on all of X, the following birational modification arises as an extension of the locus where the direct image f_*L is locally free:

Lemma 2.1.24. *Let $f : X \to Y$ be a fibration between irreducible quasi-projective varieties and L a line bundle on X such that the evaluation map $f^*f_*L \to L$ is onto. Let $\mathcal{U}_L$ be the locus where f_*L is locally free. Then there exists a birational modification $\mu : \mathcal{Y} \to Y$ which is an isomorphism on $\mathcal{U}_L$, together with $\mathcal{V}$ a vector bundle on $\mathcal{Y}$ which coincides with f_*L on $\mathcal{U}_L$.*

Moreover, let $\mathcal{X}$ be the component of the fibre product $X \times_Y \mathcal{Y}$ which dominates X, with the following commutative square.

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\mu'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y} & \xrightarrow[\mu]{} & Y \end{array}$$

Then there is a surjective map $f'^*\mathcal{V} \to \mu'^*L$ which coincides with the evaluation map $f^*f_*L \to L$ on $f^{-1}(\mathcal{U}_L)$.

Proof: By [Ro68, Theorem 3.5] there exist a birational modification $\mu : \mathcal{Y} \to Y$ and a vector bundle $\mathcal{V}$ on $\mathcal{Y}$ with a surjective map

$$\mu^*(f_*L) \to \mathcal{V}$$

whose kernel is torsion. On the locus $\mathcal{U}_L$ where f_*L is locally free, μ is an isomorphism; in particular, we have $f_*L \simeq \mathcal{V}$ on $\mathcal{U}_L$. Moreover, the pullback by μ' of the surjective map

$$f^*f_*L \to L$$

is also surjective. Let $\mathcal{K}$ denote the kernel of the above map $\mu^*(f_*L) \to \mathcal{V}$. Then in the following diagram, the top row is exact, and the vertical arrow is onto

$$\begin{array}{ccccccc} f'^*\mathcal{K} & \longrightarrow & f'^*\mu^*(f_*L) & \simeq & \mu'^*(f^*f_*L) & \longrightarrow & f'^*\mathcal{V} \longrightarrow 0 \\ & \searrow^{\alpha} & & & \downarrow & & \\ & & & & \mu'^*L & & \end{array}$$

The map α is zero, since μ'^*L is a line bundle and $f'^*\mathcal{K}$ is torsion on the irreducible variety $\mathcal{X}$. This ensures the existence of a factorization $f'^*\mathcal{V} \to \mu'^*L$ which is surjective as well. ■

2.2 Mori contractions of fibre type and submaximal length

Setup A

Let $f : X \to Y$ be an elementary Mori contraction from a smooth projective variety X. Assume that it is of fibre type, hence the exceptional locus of f is all of X. Let X_p be a general fibre of dimension

$$\dim X_p = n := \dim X - \dim Y,$$

then by the Ionescu-Wiśniewski inequality (cf. Theorem 0.0.9) the length of f is bounded from above by $\dim X_p + 1$. We assume that the length is submaximal, meaning

$$l(f) = n.$$

Remark. Under these conditions, the smooth fibres are isomorphic to quadrics thanks to the adjunction formula and [DH17, Theorem 1.3].

2.2.1 An equidimensional birational model with a smooth base

Starting from an elementary Mori contraction of fibre type and submaximal length, one might wonder whether there exists a birational model which is a quadric bundle. When the relative dimension is $n = 1$, the answer is positive, due to [Sa82, Theorem 1.13]. Here, we provide an explicit construction of a birational model and we will prove in Theorems 2.2.3 and 2.2.10 that this model is a conic bundle in codimension one.

In Setup A, let $X_y = f^{-1}(y) \subset X$ be a cycle-theoretic fibre of f with $\dim X_y = n$. We refer to [Kol96, §I.3] for the definition and some properties of the relative Chow scheme $\mathrm{Chow}(X/Y)$: the fibre X_y represents a point $[X_y]$ in a unique irreducible component $\mathcal{C}$ of $\mathrm{Chow}(X/Y)$. Let $Y_{\mathrm{equi}} \subset Y$ be the dense locus over which f is equidimensional, in other words the locus of points $p \in Y$ for which $\dim f^{-1}(p) = n$. Then the variety $\mathcal{C}$ is birational to Y via the following map:

$$y \in Y_{\mathrm{equi}} \mapsto [X_y] \in \mathcal{C}.$$

Let $\mathcal{U}$ be the universal family over $\mathcal{C}$:

$$\mathcal{U} = \{(x, [\gamma]) \in X \times \mathcal{C} \mid x \in \gamma\},$$

endowed with the natural projections $\mathcal{U} \to X$ and $\mathcal{U} \to \mathcal{C}$, making a commutative square diagram:

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathcal{C} & \longrightarrow & Y \end{array}$$

Now let $\mathcal{Y} \to \mathcal{C}$ be a resolution of the singularities, and $\mathcal{X}$ the normalization of the irreducible component of the fibre product $\mathcal{U} \times_{\mathcal{C}} \mathcal{Y}$ which dominates X. There is an equidimensional fibration $h : \mathcal{X} \to \mathcal{Y}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\varepsilon} & X \\ h\downarrow & & \downarrow f \\ \mathcal{Y} & \longrightarrow & Y \end{array}$$

and the horizontal arrows are birational. The line bundle $A = \varepsilon^*(-K_X)$ satisfies the following condition:

$$\min\{A \cdot \Gamma \mid \Gamma \subset \mathcal{X} \text{ a rational curve contracted by } f\} = n.$$

Lemma 2.2.1. *The Cartier divisor A is h-ample and $K_{\mathcal{X}} + A$ is trivial on the general fibre.*

Proof: Let $\mathcal{X}_y \subset \mathcal{X}$ be a fibre. By the Nakai-Moishezon criterion (see for instance [Kol90, Theorem 3.11]) line bundle $A|_{\mathcal{X}_y}$ is ample iff. we have $A^r \cdot W > 0$ for any r-dimensional subvariety $W \subset \mathcal{X}_y$. For any such irreducible $W \subset \mathcal{X}_y$, the restriction $\varepsilon|_W$ being finite, its image $\varepsilon(W)$ also has dimension r. Since $-K_X$ is f-ample, we have $(-K_X)^r \cdot \varepsilon(W) > 0$, hence $A^r \cdot W = \deg(\varepsilon|_W)((-K_X)^r \cdot \varepsilon(W)) > 0$.

Let now $\mathcal{F}$ be a general smooth h-fibre such that $F := \varepsilon(\mathcal{F})$ has dimension $n = \dim X - \dim Y$ and the cycle $[F]$ is a smooth point of $\mathcal{C}$. In that case ε is locally an isomorphism around $\mathcal{F}$. The support of $K_{\mathcal{X}} + A = K_{\mathcal{X}} - \varepsilon^*(K_X)$ is the exceptional locus of ε, which doesn't meet $\mathcal{F}$. ∎

We now recall the definition of a quadric bundle:

Definition 2.2.2 (Quadric bundle). *Let $f : X \to Y$ be an equidimensional fibration whose fibres have dimension n. It is a* quadric bundle *if there is a Zariski-locally trivial $\mathbf{P}^{n+1}$-bundle W over Y and an embedding $X \hookrightarrow W$ such that every f-fibre is embedded as a quadric, and f is the restriction to X of the bundle map $W \to Y$.*

The condition that W is Zariski-locally trivial over Y implies that it is the projectivization of a rank $(n+2)$-vector bundle.

Thanks to Lemma 2.2.1 we know that A is relatively ample with the length condition $l(A) = l(f)$ and $K_{\mathcal{X}} + A$ is generically relatively trivial.

The question arises whether the birational model $\mathcal{X} \to \mathcal{Y}$ introduced above Lemma 2.2.1 is a quadric bundle. In this equidimensional fibration, $\mathcal{X}$ is normal and $\mathcal{Y}$ is smooth. We will bring a positive answer when the relative dimension is equal to 1, i.e., $\mathcal{X} \to \mathcal{Y}$ is a family of curves. We will start in the case of a family of curves over a smooth base of dimension 1, shifting after that to the case where the base has dimension 2 or more.

2.2.2 A normal surface over a smooth curve

Setup A.1

We assume that $f : X \to Y$ is an equidimensional fibration of relative dimension 1 from a normal projective surface X onto a smooth projective curve Y endowed with a relatively ample polarization $A \in \mathrm{Pic}(X)$ such that

$$\min \{A \cdot \Gamma \mid \Gamma \subset X \text{ contracted rational curve}\} = 1,$$

and $K_X + A$ is trivial on the smooth fibres. This setup is obtained from Setup A via an application of Lemma 2.2.1. Here, we have dropped the notation $\mathcal{X} \to \mathcal{Y}$ and we write $X \to Y$ instead, as there is no risk of confusion.

Theorem 2.2.3. *Let $f : X \to Y$ be as in Setup A.1. Then f is a conic bundle. More precisely, there exists a rank 3 vector bundle V over Y and $X \hookrightarrow \mathbf{P}_Y(V)$ such that all the fibres of f are embedded as conics, and $A = \mathcal{O}_{\mathbf{P}_Y(V)}(1)|_X$. Moreover, the singularities of X are klt.*

Before proving this theorem, let us deduce the following corollary from it.

Corollary 2.2.4. *Under the assumptions of Setup A.1, K_X is Cartier and $K_X + A$ is relatively trivial. In particular, the singularities of X are canonical.*

Proof: By Theorem 2.2.3, there is a vector bundle V of rank 3 over Y such that $X \subset \mathbf{P}_Y(V)$ and $A = \mathcal{O}_{\mathbf{P}_Y(V)}(1)|_X$. As a hypersurface of a smooth variety, X is Gorenstein, i.e., K_X is Cartier. It follows that X has canonical singularities, as any Gorenstein variety with klt singularities is canonical.

Let $\Lambda_y \simeq \mathbf{P}^2$ denote any fibre of the projection $\mathbf{P}_Y(V) \to Y$. In particular $X_y := X \cap \Lambda_y$ is any f-fibre. By the adjunction formula one has $K_{\Lambda_y} = K_{\mathbf{P}_Y(V)}|_{\Lambda_y}$. In addition,

$$\begin{aligned} K_X|_{X_y} &= ((K_{\mathbf{P}_Y(V)} + X)|_X)|_{X_y} \\ &= ((K_{\mathbf{P}_Y(V)} + X)|_{\Lambda_y})|_{X_y} \\ &= (K_{\Lambda_y} + X_y)|_{X_y} \\ &= O_{X_y}(-1), \end{aligned}$$

since $K_{\Lambda_y} = \mathcal{O}_{\mathbf{P}^2}(-3)$ and $X_y = \mathcal{O}_{\mathbf{P}^2}(2)$ as Cartier divisors on Λ_y via the isomorphism $\Lambda_y \simeq \mathbf{P}^2$.

The equality of Cartier divisors $K_X|_{X_y} = -A|_{X_y}$ follows for every f-fibre X_y. Hence $h^0(X_y, (K_X + A)|_{X_y}) = 1$ independently of the choice of X_y, and by Grauert's theorem (see for instance [Har77, III.12.9]) $L := f_*(K_X + A)$ is a line bundle. By Lemma 2.1.22 there is an evaluation map

$$f^*L \to K_X + A,$$

whose restriction to any fibre X_y yields

$$H^0(X_y, \mathcal{O}_{X_y}) \otimes \mathcal{O}_{X_y} \to \mathcal{O}_{X_y}.$$

The above is an isomorphism, therefore $K_X + A \simeq f^*L$. ∎

Now, we introduce some intermediate results that we will use in the proof of Theorem 2.2.3.

Intermediate results

Lemma 2.2.5. *In Setup A.1, the singularities of X are klt.*

Proof: The general fibre of f is smooth and rational, by Bertini's theorem for basepoint-free Cartier divisors.. If X_y is a smooth fibre lying in the smooth locus of X, we have $-K_{X_y} = -K_X|_{X_y} = A|_{X_y}$, which is ample of degree $2 - 2g(X_y)$, yielding $g(X_y) = 0$.

Since A is Cartier and relatively ample and $A \cdot X_y = 2$, any reduced fibre has at most two irreducible components. It follows from [DH17, Lemma 3.3] that the singular points of X lying on the reduced fibres are canonical.

We assume now that X_0 is a nonreduced fibre. Since it is a hypersurface of the normal surface X, X_0 has no embedded point. Since $A \cdot X_0 = 2$, we have $X_0 = 2l$ with l a reduced and irreducible curve. The nature of the singularities being a local property, we may consider that $Y = \Delta$ is a disc around the origin in $\mathbf{C}$ with X_0 lying over the origin, so that X_0 is the only singular fibre.

Let $\Delta' \to \Delta$ be the double cover $z \mapsto z^2$ branched over the origin and X' the normalization of the fibre product $X \times_\Delta \Delta'$, with the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\alpha} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta' & \xrightarrow[2:1]{} & \Delta \end{array}$$

The branch locus of α consists in the singular points of X lying on the central fibre X_0. Indeed, α is étale outside the central fibre, and if $q \in X_0$ is a smooth point of X, then we may consider an analytic neighbourhood V of q which is isomorphic to the polydisc $\Pi = \{(x, y) \in \mathbf{C}^2 \mid |x| < 1, |y| < 1\}$ and contained in the smooth locus of X. In a suitable choice of coordinates (x, y), the restriction of f to V is the map $(x, y) \mapsto x^2$, hence $\alpha^{-1}(V)$ is the normalization of $\{(x, y, z) \in \Pi \times \Delta \mid x^2 = z^2\}$, i.e., a disjoint union of two polydiscs, which is étale over V.

Let $A' = \alpha^* A$. It is f'-ample and has degree 2 on the f'-fibres. All the fibres in X' are reduced, so the fibration f' satisfies [DH17, Lemma 3.3], ensuring that X' only has canonical singularities. Since α is unramified in codimension one, it is quasi-étale, and therefore the singularities of X are klt (see [KM98, Corollary 5.21]). ■

As a side benefit, this allows us to identify all the fibres in X.

Corollary 2.2.6. *Any reduced and irreducible fibre is smooth and rational. If $X_y \subset X$ is a singular fibre, then it is either the union of two smooth rational curves meeting transversally at a single point, or nonreduced with multiplicity 2 and a smooth rational curve as its reduction. In other words, as a 1-cycle, it is one of the following:*

- *$X_y = l_1 + l_2$ with $l_i \simeq \mathbf{P}^1$ and the two components l_1, l_2 meet transversally in a single point,*
- *$X_y = 2l$ with $l \simeq \mathbf{P}^1$.*

Proof: We recall that the general fibre is smooth and rational. Since f is an equidimensional morphism to a smooth curve, it is flat, and thus the arithmetic genus of any fibre is zero. Hence the reduced and irreducible fibres are all isomorphic to $\mathbf{P}^1$.

If X_y is nonreduced or reducible, then one of the two following equalities of 1-cycles holds in X by the fact that the Cartier divisor $A|_{X_y}$ is ample of degree 2:

(*i*) $X_y = l_1 + l_2$,

(*ii*) $X_y = 2l$,

where l_1, l_2 and l are reduced and irreducible curves.

In case (*i*), the restriction exact sequence

$$0 \to \mathcal{O}_{l_1}(-l_2 \cap l_1) \to \mathcal{O}_{X_y} \to \mathcal{O}_{l_2} \to 0 \tag{2.1}$$

coupled with $p_a(X_y) = 0$ and $h^0(X_y, \mathcal{O}_{X_y}) = h^0(l_2, \mathcal{O}_{l_2}) = 1$ yields $p_a(l_2) = 0$. Likewise, $p_a(l_1) = 0$. Then l_1 and l_2 are isomorphic to $\mathbf{P}^1$ and from the exact sequence above, we have $h^0(l_1, \mathcal{O}_{l_1}(-l_2|_{l_1})) = h^1(l_1, \mathcal{O}_{l_1}(-l_2|_{l_1})) = 0$. As a consequence $\mathcal{O}_{l_1}(-l_2|_{l_1}) = \mathcal{O}_{\mathbf{P}^1}(-1)$, meaning that l_1 and l_2 meet transversally in a single point.

In case (ii), consider $\mathcal{I}_l \subset \mathcal{O}_{X_y}$ the ideal sheaf of l in X_y and the restriction exact sequence

$$0 \to \mathcal{I}_l \to \mathcal{O}_{X_y} \to \mathcal{O}_l \to 0.$$

The curve l being reduced and irreducible, it follows that $p_a(l) = h^1(l, \mathcal{O}_l) \leq h^1(X_y, \mathcal{O}_{X_y})$.

Let Δ be an analytic neighbourhood of the point $y = f(X_y)$ which is isomorphic to the unit disc. Set $X_\Delta = f^{-1}(\Delta)$ and consider the exact sequence

$$0 \to \mathcal{O}_{X_\Delta}(-X_y) \to \mathcal{O}_{X_\Delta} \to \mathcal{O}_{X_y} \to 0. \tag{2.2}$$

By Lemma 2.2.5 the singularities of X_Δ are klt, so we may apply [KMM87, Theorem 1.2.5], ensuring that $R^1 f_*(\mathcal{O}_{X_\Delta}(-X_y))$ is the zero sheaf on Δ. The Leray spectral sequence

$$0 \to H^1(\Delta, \mathcal{O}_\Delta(-y)) \to H^1(X_\Delta, \mathcal{O}_{X_\Delta}(-X_y)) \to H^0(\Delta, R^1 f_*(\mathcal{O}_{X_\Delta}(-X_y)))$$

yields $h^1(X_\Delta, \mathcal{O}_{X_\Delta}(-X_y)) = 0$, and it follows from the exact sequence (2.2) that $h^0(X_y, \mathcal{O}_{X_y})$ does not depend on the choice of the fibre $X_y \subset X_\Delta$. As a conclusion, $h^0(X_y, \mathcal{O}_{X_y}) = 1$, $h^1(X_y, \mathcal{O}_{X_y}) = 0$ and $p_a(l) = 0$. ■

Now, we consider $V := f_*A$, which is locally free since it is torsion-free on the smooth curve Y. It has rank 3 since for a smooth f-fibre $X_y \simeq \mathbf{P}^1$ we have $h^0(X_y, A|_{X_y}) = 3$. Thanks to [Har77] and [KMM87] we are able to prove that $h^0(X_y, A|_{X_y})$ is constant across the fibres:

Lemma 2.2.7. *For every fibre $X_y \subset X$, we have $h^0(X_y, A|_{X_y}) = 3$.*

Proof: Since the singularities of X are klt, we can use the vanishing theorem by Kawamata and Viehweg ([KMM87, Theorem 1.2.5]) to show that $R^i f_* A = 0$ for all $i > 0$. From that, it follows by [Har77, Theorem III.12.11] that $h^1(X_y, A|_{X_y}) = 0$ for all X_y. Since A is flat over Y, the Euler characteristic $\chi(A|_{X_y})$ is constant and thus, we conclude $h^0(X_y, A|_{X_y}) = 3$ for all X_y. ■

Now we know that $h^0(X_y, A|_{X_y}) = 3 = \mathrm{rank}(f^*V)$ even for the singular fibres $X_y \subset X$. Therefore, we can identify the restriction of the evaluation map

$$f^*V = f^*f_*A \to A$$

to any fibre X_y as the evaluation map

$$H^0(X_y, A|_{X_y}) \otimes \mathcal{O}_F \to A|_{X_y}.$$

Lemma 2.2.8. *The evaluation map $f^*V \to A$ is surjective.*

Proof: We shall prove that the restriction of this map to any fibre X_y is surjective, in other words, that $A|_{X_y}$ is globally generated.

If X_y is a smooth fibre, then it is isomorphic to $\mathbf{P}^1$, and $A|_{X_y} = -K_{X_y}$ is globally generated.

If X_y is reduced and reducible, then it is of the form $l_1 + l_2$ with $l_i \simeq \mathbf{P}^1$ and $A|_{l_i} = \mathcal{O}_{\mathbf{P}^1}(1)$, which is globally generated. Via the isomorphism $l_1 \simeq l_2 \simeq \mathbf{P}^1$ the restriction exact sequence (2.1) becomes

$$0 \to \mathcal{O}_{\mathbf{P}^1}(-1) \to \mathcal{O}_{X_y} \to \mathcal{O}_{\mathbf{P}^1} \to 0.$$

Twisting this exact sequence by A, we obtain

$$0 \to \mathcal{O}_{\mathbf{P}^1} \to A|_{X_y} \to \mathcal{O}_{\mathbf{P}^1}(1) \to 0,$$

whose left and right members are globally generated. Since $H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}) = 0$, the map $H^0(X_y, A|_{X_y}) \to H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1))$ is surjective and thus we have the following commutative diagram where the two rows are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}) \otimes \mathcal{O}_{\mathbf{P}^1} & \longrightarrow & H^0(X_y, A|_{X_y}) \otimes \mathcal{O}_F & \longrightarrow & H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1)) \otimes \mathcal{O}_{\mathbf{P}^1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^1} & \longrightarrow & A|_{X_y} & \longrightarrow & \mathcal{O}_{\mathbf{P}^1}(1) & \longrightarrow & 0 \end{array}$$

The left and right vertical arrows are surjective and it follows by diagram chasing that the one in the middle is also surjective. Hence $A|_{X_y}$ is globally generated.

Now assume X_y is a nonreduced fibre. We have $X_y = 2l$ with $l \simeq \mathbf{P}^1$. As in the proof of Lemma 2.2.5, we take an analytic neighbourhood Δ of $y = f(X_y)$ which is isomorphic to the unit disc in $\mathbf{C}$ such that $X_y = X_0$ lies over the origin and is the only singular fibre.

Set $X_\Delta = f^{-1}(\Delta)$ and Δ' another copy of the unit disc. Let $\Delta' \to \Delta$ be the double cover ramified over 0 and X' the normalization of the fibre product $X_\Delta \times_\Delta \Delta'$, with the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\ \alpha\ } & X_\Delta \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f_\Delta} \\ \Delta' & \xrightarrow[2:1]{} & \Delta \end{array}$$

By this construction, there are involution morphisms $i_{X'}$ and $i_{\Delta'}$ on X' and Δ' respectively such that ${X'}/{i_{X'}} \simeq X_\Delta$ and ${\Delta'}/{i_{\Delta'}} \simeq \Delta$.

Besides, X' only has reduced fibres over Δ'. By the above, the line bundle $A' = \alpha^* A$ is globally generated on all the f'-fibres, so the evaluation map $f'^* f'_* A' \to A'$ is surjective, and applying [Har77, II.7.12] we obtain a morphism

$$X' \to \mathbf{P}_{\Delta'}(f'_* A')$$

such that f' is the composition $X' \to \mathbf{P}_{\Delta'}(f'_* A') \to \Delta'$.

Let $p \in \Delta'$ be a point different from the origin, $p' = i_{\Delta'}(p)$, and X_p the fibre over p. The involution $i_{X'}$ swaps the two conjugate fibres X_p and $X_{p'}$. Hence f' is equivariant under the involutions, in other words $f' \circ i_{X'} = i_{\Delta'} \circ f'$. It follows that the morphism $X' \to \mathbf{P}_{\Delta'}(f'_* A')$ which factors f' is also equivariant under the involutions. Therefore it descends to a map between the quotients

$$\psi_\Delta : X_\Delta \to \mathbf{P}_\Delta(f_* A|_{X_\Delta})$$

such that $\psi_\Delta^* \mathcal{O}_{\mathbf{P}_\Delta(f_* A|_{X_\Delta})}(1) = A|_{X_\Delta}$. As a consequence of [Har77, II.7.12], the line bundle $A|_{X_\Delta}$ is globally generated along all the fibres. In particular, it is globally generated on the central fibre X_0. ■

Corollary 2.2.9. *There exists a morphism $\psi : X \to \mathbf{P}_Y(V)$ onto a conic bundle such that $\psi^* \mathcal{O}_{\mathbf{P}_Y(V)}(1) = A$, $\psi(X)$ is normal and the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{\ \psi\ } & \mathbf{P}_Y(V) \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & Y & \end{array}$$

Proof: The existence of the morphism ψ such that $\psi^* \mathcal{O}_{\mathbf{P}_Y(V)}(1) = A$ is given by Lemma 2.2.8 and [Har77, II.7.12]. For every f-fibre X_y, we have $A \cdot X_y$ and $A|_{X_y}$ very ample; by Corollary 2.2.6 the general fibre of f is embedded as a conic in $\mathbf{P}^2$ via ψ. The base Y being smooth, we can deduce from Lemma 2.1.16 that $\psi(X)$ is a conic bundle.

Let $X_y \subset X$ be a reduced f-fibre. Then by Corollary 2.2.6, X_y is either isomorphic to a smooth conic with $A|_{X_y} = \mathcal{O}_{X_y}(1)$, or the sum of two transverse smooth rational curves $l_1 + l_2$ with $A|_{l_i} \simeq \mathcal{O}_{\mathbf{P}^1}(1)$ via the isomorphism $l_i \simeq \mathbf{P}^1$. In both cases $\psi|_{X_y}$ is bijective onto its image.

The locus in $\psi(X)$ which is covered by the reduced fibres is normal since it is Cohen-Macaulay (as a hypersurface of a smooth variety) and smooth in codimension one. As a consequence ψ is an isomorphism on the locus in X which is covered by the reduced f-fibres, as a bijective morphism onto a normal variety (Theorem 2.1.20).

It remains to prove that $\psi(X)$ is normal along any nonreduced fibre. Such a fibre is the image $\psi(X_y)$ of a nonreduced f-fibre $X_y = 2l$, with $l \simeq \mathbf{P}^1$ (Corollary 2.2.6). As in the proof of Lemma 2.2.8, we may consider an analytic neighbourhood Δ of the point $y = f(X_y)$ in Y such that Δ is isomorphic to a unit disc. Let $X_\Delta = f^{-1}(\Delta)$, $\Delta' \to \Delta$ the double cover ramified at the origin from another copy Δ' of the unit disc, and X' the normalization of the fibre product $X \times_\Delta \Delta'$. Then there are involution $i_{X'}$ on X' and $i_{\Delta'}$ on Δ' such that

$${X'}/{i_{X'}} \simeq \psi(X_\Delta), \ {\Delta'}/{i_{\Delta'}} \simeq \Delta.$$

It follows that $\psi(X_\Delta)$ is normal, as a finite quotient of a normal variety. ■

Proof of the theorem

Proof of Theorem 2.2.3: We know by Corollary 2.2.9 that $f : X \to Y$ factors through a morphism $\psi : X \to \mathbf{P}_Y(V)$ onto a normal conic bundle such that $\psi^*\mathcal{O}_{\mathbf{P}_Y(V)}(1) = A$. Every f-fibre X_y is sent via ψ to its image under the morphism induced by the linear system $|A|_{X_y}|$; since A is ample on each fibre X_y, the morphism ψ is finite. Moreover ψ is birational, since it is an isomorphism along any smooth f-fibre. As a consequence, ψ is an isomorphism as a finite and birational morphism onto a normal variety by Theorem 2.1.20. The fact that X has klt singularities is proven in Lemma 2.2.5. ■

2.2.3 Relative dimension 1 over a larger base

We would like to mention first the work of E. Romano, who proved that an equidimensional fibration $f : X \to Y$ of relative dimension 1 is a conic bundle under the assumptions that $-K_X$ is Cartier and relatively ample, Y is smooth and X is klt (see [Rm19, Proposition 1.3]).

Setup A.2

The setup here is the same as Setup A.1, but with the base Y of any dimension. Namely, we assume that $f : X \to Y$ is an equidimensional fibration of relative dimension 1 from a normal projective variety X onto a smooth projective variety Y endowed with a relatively ample polarization $A \in \mathrm{Pic}(X)$ such that

$$\min \{A \cdot \Gamma \mid \Gamma \subset X \text{ contracted rational curve}\} = 1,$$

and $K_X + A$ is trivial on the smooth fibres. Similarly to Setup A.1, this setup here is obtained from Setup A via an application of Lemma 2.2.1. Here, we have also dropped the notation $\mathcal{X} \to \mathcal{Y}$ and written $X \to Y$ instead.

With the help of Theorem 2.2.3 we are able to slightly relax the assumption about the nature of the singularities and the relative ampleness of $-K_X$ in E. Romano's result. This takes the form of Theorem 2.2.10 and Corollary 2.2.13 below.

Theorem 2.2.10. *Let $f : X \to Y$ be as in Setup A.2. Then f is a conic bundle in codimension one. More precisely, there exists a subset $X^\circ \subset X$ with canonical singularities, whose complement has codimension 2 or more, and such that the restriction of f to X° is a conic bundle with $A|_{X^\circ}$ the relative hyperplane divisor.*

Furthermore, the divisor K_X is Cartier and $K_X + A$ is relatively trivial on all of X.

Proof: Let us fix an embedding $Y \subset \mathbf{P}^N$. Since Y is smooth, by Bertini's theorem a curve $C = H_1 \cap \cdots \cap H_{\dim Y - 1}$ cut out by general hyperplane H_i is smooth. Let Y° be the locus covered by such smooth linear curves $C \subset Y$ for which the preimages $X_C = f^{-1}(C)$ are normal (such surfaces exist by [Se50, Lemma 3]). Its complement $Y - Y^\circ$ has codimension at least 2 in Y since the curves C are cut out by very ample divisors. Indeed, let us denote $Y^1 = Y - Y^\circ$; if Y^1 has codimension 1 in Y, any intersection of $(\dim Y - 1)$ hyperplane sections

$$Y \cap H_1 \cap \cdots \cap H_{\dim Y - 1}$$

cuts out a nonempty locus on Y^1. This is not possible by construction, since the intersection of such a family of general hyperplane sections of Y is a curve contained in Y°. In addition, if X° stands for the preimage of Y°, we have $\mathrm{codim}(X - X^\circ) \geq 2$ in X since f is equidimensional.

Given any such curve $C \subset Y^\circ$ with normal preimage X_C, by adjunction we have

$$K_{X_C} = K_{X^\circ}|_{X_C} + (\dim Y - 1) f^* H|_{X_C},$$

where f^*H is relatively trivial. So the surface X_C is a conic bundle over C by Theorem 2.2.3. The fibres over Y° are thus all conics.

In particular, the fibres over Y° are Gorenstein. Moreover, they all have the same Hilbert polynomial

$$d \mapsto \chi(X_y, dA|_{X_y}).$$

It follows that X° is a flat family over Y°. Being smooth, Y° is Gorenstein, and thus X° is also Gorenstein as an application of [Ma89, Theorem 23.4].

Each singular point $p \in X^\circ$ is contained in a conic bundle X_C over some smooth curve $C \subset Y^\circ$. In the surface X_C, the point p is a canonical singularity since it is klt ([DH17, Lemma 3.3]) and X_C is Gorenstein. Each surface X_C being cut out in X° by basepoint-free Cartier divisors, it follows that X° has at worst canonical singularities ([Re79, Theorem 1.13]). As a consequence, X° is a conic bundle over Y° ([Rm19, Proposition 1.3]).

It remains to be proven that K_X is locally free on all of X. Let $X_y \subset X^\circ$ be a fibre contained in a conic bundle X_C over a smooth curve $C \subset Y^\circ$. Since K_{X° is Cartier, we have by adjunction

$$K_{X_C} = K_{X^\circ}|_{X_C} + (\dim Y - 1) f^* H|_{X_C},$$

where H denotes the class of the hyperplane sections on Y°, and thus the equality of Cartier divisors

$$-K_{X^\circ}|_{X_y} = -K_{X_C}|_{X_y} = A|_{X_y}.$$

As a consequence, $K_{X^\circ} + A$ is relatively trivial over Y°. We may apply Grauert's theorem ([Har77, III.12.9]) which yields an isomorphism $f^* L^\circ \simeq K_{X^\circ} + A|_{X^\circ}$ for some line bundle L° on Y°. Since Y is smooth and $\operatorname{codim}(Y - Y^\circ) \geq 2$, the line bundle L° extends in a unique way to $L \in \operatorname{Pic}(Y)$. Therefore K_X coincides in codimension one with the line bundle $f^*L - A$ on X, and thus K_X is Cartier and $K_X + A \simeq f^*L$. ∎

Corollary 2.2.11. *Assume that the singularities of X are klt. Then the evaluation map*

$$f^* f_* A \to A$$

is surjective on X, and $X \to Y$ is a conic bundle.

Proof: By Theorem 2.2.10 the direct image f_*A is locally free of rank 3 on $Y^\circ = f(X^\circ)$ and $K_X + A$ is relatively trivial. Since X is klt, by Theorem 2.1.23 the evaluation map $f^* f_* A \to A$ is surjective.

The fact that $f : X \to Y$ follows from a straightforward application of [Rm19, Proposition 1.3]. ∎

Further results

The setup here is the same as Setup A.2. In Setup A.2, we made no assumption about the singular points of X, but we know now that there exists a locus $X^\circ \subset X$ which is canonical and such that $X - X^\circ$ has codimension 2 or more by Theorem 2.2.10.

Globally, the variety X might have worse than canonical singularities, in which case we may consider the canonical modification $\tau : X_{\text{can}} \to X$ (see [Kol13, Theorem 1.31]). This is a birational morphism from a canonical variety X_{can}, unique up to isomorphism, whose canonical divisor is τ-ample, and there exists a divisor E such that

$$K_{X_{\text{can}}} = \tau^* K_X - E.$$

Let $g : X_{\text{can}} \to Y$ be the composition $f \circ \tau$.

Lemma 2.2.12. *In Setup A.2, assume that $K_X + A$ is relatively trivial over Y. Then the divisor $-E$ is anti-effective, g-nef, semiample and* $\operatorname{Supp}(E) = \operatorname{Exc}(\tau)$.

Proof: Since $-E = K_{X_{\text{can}}} - \tau^* K_X$ is τ-ample and $\tau_* E = 0$, by the negativity lemma E is effective ([KM98, Lemma 3.39]).

The inclusion $\operatorname{Supp}(E) \subset \operatorname{Exc}(\tau)$ holds by construction, and if we assume by contradiction that the inverse inclusion does not hold, then there exists a positive-dimensional subvariety $\mathcal{Z} \subset \operatorname{Exc}(\tau)$ not contained in $\operatorname{Supp}(E)$. Let $\mathfrak{z} \subset \mathcal{Z}$ be a general fibre of the restriction $\tau|_{\mathcal{Z}}$, then $E|_{\mathfrak{z}}$ is effective since E is effective, while $-E|_{\mathfrak{z}} = K_{X_{\text{can}}}|_{\mathfrak{z}}$ is ample, a contradiction.

Now we assume by contradiction that $-E$ is not g-nef. We denote $A_{\text{can}} = \tau^* A$; up to replacing A by $A + f^* H$ for H an ample divisor on Y, we may assume that A is ample and A_{can} is nef. By Theorem 2.2.10 we have $K_X + A \simeq f^*L$ for some line bundle $L \in \mathrm{Pic}(Y)$, so $K_{X_{\mathrm{can}}} + A_{\mathrm{can}} = g^*L - E$ is not g-nef.

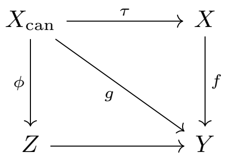
In particular, the divisor A_{can} is g-ample, and the relative nef value r (with respect to g) of the pair $(X_{\mathrm{can}}, A_{\mathrm{can}})$ is larger than 1 (Definition 2.1.13).

Now we pick a curve $\Gamma \subset X_{\mathrm{can}}$ contracted by g such that the ray $\mathbf{R}_+[\Gamma]$ is extremal in $\overline{\mathrm{NE}}(X_{\mathrm{can}})$ and $(K_{X_{\mathrm{can}}} + rA_{\mathrm{can}}) \cdot \Gamma = 0$ (such a curve exists by [Deb16, Proposition 4.21]). Since $r > 0$ and A_{can} is nef, we cannot have $K_{X_{\mathrm{can}}} \cdot \Gamma \geq 0$, or else

$$K_{X_{\mathrm{can}}} \cdot \Gamma = A_{\mathrm{can}} \cdot \Gamma = 0,$$

meaning that τ contracts Γ, which is not possible since $K_{X_{\mathrm{can}}}$ is τ-ample.

Hence Γ generates an extremal $K_{X_{\mathrm{can}}}$-negative ray, and by the relative cone theorem ([Deb16, Proposition 4.21]) there exists an associated elementary Mori contraction $\phi : X_{\mathrm{can}} \to Z$ such that the following diagram is commutative.

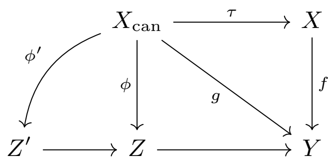


If A_{can} is not ample, then we consider the nef value morphism $\phi' : X_{\mathrm{can}} \to Z'$ of the pair $(X_{\mathrm{can}}, \overline{A})$, whose definition can be found in [And95, (1.2)], with $\overline{A} = A_{\mathrm{can}} + \phi^*M$ for some ample line bundle M on Z. We note that $\overline{A}$ is ample.

Every ϕ'-fibre G' is contracted by ϕ, or else $\phi^*M|_{G'}$ is strictly nef, which is not possible since $K_{X_{\mathrm{can}}} + rA_{\mathrm{can}}$ is nef and

$$0 = (K_{X_{\mathrm{can}}} + r\overline{A})|_{G'} = (K_{X_{\mathrm{can}}} + rA_{\mathrm{can}})|_{G'} + r\phi^*M|_{G'}.$$

Therefore the following diagram is commutative:



and thus up to a change of notations we may assume that A_{can} is ample and that ϕ is supported by $K_{X_{\mathrm{can}}} + rA_{\mathrm{can}}$. Thanks to these assumptions we may apply [And95, Theorem 2.1]. Let G be a ϕ-fibre; since $K_{X_{\mathrm{can}}}|_G$ is antiample and τ-ample, G meets each nontrivial fibre of τ in at most finitely many points. As a consequence $\dim G = \dim \tau(G) \leq 1$.

- If ϕ is birational, it follows from [And95, Theorem 2.1] that its nontrivial fibres have dimension at least 2, a contradiction.
- If ϕ is not birational, it is equidimensional of relative dimension 1. Let $G \subset X_{\mathrm{can}}$ be a general ϕ-fibre; we can also apply [And95, Theorem 2.1], to the result that $A_{\mathrm{can}}|_G$ has degree 1, a contradiction.

It remains to be proven that $-E$ is semiample, i.e., $-mE$ is basepoint-free for $m \gg 0$. This is a consequence of [HK10, Theorem 5.1], provided that $-E$ is g-nef and $-E - K_{X_{\mathrm{can}}} = -\tau^*K_X$ is g-nef and g-big. ■

Corollary 2.2.13. *Under the same assumptions as in Lemma 2.2.12, if every f-fibre contains only finitely many noncanonical singularities of X, then X is canonical and f is a conic bundle.*

Proof: We assume by contradiction that X is not canonical and that there exists a fibre X_y meeting the noncanonical locus of X in only finitely many points. Since the divisor $E \subset X_{\mathrm{can}}$ involved in the inequality

$$K_{X_{\mathrm{can}}} = \tau^*K_X - E$$

is effective and $\mathrm{Supp}(E) = \mathrm{Exc}(\tau)$ by Lemma 2.2.12, and X_y meets the noncanonical locus in finitely many points, in X_{can} we have $-E \cdot X'_y < 0$ for X'_y the proper transform of X_y. Hence $-E$ is not g-nef, which is not possible by Lemma 2.2.12.

Hence, if all the fibres meet the canonical locus of X, we deduce that X is in fact canonical, and f is a conic bundle thanks to [Rm19, Proposition 1.3]. ■

2.3 Divisorial Mori contractions of submaximal length

Setup B

Let $f : X \to Y$ be an elementary Mori contraction from a smooth projective variety X. Assume that it is birational and divisorial, with E the exceptional divisor (which is irreducible by [Deb16, Proposition 8.7.b]), and $Z := f(E)$. If E_z is a general fibre of $f|_E$ then by the Ionescu-Wiśniewski inequality (cf. Theorem 0.0.9), the length of f is bounded from above by $\dim E_z$. We assume here that the length is submaximal, in other words

$$l(f) = \dim E_z - 1,$$

for E_z a fibre of the expected dimension $n := \dim E - \dim Z$. In other words we have $l(f) = n - 1$ (in particular, n must be larger than 1).

2.3.1 Structure theorem

Before proceeding to an identification of the fibres, we need first to state the following result.

Lemma 2.3.1. *Under the assumptions of our Setup, every fibre of $E \to Z$ contains a $(-K_X)$-minimal curve.*

Proof: Let $E_z \subset E_{\text{eq}}$ be a generically reduced fibre of dimension n over a smooth point of Z_{eq}. Since E_z is locally a complete intersection in X, by the adjunction formula we have $-K_{E_z} = -K_E|_{E_z} = (-K_X - E)|_{E_z}$; in particular, $-K_{E_z}$ is ample. Let now $C \subset E_z$ be a rational curve whose degree is minimal with respect to $-K_{E_z}$ among rational curves meeting the smooth locus of E_z. Then

$$-K_{E_z} \cdot C = -K_X \cdot C - E \cdot C \geq n - 1 - E \cdot C. \tag{2.3}$$

By ampleness of $-E|_{E_z}$, it follows that $-K_{E_z} \cdot C \geq n$. Assume by contradiction that $-K_{E_z} \cdot C > n + 1$. By [Kol96, Proposition II.1.3], there exists a deformation family $\mathcal{H}_z$ of C in E_z such that

$$\dim \mathcal{H}_z \geq -K_{E_z} \cdot C + (n - 3) > 2n - 2,$$

and considering the universal family over $\mathcal{H}$,

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{ev} & E_z \\ {\scriptstyle\pi}\downarrow & & \\ \mathcal{H}_z & & \end{array}$$

where π is a $\mathbf{P}^1$-bundle, we have $\dim \mathcal{U} \geq 2n$. Fixing $p \in E_z$ a general point and considering $\mathcal{U}_p = \pi^{-1}(\pi(ev^{-1}(p)))$, we have $\dim \mathcal{U}_p \geq n + 1$, hence there exists a fibre of $\mathcal{U}_p \to E_z$ of positive dimension. In other words, there exists a point $q \in E_z$ and a positive-dimensional family of rational curves through p and q (in particular, meeting the smooth locus of E_z), all of which have minimal degree among rational curves meeting $(E_z)_{\text{smooth}}$. By the bend-and-break lemma, there exists such a member which is reducible or nonreduced, which is not possible by the minimality of $-K_{E_z} \cdot C$. As a conclusion, we have eiher $-K_{E_z} \cdot C = n$ or $-K_{E_z} \cdot C = n + 1$.

Going back to (2.3), we deduce that one of the two following cases occur:

- $-K_X \cdot C = n - 1 = l(f)$, in which case C is a $(-K_X)$-minimal curve,
- $-K_X \cdot C = n$, in which case $-E \cdot C = 1$ and by the exact sequence

$$0 \to f^*\text{Pic}(Y) \to \text{Pic}(X) \xrightarrow{\cdot C} \mathbf{Z}$$

  we have $(K_X - nE) \sim_f 0$ (in other words $-K_X$ and $-nE$ are relatively linearly equivalent).

Assume by contradiction that the second case holds. In that case, for $\Gamma \subset E$ an f-minimal curve, we have $-K_X \cdot \Gamma = n - 1$ by our length hypothesis, hence $-E \cdot \Gamma = \frac{n-1}{n}$, which is not an integer. This is a contradiction.

The conclusion follows that any generically reduced fibre of dimension n over the smooth locus of Z contains an f-minimal curve. This ensures that there exists a deformation family of f-minimal curves (unsplit by the length hypothesis and Corollary

2.1.18, hence compact) which dominates Z; since it is proper over Z, it is surjective. Hence every fibre of $E \to Z$ contains an f-minimal curve. ■

Lemma 2.3.2. *In Setup B, the $(-K_X)$-minimal curves cover the equidimensional locus $E_{\mathrm{eq}} \to Z_{\mathrm{eq}}$.*

Proof: Let $\Gamma \subset E_{\mathrm{eq}}$ be a $(-K_X)$-minimal curve. As X is smooth, we may apply [Kol96, Theorem II.1.3], which ensures the existence of a deformation family $\mathcal{H}$ of Γ in X such that

$$\dim \mathcal{H} \geq -K_X \cdot \Gamma - 3 + \dim X = n - 4 + \dim X.$$

Since each member of $\mathcal{H}$ lies in a fibre of f, and we have shown in Lemma 2.3.1 that every fibre of dimension n contains a $(-K_X)$-minimal curve, there exists such a family $\mathcal{H}$ which is surjective over Z. In particular, it admits a fibration $\mathcal{H}_{\mathrm{eq}} \to Z_{\mathrm{eq}}$, where $\mathcal{H}_{\mathrm{eq}}$ is dense in $\mathcal{H}$, such that the fibre $\mathcal{H}_z$ over a point $z \in Z_{\mathrm{eq}}$ parametrizes deformations of Γ inside E_z. Consider such a fibre E_z of dimension $n = \dim E - \dim Z$; by the above inequality we have

$$\dim \mathcal{H}_z = \dim \mathcal{H} - \dim Z \geq n - 4 + \dim X - (\dim X - 1 - n) = 2n - 3.$$

Now let $\mathcal{U}_z$ be the universal family over $\mathcal{H}_z$:

$$\begin{array}{ccc} \mathcal{U}_z & \xrightarrow{ev} & E_z \\ {\scriptstyle\pi}\downarrow & & \\ \mathcal{H}_z & & \end{array}$$

By the inequality $\dim \mathcal{H}_z \geq 2n - 3$, we have $\dim \mathcal{U}_z \geq 2n - 2$, and we assume by contradiction that ev is not surjective. Hence $ev(\mathcal{U}_z)$, which is the locus covered in E_z by the minimal curves, has dimension at most $\dim E_z - 1 = n - 1$. Hence the general fibre of ev has dimension $n - 1$ or more.

Pick a general point $x \in E_z$ and denote $\mathcal{H}_{z,x} = \pi(ev^{-1}(x))$. This is the space parametrizing the minimal curves through x, and it is birational to $ev^{-1}(x)$ via π. Let $\mathcal{U}_{z,x} = \pi^{-1}(\mathcal{H}_{z,x})$, then we have

$$\dim \mathcal{U}_{z,x} = \dim \mathcal{H}_{z,x} + 1 \geq n.$$

Since the image of ev has dimension at most $n-1$, the fibres of $\mathcal{U}_{z,x} \to E_z$ have dimension at least 1. Given a point $y \in E_z - \{x\}$, the fibre over y inside $\mathcal{U}_{z,x}$ contains a positive dimensional family of curves through the two fixed points x and y. By the bend-and-break lemma, this family admits a reducible member or a nonreduced member, which is not possible since $\mathcal{H}_z$ is unsplit by the length hypothesis $-K_X \cdot \Gamma = l(f)$ and Corollary 2.1.18.

The conclusion follows that ev is surjective onto E_z. This holds for any fibre E_z of dimension n, hence for the universal family $\mathcal{U}$ over $\mathcal{H}$,

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & E \\ \downarrow & & \\ \mathcal{H} & & \end{array}$$

the evaluation morphism $\mathcal{U} \to E$ is surjective onto the equidimensional locus E_{eq}. ■

The theorem

Back to Setup B, pick $\Gamma \subset E$ a $(-K_X)$-minimal curve and consider the Cartier divisor $-E|_E$, which is relatively ample over Z by Kleiman's criterion (see for instance [Deb16, Theorem 4.10]). Then according to the degree of $-E$ on Γ, we have the following information on the equidimensional locus of $E \to Z$:

Theorem 2.3.3. *In Setup B we have either $-E \cdot \Gamma = 1$ or $-E \cdot \Gamma = 2$. Let $E_{\mathrm{eq}} \to Z_{\mathrm{eq}}$ denote the equidimensional locus of $E \to Z$:*

- *If $-E \cdot \Gamma = 2$, then all the fibres of $E_{\mathrm{eq}} \to Z_{\mathrm{eq}}$ are normalized by $\mathbf{P}^n$. The normalization E' of E_{eq} is a family of projective spaces over the normalization Z' of Z_{eq} which is locally trivial for the analytic topology, such that the following diagram is commutative:*

$$\begin{array}{ccc} E' & \longrightarrow & E_{\mathrm{eq}} \\ \big\downarrow & & \big\downarrow \\ Z' & \longrightarrow & Z_{\mathrm{eq}} \end{array}$$

If moreover n is even, E_{eq} is isomorphic to the projectivization of a rank $n+1$ vector bundle over Z_{eq}.

- *If $-E \cdot \Gamma = 1$, there exist birational morphisms $E' \to E_{\mathrm{eq}}$ and $Z' \to Z_{\mathrm{eq}}$, a rank $n+2$ vector bundle $\mathcal{E}$ over Z', and a quadric bundle $\mathcal{Q} \subset \mathbf{P}_{Z'}(\mathcal{E})$, such that E' is the normalization of $\mathcal{Q}$ and the following diagram is commutative:*

$$\begin{array}{ccc} E' & \longrightarrow & E_{\mathrm{eq}} \\ \big\downarrow & & \big\downarrow \\ Z' & \longrightarrow & Z_{\mathrm{eq}} \end{array}$$

Moreover, the reducible fibres of $E_{\mathrm{eq}} \to Z_{\mathrm{eq}}$ are of the form $G_1 + G_2$, in which G_1 and G_2 are normalized by $\mathbf{P}^n$. The irreducible and generically reduced fibres of $E' \to Z'$ are quadrics, and the reduction of any nonreduced fibre of $E' \to Z'$ is $\mathbf{P}^n$.

Comment: An example of a non-equidimensional divisorial elementary Mori contraction with submaximal length and $-E \cdot \Gamma = 1$ is given in §2.3.2.

Identification of the general fibre

In Setup B, we start by investigating the generically reduced fibres of dimension $n = \dim E - \dim Z$.

Lemma 2.3.4. *Let $E_z \subset E$ be any generically reduced fibre of dimension n over a smooth point z of Z, and let $\Gamma \subset E_z$ be a $(-K_X)$-minimal curve. Then one of the following cases occurs:*

(i) $-E \cdot \Gamma = 2$ *and* $E_z \simeq \mathbf{P}^n$,

(ii) $-E \cdot \Gamma = 1$.

Moreover, in case (ii), if E_z is irreducible then it is isomorphic to a normal quadric.

Proof: First, consider $F \subset E$ any generically reduced fibre of dimension n over a smooth point of Z and $\Gamma \subset F$ a $(-K_X)$-minimal curve meeting the smooth locus of F. The existence of such a curve is ensured by Lemma 2.3.2.

Since $f(F)$ is a smooth point of Z, the fibre F is locally a complete intersection in X, and by adjunction we have $K_F = K_E|_F = (K_X + E)|_F$, hence $-K_F$ is ample. Moreover, since Γ meets the smooth locus of F, by [Kol96, Theorem II.1.3] we have the existence of a deformation family $\mathcal{H}$ of Γ in F such that

$$\dim \mathcal{H} \geq 2n - 4 - E \cdot \Gamma.$$

But $\mathcal{H}$ is unsplit thanks to the length hypothesis and Corollary 2.1.18, so we know thanks to [CMSB02, Theorem 0.1] that $\dim \mathcal{H}$ is bounded from above by $2n-2$, and that this bound is reached if and only if F is normalized by $\mathbf{P}^n$. Indeed, in that case $\mathcal{H}$ induces a complete family of rational curves over the normalization F' of F which have minimal degree with respect to the pullback of $-K_F$, and this implies $F' \simeq \mathbf{P}^n$ with $\Gamma \subset F$ the image of a line. In particular, $-E \cdot \Gamma$ can only equal 1 or 2.

• Let's examine the situation (*i*) $-E \cdot \Gamma = 2$. By semicontinuity, for any generically reduced fibre E_z of dimension n over a smooth point of Z and $\Gamma \subset E_z$ a minimal curve meeting the normal locus of E_z, the deformations of Γ inside E_z form at least one family of dimension $2n-2$, and therefore, by [CMSB02, Theorem 0.1], E_z is normalized by $\mathbf{P}^n$ with Γ the image of a line ℓ:

$$\nu : \mathbf{P}^n \to E_z.$$

As E_z is a local complete intersection in the smooth variety X, it is Cohen-Macaulay and Gorenstein. By Lemma 2.1.21 the conductor divisor $\mathcal{D}$ of this normalization is such that

$$\nu^* K_{E_z} \simeq K_{\mathbf{P}^n} + \mathcal{D}.$$

By adjunction, we have

$$-K_{E_z} \cdot \Gamma = (-K_X - E) \cdot \Gamma = n + 1.$$

Hence we have $-K_{E_z} \cdot \Gamma = n + 1 = -K_{\mathbf{P}^n} \cdot \ell$. As numerical equivalence implies linear equivalence on $\mathbf{P}^n$, the following equality of Cartier divisors holds:

$$\nu^* K_{E_z} = K_{\mathbf{P}^n}.$$

So the conductor $\mathcal{D}$ is trivial. Its support is the nonnormal locus of E_z (this is an application of Lemma 2.1.21, as E_z is Cohen-Macaulay and Gorenstein). In other words E_z is normal in codimension 1, and since it is Cohen-Macaulay we have $E_z \simeq \mathbf{P}^n$.

• Now let's see what happens in the situation (ii) $-E \cdot \Gamma = 1$, under the hypothesis that E_z is an irreducible fibre of dimension n above a smooth point of Z. We pick $\Gamma \subset E_z$ a minimal curve meeting the normal locus of E_z and we denote by $\mathcal{H}_z$ a deformation family of Γ inside E_z. By semicontinuity, the dimension of $\mathcal{H}_z$ can be equal to $2n-2$ or $2n-3$. Assume by contradiction that $\dim \mathcal{H}_z = 2n-2$. In particular, by [CMSB02, Theorem 0.1] the normalization of E_z is isomorphic to $\mathbf{P}^n$, with Γ the image of a line ℓ:

$$\nu : \mathbf{P}^n \to E_z.$$

From Lemma 2.1.21, the conductor divisor $\mathcal{D}$ is such that

$$\nu^* K_{E_z} \simeq K_{\mathbf{P}^n} + \mathcal{D}.$$

Moreover, since E_z is locally a complete intersection, it is Gorenstein, and by adjunction we have

$$-K_{E_z} \cdot \Gamma = (-K_X - E) \cdot \Gamma = n = -K_{\mathbf{P}^n} \cdot \ell - 1.$$

Hence the conductor divisor $\mathcal{D}$ on $\mathbf{P}^n$ satisfies $\mathcal{D} \cdot \ell = 1$, which tells us that $\mathcal{D}$ is a hyperplane of $\mathbf{P}^n$. In addition $\nu^*(-E|_{E_z})$ is also a hyperplane for the same reason.

The divisor $-E|_{E_z}$ being ample, we may consider an irreducible curve $\gamma \subset E_z$ given as a complete intersection of general divisors of the linear system $|-dE|_{E_z}|$ for $d \gg 0$ an odd integer. This curve meets transversally the image of $\mathcal{D}$ and if $\gamma' \subset \mathbf{P}^n$ stands for its proper transform, the restriction $\gamma' \to \gamma$ is the normalization of γ. By the adjunction formula

$$\begin{aligned} K_{\gamma'} + \mathcal{D}|_{\gamma'} &= (K_{\mathbf{P}^n} - d(n-1)\nu^* E + \mathcal{D})|_{\gamma'} \\ &= \nu^*(K_{E_z} - d(n-1)E)|_{\gamma'} \\ &= (\nu|_{\gamma'})^* K_\gamma, \end{aligned}$$

in other words the conductor divisor of $\gamma' \to \gamma$ is $\mathcal{D}|_{\gamma'}$. Since $\mathcal{D}$ and $\nu^*(-E|_{E_z})$ are hyperplane divisors on $\mathbf{P}^n$, The divisor $\mathcal{D}|_{\gamma'}$ has degree d^{n-1}, which is odd. This is not possible, since the conductor divisor of a curve always has an even degree, by Lemma 2.1.21.

So, under the hypothesis that E_z is irreducible and $-E \cdot \Gamma = 1$, we have $\dim \mathcal{H}_z = 2n-3$ for every deformation family $\mathcal{H}_z$ of Γ inside E_z. It remains to be proven that E_z is a normal quadric. Assume by contradiction that E_z is nonnormal. Since in that case $K_{E_z} \cdot \Gamma = nE \cdot \Gamma$, the restriction $nE|_{E_z}$ is linearly equivalent to K_{E_z}, as a consequence of Lemma ??. Moreover $-E|_{E_z}$ is ample and the normalization $\nu : E'_z \to E_z$ is such that

$$\begin{aligned} (K_{E'_z} - n\nu^* E) \cdot (-\nu^* E)^{n-1} &= (K_{E'_z} - \nu^* K_{E_z}) \cdot (-\nu^* E)^{n-1} \\ &= -\mathcal{D} \cdot (-\nu^* E)^{n-1}. \end{aligned}$$

We have $\mathcal{D} > 0$, or else E_z would be regular in codimension one and therefore normal, since it is Cohen-Macaulay. This yields

$$(K_{E'_z} - n\nu^* E) \cdot (-\nu^* E)^{n-1} < 0,$$

in which case there exists a birational morphism $E'_z \to \mathbf{P}^n$ such that $-E|_{E_z}$ is the pull-back of the hyperplane polarization, by [Hö12, Proposition 2.13]. This is not possible, as $\dim \mathcal{H}_z = 2n-3$, whereas the unique family of minimal curves (i.e., lines) in $\mathbf{P}^n$ has dimension $2n-2$. We may thus conclude that E_z is normal, with the equality of ample Cartier divisors $-K_{E_z} = -nE|_{E_z}$. Thanks to [BS95, Theorem 3.1.6], the generalization of a result by Kobayashi and Ochiai, we may conclude that E_z is isomorphic to a quadric. ∎

The case $-E \cdot \Gamma = 2$

In Setup B, we assume that we have $-E \cdot \Gamma = 2$ for every $(-K_X)$-minimal curve Γ. The goal is to exhibit a projective bundle as a birational model for the locus covered by the n-dimensional fibres. We recall the notation

$$E_{\mathrm{eq}} \to Z_{\mathrm{eq}}$$

for the equidimensional locus.

Lemma 2.3.5. *Let $E \to Z$ be an equidimensional fibration onto a normal variety with $n = \dim E - \dim Z$, such that the general fibre is a projective space.*

For such a general fibre, let ℓ be a line inside it. If there exists a relatively ample line bundle L on E such that for any contracted rational curve Γ we have

$$L \cdot \Gamma \geq d := L \cdot \ell,$$

then all the fibres are irreducible and generically reduced. Moreover, there exists a finite and birational morphism $E' \to E$ such that E' is a family of projective spaces over Z. If E is normal, then we have $E' \simeq E$.

Proof: Assume by contradiction that there exists a fibre E_z which is either reducible or not generically reduced:

$$E_z = m_1 D_1 + \cdots + m_s D_s$$

where all the m_i's are nonzero. Without loss of generality we may assume $m_1 \geq 2$ or $s \geq 2$. Either way, by Lemma 2.1.16 we have

$$d^n = (L|_{E_z})^n > (L|_{D_1})^n. \tag{2.4}$$

Now consider $C \subset E_z$ a 1-cycle obtained as a degeneration of lines in the general fibre, so that $L \cdot C = d$. We deduce that C is irreducible and reduced from the length condition on the relatively ample line bundle L, namely $L \cdot \Gamma \geq d$ for any reduced and irreducible component Γ of C.. We may assume without loss of generality that the curve C lies in D_1. By semicontinuity, a deformation family of C inside D_1 has dimension $2n-2$ or more, and [CMSB02, Theorem 0.1] ensures that D_1 is normalized by a projective space, with C the image of a line. Since $L \cdot C = d$, we obtain that the pullback of L by the normalization morphism is isomorphic to $\mathcal{O}_{\mathbf{P}^n}(d)$, and thus

$$(L|_{D_1})^n = d^n$$

which is a contradiction to the inequality (2.4).

As a result, all the fibres are generically reduced and irreducible. If E_z is a fibre, and $C \subset E_z$ is a curve obtained as a degeneration of lines, we have $L \cdot C = d$ and a deformation family of C in E_z has dimension $2n-2$ or more. By [CMSB02, Theorem 0.1] again, the normalization of E_z is isomorphic to $\mathbf{P}^n$:

$$\nu : \mathbf{P}^n \to E_z.$$

Moreover, the polarization L has degree d on the lines, so $\nu^* L \simeq \mathcal{O}_{\mathbf{P}^n}(d)$.

Now we consider the simultaneous normalization of $E \to Z$ whose existence is given by [Kol11, Theorem 12]. This is a finite birational modification $E' \to E$ such that all the fibres of $E' \to Z$ are normal; in this particular case, they are isomorphic to $\mathbf{P}^n$. Indeed, any fibre E_z of $E \to Z$ is normalized by $\mathbf{P}^n$, and if E'_z denotes the fibre of $E' \to Z$ over the same point, then the finite a birational morphism $E'_z \to E_z$ factors through

$$E'_z \to \mathbf{P}^n$$

since E'_z is normal. It follows that $E'_z \simeq \mathbf{P}^n$ since a finite and birational morphism onto a normal variety is an isomorphism (Theorem 2.1.20).

If E is normal, then $E' \to E$ is an isomorphism as a finite and birational morphism onto a normal variety, also by Theorem 2.1.20. ■

In Setup B and under the condition $-E \cdot \Gamma = 2$, we will apply Lemma 2.3.5 to $E_{\mathrm{eq}} \to Z_{\mathrm{eq}}$. If Z_{eq} is normal, then Lemma 2.3.5 ensures that the normalization of E_{eq} is a family of projective spaces over Z_{eq}. But it is in general rather difficult to check whether Z_{eq} is normal from the data of X and the divisorial Mori contraction $f : X \to Y$.

Lemma 2.3.6. *Let Z' be the normalization of Z_{eq} and E' the normalization of E_{eq}.*

$$\begin{array}{ccc} E' & \longrightarrow & E_{\mathrm{eq}} \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & Z_{\mathrm{eq}} \end{array}$$

Then the fibration $E' \to Z'$ is a family of projective spaces which is locally trivial for the analytic topology.

In addition, the normalization of any fibre of $E_{\mathrm{eq}} \to Z_{\mathrm{eq}}$ is isomorphic to $\mathbf{P}^n$.

Proof: We know from Lemma 2.3.4 that the general fibre of $E' \to Z'$ is a projective space. Moreover, the pullback of the divisor $-E|_E$ to E' satisfies the hypothesis of Lemma 2.3.5. Indeed, by Lemma 2.3.4 the minimal degree $-E|_E$ can have on contracted rational curves is reached on the lines on the fibres which are projective spaces. Hence all the fibres of $E' \to Z'$ are projective spaces, since E' is normal. Thanks to a result by Fischer and Grauert (see [FG65]), this family of projective spaces over Z' is relatively trivial in the analytic topology.

In this situation, the $(-K_X)$-minimal curves in E_{eq} are the images of lines. Let us denote by λ the normalization morphism $E' \to E_{\mathrm{eq}}$. The variety Z' being normal, by Lemma 2.1.16 the degree of $\lambda^*(-E)$ on each line contained in a fibre of $E' \to Z'$ is equal to 2.

Now we consider $E_z \subset E_{\mathrm{eq}}$ the fibre over a point $z \in Z_{\mathrm{eq}}$. Let us denote by $\nu : Z' \to Z_{\mathrm{eq}}$ the normalization of Z_{eq} and pick a point $z' \in \nu^{-1}(z)$. The fibre in E' over z' is isomorphic to $\mathbf{P}^n$, and since $\lambda : E' \to E_{\mathrm{eq}}$ is a finite morphism, it yields a finite morphism $\mathbf{P}^n \to E_z$. In this situation, the minimal curves which cover E_z (see Lemma 2.3.2) are the images of the lines of $\mathbf{P}^n$.

Since $-E \cdot \Gamma = 2$ and $\lambda^*(-E) \cdot \ell = 2$ for ℓ a line contained in any fibre of $E' \to Z'$, and the restriction of λ to ℓ is a finite morphism, say of degree δ

$$\lambda|_\ell : \ell \xrightarrow{\delta:1} \lambda(\ell),$$

for $\ell \subset \mathbf{P}^n$ a general line and $\Gamma = \lambda(\ell) \subset E_z$ a minimal curve we have

$$2 = -E \cdot \Gamma = \frac{1}{\delta}\lambda^*(-E) \cdot \ell = \frac{2}{\delta}$$

therefore $\delta = 1$. The morphism $\mathbf{P}^n \to E_z$ is thus finite along the general line $\ell \subset \mathbf{P}^n$, so it is birational. Since it is finite, it is the normalization of E_z, by the following argument: if $(E_z)'$ is the normalization of E_z, then there exists a birational and finite factorization $\mathbf{P}^n \to (E_z)'$ which is in fact an isomorphism (Theorem 2.1.20). ∎

If n is even, we may consider the Cartier divisor

$$J = -K_X + \left(\frac{n}{2} - 1\right) E$$

which has degree 1 on the $(-K_X)$-minimal curves. By Corollary 2.1.9 the divisor $E + 2J$ is relatively trivial, so $J \sim_f -\frac{1}{2}E$ is relatively ample.

Lemma 2.3.7. *If n is even, the fibration $E_{\mathrm{eq}} \to Z_{\mathrm{eq}}$ is a projective bundle, in other words E_{eq} is isomorphic to the projectivization of a vector bundle over Z_{eq}.*

Proof: The relatively ample Cartier divisor

$$J = -K_X + \left(\frac{n}{2} - 1\right) E$$

has degree 1 on the $(-K_X)$-minimal curves, and

$$K_X + (n-1)J$$

is relatively trivial by Corollary 2.1.9. Thanks to Theorem 2.1.23, if $E_z \subset E_{\mathrm{eq}}$ is any fibre, the evaluation map $f^*f_*J \to J$ is surjective along E_z.

Now let $z \in Z_{\mathrm{eq}}$ be the image of E_z and U an affine neighbourhood of z in Y. The direct image $(f_*J)|_U$ being a coherent sheaf on an affine variety, it is endowed with a surjection of the following form:

$$\mathcal{O}_U^{\oplus m+1} \to f_*J|_U.$$

Then we may take the pullback of this by f and its restriction to E_z, yielding a surjection $\mathcal{O}_{E_z}^{\oplus m} \to (f^*f_*J)|_{E_z}$. Since $(f^*f_*J)|_{E_z} \to J|_{E_z}$ is surjective as well, the composition is onto, namely

$$\mathcal{O}_{E_z}^{\oplus m+1} \to J|_{E_z}.$$

Now let e_i be the vectors of the canonical basis of $H^0(E_z, \mathcal{O}_{E_z}^{\oplus m+1})$ and σ_i their images in $H^0(E_z, J|_{E_z})$ $(1 \leq i \leq m+1)$. Then the σ_i's do not vanish simultaneously, and they induce a morphism

$$\theta : E_z \to \mathbf{P}^m$$

such that $\theta^*\mathcal{O}_{\mathbf{P}^m}(1) = J|_{E_z}$.

From Lemma 2.3.6 we know that E_z is normalized by $\mathbf{P}^n$. Let us denote $\lambda : \mathbf{P}^n \to E_z$ the normalization morphism, and $J' = \lambda^*(J|_{E_z})$. Then J' is a line bundle on $\mathbf{P}^n$ which has degree 1 on the lines, which yields $J' \simeq \mathcal{O}_{\mathbf{P}^n}(1)$. The composition $\theta \circ \lambda$ is a morphism

$$\mathbf{P}^n \to \mathbf{P}^m$$

such that $\lambda^*\theta^*\mathcal{O}_{\mathbf{P}^m}(1) = J' \simeq \mathcal{O}_{\mathbf{P}^n}(1)$. Since J' is ample and $(J')^n = 1$, $\theta \circ \lambda$ is a finite morphism onto a linear subspace of dimension n, hence an isomorphism.

From the above, we know that $\theta : E_z \to \mathbf{P}^n$ is birational and finite, hence an isomorphism.

The fibration $E_{\text{eq}} \to Z_{\text{eq}}$ is thus a family of projective spaces. It is endowed with a relatively ample polarization J whose restriction to any fibre E_z is $\mathcal{O}_{\mathbf{P}^n}(1)$ via the isomorphism $E_z \simeq \mathbf{P}^n$, and the existence of this relative hyperplane polarization ensures $E_{\text{eq}} \simeq \mathbf{P}_{Z_{\text{eq}}}(f_*J)$. ■

The case $-E \cdot \Gamma = 1$

In Setup B, we assume now $-E \cdot \Gamma = 1$ for Γ any $(-K_X)$-minimal curve. We recall that the notation $E_{\text{eq}} \to Z_{\text{eq}}$ stands for the equidimensional locus of $f|_E$ and we aim to construct a quadric bundle as a birational model for $E_{\text{eq}} \to Z_{\text{eq}}$. Let us denote $L := \mathcal{O}_{E_{\text{eq}}}(-E)$.

Lemma 2.3.8. *There exist birational morphisms $Z' \to Z_{\text{eq}}$, $E' \to E_{\text{eq}}$ with E' and Z' normal, and a quadric bundle $\mathcal{Q}$ over Z', such that E' is the normalization of $\mathcal{Q}$. In the following commutative diagram:*

$$\begin{array}{ccccc} \mathcal{Q} & \longleftarrow & E' & \longrightarrow & E_{\text{eq}} \\ & \searrow^{\pi} & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ & & Z' & \longrightarrow & Z_{\text{eq}} \end{array}$$

the normalization $E' \to \mathcal{Q}$ is birational and finite along the reduction of each irreducible component of any fibre of f'. Moreover, the reduction of each irreducible component of any reducible or nonreduced fibre of f' is isomorphic to $\mathbf{P}^n$.

In addition, if Γ is a $(-K_X)$-minimal curve in a fibre of $E_{\text{eq}} \to Z_{\text{eq}}$, and Γ' is any irreducible component of its preimage in E', then the image of Γ' in $\mathcal{Q}$ is a line.

Proof: The general fibre of $E_{\text{eq}} \to Z_{\text{eq}}$ is a normal quadric by Lemma 2.3.4.

We consider the birational modification $\mu : \mathcal{Z} \to Z_{\text{eq}}$ given by Lemma 2.1.24, and the vector bundle $\mathcal{V}$ on $\mathcal{Z}$ which coincides with f_*L over the locus where f_*L is locally free.

On the component $\mathcal{F}$ of the fibre product $E_{\text{eq}} \times_{Z_{\text{eq}}} \mathcal{Z}$ which dominates E_{eq}, in the commutative square

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\mu'} & E_{\text{eq}} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Z} & \xrightarrow[\mu]{} & Z_{\text{eq}} \end{array}$$

there is a surjective map of vector bundles

$$g^*\mathcal{V} \to \mu'^*L$$

by Lemma 2.1.24. Now let $\eta : Z' \to \mathcal{Z}$ and $\eta' : E' \to \mathcal{F}$ be the normalizations, then the pullback of the above map $g^*\mathcal{V} \to \mu'^*L$ by η' is surjective:

$$\eta'^*g^*\mathcal{V} \to \eta'^*\mu'^*L.$$

In the following commutative diagram

$$\begin{array}{ccc} E' & \xrightarrow{\eta'} & \mathcal{F} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow[\eta]{} & \mathcal{Z} \end{array}$$

we have $\eta'^*g^*\mathcal{V} \simeq f'^*\eta^*\mathcal{V}$. Hence there is a surjective map

$$f'^*\eta^*\mathcal{V} \to \eta'^*\mu'^*L$$

which induces a factorization of f' by the universal property of projectivized bundles

$$\begin{array}{ccc} E' & \xrightarrow{\chi} & \mathbf{P}_{Z'}(\eta^*\mathcal{V}) \\ & {\scriptstyle f'}\searrow \quad \swarrow & \\ & Z' & \end{array}$$

We denote the image of χ by $\mathcal{Q}$. By Lemma 2.1.16 every fibre of $\mathcal{Q} \to Z'$ is a quadric, in other words $\mathcal{Q}$ is a quadric bundle. Let us denote $\mathcal{J} = \eta'^*\mu'^*L$, then we have $\chi^*\mathcal{O}_{\mathcal{Q}}(1) = \mathcal{J}$.

Let $E'_w \subset E'$ be an irreducible fibre above a point $w \in Z'$. For a point w such that $\eta(w)$ is a smooth point of $\mathcal{Z}$, E'_w is isomorphic to its image in E_{eq} via $\mu' \circ \eta'$, which is an irreducible quadric by Lemma 2.3.4. Under the additional condition that $\eta(w)$ is outside the μ-exceptional locus, $\mu' \circ \eta'$ is an isomorphism around E'_w, and we have $\chi(E'_w) \simeq E'_w$. Therefore χ is birational, and the general fibre E'_w of $E' \to Z'$ is a quadric with $\mathcal{J}|_{E'_w}$ the hyperplane polarization.

For any $w \in Z'$, we set $z = \mu \circ \eta(w)$ and $E_z = \mu' \circ \eta'(E'_w)$. Then E_z is the fibre in E_{eq} over $z \in Z_{\mathrm{eq}}$. By construction, μ' is an isomorphism onto E_z, and η' is finite, so the morphism $E'_w \to E_z$ is finite, ensuring that $\mathcal{J} = \eta'^*\mu'^*L$ is ample on E'_w. This ensures that χ is finite, and since it is finite, the conclusion follows that E' is the normalization of $\mathcal{Q}$.

Now let $\Gamma' \subset E'_w$ be a curve obtained as a degeneration of lines in the smooth fibres of $E_{\mathrm{eq}} \to Z_{\mathrm{eq}}$. By Lemma 2.1.16 we get $\mathcal{J} \cdot \Gamma' = 1$. In particular, its image $\Gamma = \mu' \circ \eta'(\Gamma')$ in E_z is a $(-K_X)$-minimal curve. Let ℓ denote its image $\chi(\Gamma')$ in the quadric $\mathcal{Q}_w = \chi(E'_w)$. If the degree of the finite map

$$\chi|_{\Gamma'} : \Gamma' \to \chi(\Gamma') \subset Q_w$$

is equal to δ, then for the curve $\ell = \chi(\Gamma')$ we have

$$1 = \mathcal{J} \cdot \Gamma' = \delta \mathcal{O}_{\mathcal{Q}_w}(1) \cdot \ell \geq \delta$$

hence $\delta = 1$, and ℓ is a line. Since $\mathcal{J} = \eta'^*\mu'^*L$, any $(-K_X)$-minimal curve $\Gamma \subset E_z$ is the image of a degeneration of lines $\Gamma' \subset E'$, and $\chi(\Gamma')$ is a line in $\mathcal{Q}_w$.

If E'_w is irreducible and generically reduced, then $\mathcal{Q}_w = \chi(E'_w)$ is irreducible as well. Since $\mathcal{J} = \chi^*\mathcal{O}_{\mathcal{Q}}(1)$ with

$$(\mathcal{J}|_{E'_w})^n = 2 = (\mathcal{O}_{\mathcal{Q}_w}(1))^n$$

the finite morphism $E'_w \to \mathcal{Q}_w$ is birational.

Now assume E'_w is reducible or not generically reduced, and D is the reduction of any of its irreducible components. In that case the ample polarization $\mathcal{J}|_{E'_w}$ verifies

$$(\mathcal{J}|_D)^n < (\mathcal{J}|_{E'_w})^n = 2$$

so $(\mathcal{J}|_D)^n = 1$. The Cartier divisor $\mathcal{J}|_{E'_w}$ is basepoint-free since it is the pullback via the morphism χ of $\mathcal{O}_{\mathcal{Q}_w}(1)$, which is basepoint-free. It follows that the morphism $D \to \chi(D)$ induced by the linear system $|\mathcal{J}|_D|$ is birational and finite onto its image. In particular, $\chi(D)$ is isomorphic to $\mathbf{P}^n$, which is normal, so we have $D \simeq \mathbf{P}^n$ (Theorem 2.1.20). ■

Corollary 2.3.9. *If E'_w is a fibre of $E' \to Z'$, then it is either isomorphic to a normal quadric, or as a cycle it is one of the following:*

- *reducible and reduced, namely:* $E'_w = D_1 + D_2$ *with* $D_1 \simeq D_2 \simeq \mathbf{P}^n$*,*
- *nonreduced, namely:* $E'_w = 2D$ *with* $D \simeq \mathbf{P}^n$*.*

Proof: Given a fibre $\mathcal{Q}_w$ of the quadric bundle $\mathcal{Q} \to Z'$ and $E'_w = \chi^{-1}(\mathcal{Q}_w)$, there are three possibilities:

- The quadric $\mathcal{Q}_w$ is normal, in which case for D the reduction of an irreducible component of E'_w which dominates $\mathcal{Q}_w$, the morphism $D \to \mathcal{Q}_w$ is birational and finite by Lemma 2.3.8. Since $\mathcal{J}$ is f'-ample, by Lemma 2.1.16 we have

$$2 = (\mathcal{J}|_{E'_w})^n \geq (\mathcal{J}|_D)^n = \mathcal{O}_{\mathcal{Q}_w}(1)^n = 2,$$

so $E'_w = D$ is irreducible and generically reduced, and the morphism $E'_w \to \mathcal{Q}_w$ is birational and finite by Lemma 2.3.8, hence an isomorphism.

- $\mathcal{Q}_w$ is reducible, in which case we have the equality of n-cycles $\mathcal{Q}_w = Q_1 + Q_2$ where $Q_i \simeq \mathbf{P}^n$. In this case $D_i = \chi^{-1}(Q_i)$ for $i = 1, 2$ are the reductions of the two irreducible components of E'_w, and since the Q_i are normal and $D_i \to Q_i$ is birational and finite by Lemma 2.3.8 we have $D_i \simeq Q_i$. Moreover, $\mathcal{J}|_{D_i} \simeq \mathcal{O}_{\mathbf{P}^n}(1)$ through the isomorphism $D_i \simeq \mathbf{P}^n$, and we have the equality of cycles

$$E'_w = D_1 + D_2$$

since $(\mathcal{J}|_{E'_w})^n = 2$ by Lemma 2.1.16 and $(\mathcal{J}|_{D_i})^n = 1$ for $i = 1, 2$.

- $\mathcal{Q}_w$ is nonreduced, in which case $\mathcal{Q}_w = 2P$ with $P \simeq \mathbf{P}^n$. By Theorem 2.1.20, the reduction D of any irreducible component of E'_w is isomorphic to P since P is normal and $D \to P$ is birational and finite (Lemma 2.3.8). Moreover $\mathcal{J}|_D \simeq \mathcal{O}_{\mathbf{P}^n}(1)$ via the isomorphism $D \simeq \mathbf{P}^n$ since $\mathcal{J}$ has degree 1 on the lines of D. By Lemma 2.1.16 we have $(\mathcal{J}|_{E'_w})^n = 2$, whereas $(\mathcal{J}|_D)^n = 1$. As a consequence we have either the equality of cycles $E'_w = 2D$, or there exists an other irreducible component D_2 of E'_w such that $E'_w = D + D_2$ and $(\mathcal{J}|_{D_2})^n = 1$. In this case $D_2 \simeq \mathbf{P}^n$ for the same reasons as above. ■

Note that in the case where $\mathcal{Q}_w$ is reduced, $E'_w \to \mathcal{Q}_w$ is bijective by the above. Globally the normalization morphism $\chi : E' \to \mathcal{Q}$ might be bijective, but we may not conclude that it is an isomorphism. As a consequence, we may not conclude that $E' \to Z'$ is a quadric bundle. The problem comes from the existence of nonreduced fibres along which $\mathcal{Q}$ might be nonnormal.

Such an example of a nonnormal quadric bundle with a bijective normalization is given in §2.3.2.

From Lemma 2.3.8 and Corollary 2.3.9, we can deduce information on the reducible fibres of $E_{\text{eq}} \to Z_{\text{eq}}$:

Corollary 2.3.10. *Let $z \in Z_{\text{eq}}$ be a point such that the fibre $E_z \subset E_{\text{eq}}$ is reducible. Then E_z has two irreducible components, and the reduction of each component is normalized by $\mathbf{P}^n$.*

Proof: Let w be a point of $(\mu \circ \eta)^{-1}(z)$ and E'_w the fibre over it, so that $E_z = \mu' \circ \eta'(E'_w)$. Since E_z is reducible, so is E'_w and by Corollary 2.3.9 it is of the form $D_1 + D_2$ with $D_i \simeq \mathbf{P}^n$. In particular, E_z has two components G_1 and G_2 with $G_i = \mu' \circ \eta'(D_i)$ for $i = 1, 2$.

Through the isomorphism $D_i \simeq \mathbf{P}^n$ we have a morphism from $\mathbf{P}^n$ to the n-fold G_i

$$\mu' \circ \eta' : D_i \simeq \mathbf{P}^n \to G_i.$$

We know that a morphism from $\mathbf{P}^n$ to a variety of dimension n does not contract any curve. Hence $\mathbf{P}^n \to G_i$ is finite. Moreover, the restriction of the line bundle $\mathcal{J}$ to D_i is the pullback of $L|_{G_i}$, and by Lemma 2.1.16 we have

$$(L|_{G_i})^n = (\mathcal{J}|_{D_i})^n = 1,$$

so $\mathbf{P}^n \to G_i$ is a birational and finite morphism. From this we deduce that G_i is normalized by $\mathbf{P}^n$ for $i = 1, 2$. ■

Note that in that case, the intersection of the two components G_1 and G_2 might be reducible.

Proof of the theorem

Proof of Theorem 2.3.3 : In Lemma 2.3.4 we have proven that we can only have $-E\cdot\Gamma = 1$ or $-E\cdot\Gamma = 2$, and we have identified the general fibre, namely:

- if $-E\cdot\Gamma = 2$, the general fibre is $\mathbf{P}^n$,
- if $-E\cdot\Gamma = 1$, the general fibre is a normal quadric.

Afterwards, we have constructed the birational models.

- In the case $-E\cdot\Gamma = 2$, a family of projective spaces as a birational model for $E_{\mathrm{eq}} \to Z_{\mathrm{eq}}$ is constructed in Lemma 2.3.6, and we have proven in Lemma 2.3.7 that $E_{\mathrm{eq}} \to Z_{\mathrm{eq}}$ is a projective bundle if n is even.
- In the case $-E\cdot\Gamma = 1$, a quadric bundle as a birational model of $E_{\mathrm{eq}} \to Z_{\mathrm{eq}}$ is constructed in Lemma 2.3.8. In Corollary 2.3.10 we have proven that the reducible fibres of $E_{\mathrm{eq}} \to Z_{\mathrm{eq}}$ have two components whose reductions are normalized by $\mathbf{P}^n$. In Corollary 2.3.9 we have identified the fibres of $E' \to Z'$. ■

2.3.2 Examples

An example of a nonnormal quadric bundle with a bijective normalization

Consider in $\mathbf{C}\times\mathbf{P}^3$ with coordinates $(t,[x:y:z:w])$ the quadric bundle

$$Q = \left\{x^2 = t^3 q(y,z,w)\right\}.$$

with $q(y,z,w) = y^2+z^2+w^2$, and the projection $\pi : Q \to \mathbf{C}$. This contains an isolated nonreduced fibre $Q_0 = \pi^{-1}(0)$, and over a nonzero t the fibre Q_t is a smooth quadric. The singular locus of Q consists in Q_0, so Q is nonnormal.

Lemma 2.3.11. *The normalization morphism of Q is bijective.*

Proof: Let $p = [0:y:z:w] \in Q_0$ be a point such that $q(p) \neq 0$. Then locally around p, Q is of the same form as the cusp

$$x^2 = t^3$$

in an affine chart $\mathbf{C}^4_{(t,x,u,v)}$. The coordinate ring of the above is

$$A = \mathbf{C}[t,x,u,v]/(x^2 = t^3).$$

In its fraction field, we introduce $s = \frac{x}{t}$. Then s is integral over A by the relation $s^2 = t$. This s generates the extension

$$B = A[s] = C[t,x,u,v,s]/(st = x, s^2 = t) \simeq C[u,v,s].$$

Hence this affine chart Spec$A$ around p is normalized by Spec$B = \mathbf{C}^3$, and moreover the normalization morphism on this affine chart is

$$(u,v,s) \in \mathrm{Spec}B \mapsto (t,x,u,v) = (s^3, s^2, u, v) \in \mathrm{Spec}A,$$

which is bijective.

It remains to be proven that the normalization of Q is bijective around a point where q vanishes. Let now $p' \in Q_0$ be such a point with $q(p') = 0$. We have $\mathrm{Supp}(Q_0) = \{x = t = 0\}$. Moreover, since $q = y^2+z^2+w^2$, the differential of q does not vanish along $\mathrm{Supp}(Q_0)$, so q is injective in a small neighbourhood of p' inside which we may consider $u = q(y,z,w)$, so that the point p' in Q has the same local ring as the origin in SpecA', with

$$A' = \mathbf{C}[t,x,u,v]/(x^2 = t^3 u).$$

In its fraction field, we consider $s = \frac{x}{t}$. It is integral over A' by the relation $s^2 = tu$. It generates the extension

$$B' = A'[s] = \mathbf{C}[t,x,u,v,s]/(st = x, s^2 = tu) \simeq \mathbf{C}[t,u,v,s]/(s^2 = tu).$$

We recognize that SpecB' is an affine cone over a conic, which is normal. Hence it is the normalization of Q around p', and the normalization morphism goes as follows

$$(t,u,v,s) \in \mathrm{Spec}B' \mapsto (t,x,u,v) = (t, st, u, v) \in \mathrm{Spec}A',$$

which is bijective. ■

An example of a nonequidimensional divisorial elementary Mori contraction of submaximal length

Consider in $\mathbf{C}^6$ with coordinates $(x_1, x_2, x_3, x_4, \lambda, \mu)$ the cubic affine cone

$$Y = \left\{\lambda x_1^2 + \lambda x_2^2 + \mu x_3^2 + \mu x_4^2 + x_1 x_2^2 + x_1 x_3^2 + x_1 x_4^2 = 0\right\}.$$

One can think of it as a family of affine cubics $Y_{(\lambda,\mu)} \subset \mathbf{C}^4_{(x_1,x_2,x_3,x_4)}$, which is a perturbation of the family of threefold quadric affine cones

$$\left\{\lambda x_1^2 + \lambda x_2^2 + \mu x_3^2 + \mu x_4^2 = 0\right\}_{(\lambda,\mu)\in\mathbf{C}^2}.$$

Let $\varepsilon : X \to Y$ be the blow-up of Y along $\Lambda = \{x_1 = x_2 = x_3 = x_4 = 0\} \simeq \mathbf{C}^2$.

Lemma 2.3.12. *The variety X is smooth.*

Proof: Since Y is smooth outside Λ, it is enough to prove that X is smooth along $E = \varepsilon^{-1}(\Lambda)$. We have a model for X inside $\mathbf{P}^3_{[u_1:u_2:u_3:u_4]} \times \mathbf{C}^6_{(x_1,x_2,x_3,x_4,\lambda,\mu)}$ with the following equations

$$\begin{cases} \lambda u_1^2 + \lambda u_2^2 + \mu u_3^2 + \mu u_4^2 + x_2 u_1 u_2 + x_3 u_1 u_3 + x_4 u_1 u_4 = 0, \\ \det_2 \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} = 0. \end{cases}$$

Since the codimension of X in $\mathbf{P}^3 \times \mathbf{C}^6$ is 4, checking that X is smooth amounts to showing that the Jacobian of the system of equations has rank 4 everywhere on X. Since X is the blow-up of the affine cone Y along an affine subspace which contains the only singular point of Y, we only need to check that the Jacobian of X has maximal rank along E, in other words when we specify $x_i = 0$ for all i. Without loss of generality, we may assume $u_1 = 1$ and work with (u_2, u_3, u_4) affine. A straightforward calculation shows that the Jacobian with respect to the variables (x_i, λ, μ) always has rank 4, as required. ∎

Lemma 2.3.13. *The blowdown morphism $\varepsilon : X \to Y$ is a divisorial Mori contraction whose exceptional divisor E is not equidimensional onto its image Λ. The fibration $E \to \varepsilon(E)$ admits a fibre E_0 which is isomorphic to $\mathbf{P}^3$ such that $-K_X|_{E_0}$ is the hyperplane polarization.*

Moreover, the length of ε is submaximal and ε is elementary.

Proof: For $p = (\lambda, \mu) \in \Lambda$ general, the fibre $E_{(\lambda,\mu)}$ over p is the projectivization of the tangent cone of $Y_{(\lambda,\mu)}$ at p, in other words $E_{(\lambda,\mu)}$ is the quadric of $\mathbf{P}^3_{[u_1:u_2:u_3:u_4]}$ given by the equation

$$\lambda u_1^2 + \lambda u_2^2 + \mu u_3^2 + \mu u_4^2 = 0.$$

However, the fibre over the origin $(\lambda, \mu) = (0, 0)$ is the whole $\mathbf{P}^3$. The equation above is a model for E as a hypersurface of $\mathbf{P}^3 \times \Lambda$.

The general fibre E_p is embedded as a quadric surface in $\mathbf{P}^3$ and it satisfies

$$\mathcal{O}_{E_p}(2) = -K_{E_p} = (-K_X - E)|_{E_p},$$

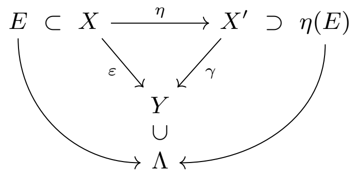
and since $-E|_{E_p}$ is ample and $-K_{E_p}$ has degree 2 on the lines, the length of the contraction ε is equal to 1. In other words:

$$l(\varepsilon) = -K_X \cdot \ell = 1 = \dim E_p - 1$$

for ℓ any line lying on the quadric E_p. Therefore the length of ε is submaximal (Theorem 0.0.9).

If we degenerate the general line $l \in E_p$ to a line l_0 in the central fibre $E_0 \simeq \mathbf{P}^3$, we have $-K_X \cdot l_0 = 1$ and via the isomorphism $E_0 \simeq \mathbf{P}^3$, the restriction $-K_X|_{E_0}$ is the hyperplane polarization.

It remains to be proven that ε is an elementary contraction. By the relative cone theorem (see for instance [Deb16, Proposition 4.21]) there exists a curve $C \subset E_0$ whose class is extremal in the relative Mori cone of $\varepsilon : X \to Y$, and an elementary contraction $\eta : X \to X'$ which contracts all the curves in the numerical equivalence class of C and fits in the following commutative diagram:



Since E_0 is isomorphic to $\mathbf{P}^3$, and η contracts a curve $C \subset E_0$, then $\eta(E_0)$ is a point of X'. The restriction of γ to $\eta(E)$ is thus a proper fibration over Λ whose central fibre is a point; by semicontinuity γ is locally an isomorphism over the origin of Y. This ensures that there exists a quadric fibre E_p for $p \neq 0$ such that $\eta(E_p)$ is a point; as a consequence both families of lines on the general fibre of $E \to \Lambda$ are contracted, and as a consequence $\eta(E) \simeq \Lambda$ and γ is an isomorphism. Therefore ε is elementary. ■

2.4 Future projects

The natural continuation of this chapter's work consists in extending the study of Setup A.1 to a similar situation, but with arbitrary relative dimension. The length condition on the relative Cartier divisor A becomes

$$l(A) := \min \{A \cdot \Gamma \mid \Gamma \subset X \text{ contracted rational curve}\} = \dim X - \dim Y.$$

This would provide a birational model for elementary Mori contractions of fibre type and submaximal length of relative dimension 2 or more. As stated earlier, we know by [DH17] that the general fibre is isomorphic to a quadric in this situation, but the identification of the singular fibres requires very fine techniques like the degeneration of lines and taking local sections through each irreducible component of reducible fibres.

There is already something we can tell about the nature of the singularities of X along the reducible f-fibres.

Theorem 2.4.1. *Let $f : X \to Y$ be an equidimensional fibration from a normal threefold to a smooth curve whose general fibre is a smooth quadric. We assume that there exists $A \in \mathrm{Pic}(X)$ f-ample, such that $A + K_X$ is trivial on the general f-fibre. Moreover, we assume that the length of the contraction is submaximal, meaning:*

$$\min \{A \cdot \Gamma \mid \Gamma \subset X \text{ contracted rational curve}\} = 2.$$

If X admits a reduced and reducible fibre $X_0 = D_0 + D_1 + \cdots$, then it is not $\mathbf{Q}$-factorial.

Note that this situation is very different from Setup A.1, in which $X \to Y$ is an equidimensional fibration from a normal surface to a smooth curve. In the situation of Setup A.1, we know by Lemma 2.2.5 that X is klt (in particular, $\mathbf{Q}$-Gorenstein).

The proof of Theorem 2.4.1 is done by contradiction and requires the following Lemma 2.4.2. The proof of Lemma 2.4.2 is broken down into Lemmas 2.4.3 and 2.4.4.

Lemma 2.4.2. *Under the same hypotheses as Theorem 2.4.1, if X is $\mathbf{Q}$-factorial, the normalization of each component D_i is $\mathbf{P}^2$. In addition, X_0 has exactly two components.*

⋆ ⋆

We begin by providing a proof for Lemma 2.4.2. The situation being local, we may assume that $Y = \Delta$ is the unit disc in $\mathbf{C}$ and that $X_t = f^{-1}(t)$ is smooth for $t \neq 0$, while X_0 is the fibre over the origin. Since we are restricted to a small analytic neighbourhood of the central fibre, we may consider a section $\sigma : \Delta \to X$ such that $\sigma(0)$ is a general point of D_0 outside of the other components D_i. Now we consider $Z \subset X$ the locus covered by the minimal rational curves through σ, that is, contracted rational curves on which A has degree 2 and which meet the section $\sigma(\Delta)$. We pick $\mathcal{H}$ a deformation family of minimal curves contracted by f, in the sense of Definition 2.1.14.

The family of 1-cycles $\mathcal{H}$ is endowed with a universal family $\mathcal{U}$ and the natural morphisms

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{ev} & X \\ {\scriptstyle q}\downarrow & & \\ \mathcal{H} & & \end{array}$$

The locus Z is then obtained as

$$Z = ev(q^{-1}(q(ev^{-1}(\sigma(\Delta))))).$$

This is a hypersurface of X: indeed, on the general fibre X_t, Z cuts out two transversal lines. Assuming X is $\mathbf{Q}$-factorial, this is a $\mathbf{Q}$-Cartier divisor of X.

Lemma 2.4.3. *Let $Z_0 = Z \cap X_0$. Then Z_0 is contained in D_0. Moreover one of the two following cases occurs.*

- *$Z_0 = D_0$, in which case the normalization of D_0 is $\mathbf{P}^2$,*
- *Z_0 has dimension 1, in which case it is either the sum of two smooth rational curves meeting transversally or a double line.*

Proof: Any family of contracted minimal rational curves is unsplit thanks to the length hypothesis, meaning that the fibres of q are irreducible and nonreduced, and the point $\sigma(0)$ lies in D_0 away from D_i, for all $i \neq 0$. Since Z_0 is set-theoretically a union of irreducible curves through $\sigma(0)$, we have $Z_0 \subset D_0$.

Now assume $Z_0 = D_0$. We pick $\mathcal{K}$ a family of minimal rational curves in D_0, all passing through $\sigma(0)$. By the hypothesis $Z_0 = D_0$, we may find such a family for which $\dim \mathcal{K} \geq 1$. Let now $\mathcal{H}_0$ be the central fibre of $\mathcal{H} \to \Delta$, i.e., a deformation family of minimal rational curves in X_0. Then $\mathcal{K}$ is naturally an irreducible subscheme of $\mathcal{H}_0$. We set $\mathcal{U}_0$ the universal family over $\mathcal{H}_0$, with the evaluation morphism ev_0 and the $\mathbf{P}^1$-bundle q_0:

$$\begin{array}{ccc} \mathcal{U}_0 & \xrightarrow{ev_0} & D_0 \\ {\scriptstyle q_0}\downarrow & & \\ \mathcal{H}_0 & & \end{array}$$

By construction, the fibre of ev_0 over $\sigma(0)$ has a component which is isomorphic to $\mathcal{K}$. The general fibre of ev_0 has thus positive dimension, yielding $\dim \mathcal{U}_0 \geq 3$. Hence the dimension of $\mathcal{H}_0$ is at least $2 = 2\dim D_0 - 2$.

Now we consider the normalization $\widetilde{D_0}$ of D_0 with the normalization morphism $\nu : \widetilde{D_0} \to D_0$, and $\widetilde{\mathcal{H}_0}$ the preimage of $\mathcal{H}_0$ under the pushforward map

$$\nu_* : \mathrm{RatCurves}^n(\widetilde{D_0}) \to \mathrm{RatCurves}^n(D_0).$$

Then $\widetilde{\mathcal{H}_0}$ is a subscheme of $\mathrm{RatCurves}^n(\widetilde{D_0})$ whose members are minimal against $\nu^* A$. As a conclusion, we have $\widetilde{D_0} \simeq \mathbf{P}^2$ thanks to [CMSB02, Theorem 0.1].

If Z_0 has dimension one, then the reduction of any of its irreducible component is by construction a minimal rational curve. Restricting A to $Z \to \Delta$, we have the equality $A \cdot Z_0 = A \cdot Z_t = 4$, so either $Z_0 = l_1 + l_2$ or $Z_0 = 2l$. In both cases the arithmetic genus of Z_0 is zero by flatness of $Z \to \Delta$. It follows that the l_i's and l are isomorphic to $\mathbf{P}^1$. ■

Lemma 2.4.4. *Now we assume that D_0 is not normalized by $\mathbf{P}^2$, and thus* $\dim Z_0 = 1$*. Then the Picard rank of D_0 is 2, and the intersection number $Z_0 \cdot D_i|_{D_0}$ is zero for all $i \neq 0$.*

Proof: Let $C \subset D_0$ be a nonminimal curve, meaning that $A \cdot C > 2$. As in the proof of Lemma 2.4.3 we consider the universal family $\mathcal{U}_0$ over $\mathcal{H}_0$ endowed with the evaluation map ev_0 and the $\mathbf{P}^1$-bundle q_0. The dimension of $\mathcal{H}_0$ must be 1, or else D_0 would be normalized by $\mathbf{P}^2$ by [CMSB02, Theorem 0.1]. Any irreducible component of $\mathcal{U}_0$ being a $\mathbf{P}^1$-bundle over a normal curve $\mathcal{L} \subset \mathcal{H}_0$, its Picard rank is 2. Since the evaluation map ev_0 is surjective, the picard rank $\rho(D_0)$ can only be 1 or 2.

Assume by contradiction that $\rho(D_0) = 1$. Then Z_0 is an ample divisor of D_0, since it is $\mathbf{Q}$-Cartier and effective; moreover Z_0 is set-theoretically a union of irreducible curves through $\sigma(0)$. The point $\sigma(0)$ does not belong to D_i as soon as $i \neq 0$, so the locus $Z_0 \cap D_i$ is either empty or finite in each case. It can not be empty by the ampleness of

Z_0, and it can not be finite since it is the intersection of the two $\mathbf{Q}$-Cartier divisors Z and D_i on the threefold X.

The conclusion follows that $\rho(D_0) = 2$. Besides, $Z \cap D_i = \varnothing$ for all $i \neq 0$. ■

We are now able to complete the proof of Lemma 2.4.2, and then that of Theorem 2.4.1.

Proof of Lemma 2.4.2 : We assume by contradiction that the normalization of D_0 is not $\mathbf{P}^2$. By Lemma 2.4.3 the dimension of Z_0 is 1, and either $Z_0 = l_1 + l_2$ or $Z_0 = 2l_1$.

- If $Z_0 = l_1 + l_2$, then Z has two components $Z_1 + Z_2$ such that $Z_j|_{X_0} = l_j$ for $j = 1, 2$. Indeed, on each fibre of $X \to \Delta$, the divisor Z cuts out two lines that are transverse to each other; the normalization of Z in this case is the union of two connected components, each being a $\mathbf{P}^1$-bundle over Δ, and the irreducible components of Z are the respective images of these connected components. In the general fibre X_t we have the equality $(Z_j|_{X_t})^2 = 0$ for $j = 1, 2$. Indeed, $Z_j|_{X_t}$ are the two lines of the ruling of $\mathbf{P}^1 \times \mathbf{P}^1$ through the point $\sigma(t)$, and

$$l_1^2 = (Z_1|_{X_0})^2 = Z_1^2 \cdot X_0 = Z_1^2 \cdot X_t = 0.$$

Likewise, $l_2^2 = 0$. Moreover, l_1 and l_2 are two effective curves meeting transversally, so that $l_0 \cdot l_1 > 0$. Since the Picard rank of D_0 is 2 (see Lemma 2.4.4), we have

$$\mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{Pic}(D_0) = \mathbf{Q} l_1 \oplus \mathbf{Q} l_2.$$

Again by Lemma 2.4.4, we have $l_j \cdot D_i|_{D_0} = 0$ for $j = 1, 2$. The equality above leads to $D_i|_{D_0} = 0$ for all $i \neq 0$. This is not possible since the central fibre X_0 is connected.

- If $Z_0 = 2l$, then by the equality

$$Z_0^2 = (Z|_{X_0})^2 = Z^2 \cdot X_t = 2,$$

we have $l^2 = \frac{1}{2}$. Up to considering the normalization of D_0, it means that there exists a normal curve $\mathcal{L}$ inside the deformation family $\mathcal{H}_0$ such that l is a member of $\mathcal{L}$, and all the members of $\mathcal{L}$ pass through a fixed singular point of D_0. Again, we consider the universal family $\mathcal{U}_0$ over $\mathcal{H}_0$. We know that $q_0^{-1}(\mathcal{L}) \subset \mathcal{U}_0$ is dominant over D_0 and smooth as a $\mathbf{P}^1$-bundle over a normal curve, and its Picard rank is 2. Besides, the restriction of the evaluation map $\mathcal{U}_0 \to D_0$ to $q_0^{-1}(\mathcal{L})$ is birationalis birational: by the hypothesis $Z_0 = 2l$, only one minimal curve of the family $\mathcal{L}$ goes through the general point $\sigma(0) \in D_0$. Hence $q_0^{-1}(\mathcal{L})$ is a resolution of the singularities of D_0, yielding $\rho(D_0) = 1$, which is not possible by Lemma 2.4.4.

Likewise, the normalization of D_i is $\mathbf{P}^2$ for all irreducible component D_i of X_0.

It remains to be proven that X_0 has exactly two components. From the length hypothesis, we have $\nu_i^* A = \mathcal{O}_{\mathbf{P}^2}(2)$, where $\nu_i : \mathbf{P}^2 \to D_i$ is the normalization morphism of D_i. Hence $A^2 \cdot D_i = \mathcal{O}_{\mathbf{P}^2}(2)^2 = 4$, whereas $A^2 \cdot X_0 = A^2 \cdot X_t = 8$. This forces $X_0 = D_0 + D_1$. ■

⋆ ⋆

Proof of Theorem 2.4.1 : Assume by contradiction that X is $\mathbf{Q}$-factorial and f admits a reducible and reduced fibre $X_0 = D_0 + D_1 + \cdots$. Then Lemma 2.4.2 applies. Let Δ be an analytic neighbourhood of $f(X_0)$ such that Δ is isomorphic to the unit disc, and $f(X_0) = 0$. If $\sigma : \Delta \to X$ is a section going through D_0 away from D_1 and Z is the locus covered by the minimal rational curves through $\sigma(\Delta)$, then we have $Z_0 = Z \cap X_0 \subset D_0$ by Lemma 2.4.3. If $Z_0 = D_0$, then Z is reducible, having D_0 as a component, but it is then possible to consider the effective $\mathbf{Q}$-Cartier divisor $Z' = Z - D_0$, which cuts out a one-dimensional locus on D_0. So we may assume that Z_0 has dimension 1.

Since D_1 is also $\mathbf{Q}$-Cartier, $Z \cap D_1$ is a curve. As a consequence, an irreducible component of the locus $Z \cap X_0$ lies in the intersection $D_0 \cap D_1$. But the curve $\sigma(\Delta)$ is contained in every component of Z by construction, whereas $\sigma(0)$ does not belong to D_1, a contradiction. ■

Thanks to [Kol13, Corollary 1.37] we know that any klt variety admits a $\mathbf{Q}$-factorial modification (without any divisorial contraction). What we do now is exhibit an example of an equidimensional contraction from a threefold to a smooth curve of length 2 such that the central fibre is reduced and irreducible. By Theorem 2.4.1 we know that this example is not $\mathbf{Q}$-factorial, although it may admit a $\mathbf{Q}$-factorial modification if it is klt.

We examine the following particular case. Consider the product $\mathbf{P}^1 \times \mathbf{P}^3$ endowed with the homogeneous coordinates $([x_0 : x_1], [y_0 : y_1 : y_2 : y_3])$, and X the hypersurface given by the equation

$$x_0 y_2 y_3 + x_1 \psi_2(\mathbf{y}) = 0,$$

where $\psi_2(\mathbf{y})$ is a homogeneous quadratic form in $\mathbf{y} = [y_0 : y_1 : y_2 : y_3]$. We point out the following facts about X:

- By adjunction, the canonical class of X is the restriction of $\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^3}(-1, -2)$. Hence $-K_X$ is relatively ample for the projection $\mathbf{P}^1 \times \mathbf{P}^3 \to \mathbf{P}^1$ with length 2.
- Assuming that the $\psi_2(\mathbf{y})$ is general, the singular fibres of the contraction $X \to \mathbf{P}^1$ are isolated. The fibre X_0 which is cut out in X by $x_1 = 0$ is a singular quadric made of two planes:
$$X_0 = \mathbf{P}^2 \cup \mathbf{P}^2 \subset \mathbf{P}^3.$$

Note that the threefold X satisfies all the hypotheses of Theorem 2.4.1, therefore it is not $\mathbf{Q}$-factorial.

We introduce the notation $\Lambda = \{x_1 = y_2 = y_3 = 0\}$. This line contains isolated singular points of X.

Question. Is the variety X klt along Λ?

As a consequence, at least locally around the central fibre, there would exist a $\mathbf{Q}$-factorialization of X.

Our best shot so far to determine whether X is klt requires to do a weighted blow-up of X along Λ and compute the discrepancies, which is the focus of Lemmas 2.4.5 and 2.4.6.

Lemma 2.4.5. *Let W be the weighted blow-up of $\mathbf{P}^1 \times \mathbf{P}^3$ along Λ with weights $(2, 1, 1)$ with respect to the toric coordinates (x_1, y_2, y_3), E the exceptional divisor of the blow-up and $\varepsilon : W \to \mathbf{P}^1 \times \mathbf{P}^3$ the blowdown map. Then we have*

$$K_W = \varepsilon^* K_{\mathbf{P}^1 \times \mathbf{P}^3} + 3E.$$

We point out first that, since $\Lambda \simeq \mathbf{P}^1$ is a line lying on a fibre of a $\mathbf{P}^3$-bundle, its normal bundle is

$$\mathcal{N}_{\Lambda/\mathbf{P}^1 \times \mathbf{P}^3} \simeq \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbf{P}^1}$$

and its projectivization $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbf{P}^1})$ is a $\mathbf{P}^2$-bundle.

However, as we are computing a weighted blow-up, the exceptional divisor E is not a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$, but rather a $\mathbf{P}(1, 1, 2)$-bundle.

Proof: We use the toric method provided in the fifth chapter of [CLS11]. A polytope for the toric variety $\mathbf{P}^1 \times \mathbf{P}^3$ is naturally obtained as the product of that of $\mathbf{P}^1$ with that of $\mathbf{P}^3$, namely a segment times a tetrahedron. We denote this polytope, or equivalently its associated fan, by the following array:

$$\Sigma_{\mathbf{P}^1 \times \mathbf{P}^3} = \mathrm{Fan}\left(\underset{\mathsf{e}_{x_0}}{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \underset{\mathsf{e}_{y_0}}{\begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \end{bmatrix}}, \underset{\mathsf{e}_{y_1}}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\mathsf{e}_{y_2}}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{\mathsf{e}_{y_3}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}, \underset{\mathsf{e}_{x_1}}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \right)$$

The correspondence between the columns and the homogeneous coordinates means that the one-dimensional cone generated by the first column encodes the vanishing locus of x_0, and so on.

Since Λ is the vanishing locus of x_1, y_2 and y_3, the weighted blow-up is encoded by the subdivision of the cone $\mathbf{R}_+\mathsf{e}_{x_1} + \mathbf{R}_+\mathsf{e}_{y_2} + \mathbf{R}_+\mathsf{e}_{y_3}$. We do this by adding the one-dimensional cone generated by the particular linear combination $2\mathsf{e}_{x_1} + \mathsf{e}_{y_2} + \mathsf{e}_{y_3}$

Hence a fan for W is given by

$$\Sigma_W = \text{Fan}\left(\underset{\mathsf{e}_{\mathtt{x}_0}}{\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}}, \underset{\mathsf{e}_{\mathtt{y}_0}}{\begin{bmatrix}0\\-1\\-1\\-1\end{bmatrix}}, \underset{\mathsf{e}_{\mathtt{y}_1}}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}}, \underset{\mathsf{e}_{\mathtt{y}_2}}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}, \underset{\mathsf{e}_{\mathtt{y}_3}}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}}, \underset{\mathsf{e}_{\mathtt{x}_1}}{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}, \underset{\mathsf{e}_{\zeta}}{\begin{bmatrix}2\\0\\1\\1\end{bmatrix}}\right)$$

This gives rise to homogeneous coordinates $(\mathtt{x}_0, \mathtt{x}_1, \mathtt{y}_0, \mathtt{y}_1, \mathtt{y}_2, \mathtt{y}_3, \zeta)$ on W whose grading is as follows in $\mathbf{Z}^3$:

	$\mathtt{x}_0$	$\mathtt{x}_1$	$\mathtt{y}_0$	$\mathtt{y}_1$	$\mathtt{y}_2$	$\mathtt{y}_3$	ζ
degree	1	1	0	0	0	0	0
in $\mathbf{Z}^3$:	0	0	1	1	1	1	0
	0	-2	0	0	-1	-1	1

(2.5)

and the blowdown morphism ε is given by the expression

$$([x_0 : x_1], [y_0 : y_1 : y_2 : y_3]) = ([\mathtt{x}_0 : \zeta^2 \mathtt{x}_1], [\mathtt{y}_0 : \mathtt{y}_1 : \zeta\mathtt{y}_2 : \zeta\mathtt{y}_3]). \tag{2.6}$$

To be precise, the first component of this map is the morphism induced by the linear system $|\mathcal{O}_W(1,0,0)|$, while the second component is induced by $|\mathcal{O}_W(0,1,0)|$.

Moreover, the exceptional divisor E of ε is the vanishing locus of ζ. By the construction, the vanishing locus of ζ does not meet that of $\mathtt{x}_0$, and we have already stated that E is a $\mathbf{P}(1,1,2)$-bundle over $\mathbf{P}^1$. The restriction of the coordinates $\mathtt{x}_1, \mathtt{y}_0, \mathtt{y}_1, \mathtt{y}_2$ and $\mathtt{y}_3$ to E yields toric coordinates on this weighted projective bundle with the following grading:

	$\mathtt{x}_1$	$\mathtt{y}_0$	$\mathtt{y}_1$	$\mathtt{y}_2$	$\mathtt{y}_3$
degree	2	0	0	1	1
in $\mathbf{Z}^2$:	0	1	1	-1	-1

(2.7)

and in these coordinates the bundle map to the base $\Lambda \simeq \mathbf{P}^1$ is $[\mathtt{y}_0 : \mathtt{y}_1]$.

Thanks to the expression of ε in coordinates, we are able to compute the difference between K_W and $\varepsilon^* K_{\mathbf{P}^1 \times \mathbf{P}^3}$. On the one hand, the canonical class of W is $\mathcal{O}_W(-2,-4,3)$ (the degree of $-K_W$ is the sum of the degrees of all the variables). On the other hand, the pullback by ε of $K_{\mathbf{P}^1 \times \mathbf{P}^3} = \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^3}(-2,-4)$ is $\mathcal{O}_W(-2,-4,0)$. Since $\mathcal{O}_W(E) = \mathcal{O}_W(0,0,1)$ we have the equality

$$K_W = \varepsilon^* K_{\mathbf{P}^1 \times \mathbf{P}^3} + 3E,$$

as required. ■

Lemma 2.4.6. *Let now $\widehat{X} \subset W$ be the proper transform of X. It satisfies*

$$\varepsilon^* X = \widehat{X} + 2E$$

and

$$K_{\widehat{X}} = (\varepsilon|_{\widehat{X}})^* K_X + E|_{\widehat{X}}.$$

Proof: From the expression of ε given in (2.6), we obtain the equation of $\varepsilon^{-1}(X)$ as the pullback of the equation of X. This equation admits ζ^2 as a factor, but not ζ^3. This proves the first equality.

In particular, the restriction of $\widehat{X}$ to E is given by such an equation:

$$\mathtt{x}_0\mathtt{y}_2\mathtt{y}_3 + \mathtt{x}_1\alpha_2(\mathtt{y}_0, \mathtt{y}_1) = 0$$

with respect to the coordinates given in (2.7). Here, α_2 is a quadratic form on $\Lambda = \mathbf{P}^1_{[\mathtt{y}_0:\mathtt{y}_1]}$. The locus cut out by $\widehat{X}$ on E is a conic bundle over Λ. Indeed, given any fixed point $[\mathtt{y}_0 : \mathtt{y}_1]$ in Λ, the above becomes a quadric of $\mathbf{P}(1,1,2)$, i.e., a conic, since $\mathtt{x}_0$ is never zero on E. There are two singular conics in this bundle, lying over the points $[\mathtt{y}_0 : \mathtt{y}_1]$ for which $\alpha_2(\mathtt{y}_0, \mathtt{y}_1) = 0$.

The rest follows by the adjunction formula. Indeed, we have

$$K_{\widehat{X}} = (K_W + \widehat{X})|_{\widehat{X}},$$

while

$$K_X = (K_{\mathbf{P}^1 \times \mathbf{P}^3} + X)|_X,$$

and it follows from Lemma 2.4.5 that $\varepsilon^* K_X = K_{\widehat{X}} - E|_{\widehat{X}}$. ■

Question. Is the proper transform $\widehat{X}$ smooth along E?

We know from the above that the defining equation for $\widehat{X}$ in W with respect to the coordinates (2.5) has the form

$$\zeta\Phi(\mathtt{x}, \mathtt{y}, \zeta) + \mathtt{x}_1\alpha_2(\mathtt{y}_0, \mathtt{y}_1) + \mathtt{x}_0\mathtt{y}_2\mathtt{y}_3 = 0,$$

where $\zeta\Phi(\mathtt{x}, \mathtt{y}, \zeta)$ is the part which admits ζ as a factor, and $\alpha_2(\mathtt{y}_0, \mathtt{y}_1)$ is a homogeneous quadric. In particular, $\Phi(\mathtt{x}, \mathtt{y}, \zeta)$ is of the form

$$\Phi(\mathtt{x}, \mathtt{y}, \zeta) = \mathtt{x}_1(\mathtt{y}_2\ell(\mathtt{y}_0, \mathtt{y}_1) + \mathtt{y}_3\ell'(\mathtt{y}_0, \mathtt{y}_1)) + \zeta\Theta(\mathtt{x}, \mathtt{y}),$$

where ℓ and ℓ' are linear forms and $\zeta^2\Theta(\mathtt{x}, \mathtt{y})$ is the part which admits ζ^2 as a factor.

However, we haven't managed so far to check whether $\widehat{X}$ is smooth from its description as a hypersurface with an equation involving the toric coordinates.

Since the coordinate $\mathtt{x}_0$ never vanishes along $E = \{\zeta = 0\}$, the restriction of $\widehat{X}$ to E is a hypersurface of the $\mathbf{P}(1, 1, 2)_{[\mathtt{y}_2:\mathtt{y}_3:\mathtt{x}_1]}$-bundle over $\mathbf{P}_{[\mathtt{y}_0:\mathtt{y}_1]}$ bundle, which given by the following equation

$$\mathtt{x}_1\alpha_2(\mathtt{y}_0, \mathtt{y}_1) + \mathtt{y}_2\mathtt{y}_3 = 0$$

with respect to the toric coordinates (2.7).

If $\widehat{X}$ is smooth, it is a resolution of X along Λ. By Lemma 2.4.6 the discrepancy is 1. So in this situation X would be klt along Λ.

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